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MODEL OF BEAM HEAD EROSION

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MODEL OF BEAM HEAD EROSION

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ABSTRACT

An analytical model of beam head dynamics is presented, leading to an estimate of the erosion rate due to the combined effects of Ohmic dissipation and scattering. Agreement with the results of a computer simulation and detailed one-dimensional computations is good in all respects except for the scaling of the erosion rate with net current.

I. BACKGROUND

The phenomenon of beam head erosion was established several years ago by analytical estimates and computer calculations of several groups.¹⁻⁴ Erosion was also observed to occur in propagation experiments⁵ at LLNL using a pulsed diode generated beam. Although the detailed structure of a pulse head was found to be quite complex, it was concluded that the net erosion rate was accurately given by the simple formula

$$\alpha \approx \frac{\Delta x}{\Delta z} = \frac{U_e I}{\gamma m c^3} \quad (1)$$

where Δx is the length of the pulse lost during propagation over distance Δz . The other quantities in Eq. (1) are:

$$\mathcal{L} \approx 3-7 = \text{dimensionless inductance} \quad , \quad (2a)$$

$$I = \text{net current in pulse body} \quad , \quad (2b)$$

$$\gamma \gg 1 = \text{relativistic factor in body} \quad . \quad (2c)$$

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Equation (1) may be written in the convenient form

$$\alpha = (0.15) \left(\frac{\rho}{5} \right) \left(\frac{I}{1 \text{ kA}} \right) \left(\frac{1 \text{ MeV}}{\epsilon} \right) \quad (1)$$

where $\epsilon = \gamma mc^2$.

The physical reasoning leading to Eq. (1) starts with the observation that Ohmic energy loss is concentrated in the neck region of the pulse head. There is a deep spike in the computed longitudinal electric field associated with the process of charge neutralization and buildup of pinch potential. Particles are effectively removed from the head as they migrate through this field spike because the energy loss they suffer causes a large increase in emittance. It is assumed that the particle energy falls to near-zero; the erosion rate is then calculated by equating the particle loss rate times initial energy to the Ohmic loss rate.

The simulation code RINGBEARER⁶ was developed and used to study erosion in greater detail than had been done before.⁷ Approximate agreement with Eq. (1) was obtained for pulse parameters similar to those of the previous studies (e.g., $I_b = 10 \text{ kA}$, $\epsilon_0 = 50 \text{ MeV}$, $\rho_0 = \text{normal density}$, $R_0 = \text{rms radius} = 0.5 \text{ cm}$). However, the scaling of α as parameters were varied away from this point did not fit Eq. (1). In particular, erosion was found to increase with injected radius and gas density. It was also observed that beam particles were lost with finite energy (typically $\sim 1/2 \epsilon_0$), in contradiction with the assumptions leading to Eq. (1). This combination of features in the simulation results lead F. Chambers to suggest⁷ that scattering was playing a significant role in erosion and that the close agreement of previous code work with Eq. (1) was due to the fortuitous selection of parameters in a narrow range. It is also likely that the 1-d models overestimate Ohmic loss by $\sim 30\%$.

The earliest models of erosion incorporated scattering but neglected energy loss, but this work was forgotten due to the success of Eq. (1). Stimulated by the unexplained simulation results, R. Briggs produced a (long forgotten) analytic estimate of erosion based on scattering and incomplete charge neutralization;³ he gives

$$\alpha \approx (\text{coeff. of order } 0.75) \left(\frac{R}{1 \text{ cm}} \right)^{1/2} \left(\frac{\rho}{\rho_{\text{normal}}} \right)^{1/2} \left(\frac{10 \text{ kA}}{I_b} \right)^{3/4} \left(\frac{1}{\gamma} \right)^{3/4} \quad (4)$$

This rate is in fair agreement with the high density simulation results and has the observed scaling with radius and density. In the general case --

energy loss and scattering included -- F. Chambers⁹ finds that the sum of Eqs. (1) and (4) yields good agreement with simulation.

The present report presents an analytical derivation of α in the general case using a simplified 1-d model. Associated beam profile quantities are also determined, for example the energy at which particles are lost, and the "coefficient of order 0.75" is evaluated. Good agreement is obtained with results from RINGBEARER and also a 1-d beam dynamics code (SNEEZE) in magnitude and scaling of α with all variables except net current I [SNEEZE yields $\alpha \sim I^{-1/2} I_b^{-1/4}$, but from the present calculation $\alpha \sim I^{1/4} I_b^{-1}$, replacing $I_b^{-3/4}$ in Eq. (4)]. The reason for this discrepancy is not yet known.

An interesting feature of this work is that the new erosion rate may be written in a form very similar to Eq. (1):

$$\alpha = \frac{Ze I}{\gamma mc^3} \frac{1}{1 - \gamma_c/\gamma} \quad (5)$$

where γ_{mc}^2 is the energy at which particles are actually lost at a "critical point," the other quantities are evaluated in the pulse body. An expression for γ_c is obtained which depends on the relative importance of scattering and Ohmic loss. It is found that γ_c vanishes as scattering becomes negligible. This point was ignored in the derivation of Eq. (1), which simply assumed $\gamma_c \ll \gamma$.

The physical basis of this work is essentially the same as that of References 1, 8, and 9, so it is discussed only briefly. The primary purpose here is to obtain the sum of rates (1) and (4) from a reasonably firm analytical base.

II. MODEL EQUATIONS

In this section the 1-d model equations are listed; they have been discussed and derived in previous reports.^{10,11,12} These equations are not solved here in their general form; this requires a 1-d propagation code. Instead, a further set of approximations is imposed (Secs. III and IV).

We consider a high energy beam ($\gamma \gg 1$), so the most useful independent variables are z and $x = ct - z$. The six coupled equations for pulse evolution follow.

(a) Energy Loss:

$$\frac{\partial}{\partial z} \gamma m c^2 = q E_z - \frac{\gamma m c^2}{\lambda_R} - \epsilon_d' \quad (6)$$

Here λ_R is the radiation length and ϵ_d' is the direct (ionization) loss rate in gas, which are taken to be constant.

(b) Longitudinal Electric Field:

$$E_z = - \frac{\partial}{\partial x} \left(\frac{\mathcal{L}I}{c} \right) \quad (7)$$

where I is the "effective current," which contains charge imbalance as well as true beam and plasma currents.

(c) Circuit Equations:

$$I + \frac{c}{4\pi\sigma} R^2 \frac{\partial}{\partial x} \frac{I}{R^2} = I_b + \pi R^2 \sigma E_z \quad (8)$$

with σ the on-axis conductivity and R is the rms radius.

(d) Scattering (Nordsieck Equation):

$$\frac{\partial}{\partial z} \log (\gamma IR^2) = \frac{4C_N q}{\lambda \gamma m c^2 I} \quad (9)$$

The Nordsieck constant is $C_N = 3.7 \times 10^{12}$ watts. This equation holds only in the pinched portion of the pulse. For the unpinch portion of the head an envelope equation will be used.

(e) Pinch Condition:

$$\langle \theta^2 \rangle \equiv \frac{E^2}{R^2} = \frac{e I}{\gamma m c^3} \quad (10)$$

where $\langle \theta^2 \rangle$ is the mean squared random angle of beam particles and E is the emittance. This equation also holds only in the pinched portion of the pulse.

(f) Conductivity Generation:

$$\frac{\partial \sigma}{\partial x} = \frac{K I_b}{\pi R^2} + (\text{avalanche term}) . \quad (11)$$

The constant K is determined from the direct loss rate (ϵ_d^1) and the mean energy required to create an ion pair ($w \approx 34$ eV/pair). The avalanche or electric breakdown effect is assumed to be present but it does not need to be specified in detail for this calculation.

The quantities γ , I , R , σ , and E_z are the principal variables of the theory and depend on both x and z . The quantities $\mathcal{U}$ and $\langle \theta^2 \rangle$ are also variables, but they may be regarded as known functions of the others in the pinched zone. We will need their values only at two points.

When the Eqs. (6) - (11) are solved (with further approximations), the erosion rate $\alpha = \Delta x / \Delta z$ appears as an "eigenvalue" determined in part by the smooth joining of beam variables between the pinched and freely expanding zones of the head. A condition to specify the "critical point" x_c at which these zones meet is also imposed. This is the point at which the quasi-adiabatic approximation underlying the Nordsiéck equation fails. An analytical representation is

$$\langle \partial R / \partial z \rangle^2 = \langle \theta^2 \rangle \quad (12)$$

at x_c . That is, there is as much transverse energy in directed flow as in random motions at this point. In the pinched zone $\langle \partial R / \partial z \rangle$ is small, while in the free expansion zone $\langle \theta^2 \rangle$ becomes small with increasing z or decreasing x .

III. PULSE PROFILE MODEL

The beam is divided into the three zones shown in Fig. 1. In the well pinched zone (pulse body), all variables are considered to be known and to evolve slowly in z according to the standard formulas for scattering and energy loss. We denote these quantities (R^* , I^* , γ^* , etc.). The point x^* is ambiguous in definition, but should be placed well behind the E_z spike which appears in the pulse neck.

In the weakly pinched zone effective current builds up from a near-zero value (I_c) to the near-final value I^* . Hence γ is reduced sharply by

Fig. 1. Beam Head Profile

Ohmic loss, and the other energy loss mechanisms may be neglected here. Scattering drives the increase in normalized emittance and is especially concentrated in effect near the critical point due to the smallness of I_c [see Eqs. (9) and (10)]. These two processes are therefore considered to be somewhat separated in their x -location, scattering is near x_c and Ohmic loss is midway between x_c and x^* .

In the free expansion zone no effective pinch force is present -- the beam radius expands rapidly due to the large transverse energy preserved by the particles as they passed through the critical point. Therefore energy loss, scattering, and the pinch effect are all neglected in computing from the free zone envelope. Continued scattering and energy loss do occur there, but the beam radius will be very large before these processes dominate over initial transverse energy.

The objective is to derive a formula for the erosion rate (α) which is a function only of the slowly changing quantities R^* , I^* , γ^* , and the atomic physics parameters λ_R and K . To facilitate the treatment we define a new longitudinal variable

$$w = x - \alpha z, \quad (13)$$

which incorporates the assumed erosion phenomena. Beam quantities in the weakly pinched and free expansion zones are rapidly varying functions of w and slowly varying functions of z (holding w fixed). We thus neglect their z variations and the transformations of derivatives are

$$(\partial/\partial x)_z \rightarrow (d/dw), \quad (14a)$$

$$(\partial/\partial z)_x \rightarrow -\alpha(d/dw). \quad (14b)$$

It is only through the transformation Eq. (14b) that α explicitly enters the theory.

Applying the above described approximations and transformations, Eqs. (6) and (7) give for the weakly pinched zone

$$-\alpha \frac{d(\gamma mc^2)}{dw} = q E_z = -q \frac{d(\mathcal{E}/c)}{dw}. \quad (15)$$

This equation may be integrated and yields, neglecting I_c compared with I^* ,

$$\alpha = \frac{q \frac{I^*}{\gamma^* mc^3}}{1 - \gamma_c/\gamma^*} \quad (16)$$

All we need to compute α is the value of γ_c , but there appears to be no easy way to find it. The problem involves energy loss, scattering, the pinch effect, conductivity generation and violation of adiabaticity in a coupled, self-regulating system. Eight simultaneous equations will be solved for the quantities α , γ_c , R_c , I_c , σ_c , $(dR/dw)_c$, $(dI/dw)_c$, and $(d\sigma/dw)_c$.

IV. APPROXIMATION OF MODEL EQUATIONS

Drastic simplifications are now made in the circuit equation [Eq. (8)] and Nordsieck equation [Eq. (9)]. First we observe that near x_c the variables σ , I , and E_z are small. Therefore we drop the first and last terms of Eq. (8) to obtain

$$\frac{d}{dw} \left(\frac{I}{R^2} \right) = \left(\frac{4\pi \sigma I_h}{c R^2} \right) \quad (17)$$

Only the process of charge neutralization is taken into account when this equation is used to compute the buildup of I . The approximations should be valid up to some point midway between x_c and x^* , and this is sufficient for our purpose. Note that Eq. (11) may be written in the form

$$dw = \left(\frac{\pi R^2}{K I_b} \right) d\sigma \quad (18)$$

which may be used to eliminate w from Eq. (17); we integrate in to get

$$I = \frac{2\pi^2 \sigma^2 R^2}{c K} \quad (19)$$

Eq. (9) may be formally integrated:

$$\ln \left(\frac{\gamma I R^2}{\gamma_c I_c R_c^2} \right) = - \int_0^w \frac{dw}{\alpha} \left(\frac{4 C_N q}{\lambda \gamma mc^2 I} \right) \quad (20)$$

where $w_c = 0$ by convention. It is assumed that most of the contribution to this integral comes from the zone close to the critical point because I is small there. Hence we substitute from Eq. (19) for I and set $\gamma = \gamma_c$ in the integrand. Using Eq. (18) to change the integration variable from w to σ , we have

$$\ln \left(\frac{\gamma_c I_c R_c^2}{\gamma_c I_c R_c^2} \right) = - \int_0^\sigma \left(\frac{d\sigma}{\alpha} \frac{\pi R_c^2}{K I_b} \right) \left(\frac{4 C_N q}{\lambda_R \gamma_c m c^2} \right) \times \left(\frac{c K}{2\pi^2 \sigma^2 R_c^2} \right) \quad (21)$$

$$= - \left(\frac{2 C_N q}{\alpha I_b \lambda_R \gamma_c m c \pi} \right) \left(\frac{1}{\sigma_c} - \frac{1}{\sigma} \right) \quad (22)$$

Neglecting σ^{-1} compared with σ_c^{-1} , Eq. (22) yields a formula connecting R_c and R^* :

$$\ln \left(\frac{\gamma_c I_c R_c^2}{\gamma^* I^* R^{*2}} \right) = \left(\frac{2 C_N q}{\alpha I_b \lambda_R \gamma_c m c \pi \sigma_c} \right). \quad (23)$$

R_c is increased compared with R^* by the three effects represented in Eq. (23): scattering near the critical point, and adiabatic adjustment of R to changing I and γ . Note that the details of the changes in I and γ did not need to be specified except near the critical point, because adiabaticity was assumed. There may be strong avalanching but the final result has the same dependence on I^* , R^* , and γ^* .

V. SUMMARY OF EQUATIONS

We now write down the eight equations satisfied by the beam variables at the critical point.

From Eqs. (10) and (12):

$$\left(\frac{q I_c}{\gamma_c m c^3} \right) = \alpha^2 \left(\frac{dR}{dw} \right)_c^2 = \frac{\alpha^2 R_c^2}{4} \left(\frac{d \ln R^2}{dw} \right)_c^2, \quad (24a)$$

with mean squared transverse velocity

$$v_c^2 = 2 \langle \theta^2 \rangle_c = \frac{\alpha^2 R_c^2}{2} \left(\frac{d \ln R^2}{dw} \right)_c. \quad (24b)$$

From Eq. (16):

$$\alpha = \frac{q \mathcal{L}^* I^*}{\gamma^* mc^3} \frac{1}{1 - \gamma_c / \gamma^*}. \quad (25)$$

From Eq. (18):

$$\left(\frac{d \ln}{dw} \right)_c = \left(\frac{K I_b}{\pi R_c^2 \sigma_c} \right). \quad (26)$$

From Eq. (19):

$$I_c = \left(\frac{2\pi^2 \sigma_c^2 R_c^2}{c K} \right), \quad (27)$$

$$\left(\frac{d \ln I}{dw} \right)_c = 2 \left(\frac{d \ln \sigma}{dw} \right)_c + \left(\frac{d \ln R^2}{dw} \right)_c. \quad (28)$$

From Eq. (20):

$$\left(\frac{d \ln I}{dw} \right)_c + \left(\frac{d \ln R^2}{dw} \right)_c = - \left(\frac{4 C_N q}{\alpha \lambda_R \gamma_c mc^2 I_c} \right) \quad (29)$$

From Eq. (23):

$$\ln \left(\frac{\gamma_c I_c R_c^2}{\gamma^* I^* R^{*2}} \right) = \left(\frac{2 C_N q}{\alpha I_b \lambda_R \gamma_c mc \pi \sigma_c} \right) \quad (30)$$

Finally, we make the definition

$$n \equiv - \frac{\left(\frac{d \ln I/dw}{d \ln R^2/dw} \right)_c}. \quad (31)$$

It is estimated (in Sec. VII) that $n \approx 0.27$. In order to calculate n use is made of the matching conditions between the free expansion and weakly pinched zones. However, it is already clear that n must be positive because I

increases and R decreases as a function of w . From Eq. (29) it is seen that $n < 1$ is also required. An examination of EMPULSE generated beam head profiles¹³ indicates $n \approx 0.5$, which is good agreement with the analytical estimate in view of the crudeness of the model assumptions.

Dimensionless variables and parameters are now introduced. The three distinct physical processes driving erosion (Ohmic loss, conductivity generation, and Nordsiéck expansion) are each characterized by a strength parameter. We define

$$T \equiv \frac{q \mathcal{L}^* I^*}{\gamma^* mc^3} = \alpha_{\text{Ohmic}} \quad , \quad (32)$$

$$U \equiv \frac{K I_b}{c} = \left(\frac{\partial \lambda_m}{\partial x} \right) \quad , \quad (33)$$

$$V \equiv \left(\frac{2 C_N q}{\lambda_R \gamma^* mc^2 I^*} \right) R^* \left(\frac{\gamma^* mc^3}{q I^*} \right)^{1/2} = \left(\frac{1}{k_\beta L_N} \right)^* \quad , \quad (34)$$

where λ_m is the magnetic decay length, k_β is the betatron frequency, and L_N is the Nordsiéck length. The parameters T and V are small compared with unity, while U is of order unity for most cases of interest. Equations (24) through (31) become

$$\left(\frac{\gamma^*}{\gamma_c} \right) \left(\frac{I_c}{I^*} \right) \frac{T}{\mathcal{L}^*} = \alpha^2 \left(\frac{R_c}{R^*} \right)^2 \left[\frac{R^*}{2} \left(\frac{d \ln R^2}{dw} \right)_c \right]^2 \quad , \quad (35a)$$

$$v_c^2 = 2 \alpha^2 \frac{R_c}{R^*}^2 \left[\frac{R^*}{2} \left(\frac{d \ln R^2}{dw} \right)_c \right]^2 \quad , \quad (35b)$$

$$\alpha = \frac{T}{1 - \gamma_c / \gamma^*} \quad , \quad (36)$$

$$R^* \left(\frac{d \ln \sigma}{dw} \right)_c = \left(\frac{R^*}{R_c} \right)^2 \left(\frac{c}{\pi R^* \sigma_c} \right) U \quad , \quad (37)$$

$$\left(\frac{I_c}{I^*}\right) = 2 \left(\frac{\pi \sigma_c R^*}{c}\right)^2 \left(\frac{R_c}{R^*}\right)^2 \left(\frac{I_b}{I^*}\right) \frac{1}{U}, \quad (38)$$

$$\frac{R^*}{2} \left(\frac{d \ln I}{dw}\right)_c = R^* \left(\frac{d \ln \sigma}{dw}\right)_c + \frac{R^*}{2} \left(\frac{d \ln R^2}{dw}\right)_c, \quad (39)$$

$$\frac{R^*}{2} \left(\frac{d \ln I}{dw}\right)_c + \frac{R^*}{2} \left(\frac{d \ln R^2}{dw}\right)_c = - \left(\frac{\gamma^*}{\gamma_c}\right) \left(\frac{I^*}{I_c}\right) \frac{V}{\alpha} \sqrt{\frac{T}{\mathcal{L}^*}}, \quad (40)$$

$$\ln \left[\left(\frac{\gamma_c}{\gamma^*}\right) \left(\frac{I_c}{I^*}\right) \left(\frac{R_c}{R^*}\right)^2 \right] = \frac{\gamma^*}{\gamma_c} \left(\frac{c}{\pi R_c \sigma_c}\right) \left(\frac{I^*}{I_b}\right) \sqrt{\frac{T}{\mathcal{L}^*}} \frac{V}{\alpha}, \quad (41)$$

$$\frac{R^*}{2} \left(\frac{d \ln I}{dw}\right)_c = - n \frac{R^*}{2} \left(\frac{d \ln R^2}{dw}\right)_c. \quad (42)$$

VI. SOLUTION OF MOIEL EQUATIONS

We first eliminate terms involving derivatives in w from Eqs. (35) through (42). From Eqs. (37), (39), and (42) we get

$$- (1 + n) \frac{R^*}{2} \left(\frac{d \ln R^2}{dw}\right)_c = \left(\frac{R^*}{R_c}\right)^2 \left(\frac{c}{\pi R^* \sigma_c}\right) U, \quad (43)$$

and from Eqs. (40) and (42)

$$- (1 - n) \frac{R^*}{2} \left(\frac{d \ln R^2}{dw}\right)_c = \left(\frac{\gamma^*}{\gamma_c}\right) \left(\frac{I^*}{I_c}\right) \frac{V}{\alpha} \sqrt{\frac{T}{\mathcal{L}^*}}. \quad (44)$$

Dividing and multiplying these equations gives, respectively

$$\delta \equiv \frac{1 - n}{1 + n} = \left(\frac{\gamma^*}{\gamma_c}\right) \left(\frac{I^*}{I_c}\right) \frac{V}{\alpha} \sqrt{\frac{T}{\mathcal{L}^*}} \left(\frac{R_c}{R^*}\right)^2 \left(\frac{\pi R^* \sigma_c}{c}\right) \frac{1}{U}, \quad (45)$$

$$(1 - n^2) \left[\frac{R^*}{2} \left(\frac{d \ln R^2}{dw}\right)_c \right]^2 = \left(\frac{R^*}{R_c}\right)^2 \left(\frac{c}{\pi R^* \sigma_c}\right) U \left(\frac{\gamma^*}{\gamma_c}\right) \left(\frac{I^*}{I_c}\right) \frac{V}{\alpha} \sqrt{\frac{T}{\mathcal{L}^*}}. \quad (46)$$

Combining Eqs. (35a) and (46) yields

$$(1 - n^2) = \left(\frac{I^*}{I_c}\right)^2 \sqrt{\frac{\mathcal{Q}^*}{T}} \alpha v \left(\frac{c}{\pi R^* \sigma_c}\right) U . \quad (47)$$

Equations (35), (38), (41), (45), and (47) are to be solved for α , R_c , γ_c , I_c and σ_c . To do this it is convenient to define a quantity X , which is a function of the parameters, T , U , and V , such that

$$\left(\frac{\gamma_c}{\gamma^*}\right) = \frac{1}{1 + X} . \quad (48)$$

The value of X will be determined in the solution of the five coupled equations. We have immediately from Eqs. (36) and (48)

$$\alpha = T \left(1 + \frac{1}{X}\right) . \quad (49)$$

Next, the combination I_c/R_c^2 may be eliminated between Eqs. (38) and (45) to obtain

$$2\delta = \left(\frac{\gamma^*}{\gamma_c}\right) \left(\frac{c}{\pi \sigma_c R^*}\right) \left(\frac{I^*}{I_b}\right) \sqrt{\frac{T}{\mathcal{Q}^*}} \frac{V}{\alpha} \quad (50)$$

We have from Eqs. (48) through (50)

$$\left(\frac{\pi \sigma_c R^*}{c}\right) = \frac{V}{2\delta} \left(\frac{I^*}{I_b}\right) \frac{X}{\sqrt{\mathcal{Q}^* T}} , \quad (51)$$

and from Eqs. (41) and (50)

$$\ell_n \left(\frac{\gamma_c}{\gamma^*}\right) \left(\frac{I_c}{I^*}\right) \left(\frac{R_c}{R^*}\right)^2 = 2\delta . \quad (52)$$

From Eqs. (47), (49), and (51) we find

$$\left(\frac{I_c}{I^*}\right)^2 = \frac{2\delta}{1 - n^2} \frac{\mathcal{Q}^* T}{X^2} (1 + X) U \left(\frac{I_b}{I^*}\right) , \quad (53)$$

and from Eqs. (48), (52), and (53)

$$\left(\frac{R_c}{R^*}\right)^2 = e^{2\delta} \left[\frac{1-n^2}{2\delta} \frac{X^2}{\mathcal{L}^* T} \left(\frac{1+X}{U} \right) \frac{I^*}{I_b} \right]^{1/2}. \quad (54)$$

Finally, eliminating I_c , σ_c and R_c from Eq. (38) we obtain

$$X^2 = \mathcal{L}^* T \frac{U}{V} \left(\frac{I_b}{I^*} \right) \left(2\delta e^{-\delta} \sqrt{\frac{\delta}{1-n^2}} \right). \quad (55)$$

The formal solution of the erosion problem is contained in Eqs. (48), (49), (51), (53), (54) and (55). It only remains to evaluate n . At this point some of the approximations assumed in the beam model may be checked. For example, from Eqs. (53) and (55)

$$\left(\frac{I_c}{I^*} \right)^2 \propto \frac{1}{X^2} (1+X) \propto V \left(1 + \sqrt{\frac{1}{V}} \right) \ll 1.$$

VII. THE FREE EXPANSION ZONE

Energy loss, scattering and pinch effect are all neglected for the free expansion zone ($w \leq 0$). The Nordsict equation is replaced by the simple envelope equation¹²

$$\frac{\alpha^2}{2} \frac{d^2 R^2}{dw^2} = v^2, \quad (56)$$

with constant mean transverse squared velocity

$$v^2 = v_c^2 = \frac{\alpha^2}{2R_c^2} \left(\frac{dR^2}{dw} \right)_c^2 \quad (57)$$

determined from the critical point condition [Eq. (35b)]. Both R and dR/dw connect smoothly across the critical point. These jump conditions, along with Eqs. (56) and (57), yield the free expansion envelope

$$R^2 = R_c^2 \left[1 + \left(\frac{d \ln R^2}{dw} \right)_c w + \frac{1}{2} \left(\frac{d \ln R^2}{dw} \right)_c^2 w^2 \right]. \quad (58)$$

The conductivity at the critical point is given by

$$\sigma_c = \int_{-\infty}^0 dw \frac{K I_b}{\pi R_c^2} = \int_{-\infty}^0 dw \frac{2K I_b}{\pi R_c^2} \left\{ 1 + \left[1 + \left(\frac{d \ln R^2}{dw} \right)_c w \right]^2 \right\}^{-1} \quad (58)$$

Substituting

$$z = 1 + \left(\frac{d \ln R^2}{dw} \right)_c w, \quad (60)$$

Eq. (59) becomes

$$\sigma_c = \frac{2K I_b}{\pi R_c^2} \frac{-1}{(d \ln R^2 / dw)_c} \int_1^{\infty} \frac{dz}{1+z^2}. \quad (61)$$

The value of the inetegral in Eq. (61) is $\pi/4$.

Equation (61) is new relation among the critical point variables; this allows an estimate of the constant (n) introduced in Sec. V. From Eqs. (26), (28), and (31) one obtains

$$\frac{\pi R_c^2 \sigma_c}{2K I_b} \left(\frac{d \ln R^2}{dw} \right)_c = \frac{1}{2} \left(\frac{d \ln R^2}{d \sigma} \right)_c = - \frac{1}{n+1}. \quad (62)$$

Hence Eq. (61) yields

$$\frac{1}{n+1} = \frac{\pi}{4}, \quad n = \frac{4}{\pi} - 1 = 0.2732. \quad (63)$$

The sensitivity of n to the details of the model is apparent when we note that in Eq. (63) $\pi \rightarrow 4$ yields $n = 0$, and $\pi \rightarrow 8/3$ yields $n = 1/2$. The erosion rate is not particularly sensitive to the value of n , and anything in the range $0 \lesssim n \lesssim 1/2$ is consistent with code results.

VIII. CONCLUSIONS AND SUMMARY

An erosion rate (α) has been derived assuming Ohmic loss and scattering are the driving mechanisms. The specific formula [Eq. (49) supplemented by Eqs. (32), (33), (34), and (55)] is in fact the sum of the previously known Ohmic and scattering driven rates [Eqs. (1) and (4)]. A discussion of the physical mechanisms is given in Refs. 1, 8, and 9. Here we conclude with a numerical evaluation of α and a comparison with the scaling inferred from the 1-d code SNEEZE.

We assume the background gas is air and set

$$K = 3.2 \times 10^{-4} , \quad (64a)$$

$$\lambda_R = (3.0 \times 10^4 \text{ cm})/\hat{\rho} \quad (64b)$$

where $\hat{\rho}$ is density in units of normal atmospheric, and the coefficients are characteristic of a 50 MeV particle (they vary very slowly with energy). Then we have parameters

$$U = (0.032) \left(\frac{I_b}{1 \text{ kA}} \right), \quad (64c)$$

$$\frac{T}{\gamma^*} = (0.059) \left(\frac{I^*}{1 \text{ kA}} \right) \frac{1}{\gamma^*}, \quad (64d)$$

$$V = (2.0) \left(\frac{R^*}{1 \text{ cm}} \right) \frac{\hat{\rho}}{\gamma^{*1/2}} \left(\frac{1 \text{ kA}}{I^*} \right)^{3/2}, \quad (64e)$$

$$X = C^{-1} \left(\frac{\mathcal{G}^* T U}{V} \frac{I_b}{I^*} \right)^{1/2}. \quad (64f)$$

Here C is the function of n

$$C = \left[2\delta e^{-\delta} \sqrt{\frac{\delta}{1-n^2}} \right]^{-1/2} \quad (65a)$$

$$\delta = \frac{1-n}{1+n}, \quad (65b)$$

See Table 1 for values of C. The calculated erosion rate is

$$\alpha = T + \frac{T}{X} \quad (66a)$$

$$= \alpha_{\text{Ohmic}} + \gamma_{\text{scattering}} \quad (66b)$$

$$= (0.059) \left(\frac{I^*}{1 \text{ kA}} \right) \frac{\mathcal{G}^*}{\gamma^*} + (1.9 \text{ C}) \left(\frac{R^*}{1 \text{ cm}} \hat{\rho} \right)^{1/2} \left(\frac{I^*}{1 \text{ kA}} \right)^{1/4} \left(\frac{1 \text{ kA}}{I_b} \right) \left(\frac{1}{\gamma^{*3/4}} \right). \quad (66c)$$

TABLE I

n	C
0.0000	1.1658
$4/\pi - 1 = 0.2732$	1.4049
0.5000	1.772
0.7500	2.658
1.0000	∞

The SNEEZE generated result is

$$\alpha_{\text{scattering}} = (2.22) \left(\frac{R^*}{I \text{ cm}} \hat{\rho} \right)^{1/2} \frac{1}{\gamma^{3/4}} \left(\frac{1 \text{ kA}}{I_b} \right)^{1/4} \left(\frac{1 \text{ kA}}{I^*} \right)^{1/2}.$$

Except for the scaling with I^* this is in good agreement with the analytical result for n in the range $0 - 0.5$. As mentioned, EMPULSE generated profiles indicate $n = 0.5$ near the point where $\langle \theta^2 \rangle \approx (\partial R / \partial z)^2$. A value $n = 0$ would show up as a point of inflection for I vs x ; this is never seen.

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