Hyperbolic-Secant Mode Coupling

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Abstract

This work presents a new solution of the coupled mode equations for a hyperbolic secant spatial variation of the coupling between two modes. An analytic expression is given for the transmission coefficient for arbitrary complex differential propagation constant and coupling strength. The expression is particularly simple in the case when the differential attenuation between the modes is negligible.

Design curves are presented in terms of normalized parameters. The hyperbolic secant coupling may be truncated and still yield virtually the same transmission as for infinite coupling length. The required coupling length is indicated by a comparison of the ideal expression with the results of numerical integration of the coupled mode equations.

Hyperbolic secant coupling can be particularly useful for the design of low-loss bends and twists in overmoded waveguide.

DISCLAIMER

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I. INTRODUCTION

There have been no known analytic solutions of the coupled mode equations for continuously and deterministically varying coupling between modes with different phase velocities [1]. The solution for constant coupling [2] is therefore often used as an approximation, and a given coupling function may be approximated by a succession of different constant coupling sections [3,4]. Constant coupling over infinitesimal lengths also provides the basis for treating random coupling [5].

The present work presents a new analytic expression for the transmission coefficient for the amplitude of a mode initially having all the power and whose coupling with another mode has a hyperbolic secant variation. This expression is valid for arbitrary differential phase and attenuation constants, and has some interesting relationships to the solutions for constant coupling and for equal phase constants, as well as to the expression derived from perturbation theory.

Aside from its theoretical interest as a new analytic solution, the solution for hyperbolic secant coupling variation can be used to design sharp low-loss bends and twists in overmoded waveguide. We examine these cases in detail in other work.

Those whose main interest is in the properties of the solution and in general design information may skip over the following section which derives the solution.

II. DERIVATION OF THE SOLUTION

The coupled mode equations for continuous coupling between two modes are [6]:

$$G_0'(z) = Kc(z)e^{\Delta\Gamma z} G_1(z) \quad (1a)$$

$$G_1'(z) = -K^*c(z)e^{-\Delta\Gamma z} G_0(z), \quad (1b)$$

where G_0 and G_1 represent the (complex) amplitudes of the two modes, and the difference in propagation constants is

$$\Delta\Gamma = \Delta\alpha + j\Delta\beta = (\alpha_0 - \alpha_1) + j(\beta_0 - \beta_1) \quad (2)$$

The propagation factors for ideal waveguide have been removed, so that Eqs. (1a) and (1b) relate only to mode conversion effects. The prime denotes differentiation with respect to z .

The coupled mode Eqs. (1a) and (1b) are valid whenever the coupling is small enough that the mode amplitudes change little in a wavelength $2\pi/\beta$ of either mode ($|Kc/\beta| \ll 1$). In this case, reflected waves can be neglected. Approximate solutions to Eqs. (1a) and (1b) derived using perturbation theory require $|Kc/\Delta\beta| \ll 1$, a much more stringent assumption that is often not valid in practical cases.

The coupling coefficient is a product of a constant K , derived from electromagnetic theory, with the coupling function $c(z)$ determined by the coupling geometry. In the case where $c(z)$ is the waveguide curvature along the waveguide axis z , K is imaginary. When $c(z)$ represents continuous offset or diameter variation, K is real. The form of (1) insures that the total power in the two modes $|G_0|^2 + |G_1|^2$ is conserved when the differential attenuation $\Delta\alpha$ is zero.

Differentiating (1a), substituting (1b) for G_1' , and using (1a) to write G_1 in terms of G_0 yields, as long as $c(z)$ is nonzero,

$$G_0'' - \left[\frac{c'(z)}{c(z)} + \Delta\Gamma \right] G_0' + |K|^2 c^2(z) G_0 = 0 \quad (3)$$

A similar equation may be obtained for G_1 by changing the sign of $\Delta\Gamma$.

If we set the arc length variable z to zero at the center of the coupling region and take $c(z)$ to have the form

$$c(z) = R_0^{-1} \operatorname{sech}(kz/2) \quad , \quad (4)$$

then Eq. (3) becomes

$$G_0'' + \left(\frac{k}{2} \tanh \frac{kz}{2} - \Delta\Gamma \right) G_0' + (|K|/R_0)^2 \operatorname{sech}^2 \left(\frac{kz}{2} \right) G_0 = 0. \quad (5)$$

A plot of the hyperbolic secant function $\operatorname{sech}(kz/2)$ is presented in Fig. 1.

To solve (4) we transform it to a differential equation whose solutions are hypergeometric functions. An equation with certain similarities to (5) and describing the propagation and reflection of obliquely incident electromagnetic waves in a plane stratified isotropic plasma has also been solved by this type of transformation [7].

The required transformation is easier to follow if we start from the hypergeometric differential equation and work back to (5). The general second order hypergeometric equation may be written as follows [8]:

$$(\Omega - \rho_1)(\Omega - \rho_2) F(\xi) = \xi(\Omega + \sigma_1)(\Omega + \sigma_2) F(\xi), \quad (6)$$

where Ω is the operator

$$\Omega \equiv \xi \frac{d}{d\xi}. \quad (7)$$

If we make the substitutions

$$\xi = -\exp(-kz) \quad (8)$$

and

$$F[\xi(z)] = f(z) G_0(z), \quad (9)$$

then (6) becomes

$$G_0'' + \left[2 \frac{f'}{f} + k \frac{(r_1 - \xi s_1)}{(1 - \xi)} \right] G_0' + \left[\frac{f''}{f} + k \frac{f'}{f} \frac{(r_1 - \xi s_1)}{(1 - \xi)} + \frac{k^2 (r_2 - \xi s_2)}{(1 - \xi)} \right] G_0 = 0, \quad (10)$$

where the prime denotes differentiation with respect to z and

$$r_1 = -\rho_1 - \rho_2, \quad r_2 = \rho_1 \rho_2, \quad (11)$$

$$s_1 = \sigma_1 + \sigma_2, \quad s_2 = \sigma_1 \sigma_2. \quad (12)$$

At this point, we choose $f(z)$ to have the special form:

$$f[\xi(z)] = (-\xi)^{-r_0/2} (1 - \xi)^{(r_0 - s_0)/2}. \quad (13)$$

Recognizing from (8) that

$$\xi/(1 - \xi) = -\frac{1}{2} \left(1 + \tanh \frac{kz}{2} \right) \quad (14)$$

and

$$\xi/(1-\xi)^2 = -\frac{1}{4} \operatorname{sech}^2 \frac{kz}{2}, \quad (15)$$

we then find that the first derivative terms in (10) and (5) can be made equal provided that

$$r_0 - r_1 = \frac{1}{2} + (\Delta\Gamma/k) \quad (16a)$$

$$s_1 - s_0 = \frac{1}{2} - (\Delta\Gamma/k) . \quad (16b)$$

With the help of a partial fraction expansion involving terms in $(1-\xi)^{-m}$, $m=0,1,2$ and again making use of (14) in (15), we find that the terms multiplying G_0 in (10) and (5) can be made equal provided that

$$\frac{1}{4} r_0^2 - \frac{1}{2} r_0 r_1 + r_2 = 0, \quad (16c)$$

$$\frac{1}{4} (r_0^2 - s_0^2) + \frac{1}{2} (s_0 s_1 - r_0 r_1) + (r_2 - s_2) = 0, \quad (16d)$$

and

$$\frac{1}{4} (s_0 - r_0) [(s_0 - r_0) + 2(1 + r_1 - s_1)] = -4|K|^2/(kR_0)^2. \quad (16e)$$

Through (16), (11), and (12) we can now relate the parameters in (6) to those of our original differential equation (5), thus essentially completing the transformation. Before we can solve explicitly for ρ_1 , ρ_2 , σ_1 , and σ_2 ,

however, we need some more information, which we can obtain by examining the form of the solutions to (6).

The solutions to (6) have the form of normal modes in the limit of large $|z|$. Near $\xi=0$ ($z=-\infty$), one solution to (6) has the form

$$F_1^-(z) = (-\xi)^{+\rho_1} {}_2F_1(\rho_1 + \sigma_1, \rho_1 + \sigma_2; \rho_1 - \rho_2 + 1; \xi), \quad (17)$$

while another independent solution $F_2^-(\xi)$ can be obtained by interchanging ρ_1 and ρ_2 in (17). The hypergeometric functions ${}_2F_1$ are expressible as power series that reduce to the value unity in the limit of small argument. From (8) we then see that the leading term in (17) is $\exp(-k\rho_1 z)$, which represents a normal mode if ρ_1 is imaginary.

Similarly, we may write the solutions for $|\xi|>1$:

$$F_1^+(\xi) = (-\xi)^{-\sigma_1} {}_2F_1(\sigma_1 + \rho_1, \sigma_1 + \rho_2; \sigma_1 - \sigma_2 + 1; \xi^{-1}). \quad (18)$$

Interchange of σ_1 with σ_2 yields $F_2^+(\xi)$. Because of the ξ^{-1} argument, these functions now reduce to unity when ξ approaches minus infinity ($z \rightarrow +\infty$). Again from (8) it is evident that the leading term in (18) has normal mode form when σ_1 is imaginary.

In view of (9), (13), and (8), we thus find the following possible limiting forms for G_0 , where we define $G_{0i}^\pm = f^{-1}(\xi)F_i^\pm$, $i=1,2$:

$$\lim_{z \rightarrow -\infty} G_{01}^-(z) = \lim_{\xi \rightarrow 0} f^{-1}(\xi)F_1^-(\xi) = \exp[k(\frac{r_0}{2} + \rho_1)z], \quad (19a)$$

$$\lim_{z \rightarrow +\infty} G_{01}^+(z) = \lim_{\xi \rightarrow -\infty} f^{-1}(\xi)F_1^+(\xi) = \exp[k(\frac{s}{2} - \sigma_1)z], \quad (19b)$$

$$\lim_{z \rightarrow +\infty} G_{02}^+(z) = \lim_{\xi \rightarrow -\infty} f^{-1}(\xi)F_2^+(\xi) = \exp[k(\frac{s_0}{2} - \sigma_2)z]. \quad (19c)$$

Since the coupling function $c(z)$ from (4) goes to zero for large $|z|$, we must pick the solutions so that G_0^+ vanishes [see (1a)]. Therefore the exponents in (19a) and (19b) must also vanish:

$$\frac{r_0}{2} + \rho_1 = 0 \quad (20)$$

$$\frac{s_0}{2} - \sigma_1 = 0. \quad (21)$$

In the usual mode conversion problems, one mode starts with all the power. We take G_0 to be this mode, and $G_{01}^-(z)$ with unit amplitude then represents the solution at $z = -\infty$. There are no other waves present in this limit, since the coupled mode equations (1) do not allow for reflections. The amplitude of $G_{01}^+(z)$ at $z = +\infty$ then represents the transmission coefficient for G_0 . Because $G_{02}^+(z)$ has a nonvanishing exponent at $z = +\infty$, it cannot be part of the solution for G_0 in that limit. In fact, we will see that the exponent is negative and hence G_{02}^+ goes to zero as z goes to infinity.

To find the transmission coefficient T , we need to find an analytic solution of (6), valid for all ξ , that will reduce to F_1^- for $|\xi| < 1$ and to a linear combination of F_1^+ and F_2^+ for $|\xi| > 1$. Such a solution can be found in terms of Meijer's G-functions, which are expressible as Barnes contour integrals [9]. The desired solution to (6) is

$$F(\xi) = \frac{\Gamma(1 - \rho_1 - \rho_2)}{\Gamma(\rho_1 + \sigma_1)\Gamma(\rho_1 + \sigma_2)} \frac{1}{2\pi j} \int_{L_C} \frac{\Gamma(\rho_1 - w)\Gamma(\sigma_1 + w)\Gamma(\sigma_2 + w)}{\Gamma(1 - \rho_2 + w)} (-\xi)^w dw, \quad (22)$$

where the contour L_C goes from $w = -j\infty$ to $w = +j\infty$ in a manner such that all the poles of $\Gamma(\rho_1 - w)$ lie to the right of the contour, and all the poles of

$\Gamma(\sigma_i + w)$, $i=1,2$ lie to the left of the contour. It is assumed that $1 - \sigma_m - \rho_n$ is not a positive integer for $m,n=1,2$. The integral in (21) converges if $|\arg \xi^{-1}| < \pi$. Notice from (8) that (22) resembles a bilateral inverse Laplace transform for F considered as a function of z .

To evaluate (22) for $|\xi| < 1$, we may close the contour L_C in the right-half w -plane without affecting the value of the solution. The evaluation of this closed contour integral in terms of the residues of gamma functions yields the infinite power series denoted by (17). Similarly, we may close the contour in the left-hand w -plane when $|\xi| > 1$. The coefficient of the power series denoted by (18) is then the transmission coefficient T , which we find to be:

$$T = \frac{\Gamma(1 - \rho_2 + \rho_1) \Gamma(\sigma_2 - \sigma_1)}{\Gamma(1 - \rho_2 - \sigma_1) \Gamma(\sigma_2 + \rho_1)} = G_0(+\infty) \quad (23)$$

To find T explicitly in terms of the parameters of the coupling function (4), we need now only solve (16) with (20) and (21) to obtain the quantities appearing in (23). It will turn out that the ρ_i and σ_i cannot all be specified uniquely since we have an insufficient number of constraints, but T nevertheless is uniquely determined.

First of all, (16c) and (16d) are consistent with, but less restrictive than, (20) and (21). Hence we are left with (16a), (16b), and (16e). Using (20), (21), and the definitions (11) and (12), we find from (16a) and (16b)

$$\sigma_2 - \sigma_1 = \frac{1}{2} - \frac{\Delta\Gamma}{k} = 1 - \rho_2 + \rho_1, \quad (24)$$

while from (16e) we obtain

$$(\sigma_1 + \rho_1) (1 - \rho_2 - \sigma_2) = -4|K|^2 / (kR_0)^2 \quad (25)$$

Combining (24) and (25) yields

$$(\sigma_1 + \rho_1) = \pm 2|K|/(kR_0) , \quad (26a)$$

and

$$(1 - \rho_2 - \sigma_2) = \mp 2|K|/(kR_0) . \quad (26b)$$

Adding the expressions represented by (24), we find

$$(\sigma_1 + \rho_2) - (\rho_1 + \sigma_2) = 2\Delta\Gamma/k , \quad (27)$$

while subtraction of (26b) from (26a) yields (if we choose the upper signs)

$$(\sigma_1 + \rho_2) + (\rho_1 + \sigma_2) = 1 + 4|K|/(kR_0) . \quad (28)$$

Combination of (27) and (28) finally yields

$$1 - \rho_2 - \sigma_1 = \frac{1}{2} - \Delta\Gamma - 2|K|/(kR_0) , \quad (29a)$$

and

$$\sigma_2 + \rho_1 = \frac{1}{2} - \Delta\Gamma + 2|K|/(kR_0) . \quad (29b)$$

If we had chosen the lower signs in (26), the sign of the $2|K|/(kR_0)$ terms in (29) would be reversed, but T remains the same.

The final result for T is, from (23), (24), and (29):

$$T = \frac{\Gamma^2 (1/2 + A - jB)}{\Gamma(1/2 + A - jB + C) \Gamma(1/2 + A - jB - C)}, \quad (30)$$

where we have defined the normalized quantities

$$A \equiv -\Delta\alpha R_0 \theta / 2\pi, \quad (31a)$$

$$B \equiv \Delta\beta R_0 \theta / 2\pi, \quad (31b)$$

$$C \equiv |K| \theta / \pi. \quad (31c)$$

Here we have also defined

$$\Theta \equiv \int_{-\infty}^{+\infty} c(z) dz \approx 2\pi / (kR_0), \quad (32)$$

making use of (4).

In the case where $c(z)$ represents the curvature in a bend, Θ in (32) is simply the total bend angle, and R_0 from (4) is the radius of curvature at the center of the bend $z=0$. In the general case, define

$$1/R(z) \equiv c(z) = (d\theta/dz) = (1/R_0) \operatorname{sech}(\pi z / (R_0 \theta)) \quad (33)$$

using (4) and (32), where

$$1/R_0 = (d\theta/dz)_{z=0}. \quad (34)$$

The case of a twist of total angle θ in rectangular waveguide, for example, can easily be treated with this notation. In that case, $1/R(z)$ is the torsion of the twist.

In order that the residual solution G_{02}^+ vanish as z approaches $+\infty$, as discussed earlier, the real part of the exponent in (19c) must be negative. From (21), (24), (31a), and (2), this requires that

$$0 < \operatorname{Re} \left(\frac{1}{2} + A - jB \right) = \frac{1}{2} + A. \quad (35)$$

Notice that this condition also ensures that T in (3) has no poles. In the usual cases of interest, $\Delta\alpha$ is negative because the desired mode G_0 has lower loss than the spurious mode G_1 . Then (35) is automatically satisfied.

III. DISCUSSION OF SOLUTION

In the case of zero differential attenuation ($\Delta\alpha = 0 = A$), a simple exact expression can be found from (30) for the transmitted power $|T|^2$. Write $|T|^2 = T^*T$, note that $\Gamma^*(x^*) = \Gamma(x)$ for any complex x , and use the identity

$$\Gamma\left(\frac{1}{2} + x\right) \Gamma\left(\frac{1}{2} - x\right) = \pi / \cos \pi x. \quad (36)$$

Expanding the resulting products of cosines using ordinary trigonometric identities, we find

$$1 - |T|^2 = \sin^2(\pi C) / \cosh^2(\pi B), \quad \Delta\alpha = 0. \quad (37)$$

Identifying $|T| = |G_0(+\infty)|$ from (23) and using the fact that

$$|G_0|^2 + |G_1|^2 = 1, \quad \Delta\alpha = 0, \quad (38)$$

we see that

$$|G_1(+\infty)|^2 = 1 - |T|^2 \quad (39)$$

Hence (37) represents the power coupled from G_0 to G_1 .

The behavior of the transmitted power $|T|^2$ is depicted in Fig. 2 as a function of C for various B . Notice from (37) that $|T|$ is periodic in C , with unity period. In the case of degenerate modes ($\Delta\beta = 0 = B$), (37) becomes

$$1 - |T|^2 = \sin^2(\pi C); \Delta\alpha, \Delta\beta = 0 \quad (40)$$

This result can be obtained directly from the coupled wave equations (1) using definition (31c) for C , and holds for arbitrary coupling $c(z)$. The interesting feature of (37) is that the factor (40) remains intact, and the denominator is independent of C . This result contrasts with the solution in the case of constant coupling [2], where a factor containing B alone cannot be separated out.

In the present case, progressively increasing B from 0 monotonically decreases the amount of power coupled to G_1 . Figure 3 shows that when B reaches unity or more, very little power is coupled to G_1 , and the transmitted power $|T|^2$ is virtually unity regardless of C .

When $C/B = 2|K|/(\Delta\beta R_0)$ is small and $A = 0$, perturbation theory may be used to find an estimate for $|T|^2$. Small C/B is equivalent to a small amount of coupling in one beat wavelength $2\pi/\Delta\beta$. The perturbation solution for arbitrary $c(z)$ is [6]

$$1 - |T|^2 \approx \left| \int_{-\infty}^{+\infty} e^{-j\Delta\beta z} |K|c(z)dz \right|^2, \Delta\alpha = 0 \quad (41)$$

which in the present case becomes, using (4) and (31)

$$1 - |T|^2 = (\pi C)^2 / \cosh^2(\pi B), \quad \Delta\alpha = 0. \quad (42)$$

Notice that this in general overestimates the power coupled to G_1 , but reduces to the exact result (37) when $C = |K|B/\pi$ is small.

No simple trigonometric formula can be derived for $|T|^2$ in the general case when $\Delta\beta = 0$, but $\Delta\alpha \neq 0$. The curves in Fig. 4 were calculated directly from (30) using a computer program for numerical evaluation of the gamma functions [10]. These results were checked against simple formulas which can be derived for the cases $A = 0.5$ and 1.0 using (36). Notice that for small total coupling C , increasing A from zero increases the transmitted power by breaking the degeneracy $\Delta\Gamma = 0$ between the two modes in a manner similar to Fig. 2. For large C , however, the power never returns completely to G_0 because of the large damping in the coupled mode G_1 .

In most practical cases the differential attenuation is small, and its effect is a small correction to $|T|^2$. In such cases it is convenient to define an "added heat loss" (A.H.L.) as

$$\text{A.H.L.} \equiv \ln \left| \frac{T(0, B, C)}{T(A, B, C)} \right|, \quad (43)$$

where the transmission coefficient T is considered as a function of the three normalized parameters of (31).

When B is large and both A and C are small, an estimate of A.H.L. can be obtained by expanding the gamma functions in (30) using Stirling's approximation. The result is

$$A.H.L. \approx (C/B)^2 . \quad (44)$$

This result agrees with the following formula for A.H.L. that can be derived from perturbation theory [11], whenever the loss considered as a function of B is maximum near $B = 0$ and is small for all $|B|$ greater than some reasonably small fraction of the actual B:

$$A.H.L. \approx \frac{(\Delta\alpha)}{(\Delta\beta)^2} \int_{-\infty}^{+\infty} |K|^2 c^2(z) dz . \quad (45)$$

Equation (44) can be obtained by performing the integration in (45) using (4) and the definition (31).

A comparison between the approximate expression (44) for A.H.L. and the exact expression (43) evaluated numerically is depicted in Fig. 5 for relatively small coupling C. As in the case of zero $\Delta\alpha$, the perturbation results again overestimate the loss, but reduce to the exact result for small $(C/B)^2$.

Figure 6 reproduces the exact results of Fig. 5 and compares these with additional results for larger C and A. Notice that the value of A does not affect the added heat loss to A ratio when C is an integer. From (37) it can be seen that this corresponds to the case of zero transmission loss for arbitrary B when $A=0$. When C takes on half integral values, on the other hand, the loss from (37) is maximum. In this case, increasing A for very small B tends to decrease the degeneracy between modes and also to decrease the transmission loss (see the discussion of Fig. 4). In such cases, the added heat loss can even become negative for larger A, as Fig. 6 shows.

In practice it is of course necessary to truncate the hyperbolic secant coupling function $c(z)$ at some finite length. To achieve the same total coupling (which would be the bend angle for curvature coupling, for example), it is necessary to scale the value of θ used in the coupling function (33). If we define L as the total coupling length, and θ_s as the scaled coupling parameter, then we require that

$$\int_{-\infty}^{+\infty} c(z) dz = \frac{1}{R_0} \int_{-L/2}^{L/2} \text{sech}(\pi z / R_0 \theta_s) dz = \theta . \quad (46)$$

Evaluation of the integral in (46) yields an implicit expression for θ_s :

$$(2\theta_s / \pi) \arctan \left(\sinh \frac{\pi}{2} \frac{L}{R_0 \theta_s} \right) = \theta . \quad (47)$$

When $L/(R_0 \theta_s)$ is large, which is usually required for low loss, we may expand the functions in (47) in terms of simple exponentials. Expanding the result about $\theta_s = 0$, we find

$$\frac{\theta_s}{\theta} \approx \frac{1}{1 - 1/\left(\frac{\pi}{4} e^x - x\right)} , \quad (48)$$

where

$$x \equiv \frac{\pi}{2} \frac{L}{R_0 \theta} . \quad (49)$$

Figure 7 contains a plot of (48) compared with the exact result obtained numerically for the case $\theta = \pi/2$. Whenever $L/R_0 > 2.5$, which is usually required, the approximate expression is very good.

In Figs. 8 and 9 we provide a basis for choosing the truncated coupling length L . These figures consider the case $\theta = \pi/2$ and $A = 0$, $B = 0.5$. For both integral and half-integral C , which correspond to minimum and maximum loss as $L \rightarrow \infty$, respectively, L/R_0 must reach about 6 before the loss (coupled power) with truncated coupling approaches the loss for infinite coupling. The actual loss in these figures was calculated numerically from the coupled mode equations assuming a sharp truncation in $c(z)$ given by (4) at $z = \pm L/2$. The straight lines in the figures were calculated from (37).

The actual value of $(L/R_0\theta)$ required for satisfactory results depends on the ratio $|C/B|$. For small values of this ratio, the coupling at the truncation points is small and does not strongly affect the overall loss. Hence $L/(R_0\theta)$ can be decreased to make a shorter coupled section. For larger $|C/B|$, the coupling $c(z)$ must be reduced to a small value before truncation, and this implies larger $L/(R_0\theta)$.

Truncation of $c(z)$ at quite small $L/(R_0\theta)$ may be acceptable, if θ , B , and C are all fixed. Values of $L/(R_0\theta) = 1.25$ and 1.5 , respectively, would be fine for the cases considered in Figs. 8 and 9. With such small $L/(R_0\theta)$, however, there is not much difference between the truncated hyperbolic secant coupling and truncated constant coupling (see Fig. 1). The transmitted power with such coupling is always very sensitive to the parameters θ , B , and C , as Figs. 8 and 9 indicate. Choice of larger $L/(R_0\theta)$ reduces this sensitivity.

If the truncation is not so abrupt, on the other hand, the overall coupling length can again be reduced. In general, the strength of the coupling between modes is roughly inversely proportional to the order of the lowest order derivative in the coupling function that is discontinuous. That is, a parabolic taper to zero in $c(z)$ near $z = \pm L/2$ is better than a linear taper, which in turn is better than an abrupt truncation. Integration by

parts of the integral in the perturbation solution (41) quickly shows this to be true, provided that the length of the taper exceeds a beat wavelength $2\pi/\Delta\beta$.

The hyperbolic secant function has no discontinuities in any derivative of $c(z)$. Hence it may be viewed as a limiting form of polynomial coupling functions, and with suitable truncation may provide near-minimum lengths for a given loss over a wide range of parameters. Application of this coupling function to the design of low-loss broadband bends and twists in overmoded rectangular waveguide and bends in corrugated circular waveguide is considered in other work.

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Figure Captions

Fig. 1 Hyperbolic Secant Coupling.

Fig. 2 Transmitted power as a function of the normalized total coupling C when $\Delta\alpha = 0$.

Fig. 3 Transmitted power as a function of the normalized propagation constant when $\Delta\alpha = 0$.

Fig. 4 Transmitted power as a function of C when $B = 0$.

Fig. 5 Ratio of the Added Heat Loss (A.H.L.) to the normalized differential attenuation constant A , as calculated by the exact formula (43) and by perturbation theory (44) for $A = .0025$.

Fig. 6 Ratio of A.H.L. to A as a function of B for various C and A .

Fig. 7 Approximate (48) and exact ratios of the scaled and desired angles θ_s and θ as a function of the normalized coupling length.

Fig. 8 Transmitted power versus normalized coupling length for $\theta = \pi/2$, $A = 0$, $B = 0.5$, $C = 2.5$. The straight line is the value for infinite coupling length obtained from (37).

Fig. 9 Transmitted power versus normalized coupling length for $\theta = \pi/2$, $A = 0$, $B = 0.5$, $C = 3.0$. The straight line is the value obtained from (37).

Fig. 1

82E0081

Fig. 2

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Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

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Fig. 8

Fig. 9