DOE/ER/03227--T2

DOE/ER/03227--T2

DE82 013385

A CLASS OF POTENTIALS WITH EXTREMELY NARROW RESONANCES

I. Case with Discrete Rotational Symmetry

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ABSTRACT :

We consider a quantum-mechanical particle in three dimensions, subject to point interactions (Fermi pseudopotentials) placed on the vertices of a regular plane polygon. If N , the number of vertices, tends to infinity and the distance between two consecutive interaction centers tends to a constant, we show that our system has resonances that tend exponentially fast to the real axis. We discuss several conjectures on the generality of this result and stress its relevance as a simplified model for the Yagi-Uda antenna array of classical electromagnetism.

JANUARY 1981

CPT-81/PE.1291

POSTAL ADDRESS : C.N.R.S. - LUMINY - CASE 907
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I. INTRODUCTION

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From time to time, progress in quantum mechanics and that in electromagnetic theory stimulate each other. One example is the decay of a compound nuclear state [1] and the behavior of an imperfectly conducting coaxial line [2]. It is the purpose of the present series of papers to study the quantum-mechanical analog of one of the most intriguing and practical phenomena in classical electromagnetism: the Yagi-Uda antenna array [3].

Although it was invented over half a century ago and was used almost universally for television reception, the Yagi-Uda antenna array has defied a complete theoretical analysis despite the many excellent papers on this topic. It was found several years ago by extensive numerical calculation that the Yagi-Uda array of circular loops has, under certain circumstances, two distinct but very close resonances of virtually identical current distributions [4]. No theoretical analysis or experimental measurement is precise enough to answer the question whether the same phenomenon occurs for the Yagi-Uda array of dipole antennas.

On the side of quantum mechanics, the Fermi pseudopotential [1] has been known for several decades. In particular, it has been extensively used in many-body problems [5]. Recently, the concept of a point interaction has been advanced and used in a mathematically rigorous way [6]-[10]. It seems intuitively obvious that this point interaction cannot be different from the Fermi pseudopotential. This is indeed the case and this point will be discussed in detail in Sec. II.

One of the applications of the point interaction is to an ideal polymer [14], [15], [9], i.e., an infinite one-dimensional array of equidistant point interactions. It is found that such an array has resonances of zero width, i.e., on the positive energy axis. This must be the quantum-mechanical analog of the Yagi-Uda

array. However, because of the scalar nature of the Schrödinger equation as compared to the vector nature of Maxwell's equations, and because of the simplicity of the Fermi pseudopotential as compared to the theory of linear antennas [11], this quantum-mechanical analog is much easier to understand.

Resonances of zero width are disturbing objects, and their occurrence in the case of the ideal polymer just mentioned must be attributed to the infinite length of the polymer. The case of the semi-infinite polymer can clearly be solved exactly by the method of the Wiener-Hopf sum equation, and the case of a long but finite polymer can be treated approximately by the same technique, as used for example in the two-dimensional Ising model [12]. The resonances in this finite case are not especially sharp, due physically to the radiation along the direction of the linear polymer. It is therefore natural to ask whether there is any other configuration which involves a large but finite number of pseudopotentials and yet has extremely narrow resonances.

A moment's reflection leads to the idea that a very narrow resonance should occur if the ideal polymer is bent into a closed loop. Since any abrupt bend would lead to radiation and thus increase the width, we shall investigate in this series of papers the case of pseudopotentials distributed along or near a closed analytic space curve. In such cases, the width of the resonances may be expected to be quite small. In this paper, we begin with the simplest possible configuration: the circle in three-dimensional space.

The present investigation in the problem of quantum mechanics also suggests strongly that similar extremely narrow resonances occur in various antenna arrays. In particular, the circular array of dipole antennas must exhibit such a resonance. It is curious that, in all investigations of the circular array that we are able to find, the lengths and spacings are such that these narrow resonances fail to appear. It would be very interesting to study, either numerically or experimentally, a circular array with an extremely narrow resonance.

II. POINT INTERACTIONS AND FERMI PSEUDOPOTENTIAL

We shall be concerned with nonrelativistic one-particle Hamiltonians that correspond to the intuitive idea of a particle undergoing interactions only at a discrete set of points. An obvious way of trying to implement this idea is to introduce potentials that are sums of Dirac δ -functions located at various points in space. The question then arises whether such potentials define self-adjoint operators in the physical Hilbert space.

The answer is independent of the number of scattering centers, but is strongly dependent on the number of degrees of freedom, i.e., on the dimension of the space in which the particle is moving. For the moment, consider one interaction center located at the origin. In one dimension, the sum $-\frac{d^2}{dx^2} + g\delta(x)$ is defined, e.g., in the sense of quadratic forms in Hilbert space, and gives rise to a self-adjoint operator in physical Hilbert space $\mathcal{H}_0 = L^2(\mathbb{R})$ for all real values of the coupling constant g .

If the number of dimensions is four or more, it is not possible to construct a model that corresponds to the idea of point interactions. The reason for this is, basically, that the function $\frac{1}{p^2 + 1}$ is not square integrable over four-dimensional space. (See, for example, ref. [10].)

In two or three dimensions, an intermediate situation obtains. (From now on, we consider only the three-dimensional case; see [10] for two dimensions.)

A sum

$$-\Delta + g\delta(x) \quad (g \in \mathbb{R}, g \neq 0) \quad (2.1)$$

cannot define an operator in physical Hilbert space $\mathcal{H}_0 = L^2(\mathbb{R}^3)$. Nevertheless, there exists a one-parameter family of self-adjoint operators $H_a^{\pm=0}$ labelled by a number a that has the dimension of length and having all the properties described

at the beginning of this section. An operator belonging to this family can be described in several equivalent ways:

(i) As an object of the form (2.1) but with truly infinitesimal g (in the sense of nonstandard analysis - see [7]).

(ii) As a self-adjoint extension of the symmetric operator $\dot{T}$ obtained by restricting $-\Delta$ to the space of smooth square integrable functions vanishing at 0. In contrast to the situation in four or more dimensions, such an extension need not coincide with $-\Delta$. The reason is, essentially, that the function $\frac{1}{p^2 + 1}$ is square integrable over three-dimensional space. It belongs to the domain of the adjoint $(\dot{T})^+$ of $\dot{T}$ without being in the domain of $\dot{T}$. (See, for example, [13, p. 1222] for a general discussion that is well adapted to our special case, and [8] for the example considered here.)

(iii) A third way of introducing the operator $H_a^{\pm=0}$ is to write down an explicit formula for its resolvent $Q(E) = (H_a^{\pm=0} - E)^{-1}$. This formula, Eq. (2.2) below, involves the orthogonal projection operator $P_{\vec{r}=0}^E$ on the one-dimensional space spanned by the Hilbert space vector $(e^{ik|\vec{r}|})/4\pi|\vec{r}|$. Here $k = \sqrt{E}$ and E is not on the positive real axis. $Q(E)$ is then defined by:

$$Q(E) = (T - E)^{-1} + \frac{4\pi a}{1 + ika} P_{\vec{r}=0}^E, \quad (2.2)$$

where $T = p^2$ is the kinetic energy operator (see [10]).

(iv) In the literature of physics, the traditional expression for $H_a^{\pm=0}$ is:

$$H_a^{\pm=0} = -\nabla^2 + 4\pi a\delta(\vec{r}) \frac{\partial}{\partial r}, \quad (2.3)$$

with $r = |\vec{r}|$. (See, for example, ref. [1].) The second term on the right-hand-side of (2.3) is known as the Fermi pseudopotential.

In order to discuss this operator, we shall first specify its domain.

Roughly speaking, it consists of smooth functions with a prescribed singularity

at the origin. More precisely, it consists of functions $\psi(\vec{r})$ that can be written in the form

$$\psi(\vec{r}) = \phi(\vec{r}) - \frac{a}{|\vec{r}|} \phi(0) \quad (2.4)$$

in some neighborhood of the origin. The function $\phi(\vec{r})$ is required to satisfy

$$\int (1 + p^2)^2 |\tilde{\phi}(p)|^2 d^3 p < \infty, \quad (2.5)$$

where $\tilde{\phi}(p)$ is the Fourier transform of $\phi(\vec{r})$. Away from the origin, $\phi(\vec{r}) = \phi_1(\vec{r})$

where ϕ_1 also satisfies (2.5).

The individual terms in (2.3), when acting on a function of the form (2.4), map it into a space that is larger than $\mathcal{H}_0$ and contains the δ -function. However, the sum (2.3) maps ψ into $\mathcal{H}_0$ and is self-adjoint on the domain (2.4)

The expressions (2.2) and (2.3) can be generalized to N interaction centers. Furthermore, (2.2) can be analytically continued into unphysical sheets, to allow a discussion of resonances.

III. FORMULATION OF THE PROBLEM

Let C be a rectifiable closed curve of length L in three dimensions and RC be a curve similar to C and of length RL. On RC there are N points located at $\vec{r}_i$. [More generally, these points can be close to the curve RC, but this generalization will not be considered in this paper.] With the mass chosen to be $\frac{1}{2}$ and $\hbar = 1$, we consider the Hamiltonian

$$H_{a_1 \dots a_n}^{\vec{r}_1 \dots \vec{r}_n} = -\nabla^2 + \sum_{i=1}^N 4\pi a_i \delta(\vec{r} - \vec{r}_i) \frac{\partial}{\partial R_i} R_i, \quad (3.1)$$

where $R_i = |\vec{r} - \vec{r}_i|$. It is the purpose here to study the narrow resonances of H in the limiting case when N and R are both large. Restrictions on the locations of the N points on RC are to be expected.

The Schrödinger equation is

$$H\psi = k^2\psi, \quad (3.2)$$

which is more explicitly

$$(\nabla^2 + k^2)\psi = \sum_{i=1}^N A_i \delta(\vec{r} - \vec{r}_i), \quad (3.3)$$

where

$$A_i = 4\pi a_i \left. \frac{\partial}{\partial R_i} (R_i \psi) \right|_{R_i=0}. \quad (3.4)$$

In the absence of an incident field, the solution of (3.3) is:

$$\psi(\vec{r}) = - \sum_{i=1}^N A_i (4\pi R_i)^{-1} e^{ikR_i}. \quad (3.5)$$

Substitution into (3.4) then yields the following linear equations for the

determination of A_1 and hence ψ from (3.5):

$$(1 + ika_1)A_1 + a_1 \sum_{j \neq 1} r_{1j}^{-1} e^{ikr_{1j}} A_j = 0, \quad (3.6)$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$. Equation (3.6) can also be obtained easily with the formalism of point interactions.

Given a_1 and $\vec{r}_1$, (3.6) can admit a nontrivial solution only for certain values of k ; these are the resonances. In general, the resonances are not especially close to the positive real axis. We are, however, especially interested in those cases when the resonances are very near the positive real axis. In particular, such a resonance is the underlying reason why the Yagi-Uda antenna array is of such great practical importance.

IV. ASYMPTOTIC BEHAVIOR OF RESONANCE WIDTH

In this first paper, only the simplest case is to be studied. It is clear that the simplest case corresponds to a circle C , with the N points uniformly distributed, and $a_1 = a > 0$ independent of 1 . It will be shown in this section that, in the limit $R \rightarrow \infty$ and $N \rightarrow \infty$ with fixed N/R , there are resonances when the widths are exponentially small in R .

It is convenient to choose $L = 2\pi$ so that C is a unit circle, and the radius of RC is R . With this choice,

$$r_{ij} = 2R \sin \pi |i - j|/N \quad (4.1)$$

Since (3.6) in the present case is cyclic, viz.,

$$(1 + ika)A_1 + a \sum_{j \neq 1} r_{1j}^{-1} e^{ikr_{1j}} A_j = 0, \quad (4.2)$$

A_1 can be chosen to be

$$A_j = \text{constant } e^{2\pi imj/N}$$

without loss of generality, where m is an integer such that $0 \leq m < N$. The equation for k is then simply:

$$1 + ika + a \sum_{j=1}^{N-1} r_{0j}^{-1} e^{ikr_{0j}} e^{2\pi imj/N} = 0 \quad (4.3)$$

This equation is to be solved asymptotically in the limit of interest. Define

$$F(k) = ik + \sum_{j=1}^{N-1} r_{0j}^{-1} e^{ikr_{0j}} e^{2\pi imj/N} \quad (4.4)$$

We shall first give for $F(k)$ an asymptotic evaluation which is not sufficiently accurate for our purpose. This asymptotic evaluation will then be improved.

When j is increased by 1, the increase in phase is

$$k(r_{0j+1} - r_{0j}) + 2\pi m/N \quad (4.5)$$

We assume for the moment that this phase change is never a multiple of 2π . Under this circumstance, when $R \rightarrow \infty$ and $N \rightarrow \infty$ with fixed N/R , $F(k)$ can be approximated by:

$$F(k) \sim ik + \sum_{j=1}^{\infty} (2R\pi j/N)^{-1} e^{2\pi i(kR+m)j/N} + \sum_{j=1}^{\infty} (2R\pi j/N)^{-1} e^{2\pi i(kR-m)j/N} \quad (4.6)$$

Direct evaluation then gives simply:

$$F(k) \sim -\frac{N}{2\pi R} \ln[2[\cos(2\pi kR/N) - \cos(2\pi m/N)]] \quad (4.7)$$

Equation (4.5) in the form

$$a^{-1} + F(k) = 0 \quad (4.8)$$

can admit real solutions for k provided that the approximation (4.7) is used.

Note that (4.8) with (4.7) is the same as the equation that determines the real resonances for the idealized polymer [9].

Since the width of a resonance is determined by the imaginary part of k , the above approximation is not accurate enough to determine the widths of the narrow resonances. Let k_0 be a real solution of (4.8) with the approximation (4.7), and $k = k_1 + ik_2$ be the corresponding exact solution of (4.8). It is

seen from the above consideration that $k_1 \sim k_0$ and k_2 is small. Because of the smallness of k_2 , let the $F(k)$ of (4.8) be expanded about k_1 to give:

$$a^{-1} + F(k_1) + ik_2 F'(k_1) = 0 \quad (4.9)$$

and hence k_2 is given approximately by:

$$k_2 = -\text{Im } F(k_1) / \text{Re } F'(k_1) \quad (4.10)$$

Since the denominator is given adequately by (4.7), it remains to calculate $\text{Im } F(k_1)$ approximately in the limit $R \rightarrow \infty$ and $N \rightarrow \infty$ with fixed N/R . Let

$$G(k_1) = \text{Im } F'(k_1) \quad (4.11)$$

Since

$$\text{Im } F(0) = 0 \quad (4.12)$$

$\text{Im } F(k_1)$ is given by:

$$\text{Im } F(k_1) = \int_0^{k_1} G(k'_1) dk'_1 \quad (4.13)$$

But it follows from (4.4) that

$$G(k_1) = \text{Re} \left[1 + \sum_{j=1}^{N-1} e^{ik_1 r_{0j}} e^{2\pi imj/N} \right] = \text{Re} \sum_{j=0}^{N-1} e^{ik_1 r_{0j}} e^{2\pi imj/N} \quad (4.14)$$

Application of the Poisson summation formula leads to

$$G(k_1) = \frac{N}{2\pi} \sum_{n=-\infty}^{\infty} \text{Re} \int_0^{2\pi} d\theta e^{2ik_1 R \sin \theta/2} e^{i(m+nN)\theta}$$

$$= N \sum_{n=-\infty}^{\infty} J_{2(m+nN)} |2k_1 R|^{(2k_1 R)} \quad (4.15)$$

where J denotes the Bessel function.

This derivation of (4.15) is the main step of the present paper. It is a property of the Bessel function that $G(k_1)$ is exponentially small provided that

$$|\frac{m}{N} + n| > k_1 R/N \quad (4.16)$$

for all n . In this case, since $0 \leq m < N$,

$$\begin{aligned} G(k_1) &\sim N J_{2m}^{(2k_1 R)} + N J_{2(N-m)}^{(2k_1 R)} \\ &\sim \frac{1}{2} N \pi^{-1/2} (m^2 - k_1^2 R^2)^{-1/4} \exp 2[(m^2 - k_1^2 R^2)^{1/2} - m \cosh^{-1} \frac{m}{k_1 R}] \\ &\quad + \frac{1}{2} N \pi^{-1/2} [(N-m)^2 - k_1^2 R^2]^{-1/4} \exp 2\{[(N-m)^2 - k_1^2 R^2]^{1/2} \\ &\quad - (N-m) \cosh^{-1} \frac{N-m}{k_1 R}\} \end{aligned} \quad (4.17)$$

Accordingly, the width of the narrow resonance is determined by:

$$\begin{aligned} \text{Im } k &\sim -\frac{1}{4} k_1 N \pi^{-1/2} [\cos(2\pi k_1 R/N) - \cos(2\pi m/N)] \csc(2\pi k_1 R/N) \\ &\quad \times \{ (m^2 - k_1^2 R^2)^{-3/4} \exp 2[(m^2 - k_1^2 R^2)^{1/2} - m \cosh^{-1} \frac{m}{k_1 R}] \\ &\quad + \text{same term with } m \rightarrow N-m \} \end{aligned} \quad (4.18)$$

Note that, because of (4.16), the exponentials in (4.18) are both small.

V. DISCUSSIONS

In our result (4.18), the width of the resonance $-\text{Im } k$ is expressed in terms of the location of the resonance $k_1 = \text{Re } k$, which is in turn given approximately by:

$$k_1 \sim k_0 = \frac{N}{2\pi R} \cos^{-1} \left[\cos \frac{2\pi m}{N} + \frac{1}{2} e^{\frac{2\pi R}{Na}} \right] \quad (5.1)$$

It is therefore natural to replace all the k_1 on the right-hand-side of (4.18) by k_0 as given by (5.1).

The question may be raised whether this replacement of k_1 by k_0 is legitimate. The reason for this question is that, in (4.18), R appears in the exponent. For example, if $k_1 - k_0 = O(R^{-1/2})$, then this replacement leads to a factor of the form $\exp(\text{constant } R^{1/2})$ and hence is incorrect. If $k_1 - k_0 = O(R^{-1})$, then it leads to a constant factor and hence is still incorrect. But, if $k_1 - k_0 = o(R^{-1})$, then the replacement is legitimate. It is therefore essential to return to the calculation of the preceding section and ascertain the accuracy of (5.1). The result is:

$$k_1 - k_0 = O(R^{-2}) \quad (5.2)$$

and hence all the k_1 on the right-hand-side of (4.18) can be replaced by k_0 .

Attention is now turned to the factor in the braces of (4.18). Unless m is very close to $N-m$, or more precisely,

$$m = \frac{1}{2} N + O(1) \quad (5.3)$$

the two terms are of different orders of magnitude. If $m < N/2$, the first term dominates; if $m > N/2$, the second term dominates. Without loss of generality,

consider $m < N/2$ and assume that m is not close to $N/2$ so that (5.3) does not hold; then the second term may be neglected and the result for the width of the resonance is:

$$\text{Im } k \sim -\frac{1}{8} k_0 N \pi^{-1/2} (N^2 - k_0^2)^{-3/4} \csc(2\pi k_0 R/N) \exp 2[(m^2 - k_0^2)^{1/2} - m \cosh^{-1} \frac{m}{k_0 R} - \frac{2\pi R}{Na}] \quad (5.4)$$

On the other hand, when (5.3) holds, both terms need to be kept and (4.18) reduces to:

$$\text{Im } k \sim -k_0 N (2\pi)^{-1/2} (N^2 - 4k_0^2)^{-3/4} \csc(2\pi k_0 R/N) \exp [(N^2 - 4k_0^2)^{1/2} - N \cosh^{-1} \frac{N}{2k_0 R} - \frac{4\pi R}{Na}] \cosh [(2m - N) \cosh^{-1} \frac{N}{2k_0 R}] \quad (5.5)$$

Although narrow resonances are to be expected, it is surprising that the widths are exponentially small. That the pseudopotentials be located on a circle is clearly not a necessary condition; we speculate in this section about the possible appearance of such exponentially narrow resonances under more general circumstances.

(A) Arrangements of pseudopotentials on a curve with corners are most probably not acceptable. It is natural to believe that the underlying closed curve must be analytic or at least infinitely differentiable.

(B) In this paper we have considered only the simplest case, - when a discrete rotational symmetry obtains. In general, on an analytic curve there is no precise meaning to the concept of equidistant points. The distance can be measured either along the curve or along the straight line joining the neighboring points. It therefore follows that the equidistant aspect of the

present problem cannot be essential. In other words, small variations in the distances between neighboring points cannot ruin the exponential smallness of the width.

(C) Similarly, the pseudopotentials need not be located exactly on the curve RC. In other words, small variations in the positions perpendicular to the curve are most likely also acceptable. However, it is much less clear how much deviation is permitted. In particular, is the deviation required to be small as a power of R or as an exponential of R ?

(D) It is also not clear whether the analytic curve is allowed to be self-intersecting. Although without a good argument, we nevertheless conjecture that, if the analytic curve is self-intersecting without being self-tangent, the widths of its resonances are not exponentially small. It would be most interesting to determine whether this conjecture is true or false.

(E) It is even less clear whether the following Property A of the analytic curve needs to be avoided in order to have exponentially small widths.

Property A: A differentiable curve C is said to have Property A if there exists at least one point P on C such that the tangent to C at P intersects C at a point different from P .

(F) For the circle studied here, the radius of curvature is a constant. In general, this is of course not true. If the pseudopotentials are in some sense equidistant and of equal strength, presumably the exponent in the asymptotic formula for the resonance width is determined by the minimum radius of curvature. In the case of unequal strength and/or unequal distance, is there a simple combination that determines this exponent?

We conclude with a comment on the use of the pseudopotential. A pseudopotential has zero range. As seen from the case of a Yagi-Uda array of circular loops, finite ranges in the directions perpendicular to the direction of the

analytic curve are probably acceptable for exponentially small widths. Finite ranges in the direction of the analytic curve are probably not acceptable unless such ranges are exponentially small. Thus there are two contributions to the width: the effect studied here and the finite range of the potentials. Physically, such finite ranges lead to radiation loss perpendicular to the curve and hence increase the width of resonance. This range effect is being studied in collaboration with S. Albeverio and R. Høegh-Krohn.

In Part II of this paper, higher correction to this narrow width will be given. Beginning with Part III, the more general and more interesting case of adopotentials on an analytic curve will be analyzed asymptotically.

ACKNOWLEDGMENTS

We are greatly indebted to Professor Ronold W. P. King and Dr. Ralph D. Kodis for their most helpful discussions. This work was supported in part by the U.S. Department of Energy under Contract DE-AS02-76ER03227 and in part by the Joint Services Electronics Program under Contract N00014-75-C-0648, both with Harvard University.

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