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**Quadric Solids and
Computational Geometry**

By J. D. Emery

Published July 25, 1980

Topical Report

Prepared for the United States Department of Energy
Under Contract Number DE-AC04-76-DP00613.

**Kansas City
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QUADRIC SOLIDS AND COMPUTATIONAL GEOMETRY

BDX-613-2484, Topical Report, Published July 25, 1980

Prepared by J. D. Emery

As part of the CAD-CAM development project, this report discusses the mathematics underlying the program QUADRIC, which does computations on objects modeled as Boolean combinations of quadric half-spaces. Topics considered include projective space, quadric surfaces, polars, affine transformations, the construction of solids, shaded image, the inertia tensor, moments, volume, surface integrals, Monte Carlo integration, and stratified sampling.

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INTRODUCTION

Quadric solids provide a rich collection of sets for constructing models of 3-dimensional objects. They include the sphere, the ellipsoid, the block, the cone, the wedge, and various cylinders. Many common objects are combinations of these shapes: rectangular blocks, circular cylinders, and spheres. Polyhedra are also quite common and can be built from planar half-spaces and wedges, which are quadric solids. Any common object can be approximated by a polyhedron, but because it has planar faces it cannot model local shape. An advantage of using quadric surfaces is that they can locally approximate curvature and shape (see O'Neil, p. 202), so that an object can be represented by a relatively small number of primitive solids. For computational reasons this is the task: not to find just a class of approximately primitive objects, but to find a class that requires few primitives to model common objects. Spheres, then, are not suitable, because although all objects are collections of spheres, at least if atoms are assumed spherical, Avagadro's number is so huge that it is impossible to represent an object by spherical models of its atoms. On the other hand, spheres do not locally approximate shape so that an object cannot in general be modeled by a small number of them.

Computations with quadric solids are simple. The calculation of a normal direction at a point on a quadric surface is accomplished by a matrix multiplication, the intersection points of a straight line with the solid require only the solution of a quadratic equation, and the new equation of a translated or rotated quadric surface is easily found.

Surfaces of revolution are commonly found, but, unfortunately, most are not quadric surfaces. Further, a large number of quadrics is frequently required to get an acceptable approximation of them. This is true of the torus. A chain consisting of a few ellipsoids, for example, makes a lumpy torus.

The advent of the computer gave rise to a renaissance in many areas of computing. Many well-known algorithms and calculating methods were of little interest simply because they were too burdensome for the human computer. Thus, techniques that could be carried out easily by hand were emphasized. But now those calculations that were too lengthy for hand calculation are suitable for computer graphics. Frequently, the graphics novice does not use these tools but instead naturally uses the calculations taught him in high school and college. These calculations are often only the small amount of analytic geometry found in a calculus course and deal, for example, with the simultaneous solution of a system of equations in a fixed coordinate system. For many calculations a vector approach is superior, because it tends to be independent of the coordinate system.

Most people are familiar with vectors, but many are still unaware of some very useful material: projective geometry, projective space, and homogeneous coordinates. Projective space is very important in certain advanced areas of mathematics, such as in algebraic geometry. But even among mathematicians, the study of algebraic geometry is not universal. A course offered in projective geometry at American universities and colleges is usually designed for future teachers of high school mathematics. This course, or a course called modern geometry, is taught to show the future teacher of Euclidean geometry that there are many geometries and that not all are Euclidean. Neither this projective geometry course nor the books designed to teach it emphasizes the beautiful simplifications and generalizations of calculations that come from treating points as belonging to projective space. The few pages that discuss projective space in Birkhoff and MacLane are more illuminating than many more pages of most elementary books on projective geometry.

The purpose of these notes is first to describe a modeling system that uses quadric solids as primitive objects and, second, to present some elementary ideas of projective geometry that are useful for computing. These elementary ideas do not seem to be readily available. The modeling system is implemented in a computer program called QUADRIC. It produces views of 3-dimensional objects and calculates volumes, moments, and inertia tensors.

The idea of representing objects as primitive solids has recently become fashionable. Quadric solids are obvious candidates for the primitives. Although there are several papers dealing with quadric surfaces (see Leven), I have seen none describing quadric solid models.

The model described here represents an object as a Boolean combination of quadric half-space primitives. The object is viewed by scanning its surface and calculating an illumination intensity at each surface point. To construct intensity values, the 2-dimensional picture plane is scanned. A surface normal is calculated where the line through the picture point pierces the object. The intensity value assigned is determined from the inner product of the normal at the first pierce point with an illumination vector. Because the primitives are quadric solids, this can be done easily.

A quadric surface is the set of real zeroes of a quadratic form in four dimensions. Such a set of zeroes is called a variety in algebraic geometry. A quadratic form is a polynomial in which each term is of the second degree. The set of points where the form takes nonpositive values is called a lower half-space, and an upper half-space is the set where the form takes nonnegative values. Thus, a quadric form defines two quadric half-spaces. Examples of half-spaces are the points inside an ellipsoid and the points outside two parallel planes.

It is easy to find the intersection points of a quadric surface with a straight line. The parameters of the intersection points on the line will be the roots of a quadratic equation. Surface normals are also easily calculated. Our task is to compute these entities on the assembled object; however, the casual reader may want to read the Introduction and Overview sections.

The objects are built using the standard set operators, union, intersection and complementation. Some modeling schemes, such as the PADL system, which was developed at the University of Rochester, use regularized set operators. They guarantee that the resulting objects are topologically closed and homogeneously 3-dimensional. Using standard operators, it is possible to get a set that is not closed. Because we have only an approximate model of the surface and do not explicitly calculate the boundary, the use of regularized operations does not seem appropriate. For intersection points, which are candidates to be on the boundary, we construct a small line segment in the direction of the normal that contains the point. If one end of the segment is inside and the other outside, the point is a boundary point of the object.

OVERVIEW

I start with 2-dimensional projective space because it is easy to visualize. Projective 2-space is the set of lines in 3-dimensional space which pass through the origin. Thus, projective 2-space is the set of one-dimensional subspaces of a 3-dimensional vector space. A plane that does not pass through the origin is isolated and called the affine plane. A coordinate system is usually selected so that the affine plane is the plane $z=1$. Projective space can be illustrated by considering the intersection of a cone and a plane. The intersection curve is called a conic section. When the cone is rotated, the conic section is transformed to a new conic section. A circle may go into a hyperbola, a parabola, or an ellipse. The lines on the cone are points of projective 2-space. The intersection of these lines with the affine plane are the representation of the points in affine space. However, the projective points--i.e., lines--and the affine points are to be considered the same objects, namely 2-dimensional points. The components of any vector which lies in the subspace representing a projective point are called the homogeneous coordinates of the point. Thus, homogeneous coordinates are determined only up to a scalar multiple. A line in 3-space that is parallel to the affine plane meets it at infinity. Thus, the projective point in the direction of this line is called a point at infinity; its z coordinate will be zero. The property of being a point at infinity depends on the particular affine plane selected. Now consider a 2-dimensional line. We may view it as a line lying in the affine plane. The points of this line in projective space will be the locus of lines that pass through it and the origin. It is a plane in 3-space. Since any two planes with a point in common must coincide or meet in a line, it follows that any 2-dimensional lines in projective space meet in a point. Two parallel lines meet at a point at infinity. Notice that 2-dimensional projective space contains more points than affine space; that is, it contains the points at infinity. In 2-dimensional space, an affine transformation is the sum of a linear transformation and a translation. If the linear transformation is orthogonal--i.e., a rotation--then the affine transformation is a rigid motion or a congruence. Now any nonsingular linear transformation maps the set of 1-dimensional subspaces onto itself. So we may consider such a mapping as taking 2-dimensional projective points to projective points. Such a mapping is thus called a projective transformation. These transformations distinguish Euclidean, affine, and projective geometry. This is the thesis of Felix Klein, who, in his famous Erlangen

Program, defined a geometry to be the study of the invariants of a group of transformations. Thus, the concept of angle is a property of Euclidean space because angle is invariant under a rigid motion. Parallelism belongs to affine geometry because it is preserved under an affine transformation. Being a conic section is a property of projective geometry because it is preserved under a projective transformation. Being an ellipse, however, is not a projective property. There is a projective transformation that takes an ellipse to a different type of conic section. We come now to one of the major reasons for using homogeneous coordinates for calculations. An affine transformation is a special case of a projective transformation. Since a projective transformation is, in essence, a linear transformation in a higher dimensional space, it can be accomplished by matrix multiplication. Thus, an affine transformation can be accomplished by matrix multiplication. This is not the only advantage to using homogeneous coordinates. Quadrics are represented homogeneously as quadratic forms and can be represented as symmetric matrices. Perspective views are projections from projective 3-space to projective 2-space. The intersection point of two lines can be calculated without considering their slopes or parallelism.

The idea of the polar of a point with respect to a quadric plays a large role in the theory of quadrics. A quadratic form is represented by a symmetric matrix. There is an associated bilinear form, or functional. A bilinear functional is a mapping from the Cartesian product of a vector space to the real numbers, which is linear in each variable. For a fixed point, or pole p , the polar is the set of points such that the bilinear form corresponding to the quadric vanishes on any pair of points consisting of p and a point from the set. In two dimensions the polar is a line; in three dimensions, it is a plane. If p is on the surface of the quadric, the polar is the tangent plane. This gives the normal vector. If p is at infinity, the polar is a diameter of the quadric.

The cross ratio is an important invariant in projective geometry. It is a certain ratio of distances between four collinear points. The cross ratio is used to prove several properties of projective space.

We require the intersection of a line and a quadric. A line is defined by any two distinct points, including the case of points at infinity. The intersection points are determined by the roots of a quadratic equation. There are several cases: there may be two intersection points, there

may be no real intersection points, there may be a tangent point, or the line may be a ruling. A ruling is a straight line that lies entirely in the surface of the quadric. For example, bilinear interpolation on a rectangle produces a ruled quadric surface.

Conic arcs have been used in various applications of geometry. For example, they are used in the APT program, which generates machine tool paths. A conic arc is simply a portion of a conic section. A projective transformation exists that takes four points to four points. Thus, if we desire a given conic arc defined by four points, we construct the transformation that takes a given parabola, which is situated so that its points are a function of one parameter, to the desired arc. Then we have a parametric representation of the arc. One advantage is that sines and cosines do not have to be calculated for the points of a circular arc.

Surface normals can be considered points at infinity. A vector field is a mapping that assigns a vector to a given point. The surface normal is a vector field defined on the surface. But this field is really only a direction field, and a direction is essentially a point at infinity. Thus, we may consider a surface normal as a point at infinity.

Three-dimensional projective space is defined in the same way as 2-dimensional space. It consists of all 1-dimensional subspaces of a 4-dimensional vector space. The affine plane becomes a 3-dimensional hyperplane. Since we cannot climb into 4-dimensional space, we cannot visualize projective space from outside as we did in the 2-dimensional case.

Objects are constructed from quadric half-space primitives. There are only two bounded primitives, namely, the sphere and the ellipsoid. We create bounded objects by suitably combining the primitives. To each set corresponds a characteristic function which takes a value of true or false on each point of space. Its value at a point x is true if and only if x is in the set. The mapping, which takes a set to its characteristic function, union to "and," intersection to "or," and complementation to "not," is a Boolean algebra isomorphism from sets to the algebra of propositional functions. The latter algebra is represented in the FORTRAN computer language. Thus, in our program objects are represented as logical expressions. Actually, algebraic expressions are binary trees (see Knuth), so we may consider our object a binary tree with primitives at the terminal nodes and operators at the other nodes.

We determine the characteristic function of a primitive from the corresponding quadratic form. Then the characteristic function of the object comes directly from the definition of the object as a logical expression. From topology we may show that a point is on the surface of the object only if it is on the surface of some primitives, and the normal to the surface at a point on the surface is equal to the normal of some primitive. If the surface point lies on more than one primitive, then either the primitives are tangent or they are not. In the former case they have the same normal, in the latter case the point lies on the intersection curve, and hence, is on the boundary between regions where the normal belongs to one or the other primitives. Thus either can be assigned. Usually, when a point lies on several primitives, it is not important how the normal is assigned, since the intersection of the primitives will be of measure zero and so is negligible in any view or integration. A slight problem is the specification of the primitives.

Although all primitives are specified by symmetric 4-by-4-matrices it may be difficult in practice to determine the coefficients from their ordinary characterization. The primitives are most easily specified by entities such as radius, center, major axis, and orientation. We know the matrices of certain well-situated primitives. For example, we know the matrix of the sphere of radius one, which has its center at the origin. To obtain the matrix of a more complex primitive, we transform a simple primitive to the more complex configuration. The transformations employed are rotation, scaling, and translation. Thus, for example, by scaling, the unit sphere can be made an ellipse. It then can be rotated and translated. The new matrix of the transformed quadric can be obtained by multiplying the original matrix on the left and the right by matrices related to the transformations.

The motivation for constructing the modeling system was the desire for a system that would compute the volume, center of gravity and the inertia tensor of the object. Since the characteristic function is available, a Monte Carlo integration method is suggested. This technique consists of assigning a probability measure to a bounded space containing the object and obtaining the value of the integral as the expectation of some random variable. For example, the volume integral is defined as the expectation of the characteristic function. The expectation is estimated from a random sample of the random variable. In the case of the volume, the variance of the characteristic function is quite high. Thus, a large number of samples is required so

that the sample variance, which is a measure of the error of the estimated integral, is small. Stratified sampling in general will decrease the sampling variance. In stratified samples, we break up the domain into smaller subdomains so that the variance is smaller in each subdomain. For example, if a small block is chosen completely inside the object, then of course the variance of the characteristic function is zero. Likewise, the variance of the characteristic function completely outside the function is zero. Thus, only those blocks that contain boundary points contribute to the variance. Stratified sampling should decrease the overall variance and so fewer sample points should be required for the error to be within a given tolerance. When only one point is sampled from each subdomain, we have essentially a Riemann sum for the integral.

In general, the amount of work required to evaluate an integral numerically is proportional to the number of function evaluations required in one dimension raised to a power equal to the dimension of the space. This is true at least when the multiple integral is computed by iterated integration. Hence, if 100 function evaluations are required in one dimension, 1,000,000 evaluations are required in three dimensions, so it pays to reduce the dimension. By applying the divergence theorem, the integral over a volume can usually be reduced to the integral over a surface; so we need a technique for calculating surface integrals. Of course, to integrate on a surface, we must have some model of the surface; but in this modeling system we do not have such an explicit model. We have a multiple surface patch defined by projection. When the object is projected to two dimensions, to every point in the 2-dimensional region corresponds a set of points where the projection line pierces the object. This defines a multiple valued function.

A manifold is the mathematical generalization of a surface; it is a space together with a set of coordinate systems or surface patches. For a 2-dimensional manifold a surface patch is a 1-1 mapping from the manifold to an open set in 2-dimensional Euclidean space. A complete set of patches that covers the manifold is called an atlas. The projections from several projection points of our object will be a set of patches that completely covers the surface of the object. However, when we include all pierce points, these mappings are not 1-1 so they are not legitimate coordinate systems. Still, we can do integration on these multilayer patches because integration is loosely an adding up of all infinitesimal surface elements, and we may add the elements

in any order. We can add the elements corresponding to pierce points of a given projection line all at once. There is a difficulty: the set of patches that define a manifold are not disjointed so that when we add the results of integrating each patch, we may well be adding a given surface element many times. We need a partition of unity. A partition of unity is a set of weighting functions applied to each patch so that the sum of the weighted patch integrals is the whole surface integral (see Spivak). When our surface patches derive from projections in the x , y , and z directions, a natural partition of unity is the set of squares of the components of the surface normal vectors. The individual layers of each view must be considered as separate patches. So we have reduced our task to the calculation of a 2-dimensional integration over a rectangular domain. The integrand in general will be discontinuous and we must choose a suitable integration algorithm. Gaussian quadrature is not suitable because the integrand will be discontinuous. The generalized trapezoid method will work but may require more calculation and give less accuracy than necessary.

Since the moment of inertia about any axis can be obtained from the inertia tensor, it is economical to compute the inertia tensor rather than moments of inertia. However, there are two different tensors called the inertia tensor; one definition is given in mathematics books and another in some physics books. In order to clarify what is being calculated, I have included material concerning angular momentum, angular velocity, moments of inertia and their relation to the inertia tensor. We now begin a more detailed treatment of the material that heretofore has been scanned.

PROJECTIVE SPACE AND THE CONIC SECTIONS

Projective geometry is related to the theory of perspective, which was developed by Italian painters during the Renaissance. Consider the eye as the origin in a 3-dimensional coordinate system. Let a ray go out from the eye. Suppose this ray impinges on some point of an object. In some sense the ray represents the point. Let a picture plane be interposed between the eye and the object. Then the point where the ray intersects the picture plane is a projection of the object point to its picture image. Three-dimensional space is projected to 2-dimensional space. The concern of the artist is to accurately represent the object as it appears to the eye. He is concerned with vanishing points and lines, which are the images of points and lines at infinity and will be discussed in the section on

perspective. Here we confine ourselves to the 2-dimensional case. The artist Albrecht Dürer, who dabbled a bit in geometry (see Panofsky), constructed a device for recording the perspective image of an object in the picture plane by recording the position on the plane of a chord stretched from a fixed point to a point on the object, which can be seen in one of his works. The idea of projective geometry is to consider the above ray or chord a 2-dimensional point.

Projective space is formally defined as the set of 1-dimensional subspaces of a vector space. It can be illustrated by considering those plane curves which are called conic sections. Figure 1 shows a cone with its axis in the z direction and with a 90-degree vertex angle.

Figure 1

This cone meets the plane $z=1$ in a circle. A point on the surface of the cone satisfies the equation

$$x^2 + y^2 - z^2 = 0$$

The circle in the plane has the equation

$$x^2 + y^2 - 1 = 0$$

The first equation can be written as

$$(x, y, z) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or as

$$P^T A P = 0$$

where

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The function $Q(P) = P^T A P$ is a quadratic form. The cone is the set

$$S(Q) = \{P : Q(P) = 0\}$$

Let L be a nonsingular linear transformation with matrix B . Then L maps the set $S(Q)$ to

$$\begin{aligned} L(S(Q)) &= \{P : Q(L^{-1}P) = 0\} \\ &= \{P : (B^{-1}P)^T A (B^{-1}P) = 0\} \\ &= \{P : P^T (B^{-1})^T A B^{-1} P = 0\} \\ &= \{P : L(Q)(P) = 0\} \end{aligned}$$

where $L(Q)(P) = Q(L^{-1}(P))$. $L(Q)$ is a quadratic form with symmetric matrix

$$(B^{-1})^T A B^{-1}$$

Let L be the transformation of a rotation about the x -axes by an angle a . Let $\cos(a) = c$ and $\sin(a) = s$. Then the matrix of L is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

and $B^{-1} = B^T$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

Then $L(Q)$ has matrix

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & -s & -c \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^2 - s^2 & -2cs \\ 0 & -2cs & s^2 - c^2 \end{bmatrix} \end{aligned}$$

When the angle a is 45 degrees the matrix of $L(Q)$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

The equation of the rotated cone is

$$\begin{aligned} (x, y, z) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= (x, y, z) \begin{bmatrix} x \\ -z \\ -y \end{bmatrix} \\ &= x^2 - 2yz = 0 \end{aligned}$$

The equation of its intersection with the plane $z = 1$ is

$$x^2 - 2y = 0$$

or

$$y = (1/2)x^2$$

This is the equation of a parabola.

When the angle α is 90 degrees, the equation of the cone becomes

$$(x, y, z) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 - y^2 + z^2 = 0$$

The equation of its intersection with the plane $z = 1$ is

$$y^2 - x^2 = 1$$

This is the equation of a hyperbola. The plane that cuts the cone is called the affine plane. For each point in the plane there is a corresponding line through it and the origin. The set of lines that pass through the origin is called projective space. Each line is considered a point of the space. Those points in projective space--i.e., lines in 3-dimensional Euclidean space--that are parallel to the affine plane are called points at infinity, because they meet the plane only at infinity. Thus, there are more projective points than affine points. The coordinates of any vector in the direction of a line that passes through the origin are the homogeneous coordinates of the projective point represented by the line. Since the coordinates of any scalar multiple of the vector are also homogeneous coordinates of the projective point, homogeneous coordinates are determined only up to a scalar multiple. So, for examples, $(1, 2, 3)$ and $(2, 4, 6)$ are homogeneous coordinates of the same point. A point at infinity has coordinates of the form $(x, y, 0)$. If a point with coordinates (x, y, z) is not an ideal point--e.g., point at infinity, then its affine coordinates are $(x/z, y/z)$. One of the properties of projective space is that every pair of distinct lines meet in exactly one point. By a line we mean the locus of points that satisfy an homogeneous equation of the form

$$ax + by + cz = 0$$

The corresponding affine equation is

$$ax + by + c = 0$$

Then, lines that are parallel in the affine plane meet at a point at infinity. For example, the parallel lines with affine equations

$$ax + by + c = 0$$

and

$$ax + by + d = 0$$

meet at the point which has coordinates $(-b/a, 0)$. Affine rotations and translations are represented as linear transformations in projective space. A rotation has matrix

$$R = \begin{bmatrix} \cos(a) & \sin(a) & 0 \\ -\sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus

$$\begin{bmatrix} \cos(a) & \sin(a) & 0 \\ -\sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(a)x + \sin(a)y \\ -\sin(a)x + \cos(a)y \\ 1 \end{bmatrix}$$

The following calculation represents an affine translation

$$T_p = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ 1 \end{bmatrix}$$

The product of the two matrices above then both rotates and translates. This shows how affine transformations can be calculated using homogeneous coordinates.

A projective point is a set of all scalar multiples of a vector. It is a one dimensional subspace of a vector space. In general, projective n -space is the set of 1-dimensional subspaces contained in a vector space of dimension $n+1$. The nonsingular linear transformations on the vector space map the set of 1-dimensional subspaces onto themselves. These linear mappings are called projective transformations when they are applied to projective space. Considering the previous example, it is easy to see that an affine transformation is a special case of a projective transformation.

We now introduce quadratic forms. Let A be a symmetric matrix and let p be a coordinate vector.

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then $P^T A P$ is a quadratic form. For example if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Then

$$P^T A P = x^2 + y^2 + z^2 + 4xy + 6xz + 10yz$$

All quadratic forms can be obtained in this way. Corresponding to each quadratic form is an associated bilinear form or tensor of rank two. It is a function of two vector variables and is linear in each. It is given as

$$B(P, Q) = P^T A Q$$

A conic section S is the set of zeroes of a quadratic form in 2-dimensional space. Thus

$$S = \{P : P^T A P = 0\}$$

Given a point Q , the polar of q with respect to a quadric represented by a matrix A is the following set.

$$\{P : B(P, Q) = 0\}$$

Since the equation is linear, the polar of a point is a line in the 2-dimensional case and a plane in the 3-dimensional case. Directly from the definition, it is seen that P is on the polar of Q if and only if Q is on the polar of P . Properties of the polar can be obtained by using the concept of the cross ratio of four collinear points.

CROSS RATIO

Given four collinear points in affine space, the cross ratio is defined in terms of the directed distances between the points. Affine space is embedded in projective space, and the cross ratio can be extended to apply to the points of this space. It can be defined without the non-projective concept of distance. The importance of cross ratio is that it is invariant under projective transformations.

Before we define the cross ratio, a few remarks on notation are in order. A point may refer to several distinct concepts. A point may be an $n+1$ dimensional vector, or it may be the span of such a vector; that is, the 1-dimensional subspace generated by the vector, or it may be the corresponding affine point, which is the intersection of the subspace with the affine hyperplane. To complicate things further, we must distinguish between a vector and its corresponding coordinate vector. Its coordinates depend on the chosen basis of the vector space. To distinguish all of these things would result in too many symbols. So we will use the same symbols for all of the above ideas and hope that context will give the desired meaning. For example, if we add two points, they are vectors, for addition is not defined for projective points.

Let A, B, C , and D be four collinear points. In affine space, define the cross ratio as

$$(d(C,A)/d(C,B))/(d(D,A)/d(D,B))$$

where $d(X,Y)$ is a directed distance from X to Y . Now let A, B, C , and D be vectors representing the points. Numbers a and b can be found so that aC and bD lie on the straight line connecting A and B . Then there exists a t such that

$$aC = (1-t)A + tB$$

and

$$aC - B = (1-t)A - (1-t)B$$

Then

$$d(C,A)/d(C,B) = (A-aC)/(B-aC) = -t/(1-t)$$

It follows that if $C = c_1A + c_2B$ then

$$d(C,A)/d(C,B) = -c_2/c_1$$

Similarly, if $D = d_1A + d_2B$, then

$$d(D,A)/d(D,B) = -d_2/d_1$$

So the cross ratio is

$$(c_2/c_1)/(d_2/d_1) = (c_2d_1)/(c_1d_2)$$

Also, this ratio is independent of which representative vectors are chosen for the points. For example, if A is

replaced by a new vector that is a multiple of A, then c and d are multiplied by the same constant. So the cross ratio is not changed. Thus, the cross ratio is defined independently of a distance function or metric. The preservation of the cross ratio by a projective transformation follows because it is a linear transformation of the vector space.

PROPOSITION. A projective transformation preserves the cross ratio.

PROOF. Let A, B, C, and D be representative vectors of four distinct collinear points. Suppose

$$C = c_1A + c_2B \text{ and } D = d_1A + d_2B$$

Then, if L is the linear transformation corresponding to the given projective transformation, we have

$$L(C) = c_1L(A) + c_2L(B) \text{ and}$$

$$L(D) = d_1L(A) + d_2L(B)$$

So the points $L(A), L(B), L(C)$, and $L(D)$ have the same cross ratio.

A set of collinear points is called a range of points. A range of four points is called a harmonic range when its cross ratio is equal to -1. The name can be explained in the following way. Suppose the distance from A to C is $1/3$, the distance from B to C is $-1/6$, and the distance from B to D is $1/2$. Then the cross ratio is -1 so the points constitute a harmonic range. The distance from A to D is 1, the distance from A to B is $1/2$, and the distance from A to C is $1/3$. Thus, vibrating strings of these lengths produce the first, second, and third harmonic tones.

A point Q is called the harmonic conjugate of P with respect to A and B when $(AB, PQ) = -1$.

PROPOSITION. If a line through P meets a quadric at A and B, then the harmonic conjugate Q lies on the polar of P.

PROOF. Suppose $P = p_1A + p_2B$ and $Q = q_1A + q_2B$ then $p_2q_1 = -p_1q_2$ so

$$q_2P = q_2p_1A + q_2p_2B$$

and

$$p_2 Q = p_2 q_1 A + p_2 q_2 B$$

Subtracting, we find $A = (1/2)(P/p_1 + Q/q_1)$. Similarly,

$$B = (1/2)(P/p_2 + Q/q_2)$$

Let f be the bilinear form of the quadric. Since A is on the quadric, we have

$$0 = f(A, A) = (q_1/p_1)^2 f(P, P) + 2(q_1/p_1) f(P, Q) + f(Q, Q)$$

Similarly

$$0 = (q_2/p_2)^2 f(P, P) + 2(q_2/p_2) f(P, Q) + f(Q, Q)$$

Subtracting and noting that $q_1/p_1 = -q_2/p_2$ we find that $f(P, Q)$ must be zero.

This means that Q is on the polar of P .

COROLLARY. If a line through P is tangent to a quadric at Q , then Q is on the polar of P .

INFORMAL PROOF. Rotate the line through P slightly if necessary, so that it intersects the quadric in two near points. The harmonic conjugate of P is between these two points and from above is on the polar of P . As the line approaches the tangent line, this point approaches Q .

CONSTRUCTION. To construct a tangent to a conic from a point P , find the intersection of the conic with the polar of P .

COROLLARY. The locus of the harmonic conjugates of p with respect to the intersection points of lines through p with the conic is the polar of P .

EXAMPLE.

Let lines PQ and PR be tangent to the ellipse. We conclude that Q and R are on the polar of P. Thus, the line defined by Q and R is the polar of P. It also follows that S is on both the polar of P and the polar of P'. Thus, the polar of S is the line through P and P'.

A diameter of a quadric is the polar of a point at infinity. Thus "diameter" is an affine idea because "point at infinity" only makes sense when some affine hyperplane has been identified.

We may represent a line through P and Q by $P + tQ$ where the parameter t varies from minus to plus infinity. t equal to infinity corresponds to the point Q. We may think of this as the line through P in the direction Q. When a line meets a quadric in two finite intersection points R and S, the line segment from R to S is called the chord determined by the line. A diameter in the direction Q is the polar of Q where Q is a point at infinity.

Consider the intersection of the line $P + tQ$ with a quadric. There are several cases. Let the bilinear form be f . Then

$$\begin{aligned} 0 &= f(P + tQ, P + tQ) \\ &= f(P, P) + 2tf(P, Q) + t^2f(Q, Q) \end{aligned}$$

CASE 1. $f(Q, Q) = 0, f(P, Q) \neq 0$
Q is one intersection point and $t = -f(P, P)/(2f(P, Q))$ gives the other.

CASE 2. $f(Q, Q) = 0, f(P, Q) = 0, f(P, P) \neq 0$ the two intersection points coincide. They equal Q.

CASE 3. $f(Q, Q) = 0, f(P, Q) = 0, f(P, P) = 0$ the line is a ruling. All points lie on the conic.

CASE 4. $f(Q, Q) \neq 0$ there are two roots which may be real, or complex and may coincide.

PROPOSITION. A diameter in the direction Q bisects all chords in the direction Q.

PROOF. Since Q is a point at infinity and the line determines a chord with two finite intersection points R and S, from above we must have $f(Q, Q) \neq 0$. For, otherwise, Q would be an intersection point. Using the quadratic formula for the two roots, substituting in $P + tQ$, adding and

dividing by two, we find

$$(R + S)/2 = P - (f(P,Q)/f(Q,Q))Q$$

Then

$$f((R + S)/2, Q) = f(P, Q) - f(P, Q)f(Q, Q)/f(Q, Q) = 0$$

So the midpoint of the chord lies on the polar of Q , which is the diameter determined by Q .

EXAMPLE. Consider the parabola $x^2 - 2yz = 0$. Let

$$P = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The matrix of this conic is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Q does not determine a chord because the line does not meet the conic in two finite intersection points. In fact, Q , which is a point at infinity, lies on the conic. The polar of Q has line coordinates given by

$$Q^T A = (0, 0, -1)$$

The equation of the polar of q , which is a diameter of the parabola, is

$$-z = 0$$

This is the line at infinity.

EXAMPLE. Consider the same example as above but let

$$Q = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Then the polar of Q has equation $x - z = 0$. Thus, the affine equation of the diameter corresponding to Q is $x = 1$.

TANGENT LINE

Let P and Q be distinct points. Let P be on the conic.

PROPOSITION. $P + tQ$ is tangent to the conic at P if and only if $f(P,Q) = 0$.

PROOF. First, suppose $P + tQ$ is tangent at P . Then the intersection points coincide and equal P . Since P is on the quadric, $f(P,P) = 0$. But the intersection equation is then

$$2tf(P,Q) + t^2f(Q,Q) = 0$$

and its roots must be zero. So $f(P,Q) = 0$.

Now suppose $f(P,Q) = 0$. It is given that $f(P,P) = 0$. The intersection equation is

$$t^2f(Q,Q) = 0$$

If $f(Q,Q) \neq 0$, then the roots of the equation are zero, so P is a double intersection point. If $f(Q,Q) = 0$ the line is a ruling. Hence, in both cases, the line is the tangent line.

COROLLARY. The tangent line $P + tQ$ at P is the polar of P .

PROOF. Since the line passes through P , we have $f(P,P) = 0$ so P is on the polar of P . From above $f(P,Q) = 0$ so Q is on the polar of P . Therefore, the line through P and Q is the polar of P .

We have taken the tangent line as the line that intersects the conic at two coinciding points. With this definition the tangent line does not have points on both sides of the conic curve. For consider the set of points where $f(P,P) < 0$ and the set where $f(P,P) > 0$. These sets are bounded by the conic curve. We may consider one the inside and the other the outside of the curve. Suppose P and Q are on opposite sides of the conic. Then $f(P,P)f(Q,Q) < 0$. Then the intersection equation of the line through P and Q has positive discriminant. So the line meets the conic in two real points that do not coincide. Hence, the line through P and Q is not a tangent line.

If the point Q is on the polar of P , then

$$f(P,Q) = P^T A Q = 0$$

Suppose

$$P^T A = (a, b, c) \quad \text{and} \quad Q = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then the first vector is called the line coordinate vector and the equation of the polar is

$$ax + by + cz = 0$$

The 2-dimensional vector

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

is normal to the line. For if

$$\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \quad \text{are on the line, we have by subtraction}$$

$$(x - x_2)a + (y_1 - y_2)b = 0$$

EXAMPLE. Consider the hyperbola $x^2 - y^2 = 1$

$$\text{Let } Q = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Then } f(Q, Q) = f(P, Q) = 0$$

$$\text{Since } AQ = (1, -1, 0)$$

The equation of the tangent line at the ideal point Q is $x - y = 0$. It is an asymptote of the hyperbola.

The set of centers is the intersection of all diameters. Thus, P is a center if $f(P, Q) = 0$ for all ideal points Q. Thus, the set of centers is the set of all points P such that

$$AP = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If A is nonsingular, then the conic has only one center.

EXAMPLE. Consider the conic $(x - 1)^2 + y^2 = 1$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Hence the center is $x = 1, y = 0, z = 1$.

PARAMETRIC CONIC ARCS

A conic arc is simply a portion of a conic curve. It is desirable that such a curve be represented parametrically. This can be done so that each coordinate is a rational function of a single variable. To construct such a function, we map a certain parabola to the required arc, using a projective transformation.

PROPOSITION. If L and M represent the same projective transformation, L is a scalar multiple of M.

PROOF. We shall write $[v]$ for the subspace spanned by the vector V. Let V and U be linearly independent. Let P be the given projective transformation. Then

$$P([U]) = [L(U)] = [M(U)]$$

So there is a number a so that $L(U) = aM(U)$. Similarly, there is a number b so that $L(V) = bM(V)$. Then

$$\begin{aligned} P([U + V]) &= [L(U) + L(V)] \\ &= [M(aU) + M(bV)] \\ &= P([aU + bV]) \end{aligned}$$

But a projective transformation is one to one. Therefore, there is a c, so that

$$U + V = c(aU + bV)$$

Then by the linear independence of U and V, we have

$$ca = 1 = cb$$

It follows that $a = b$ and $L = aM$.

Now we shall show that there is a projective

transformation mapping four points to four points.

PROPOSITION. Let $A, B, C,$ and D be distinct points, no three of which are collinear. Let $A', B', C',$ and D' be points with the same properties. Then there is a unique projective transformation taking the first four to the second four.

PROOF. Consider the points as coordinate vectors. There are numbers a, b, c so that

$$D = aA + bB + cC$$

This is true because $A, B,$ and C are a basis of the vector space. Similarly there are numbers a', b', c' so that

$$D = a'A + b'B + c'C$$

Define a'', b'', c'' by

$$a'' = a'/a, b'' = b'/b, \text{ and } c'' = c'/c$$

None of $a, b,$ or c are zero because no three of the given points are collinear. Define a matrix M as a product of partitioned matrices.

$$M = \begin{bmatrix} a''A' & b''B' & c''C' \end{bmatrix} \begin{bmatrix} A & B & C \end{bmatrix}^{-1}$$

Then the projective transformation defined by M maps the first three unprimed points to the first three primed points. Also, D is mapped to D' as can be seen as follows.

$$\begin{aligned} MD &= aMA + bMB + cMC \\ &= aa''A' + bb''B' + c'c''C' \\ &= a'A' + b'B' + c'C' = D' \end{aligned}$$

It remains to show that the projective transformation defined by M is unique. From above, we have

$$MA = a''A', MB = b''B', MC = c''C', \text{ and } MD = D'$$

Suppose there is a matrix N and numbers $\bar{a}, \bar{b}, \bar{c},$ and $\bar{d}$ so that

$$NA = \bar{a}A', NB = \bar{b}B', NC = \bar{c}C', \text{ and } ND = \bar{d}D'$$

Then

$$\begin{aligned} ND &= N(aA + bB + cC) \\ &= \bar{a}a A' + \bar{b}b B' + \bar{c}c C' \end{aligned}$$

and $ND = \bar{d}D'$ so

$$D' = (aa/d)A' + (bb/d)B' + (cc/d)C'$$

So $a\bar{a}/\bar{d} = a'$, $b\bar{b}/\bar{d} = b'$, and $c\bar{c}/\bar{d} = c'$

Then $\bar{a} = \bar{d}(a'/a) = \bar{d}a''$, $\bar{b} = \bar{d}b''$, and $\bar{c} = \bar{d}c''$. So $N = dM$ and N and M generate the same projective transformation.

We can now construct the rational parametric arc. The parabola with equation $x - y^2 = 0$ and matrix

$$\begin{bmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

is to be mapped to the required conic arc. If we take $t = y$ then

$$\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

is a point on the parabola. Points A, B, and D lie on the parabola and are given as

$$A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix}$$

C is the intersection point of the tangent lines at A and B. The tangent line at B has equation

$$(x, y, 1) \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (1/2)x - y + 1/2 = 0$$

so

$$c = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$$

For future reference note that

$$D = aA + bB + cC$$

where $a = 1/4$, $b = 1/4$, and $c = 1/2$

It takes five points to determine a conic. Here we do have five pieces of information, namely three points which lie on the conic and two tangents.

This parabola is the set $\{P:f(P,P) = 0\}$. A projective transformation M maps it to the conic $\{P':f'(P',P') = 0\}$ where $f'(P',P') = f(M^{-1}(P'),M^{-1}(P'))$. By the construction above, let M be the projective transformation that takes the four points defining the parabola to the four primed points defining some conic arc. From the definition of f' we see that

$$f'(C',A') = f(C,A) = 0$$

and

$$f'(C',B') = f(C,B) = 0$$

So C' is the intersection of the tangents to the conic arc at A' and B' . A parametric equation of the arc from A' to B' is given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}, \text{ for } 0 \leq t \leq 1$$

EXAMPLE. Let the arc be a half circle.

$$A' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, B' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, D' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For this example we see by inspection that

$$C' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

That is, the intersection of the tangents at A' and B' , is the point at infinity in the direction of the y-axis. In general we may find the polars of A' and B' and then calculate the cross product of their line coordinates to get C' . We need the coordinates a' , b' , c' of D' with respect to the basis A' , B' , C' . We have

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} | & | & | \\ A' & B' & C' \\ | & | & | \end{bmatrix}^{-1} D' \\ = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

We know a, b, c from above so

$$a'' = a'/a = (1/2)/(1/4) = 2, \quad b'' = b'/b = 2, \quad c'' = c'/c = 2$$

Now

$$M = \begin{bmatrix} | & | & | \\ a''A' & b''B' & c''C' \\ | & | & | \end{bmatrix} \begin{bmatrix} | & | & | \\ A & B & C \\ | & | & | \end{bmatrix}^{-1} \\ = 2 \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1/2 \\ 1 & 1 & 1 \end{bmatrix} \\ = 2 \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ -2 & 2 & 0 \end{bmatrix} \\ = \begin{bmatrix} 0 & 4 & -2 \\ -4 & 4 & 0 \\ 4 & -4 & 2 \end{bmatrix}$$

Finally, we get

$$\begin{bmatrix} x' \\ y' \\ x' \end{bmatrix} = M \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

So, in the affine plane we have

$$\bar{x}' = (2t - 1)/(2t^2 - 2t + 1)$$

and

$$\bar{y}' = (2t - 2t^2)/(2t^2 - 2t + 1)$$

Although there is much calculation involved here, it is still advantageous to calculate the rational parametric form when the arc is frequently displayed or plotted.

STRAIGHT LINES IN PROJECTIVE SPACE

The equation of a line is of the form

$$ax + by + cz = 0$$

The coordinates of the line

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

are perpendicular to the coordinates of the point

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

If A and B are coordinate vectors of two lines, then the cross product is the coordinate vector of the intersection point because it is perpendicular to both. In the same way, if X and Y are coordinate vectors of two points, then their cross product is the coordinate vector of the line containing both. This is an example of duality in projective space.

EXAMPLE. Find the intersection point of the two lines with equations

$$x + y = 0 \text{ and } 2x + y = 1$$

The cross product is

$$\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$$

So the affine coordinates of the intersection point are $(-2, 5)$.

EXAMPLE. Find the line through the points

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The cross product is

$$\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

So the equation is

$$-x + 2y = 1$$

EXAMPLE. Find the line passing through $(1, 1)$ and make an angle of 30 degrees with the x-axis. The line passes through the point at infinity given by

$$\begin{bmatrix} \cos(30) \\ \sin(30) \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{bmatrix}$$

and the point

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Taking the cross product, we find the line to be

$$x - \sqrt{3} y = 1 - \sqrt{3}$$

QUADRIC SURFACES

Quadric surfaces are the analogue of conic sections in 3-dimensional space. This space is constructed in a 4-dimensional vector space so there are four coordinates for each point. Again, corresponding to each quadric is a bilinear form f . The quadric surface is the set of points P where $f(P,P) = 0$. The properties and propositions that have been treated for 2-space are essentially the same in 3-space.

The equation for the intersection of a line $P + tQ$ is still

$$0 = f(P + tQ, P + tQ) = f(P,P) + 2tf(P,Q) + t^2f(Q,Q)$$

and the various cases are the same as for the conics. If P is a point, then

$$(P,Q) = 0$$

is the equation of a plane where (P,Q) is the inner product of P and Q . Q is the coordinate vector of the plane. We have a duality between points and planes. For example, three points determine a plane; dually three planes determine a point. In two dimensions the duality was between points and lines.

The first three coordinates of the plane give a vector normal to the plane in affine space where the affine hyperplane has equation $w = 1$. The homogeneous coordinates of a point are written as x, y, z, w . If P is a point, then the set of points Q where

$$f(P,Q) = 0$$

is the polar of P . In 3-space, the polar is a plane. If A is the symmetric matrix of the quadric

$$f(P,Q) = P^T A Q = Q^T A P = f(Q,P) = (Q,AP)$$

If P is on the quadric surface, then the polar of P is the tangent plane. This can be seen by examining the equations of intersection of lines in the tangent plane passing through P . The equation of the tangent plane is then

$$(Q,AP) = 0$$

and AP is the coordinate vector of this plane. The first

three coordinates of AP are the coordinates of a normal vector.

A diametrical plane in the direction Q, where Q is a point at infinity, is the polar of Q. The diametrical plane is the analogue in 3-space of the diameter in 2-space. A center is a point in the intersection of all diametrical planes. Points on a quadric surface are called singular points if they are centers, and regular points otherwise. An example of a singular point is the apex of a cone. The statement above that the polar of a point on the surface is the tangent plane must be qualified. It holds only for regular points. Obviously, the apex of a cone has no well-defined tangent plane.

As in the 2-dimensional case, a diametrical plane in the direction Q bisects all chords in the direction Q. The 2-dimensional proof holds.

QUADRIC SURFACES AND THEIR MATRICES

There are 17 quadric surfaces (see Olmsted). Some are complex and some are degenerate. The names, equations, and matrices of some of the real nondegenerate quadric surfaces are:

(1) Sphere: $x^2 + y^2 + z^2 - w^2 = 0$

Matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2) Hyperbolic paraboloid: $x^2 - y^2 - 2wz = 0$

Matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

(3) Elliptic paraboloid: $x^2 + y^2 - 2zw = 0$
matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

(4) Parabolic cylinder: $x^2 - 2yz = 0$

matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(5) Hyperbolic cylinder: $x^2 - y^2 - w^2 = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(6) Intersecting planes: $x^2 - y^2 = 0$
matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(7) Parallel planes: $x^2 - w^2 = 0$
matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(8) Circular cylinder: $x^2 + y^2 - w^2 = 0$
matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(9) Hyperboloid of two sheets: $x^2 + y^2 - z^2 + w^2 = 0$
matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(10) Cone: $x^2 + y^2 - z^2 = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(11) Hyperboloid of one sheet: $x^2 + y^2 - z^2 - w^2 = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

TRANSFORMATIONS

A one to one linear transformation maps a quadric surface to a new quadric surface. Suppose

$$S = \{P: f(P, P) \leq 0\}$$

then

$$\begin{aligned} LS &= \{LP: f(P, P) \leq 0\} \\ &= \{Q: f(L^{-1}Q, L^{-1}Q) \leq 0\} \\ &= \{Q: (L^{-1}Q, AL^{-1}Q) \leq 0\} \\ &= \{Q: (Q, L^{-1T}AL^{-1}Q) \leq 0\} \\ &= \{Q: f^*(Q, Q) \leq 0\} \end{aligned}$$

where f^* is a bilinear form with matrix $L^{-1T}AL^{-1} = \bar{A}$

The matrices for the scaling, translating and rotation transformations are given below.

(1) Scaling:

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse:

$$\begin{bmatrix} 1/a & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2) Affine rotation: Let $c = \cos(\theta)$ and $s = \sin(\theta)$. The sense of the rotations about the unit coordinate vectors is given by the right-hand rule. With the thumb of the right hand extended in the direction of the vector, positive angle is the direction in which the fingers curl while grasping the vector.

Rotation about the x-axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about the y-axis:

$$\begin{bmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about the z-axis:

$$\begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now if R is a rotation matrix then

$$R^{-1}(\theta) = R(-\theta)$$

and for an orthogonal matrix $R^T = R^{-1}$, so the matrix of the transformed quadric surface is

$$\bar{A} = R(\theta)AR(-\theta)$$

(3) Affine translation

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translates by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and T^{-1} is

$$\begin{bmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\bar{A}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a & -b & -c & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

EXAMPLE.

Given the sphere with matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The scaling transformation matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Maps the sphere to an ellipse with matrix

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The translation matrix T shifts it by 2

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix of the shifted conic becomes

$$(T^{-1})^T (S^{-1})^T A S^{-1} T^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$$

The equation of the shifted ellipsoid is

$$x^2/4 + y^2 + z^2 + 1/2 xw + 1/2 wx = 0$$

Suppose it is now rotated about the z-axis by angle 45 degrees. The rotation matrix is

$$R_z(45) =$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix of the quadric becomes

$$R_z(45) (T^{-1})^T (S^{-1})^T A S^{-1} T^{-1} R_z(-45)$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/(4\sqrt{2}) & 1/(4\sqrt{2}) & 0 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/(2\sqrt{2}) & 1/(2\sqrt{2}) & 0 & 0 \end{bmatrix}$$

The equation of the shifted rotated quadric is

$$(5/8)x^2 + (5/8)y^2 - (3/4)xy + z^2 + xw/2 + yw/\sqrt{2} = 0$$

CONSTRUCTION OF SOLIDS

A primitive solid is the set of points "inside" or "outside" a quadric surface. If f is the bilinear form of the quadric surface, the primitive solid is

$$\{P: f(P,P) \leq 0\}$$

A solid object is constructed by combining the primitive solids using the set operators union, intersection, and complementation. The characteristic function $C(S)$ corresponding to the set S is a logical valued function mapping points to a value true or false. $C(S)(P)$ is true if P is in S ; otherwise, it is false. By definition

$P \in A \cup B$ if and only if $P \in A$ or $P \in B$

So the characteristic function of $A \cup B$ is

$C(A \cup B) = C(A) \vee C(B)$ where $\vee$ is the logical

"or" operator. Similarly

$C(A \cap B) = C(A) \wedge C(B)$ and $C(A) = \neg C(\neg A)$

where $\wedge$ is the logical "and" operator and $\neg$ is the negation operator. Objects are thus defined as logic-valued functions whose domain is 3-space. These functions may be represented in the FORTRAN language.

PLANE IMAGE

A shaded image is produced by finding the intersection of the object and a line, which passes through the projection point. With L such a line, its intersection with each primitive is calculated. Corresponding to each such point is a normal direction. A small line segment of fixed length, in the normal direction with center at the intersection point, determines when the point is on the surface of the object. It is on the object when one endpoint is inside and the other is outside the object. This is determined by the characteristic function.

One or more light sources illuminate the object. The inner product of these light source vectors with the surface normal give an intensity function defining the shading. The point on the surface is projected to 2-space with the calculated shading value. Note that the projection point may be at infinity.

MONTE CARLO INTEGRATION

An integral on a 3-dimensional set can be evaluated by enclosing the set in a hexahedron and generating a random sample from a uniform distribution. Suppose the integral

$$\int_S f(x) dm$$

is to be evaluated. We write x to denote a point. Let B be a rectangular hexahedron enclosing S . Let m be the volume measure (Lebesgue measure). Define a probability measure on the set B by

$$p(S) = m(S)/m(B)$$

Let $C(S)$ be the characteristic function of the set S . $C(S)(x)$ is 1 if x is in the set S and 0 otherwise. Let g be the product of f and $C(S)$. Then g is a random variable defined on the enclosing set B and its expectation is

$$E(g) = \int_B g dp = \int_S f dp = (1/m(B)) \int_S f dm$$

Thus

$$\int_S f dm = m(B) E(g)$$

Let $x_1, \dots, x_n$ be a sample of independent uniformly distributed random variables. Then $g(x_1), \dots, g(x_n)$ is also independent. Let $\bar{g}$ be the sample mean

$$\bar{g} = (g(x_1) + \dots + g(x_n))/n$$

Then the expectation is

$$E(\bar{g}) = E(g)$$

The variance is

$$\text{var}(\bar{g}) = E((\bar{g} - E(g))^2) = E(\bar{g}^2) - E(\bar{g})^2 = \text{var}(g)/n$$

By the central limit theorem, the random variable

$$(\bar{g} - E(\bar{g}))/\sqrt{\text{var}(\bar{g})} = (\bar{g} - E(g))/\sqrt{\text{var}(g)/n}$$

has for sufficiently large n approximately the standard normal distribution. Now

$$\frac{1}{\sqrt{2\pi}} \int_{-1.96}^{1.96} \exp(-x^2/2) dx \approx .95$$

So

$$\text{prob}(|\bar{g} - E(g)| \leq 1.96\sqrt{\text{var}(g)/n}) \approx .95$$

This gives a 95% confidence interval for the estimated value of the mean. Multiplying by $m(B)$ we get a confidence interval for the integral.

$$\text{Prob}(|m(B)\bar{g} - \int_S f dm| \leq 1.96m(B)\sqrt{\text{var}(g)/n}) \approx .95$$

We specifically want to evaluate the integrals:

- (1) Volume; $f=1$
- (2) Moments; $f=x_i, i=1,2,3$
- (3) Inertia tensor: $f = x_i x_j, i,j = 1,2,3$

EXAMPLE. Let S be a sphere of radius a . Let B be the cube of side $2a$ centered at the origin. Then $m(B) = 8a^3$ and $m(S) = (4/3)\pi a^3$. Let $f = 1$ and $g = C(S)$. Then the expectation of g is

$$E(g) = \int_S g(x) dp = \int_S dp = p(S) = ((4/3)\pi a^3)/(8a^3) = \pi/6$$

The variance of g is

$$v(g) = \int_S g^2 dp - E(g)^2 = E(g)(1-E(g))$$

Let us choose the sample size n so that the relative error of the estimated volume is less than $1/100$. The error is

$$1.96 m(B)\sqrt{E(g)(1-E(g))/n}$$

We can get a relative error by dividing by the volume $m(B)E(g)$. We get

$$1.96 \sqrt{(1-E(g))/E(g)/n} < .01$$

$$(1/E(g) - 1) < (.01/1.96)^2 n$$

$$n > (1.96)^2 ((6/\pi) - 1) = 34953$$

This is a large number of samples.

How should B be chosen? For volume the error is

$$1.96 m(B) \sqrt{[m(S)/m(B)][1-(m(S)/m(B))]/n}$$

$$= 1.96 \sqrt{m(S)[m(B) - m(S)]/n}$$

So we must minimize a function of the form

$$h(x) = \sqrt{c(x-c)} \quad \text{where } x \geq c$$

Because

$$dh/dx = (1/2)c/\sqrt{c(x-c)}$$

The function is increasing. The minimum occurs at $x = c$. It follows that the box B should be as small as possible.

Usually we do not know the variance $v(g)$, so we use the sample variance as an estimate. It is defined as

$$\begin{aligned} S &= ((g(P_1) - \bar{g})^2 + \dots + (g(P_n) - \bar{g})^2)/n \\ &= (g(P_1)^2 + \dots + g(P_n)^2)/n - \bar{g}^2 \end{aligned}$$

An unbiased estimator of the variance of g is (p. 165 Brunk)

$$(n/(n-1))S^2$$

So

$$\int_S f dm = m(B)g + 1.96m(B) \sqrt{S^2/(n-1)}$$

EXAMPLE. Calculate the x_1 moment of the sphere in the previous example. Then $f = x_1$ and

$$E(g) = \int_B g dp = (1/8a^3) \int_S x_1 dm = 0$$

By symmetry the variance is

$$(1/8a^3) \int_S x_1^2 dm = (\pi/8a^3) \int_{-a}^a x_1^2 (a^2 - x_1^2) dx = \pi a^2/30$$

So the error is

$$(1.96a^4 \sqrt{\pi/(30n)})$$

In this case we can't divide by the moment to get a relative error because the moment is zero. But a moment is the product of a volume and a length, so a relative error results from dividing by a^4 .

EXAMPLE. We calculate the inertia tensor components for the sphere. By symmetry

$$\int_S x_i x_j dm = 0 \text{ for } i \neq j \text{ and}$$

$$\begin{aligned} \int_S x_i^2 dm &= \int_{-a}^a x_i^2 (a^2 - x_i^2) dx = \pi [(x_i^3 a^2)/3 - x_i^5/5] \Big|_{-a}^a \\ &= 2\pi a^5 (1/3 - 1/5) \\ &= (4/15)\pi a^5 \end{aligned}$$

STRATIFIED SAMPLING

It may be possible to decrease the error by doing stratified sampling. Divide the domain B into m subdomains. Introduce a probability measure p_i on each B_i by

$$p_i(S_i) = m(S_i)/m(B_i)$$

Then

$$\int_{S \cap B_i} f dm = m(B_i) \int_{B_i} g_i dp_i = E(g_i) m(B_i)$$

where g_i is the product of f and the characteristic function $C(S_i)$ of S_i . Then

$$\int_S f dm = m(B_1) \int_{S \cap B_1} g_1 dp_1 + \dots + m(B_m) \int_{S \cap B_m} g_m dp_m$$

Let $h_i = m(B_i) g_i$. Then defining $h = h_1 + h_2 + \dots + h_m$

$$\int_S f dm = E(h) = E(h_1) + \dots + E(h_m)$$

Also

$$\text{var}(h) = \text{var}(h_1) + \dots + \text{var}(h_m)$$

Let $h_{i1}, \dots, h_{im}$ be an independent sample of random variables having the distribution of h_i . We take k samples from each stratum. Let

$$b_i^2 = \text{var}(h_i) \text{ and}$$

$$\begin{aligned} B^2 &= \text{var}(h_{11}) + \dots + \text{var}(h_{mk}) \\ &= k(b_1^2 + \dots + b_m^2) \end{aligned}$$

Let $a_i = E(h_i)$

and $A = k(a_1 + \dots + a_m)$

Consider

$$\begin{aligned} & ((h_{11} - a_1) + (h_{12} - a_1) + \dots + (h_{mk} - a_k)) / B \\ &= (h_{11} + \dots + h_{mk} - k(a_1 + \dots + a_m)) / \sqrt{k(b_1^2 + \dots + b_m^2)} \\ &= ((h_{11} + \dots + h_{mk}) / k - \int f dm) / \sqrt{(b_1^2 + \dots + b_m^2) / k} \end{aligned}$$

Let $y_k = (h_{11} + \dots + h_{mk}) / k$ and $v(k) = \sqrt{(b_1^2 + \dots + b_m^2) / k}$. Then the Lindeberg condition is satisfied (p.455 Eisen) and as $k \rightarrow \infty$

$$p((y_k - \int f dm) / v(k) \leq x) \rightarrow \sqrt{1/2\pi} \int_{-\infty}^x \exp(-t^2/2) dt$$

So

$$p(|y_k - \int f dm| \leq 1.96 v(k)) \approx .95$$

y_k is a better estimate of the integral than $m(B)\bar{g}$ provided

$$v(k) < m(B) \sqrt{\text{var}(g) / (mk)} \quad (\text{There are } mk \text{ samples})$$

Suppose $\text{var}(g_i) = \text{var}(g)$ or else is zero. Then

$$b_i^2 = \text{var}(h_i) = m(B_i)^2 \text{var}(g_i) = \begin{cases} m(B_i)^2 \text{var}(g) \\ 0 \end{cases}$$

Suppose j of the b_i^2 are not zero and suppose the strata are equal so that $m(B_i) = m(B)/m$. Then

$$\begin{aligned} v(k) &= \sqrt{(b_1^2 + \dots + b_k^2) / k} = \sqrt{(j(m(B)^2 / m^2) \text{var}(g) / k)} \\ &= \sqrt{(j/m)} m(B) \sqrt{\text{var}(g) / (mk)} \\ &\leq m(B) \sqrt{\text{var}(g) / (mk)} \end{aligned}$$

Thus, when $j < m$, because many strata will be completely inside or outside the object and hence will have zero variance, y should be a better estimate of the integral.

POINT TRANSFORMATIONS AND COORDINATE TRANSFORMATIONS

Let

$$u_1, u_2, u_3 \quad \text{and} \quad u'_1, u'_2, u'_3$$

be bases of a vector space and let c and c' be corresponding coordinate maps. That is, for example, c maps a vector to its coordinate vector with respect to the unprimed basis. Let A be the change of coordinate matrix so that

$$c'(P) = Ac(P) \quad \text{and} \quad c(P) = A^{-1}c'(P)$$

Let T be a linear transformation defined by

$$T(u_i) = u'_i$$

Then

$$c(P) = c'(TP) = Ac(TP)$$

Thus

$$c(TP) = A^{-1}c(P)$$

And

$$c'(TP) = c(P) = A^{-1}c'(P)$$

Hence the matrix of T is A^{-1} in both bases. In summation we have shown that when an unprimed basis is transformed to a primed basis, the matrix of the coordinate transformation from the unprimed to the primed coordinate system is the inverse of the matrix of the point transformation.

CROSS PRODUCT, AXIAL VECTORS

The cross product of two vectors in a Euclidean space is defined by its components with respect to an orthogonal coordinate system by

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} u_1 & u_2 & u_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= \epsilon_{ijk} a^i b^j u^k \end{aligned}$$

is the permutation symbol and u_1, u_2, u_3 are unit coordinate vectors. The k th component of $\mathbf{a} \times \mathbf{b}$ is

$$c_k = \epsilon_{ijk} a^i b^j$$

Let there be a new primed coordinate system. In general

$$\epsilon_{ijk} \partial x^i / \partial x'^p \partial x^j / \partial x'^q \partial x^k / \partial x'^r = \det(\partial x^i / \partial x'^j) \epsilon_{pqr}$$

(Lass page 8). The above definition of $\mathbf{a} \times \mathbf{b}$ depends on the coordinate system. But if the coordinate systems are related by a proper orthogonal transformation, then

$$\det(\partial x^i / \partial x'^j) = 1$$

So the permutation symbol transforms as a tensor. Then the components of $\mathbf{a} \times \mathbf{b}$ are obtained by contracting the product of a tensor and two vectors and is thus a Cartesian vector.

A second order tensor whose components obey

$$c_{ij} = -c_{ji}$$

is called antisymmetric. An antisymmetric tensor defines a Cartesian vector. The components of this vector are

$$c_k = -(1/2) \epsilon_{ijk} c_{ij}$$

These are the components of a Cartesian vector because it is the contraction of a Cartesian tensor and a general tensor. We find

$$c = \begin{bmatrix} c_{32} \\ -c_{31} \\ c_{21} \end{bmatrix}$$

A mnemonic is

$$\begin{bmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix}$$

$\uparrow$ $\leftarrow$
 c_{31} c_{32}

c is called an axial vector.

let d_r be the r th component of the cross product $c \times x$, then

$$d_r = \epsilon_{pqr} (-1/2) \epsilon_{ijp} c_{ij} x^q$$

We claim that

$$\epsilon_{pqr} (-1/2) \epsilon_{ijp} c_{ij} = c_{rq}$$

Now

$$\begin{aligned} & -(1/2) \epsilon_{pqr} \epsilon_{pij} c_{ij} \\ &= \sum_{p=1}^3 (-1/2) \epsilon_{pqr} \epsilon_{pij} c_{ij} \\ &= \sum_{p \neq q, r} (-1/2) \epsilon_{pqr} \epsilon_{pij} c_{ij} \\ &= \sum_{p \neq q, r} ((-1/2) \epsilon_{pqr} \epsilon_{pqr} c_{qr} - (1/2) \epsilon_{pqr} \epsilon_{pqr} c_{rq}) \\ &= \sum_{p \neq q, r} ((-1/2) c_{qr} + (1/2) c_{rq}) = c_{rq} \end{aligned}$$

$$\text{So } d_r = c_{rq} x^q$$

If L is the linear transformation which has components c_{rq} , then the value to which a vector x is mapped is given by a cross product.

$$L(x) = c \times x$$

where c is the axial vector defined by the antisymmetric tensor with components c_k .

ANGULAR VELOCITY

Consider a rigid body with the point at the origin fixed.

Let P be a point in the body and let u_1, u_2, u_3 and u'_1, u'_2, u'_3 be two bases of the vector space. The unprimed basis is fixed in space and the primed basis is fixed in the body. Then

$$u'_i = R(u_i)$$

where R is an orthogonal transformation which varies with time. Then the coordinates of P are related by

$$x'_i = a_{ij} x_j$$

where (a_{ij}) is the inverse of the matrix $(\bar{a}_{ij})$

of R . By differentiating and noting that the components of P are constant in the primed system, we find

(Lass, p. 46)

$$dx_i/dt = w_{ij} x_j$$

for $w_{kj} = -\bar{a}_{ki} da_{ij}/dt$. By differentiating the norm of P , which is invariant, we find that (w_{ij}) is skew symmetric ($w_{ij} = -w_{ji}$). So it can be represented as an axial vector.

$$c(w) = \begin{bmatrix} w_{32} \\ -w_{31} \\ w_{21} \end{bmatrix} = \begin{bmatrix} w_{32} \\ w_{13} \\ w_{21} \end{bmatrix}$$

and

$$v = d(c(P))/dt = c(w) \times c(P)$$

w is the angular velocity vector. It transforms like a vector under an orthogonal transformation.

EXAMPLE. Let R have matrix $(\bar{a}_{ij}) =$

$$\begin{bmatrix} c(t^2) & -s(t^2) & 0 \\ s(t^2) & c(t^2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $(a_{ij}) = (\bar{a}_{ij})^T =$

$$\begin{bmatrix} c(t^2) & s(t^2) & 0 \\ -s(t^2) & c(t^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $da_{ij}/dt =$

$$2t \begin{bmatrix} -s(t^2) & c(t^2) & 0 \\ -c(t^2) & -s(t^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $(w_{ij}) =$

$$\begin{aligned} & -2t \begin{bmatrix} c(t^2) & -s(t^2) & 0 \\ s(t^2) & c(t^2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s(t^2) & c(t^2) & 0 \\ -c(t^2) & -s(t^2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & \equiv -2t \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So

$$c(w) = \begin{bmatrix} 0 \\ 0 \\ 2t \end{bmatrix}$$

Suppose $P = u_1'$. Then

$$\begin{aligned} c(P) &= (\bar{a}_{ij}) c'(P) \\ &= (\bar{a}_{ij}) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} c(t^2) \\ s(t^2) \\ 0 \end{bmatrix}$$

Now $dc(P)/dt =$

$$2t \begin{bmatrix} -s(t^2) \\ c(t^2) \\ 0 \end{bmatrix}$$

And $c(w) \times c(P) =$

$$\begin{bmatrix} 0 \\ 0 \\ 2t \end{bmatrix} \times \begin{bmatrix} c(t^2) \\ s(t^2) \\ 0 \end{bmatrix} = 2t \begin{bmatrix} -s(t^2) \\ c(t^2) \\ 0 \end{bmatrix}$$

This is the velocity of the point.

ANGULAR MOMENTUM AND THE INERTIA TENSOR

The angular momentum vector is defined as (Lass, p.115)

$$\mathbf{L} = \int \mathbf{r} \times \mathbf{v} dm$$

$\mathbf{r}$ is the position, $\mathbf{v}$ the velocity and m the mass.
Its k th component L_k is

$$\int \epsilon_{ijk} x^i x^j dm$$

$$\begin{aligned} \text{Then } dL_k/dt &= \int \epsilon_{ijk} x^i \ddot{x}^j dm + \int \epsilon_{ijk} \dot{x}^i \dot{x}^j dm \\ &= \int \epsilon_{ijk} x^i \ddot{x}^j dm \end{aligned}$$

The second term vanishes because the cross product of parallel vectors is zero. The torque vector $\mathbf{T}$ is

$$\int \mathbf{r} \times \mathbf{f} dx = \int \mathbf{r} \times d^2 \mathbf{r} / dt^2 dm$$

Its k th component is

$$T_k = \int \epsilon_{ijk} x^i \ddot{x}^j dm$$

Thus $dL/dt = T_k$. And because $\dot{x}^j = w_p^j x^p$, we have

$$L = \int \epsilon_{ijk} x^i w_p^j x^p dm = \epsilon_{ijk} w_p^j \int x^i x^p dm$$

The components of the inertia tensor are $I^{ip} =$

$$\int x^i x^p dm$$

The moment of inertia about an arbitrary axis can be found from the inertia tensor. Given a straight line through the origin, let p be the distance function whose value at a point is the distance of the point from the line. The moment of inertia about the line is then defined as (McConnell, p.233)

$$\int p^2 dm$$

Suppose $\mathbf{u}$ is a unit vector in the direction of the line; then

$$\begin{aligned} p^2 &= x^i x^i - (\mathbf{r} \cdot \mathbf{u})^2 \\ &= \delta(i,j) x^i x^j - (\delta(i,j) x^i u^j)^2 \end{aligned}$$

So the moment of inertia is

$$\begin{aligned}
 & \int (\delta(i,j)x^i x^j - \delta(i,j)x^i u^j \delta(k,p)x^k u^p) dm \\
 &= \delta(i,j) I^{ij} - \delta(i,j) \delta(k,p) u^j u^p I^{ik} \\
 &= \text{trace}(I) - I^{ik} u^i u^k \\
 &= \text{trace}(I) \delta(i,k) u^i u^k - I^{ik} u^i u^k \\
 &= (\text{trace}(I) \delta(i,k) - I^{ik}) u^i u^k \\
 &= J^{ik} u^i u^k
 \end{aligned}$$

The tensor J is also called the inertia tensor in some physics books (Goldstein). The explicit components of J are

$$\begin{aligned}
 J_{11} &= \int [(x^2)^2 + (x^3)^2] dm \\
 J_{22} &= \int [(x^1)^2 + (x^3)^2] dm \\
 J_{33} &= \int [(x^1)^2 + (x^2)^2] dm \\
 J_{ij} &= - \int x^i x^j dm \quad \text{for } i \neq j
 \end{aligned}$$

If f is the bilinear form corresponding to the symmetric matrix J then the moment of inertia about a unit vector u is $f(u,u)$.

SURFACE INTEGRALS

When projected to 2-space the surface element, $d\sigma$ goes to $ds = d\sigma \cos(\theta)$ where θ is the angle between the surface normal and the projection direction. With u a unit vector in the projection direction

$$d\sigma = ds / (n \cdot u)$$

A general surface integral is

$$\int f(x,n) d\sigma$$

f is a function of position x and normal n . Suppose p is the projection function. Then

$$\int_A f(x,n) d\sigma = \int_{p(A)} f(p^{-1}(y), n(p^{-1}(y))) (ds / (n \cdot u))$$

Now suppose $A_1, \dots, A_n$ covers the surface and p is the plane projection from A_i to 2-space. Suppose there are functions g_i defined on the surface, each g_i vanishing outside of A_i , having value between 0 and 1, and summing to 1. Such a set of functions is called a partition of unity subordinate to the cover $A_1, \dots, A_n$. Then the surface integral is

$$\int f d\sigma = \sum_{i=1}^k \int_{p(A_i)} g_i(p_i^{-1}(y)) f(p_i^{-1}(y), n) ds / (n \cdot u)$$

When p_i is the projection in the x_i direction, let g_i be the function whose value at a point is the square of the i th component of the unit normal vector to the surface. In general p_i is a many-to-one function, and when we define $A(i, k)$ to be the k th layer of the surface which is pierced by the projecting ray, the functions

$$f_i: A(i, k) \rightarrow \mathbb{R}^2$$

are surface patches and the

$$g_i: A(i, k) \rightarrow \mathbb{R}$$

constitute a partition of unity.

A volume integral can usually be changed to a surface integral by using the divergence theorem. For example, suppose

$$A = (1/3) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then $\text{vol}(B) =$

$$\begin{aligned} & \int_B dv \\ &= \int_B \text{div}(A) dv \end{aligned}$$

$$= \int_{\partial B} A \cdot n d\sigma = (1/3) \int_{\partial B} (x n_1 + y n_2 + z n_3) d\sigma$$

PERSPECTIVE PROJECTION

A perspective projection is a mapping from 3-dimensional space to 2-dimensional projective space. The picture plane is denoted by S and has equation

$$ax + by + cz + dw = 0.$$

The plane has coordinate vector $R =$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

With Q a point in the plane we may write the equation above as $(R, Q) = 0$. We denote the projection point by $P \notin S$. the perspective projection maps a general point Q to the point of intersection of S and the line passing through P and Q . This point $Q + tP$ satisfies the equation

$$(R, Q + tP) = 0$$

Then $(R, Q) + t(R, P) = 0$. The perspective transformation L is then given by

$$L(Q) = Q - ((R, Q)/(R, P))P$$

L is a linear transformation on the 4-dimensional vector space because given any points Q_1 and Q_2 , we have

$$L(Q_1 + Q_2) = L(Q_1) + L(Q_2) \text{ and } L(aQ) = aL(Q)$$

S is the subspace of vectors orthogonal to R . Thus, the 4-dimensional vector space V in which the projective space is embedded is the direct sum of R and S (see Halmos).

$$V = R \oplus S$$

When $T \in S$ then $(T, R) = 0$ so $L(T) = T$. Thus

$$L^2(Q) = L(L(Q)) = L(Q).$$

So L is a projection operator that maps V into S (see Halmos). The set of 1-dimensional subspaces of S is the 2-dimensional projective space to which projective 2-space

is mapped. We have now defined the perspective mapping; it remains to choose a basis of the vector space V that is suitable for calculation. Let u_1, u_2, u_3, u_4 be the given basis of V . A point is represented as

$$xu_1 + yu_2 + zu_3 + wu_4$$

We need a basis u'_1, u'_2, u'_3, u'_4 so that each of u'_1, u'_2, u'_3 are in the subspace S . Suppose the coordinate vector of the plane S , which is R , is defined by a latitude angle ϕ a longitude angle θ and the distance of the plane S from the origin in Euclidean space, which is $-d$. The latitude angle takes values with magnitude less or equal to π divided by two. The longitude angle takes values with magnitude less or equal to π . Then we have $R =$

$$\begin{bmatrix} c(\phi)c(\theta) \\ c(\phi)s(\theta) \\ s(\phi) \\ d \end{bmatrix}$$

We have let c and s stand for $\cos$ and $\sin$. It is seen that the x -axis is the direction of zero longitude and latitude. It is the intersection of the Greenwich meridian with the equator. We let u'_1 be the vector

$$\begin{bmatrix} s(\theta) \\ -c(\theta) \\ 0 \\ 0 \end{bmatrix}$$

u'_1 lies in S and is parallel to the xy -plane. Let n be the vector obtained by changing the last component of R to 0. It is then the normal to the plane S . If we consider the picture plane a camera, n is the direction to which the camera points. The camera is usually held so that the lower edge of the camera is parallel to the ground. This corresponds to choosing u'_1 parallel to the xy -plane. We have taken u'_1 to be the direction of the photographer's extended right hand as he peers through the camera. The second basis vector u'_2 is to be the vertical direction in the picture plane. It is parallel to a vertical edge of the camera and, depending upon how the camera is held, may be tilted with the ground. We take $u'_2 = u'_1 \times n =$

$$\begin{bmatrix} -c(\theta)s(\phi) \\ -s(\theta)s(\phi) \\ c(\phi) \\ 0 \end{bmatrix}$$

We see that u_2' is perpendicular to R , thus is in S , and is perpendicular to u_1' . The next element of the primed basis will be a point at the origin in the picture plane. The origin in affine 3-space is the point

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

It is the point that corresponds to the origin of the vector space with basis u_1, u_2, u_3 , to which affine 3-space is naturally mapped. The vector in Euclidean space from the origin to a point on the plane S which is closest to the plane is $-dn$. The Euclidean vector space is mapped into affine space by adding u_4 . We divide by $b = \sqrt{1 + d^2}$ to obtain a unit vector and we get $u_3' =$

$$\begin{bmatrix} -(d/b)c(\emptyset)c(\theta) \\ -(d/b)c(\emptyset)s(\theta) \\ -(d/b)s(\emptyset) \\ 1/b \end{bmatrix}$$

Notice that we are again abusing notation and are, for example, using S for the plane in 3-space and also for the corresponding 3-dimensional subspace of the 4-dimensional vector space V . We find that u_3' is perpendicular to R and so is in the subspace S . Finally, we let $u_4' = R/b$. Then u_1', u_2', u_3', u_4' is an orthonormal basis of V . Suppose we compute the projection of a general point Q and find

$$L(Q) = xu_1 + yu_2 + zu_3 + wu_4$$

We want the primed coordinates of this point. We have

$$u_1' = s(\theta)u_1 - c(\theta)u_2$$

$$u_2' = -c(\theta)s(\emptyset)u_1 - s(\theta)s(\emptyset)u_2 + c(\emptyset)u_3$$

$$u_3' = -(d/b)c(\emptyset)s(\theta)u_1 - (d/b)c(\emptyset)s(\theta)u_2 - (d/b)s(\emptyset)u_3 + (1/b)u_4$$

$$u_4' = (1/b)c(\emptyset)c(\theta)u_1 + (1/b)c(\emptyset)s(\theta)u_2 + (1/b)s(\emptyset)u_3 + (d/b)u_4$$

We let the coefficients be (a_{ij}) then

$$u_i' = a_{ij}u_j$$

using the summation convention. Then

$$x'_i u'_i = x'_i a_{ij} u_j = x_j u_j$$

So

$$x'_i a_{ij} = x_j$$

The matrix equation is $A^T x' = x$. A is orthogonal so

$$A^T = A^{-1}. \text{ Then}$$

$$A^{-1} x' = x \text{ or } Ax = x'$$

Let B be the matrix of the first three rows of A. Then

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = BL \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Because u'_4 is perpendicular to S, the coordinate of u'_4 in the projection will be zero. If the projected point is not a point at infinity ($z' = 0$) then the affine coordinates of the projected point will be

$$\bar{x} = x'/z', \bar{y} = y'/z'$$

EXAMPLE. Isometric projection.

The picture plane has coordinate vector R =

$$\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

The projection point is at infinity P =

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Then the longitude is $\theta = -3\pi/4$ and the latitude is $\phi = -\arcsin(\sqrt{3}/3)$. Hence $s(\theta) = c(\theta) = -\sqrt{2}/2$, $s(0) = -\sqrt{3}/3$ and $c(0) = \sqrt{6}/3$

Then B =

$$\begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ -\sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We find $(R,P) = -3$ and $(R,Q) = -(x + y + z)$

So $L(Q) = Q - ((x + y + z)/3)P$

Therefore

$$L(Q) = (1/3) \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

and $BL(Q) =$

$$(\sqrt{6}/3) \begin{bmatrix} -\sqrt{3}/2 & \sqrt{3}/2 & 0 & 0 \\ -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

The images of the unit points

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

are

$$\sqrt{6}/3 \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \\ 1 \end{bmatrix}, \sqrt{6}/3 \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \\ 1 \end{bmatrix}, \sqrt{6}/3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The image of the coordinate axes are as shown

The lengths are forshortened by the factor $(\sqrt{6}/3)$. In conventional isometric drawing, true lengths are laid

off along the coordinate axes images so that the matrix is

$$\begin{bmatrix} -\sqrt{3}/2 & \sqrt{3}/2 & 0 & 0 \\ -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In perspective drawing, the horizon is defined as the intersection of the xy-plane with the plane at infinity $w=0$. A horizon point cannot be reached in 3-space, although we may "see" it. A line at infinity may project to a finite line in the picture plane. Similarly, the point at infinity where two parallel lines meet may project into a finite point in 2-dimensional space. Such a point is called a vanishing point. For example, the point

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

is the point at infinity where the lines meet, which are in the xy-plane and which make angles of 45 degrees with the x and y axes. Thus the vanishing point is $BL(Q)$ in the picture plane. The horizon in the picture plane is determined by two vanishing points corresponding to two sets of parallel lines in the xy-plane. When the projection point P is at infinity as in isometric projection, parallel lines in 3-space are mapped to parallel lines in 2-space so there are no vanishing points.

If the line through P and Q is parallel to the picture plane, the projection of Q will be a point at infinity in the picture plane.

Multiplying by the scalar (R,P) the projective transformation L may be written as

$$L(Q) = (R,P)Q - (R,Q)P = (R^T P I - P R^T)Q$$

The general transformation equation is therefore

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = B(R^T P I - P R^T) \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

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SPECIAL PROJECTS: Quadric

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