PLANAR NUMERICAL COSMOLOGY

I. The Differential Equations

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ABSTRACT

This paper is the first in a series describing a method for evolving planar cosmological models on a computer. In this paper we set forth our coordinate conditions and variables, and discuss the way our special gauge simplifies the numerical evolution. We write down the Einstein equations that govern the gravitational field in our cosmologies, as well as the hydrodynamical equations for the fluid sources. Later papers in this series will present the differencing of these equations, tests of the code, and evolutions of dynamical inhomogeneous cosmological models.

1. INTRODUCTION

The chief problems of modern cosmology center on understanding how the universe evolved into its present state. For instance, we need to know how manifestly nonlinear objects such as galaxies formed. Under what conditions is linear perturbation analysis a good guide to understanding the physics of the early universe? How do strong gravitational waves and large density fluctuations behave? And what physical processes occurred at early times to produce the large scale structure of the universe we observe today? The answers to such questions can be sought through a study of dynamical inhomogeneous and anisotropic models of the early universe.

To probe the physics of the early universe, we have built a computer code on which we can simulate a wide variety of physical conditions. This numerical study of cosmology uses techniques developed by numerical relativists over the past decade to solve the Einstein equations coupled to the hydrodynamical equations. In particular, we use the ADM or "3 + 1" formalism, in which four-dimensional spacetime is split into a set of three-dimensional spacelike slices threaded by a congruence of curves that give the time direction. To solve a problem in numerical relativity we first set data that satisfy the constraint equations on an initial slice. We then propagate these data using the evolution equations to produce a four-dimensional spacetime.

Previous work in numerical cosmology has looked at cosmological perturbations as well as gravitational and hydrodynamical shocks. Initial data for "a Type I cosmology at each point" have been given by Centrella and Matzner (1979). This data set was used to evolve cosmological perturbations for vacuum plane-symmetric models by Centrella and Smarr (1978) (constant mean curvature slicing) and Centrella (1980b) (unit lapse slicing). Constant mean curvature slicing was also used by Piran (1980) to evolve perturbations in cylindrical systems with matter. Nadejin, Novikov, and Polnarev (1978; see also Novikov and Polnarev 1979) evolved strong density perturbations to form primordial black holes in spherical cosmologies. The propagation of gravitational shocks has also been studied by Centrella (1979; 1980a) using the above initial data in the limit of two Type I cosmologies abutted at the x-y plane, for both vacuum and fluid models. Finally, the formation of hydrodynamical shocks in cosmology has been studied by Anile, Miller and Motta (private communication).

The computer code described in this paper and its sequels is designed with the experience of previous work in numerical relativity and cosmology, as well as numerical hydrodynamics, in mind. We want a code that can handle a wide variety of physical conditions such as strong gravitational waves, strong shocks, and the coupling of gravity to matter. At the same time we want to strike a balance between simplicity and generality. To achieve this balance we started with a one-dimensional (one spatial dimension plus time) planar code and exploited the symmetries of the model to simplify the equations. We have, however, used gauge conditions that will be useful in more general models and have employed explicit differencing techniques that can readily be generalized to multi-dimensional codes. The resulting code has variables which depend on one spatial coordinate plus time and describe one polarization of gravitational waves. A shift vector is used to simplify the three-metric and keep it diagonal. The spacelike slices have periodic boundary conditions and constant mean curvature [equation (2.4)]. The "matter" content of these models is a perfect fluid.

Thus, our code will allow us to study a universe with an overall Hubble expansion, like the flat Friedmann-Robertson-Walker (FRW) models, as well as models with large scale inhomogeneities in the scale factor, such as will occur in a region of very strong gravitational field. Our code also allows for models with large anisotropy, such as the Kasner model, in which the anisotropy can be interpreted as long wavelength gravitational waves. Inhomogeneous cosmologies dominated by gravitational waves as well as models with peculiar velocities of the matter with respect to the Hubble flow can also be handled by our code.

In this paper we set up the basic equations for studying the physics of the early universe on a computer. We describe our coordinate conditions and variables in Section II, including a discussion of our gauge and its use in simplifying the field equations. The Einstein equations are discussed in Section III and are written in our variables in Section IV. The hydro equations for the fluid source are given in Section V. The differencing of these equations, tests of the code, and evolutions of dynamical, inhomogeneous and anisotropic cosmologies will be presented in later papers in this series.

II. COORDINATE CONDITIONS AND VARIABLES

Our spacetime coordinates are a time coordinate t and three spatial Cartesian coordinates x , y , and z . In this one-dimensional code, all quantities

are functions of z and t only. We impose periodic boundary conditions by wrapping each of the x , y , and z axes back onto itself. Our three-dimensional spacelike slices are therefore closed and are topologically three-tori.* Waves will travel around such a closed universe along the z axis.

In the "3 + 1" notation we write the four-dimensional spacetime metric

$${}^{(4)}g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_x & 0 & \beta_z \\ \beta_x & \gamma_{ij} & 0 & 0 \\ 0 & 0 & \gamma_{ij} & 0 \\ \beta_z & 0 & 0 & 0 \end{pmatrix}, \quad (2.1)$$

where Greek indices $\mu, \nu = 0, 1, 2, 3$, Latin indices $i, j = 1, 2, 3$, and the speed of light $c = 1$. The superscript ${}^{(4)}$ indicates that this geometrical quantity belongs to the full four-dimensional spacetime. For background and notation see Smarr and York (1978) or York (1979) and references therein. The lapse function α measures the orthogonal proper time interval Δt between neighboring slices a distance Δt apart. The shift vector $\beta^i = (\beta^x, 0, \beta^z)$ moves the spatial coordinates as the data are evolved from one slice to the next. The three-metric of the slices is γ_{ij} and is taken to have the diagonal form

$$\gamma_{ij} = A^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & h^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.2)$$

The conformal factor A is a generalization of the usual scale factor R in the homogeneous and isotropic FRW models, where $h = 1$. The function h is an anisotropic metric shear variable.

The extrinsic curvature K^i_j of the three-slices is taken to have the form

$$K^i_j = \begin{pmatrix} K^x_x & 0 & K^x_z \\ 0 & K^y_y & 0 \\ K^z_x & 0 & K^z_z \end{pmatrix}. \quad (2.3)$$

*The analytic vacuum T^3 Gowdy models (Gowdy 1971; Berger 1974) also have closed spacelike slices which are three-tori; these slices, however, are not constant mean curvature slices.

Note that $K^x_z = K^z_x$, since $\gamma_{xx} = \gamma_{zz}$ and $\gamma_{xz} = 0$. The remaining extrinsic curvature components are related by $K^x_x + K^y_y + K^z_z = \text{trace}(K_{ij}) \equiv \text{tr}K$. We will use constant mean curvature slices

$$\text{tr}K = \text{tr}K(t) \quad (2.4)$$

and will solve for $\text{tr}K$ on each slice using the lapse equation. Thus there are only two independent diagonal components of K^i_j , which we take to be K^z_z and

$$K_1 \equiv K^x_x - K^y_y. \quad (2.5)$$

K_1 is an anisotropic shear variable and, together with h , is the chief gravity wave variable in the code.

In summary, the three-metric (2.2) features a conformal factor A which gives the overall length scale of space, and a metric shear variable h which represents one gravitational degree of freedom, i.e. gravitational waves having a single polarization state.[†] Similarly, the two independent diagonal components of the extrinsic curvature are K^z_z and K_1 , defined in (2.5). [The off-diagonal component K^z_x is related to matter motion in the x -direction, $S_x(z,t)$; cf. equation (4.6)]. Our choice of gravitational wave variables $h = (\gamma_{yy}/\gamma_{xx})^{1/2}$ and K_1 leads to a good set of wave equations (4.8) and (4.9). This choice is not unique; however, experimentation with different sets of wave variables convinced us that the set of h and K_1 is the optimal choice for our numerical work.

The lapse function α has been left free to specify the slicing condition which in our case is that of constant mean curvature slicing (2.4). Such slices have "singularity avoidance" properties similar to those of maximal ($\text{tr}K \equiv 0$) slices (Eardley and Smarr 1979; Marsden and Tipler 1980). We therefore expect that these slices will produce good strong field calculations in the early universe by wrapping around the cosmic singularity, thus allowing us to continue integrating the field equations when running the calculations toward singularities.

[†]An off-diagonal component of γ_{ij} is needed to treat gravitational waves with two polarization states.

The shift vector β^i is used to simplify the form of the three-metric. The many possible terms of the Ricci tensor R_{ij} argue for the simplest three-metric possible, and so we have chosen to make γ_{ij} diagonal. However, to preserve this diagonal form in time, we have to adjust our coordinate trajectories using β^i , which results in two non-zero shift vector components in the four-metric (2.1). Such a trade of kinematical variables for dynamical ones leads to simpler equations in general. See, for example, the discussion in Smarr (1979). The consistency of the metric forms (2.1) and (2.2) with the Einstein equations will be demonstrated in Section IV.

A particular choice of the lapse and shift fixes a computational coordinate frame or gauge within which the dynamical variables are evolved. Our choice -- constant mean curvature slicing given by α and the use of β^i to keep γ_{ij} diagonal -- is a special case of the gauge used by Wilson (1979) and has proved to be highly effective (using maximal slicing) in calculations of axisymmetric gravitational waves, stellar collapse, cylindrical systems, and black holes (Eppley 1979; Dyckema 1980; Shapiro and Teukolsky 1980; Piran 1979; Wilson 1979).

Our choice of gauge and variables was also designed to help us understand complex inhomogeneous, anisotropic cosmologies in terms of a simple, homogeneous, isotropic FRW model (see, e.g., Weinberg 1972). Our code generalizes the flat ($k = 0$) FRW cosmology, for which $A(z,t) = R(t)$ (the usual cosmic scale factor) and $R^2_{,x} = 0$, $h = 1$, and $K_1 = 0$ (since the flat FRW model is isotropic and shear-free in these coordinates). In addition, $\text{tr}K = \text{tr}K(t)$ is the usual slicing of this FRW model. Therefore, we can easily recognize this FRW model in the results of our calculations and we can analyze more general cosmologies by comparing them to this simple one. In particular, if we see a perturbation in our models with $\text{tr}K \neq \text{tr}K(t)$, we know we are not really in the homogeneous FRW model in bad coordinates.

III THE EINSTEIN FIELD EQUATIONS

As is well known, the Einstein equations $G_{\mu\nu} = T_{\mu\nu}$ ($8\pi G = 1$) in the "3 + 1" notation split into two distinct sets: constraint equations and evolution equations (see the discussion in Smarr and York 1978). The constraint equations provide conditions which must be satisfied by the field variables on each slice, while the evolution equations determine the time development of these field variables. Both types of equations have characteristic properties which must be taken into account to produce good numerical spacetimes.

The matter source for the Einstein equations is the stress energy tensor $T_{\mu\nu}$. We will consider only perfect fluid sources which can be decomposed in the standard fashion:

$$T_{\mu\nu} = (\rho + e + P)u_\mu u_\nu + P^{(4)}g_{\mu\nu}. \quad (3.1)$$

We also define

$$\sigma = \rho + e + P. \quad (3.2)$$

Here, u_μ is the four-velocity of the fluid, ρ is the baryon mass density, e is the internal energy per volume, and P is the fluid pressure. The usual specific internal energy (per unit mass) is $\epsilon = \frac{e}{\rho}$. The Einstein equations in the "3 + 1" notation are projected onto n_μ , the normal unit four-vector to the time slices. The source terms for the Einstein equations are therefore those components seen by the observers at rest in the time slices n_μ . That is, to the n_μ observers, $T_{\mu\nu}$ appears to be an imperfect fluid with components

$$T_{\mu\nu} = dn_\mu n_\nu + j_\mu n_\nu + n_\mu j_\nu + S_{\mu\nu}, \quad (3.3)$$

where

$$d = n^\mu n^\nu T_{\mu\nu}, \quad (3.4)$$

$$j_\mu = n^\delta h_\mu^\beta T_{\beta\delta}, \quad (3.5)$$

and

$$S_{\mu\nu} = h_\mu^\delta h_\nu^\beta T_{\delta\beta}. \quad (3.6)$$

The projection operator is $h_{\mu\nu} = {}^{(4)}g_{\mu\nu} + n_\mu n_\nu$ (Smarr 1975; York 1979). In our variables (cf. Section V) these quantities are

$$d = \sigma w^2 - P, \quad (3.7)$$

$$j_i = -\sigma w^2 V_i, \quad (3.8)$$

and

$$S_{ij} = P\gamma_{ij} + \sigma w^2 V_i V_j. \quad (3.9)$$

See the discussion in Smarr, et al. (1980). Note that $w = \alpha u^t = (1 - V_i V_j \gamma^{ij})^{-1/2}$ is the general relativistic generalization of the special relativistic "gamma factor." Thus

$$u_i = w V_i. \quad (3.10)$$

In the standard FRW cosmologies, the global rest frame n_μ and the fluid four-velocity u_μ coincide. For general inhomogeneous cosmologies, however, we are interested in peculiar velocities of the matter relative to an overall Hubble expansion. In these cases, n_μ and u_μ will not coincide; the peculiar velocities are given by V_i [see equation (3.10)]. Since the Einstein equations are projected onto the geometric global rest frame n_μ while the fluid equations (Section V) are projected onto the fluid rest frame u_μ , it is necessary to boost back and forth between u_μ and n_μ to get the hydro sources (3.7) - (3.9) for the field equations.

The evolution of these matter terms must be carried out simultaneously with the evolution of the gravity terms. In this section and the next we concentrate on the evolution of the gravitational field; the hydro equations which evolve the matter terms are the topic of Section V.

The four general relativistic constraint equations are elliptic equations and must be satisfied by the gravitational field and its sources on each slice. The constraint equations therefore also serve as the initial value equations of relativity. The Hamiltonian constraint $G^0_0 = T^0_0$ becomes

$$R + (\text{tr} \underline{K})^2 - K^i_j K_i^j = 2d, \quad (3.11)$$

where R is the Ricci scalar formed from the three-metric γ_{ij} . The three momentum constraint equations become

$$D_i (\text{tr} \underline{K}) - D_j K^j_i = j_i, \quad (3.12)$$

where D_i is the covariant derivative formed from γ_{ij} .

The rest of the Einstein equations are propagation equations and give the time development of the gravitational field variables γ_{ij} and K^i_j . The evolution equations of the three-metric are

$$\partial_t \gamma_{ij} = -2\alpha \gamma_{ik} K^k_j + \gamma_{jk} D_i \beta^k + \gamma_{ik} D_j \beta^k \quad (3.13)$$

and the evolution equations of the extrinsic curvature are

$$\begin{aligned} \partial_t K^m_j &= \beta^c \partial_c K^m_j - K^c_j \partial_c \beta^m + K^m_c \partial_j \beta^c \\ &+ \alpha (\text{tr} K) K^m_j + \alpha \gamma^{im} R_{ij} - \gamma^{im} D_i D_j \alpha \\ &- \alpha \gamma^{im} S_{ij} + 1/2 \alpha \text{tr} S \delta^m_j. \end{aligned} \quad (3.14)$$

Here, $\partial_t = \partial/\partial t$ and $\partial_c = \partial/\partial x^c$. R_{ij} is the Ricci tensor of γ_{ij} and $\text{tr} S = \text{trace}(S_{ij})$.

To complete the set of equations needed to construct a spacetime on a computer, we also require the values of α and β^i . In general, we can freely specify these kinematical variables; i.e. we are free to choose our coordinate system. However, experience has shown that it is better to use the Einstein equations to derive differential equations for the lapse and shift, thereby using the coordinate freedom inherent in α and β^i to simplify the numerical evolution.

The lapse function α controls the slicing of spacetime. Taking the trace of equation (3.14) and using (3.11) gives the lapse equation

$$\gamma^{ij} D_i D_j \alpha = \alpha \frac{\rho_\alpha}{2} + \beta^i D_i (\text{tr} K) - \partial_t (\text{tr} K), \quad (3.15)$$

where $\frac{\rho_\alpha}{2} = \kappa^i_j K_i^j + \frac{1}{2} (d + \text{tr} S)$. The lapse equation is solved for α . It is an elliptic equation that is second order and linear in α and its first two derivatives. In constructing our numerical spacetimes we regard $\text{tr} K$ as a freely specifiable function on each slice. Equation (3.15) therefore gives us a particularly convenient, although not a unique, way of solving for α .

Our remaining coordinate freedom lies in the shift vector β^i . As described in Section II, we will use the shift vector to move our spatial coordinates in such a way as to keep γ_{ij} simple. This prescription will lead to differential equations for β^i which will be derived below in the course of writing the Einstein equations in our variables.

IV. THE FIELD EQUATIONS FOR PLANAR NUMERICAL COSMOLOGIES

Our task now is to write equations (3.11) - (3.15) in terms of the coordinate conditions and variables introduced in Section II. This procedure will demonstrate the consistency of our gauge, through α and β^i , with our choice of three metric (2.2) and extrinsic curvature (2.3) - (2.5).

Numerical studies of black holes have clearly demonstrated that our primary kinematical freedom is the choice of slicing through α . By imposing constant mean curvature $\text{tr}\underline{K}(t)$ on each slice, the term in β^i in equation (3.15) drops out, thus decoupling the choice of the shift vector from the solution for α . The lapse equation becomes

$$\partial_z \partial_z \alpha + \left(\frac{\partial_z A}{A} + \frac{\partial_z h}{h} \right) \partial_z \alpha - \alpha A^2 [K^i_j K^j_i + \alpha (w^2 - 1/2) + p] + A^2 \partial_t (\text{tr}\underline{K}) = 0. \quad (4.1)$$

The rest of the kinematical freedom in this formalism is the choice of shift vector β^i . As noted above we will use the shift vector to move our coordinates as we evolve in time to preserve the form (2.2) of the three-metric γ_{ij} . The component β^z is used to keep $\gamma_{zz} = \gamma_{xx} = A^2$. To preserve this relation in time we need $\partial_t \gamma_{zz} = \partial_t \gamma_{xx}$. Equating these two components of (3.13) leads to a first order equation for β^z :

$$\partial_z \beta^z = 1/2 \alpha (3K^z_z - \text{tr}\underline{K} - K_1). \quad (4.2)$$

Similarly, the component β^x is used to keep $\gamma_{zx} = 0$. The condition $\partial_t \gamma_{zx} = 0$ then gives, via (3.13)

$$\partial_z \beta^x = 2\alpha K^z_x. \quad (4.3)$$

To maintain the condition $\gamma_{yz} = 0$ in time requires that $\partial_t \gamma_{yz} = 0$, or $\partial_z \beta^y = 2\alpha K^y_z$. We have chosen to use $\beta^y = 0$, and so $K^y_z = 0$. Finally, the relation $\gamma_{xy} = 0$ requires $\partial_t \gamma_{xy} = 0$, or $K^x_y = 0$. These choices are seen to be consistent by substitution into equation (3.14). We have now shown the major part of the consistency among the variables β^i , γ_{ij} , and K^i_j .

We now turn our attention to the constraint equations (3.11) and (3.12). Once these equations are satisfied on an initial spacelike slice, they must continue to hold on succeeding slices that are reached using the evolution equations. Of course, the numerical solution of these equations introduces errors which cause the values of the field variables to drift from their true analytic values with time. In order to produce good spacetimes, numerical relativists have two basic choices: either evolve all the field variables using their propagation equations and check the solution of the constraints on each slice to keep track of the drift, or impose the constraint equations on each slice while evolving the minimum number of field variables using their propagation equations.

We use a fully-constrained evolution scheme, in which certain metric and extrinsic curvature variables are found by solving the constraint equations on each slice (Pachner 1975; Piran 1980). Experience has shown that this procedure leads to greater stability than simply evolving all the variables using the propagation equations. We will demonstrate this point when we discuss tests of the code in the second paper in this series (Centrella and Wilson 1982).

The Hamiltonian constraint (3.11) becomes, in our variables,

$$\partial_z \partial_z h + 2 \frac{\partial_z A}{A} \partial_z h = h \left[-2 \frac{\partial_z \partial_z A}{A} + \left(\frac{\partial_z A}{A} \right)^2 - \frac{A^2}{2} (K_i^j K_j^i - (\text{tr} K)^2 + 2d) \right], \quad (4.4)$$

where $d = w^2(p + \epsilon) + p(w^2 - 1)$. This is a linear equation for h and is solved for h on each slice. Usually, following York, we would like to solve the Hamiltonian constraint for the conformal factor A and evolve only the wave variables h and K_i^j using the evolution equations. However, equation (4.4) is nonlinear in A and has proved difficult to solve numerically; we have therefore chosen to solve it for h to obtain a better numerical evolution.

The two nontrivial momentum constraints are, from (3.12),

$$\partial_z K_z^z + (3K_z^z - \text{tr} K) \left(\frac{\partial_z A}{A} + \frac{1}{2} \frac{\partial_z h}{h} \right) + \frac{1}{2} \frac{\partial_z h}{h} K_l^l = S_z, \quad (4.5)$$

which is solved for K_z^z , and

$$\partial_z K_x^z + \left(3 \frac{\partial_z A}{A} + \frac{\partial_z h}{h} \right) K_x^z = S_x, \quad (4.6)$$

which is solved for K_x^z . The momentum variables S_i are defined in Section V.

Our remaining two variables are A and K_1 . They are propagated forward in time using their evolution equations

$$\partial_t A = \beta^z \partial_z A - 1/2 \alpha (tr \underline{K} + K_1 - K^z_z) \quad (4.7)$$

and

$$\partial_t K_1 = \beta^z \partial_z K_1 + \alpha (tr \underline{K}) K_1 - 2\alpha (K^z_x)^2 \quad (4.8)$$

$$+ \frac{1}{A^2 h} \left[\partial_z \alpha \partial_z h + \alpha \left(\partial_z \partial_z h + \frac{\partial_z A}{A} \partial_z h \right) \right] - \frac{\alpha}{A^2} \frac{s_x^2}{\sigma w^2}.$$

Equations (4.1) - (4.8) are a consistent set of equations which completely determines the gravitational field, given the source terms in $T_{\mu\nu}$. We will, however, also use the propagation equation for h ,

$$\partial_t h = \alpha h K_1 + \beta^z \partial_z h, \quad (4.9)$$

to evolve h to each new slice. This updated value for h is then used as a "first guess" for the iterative solution of the Hamiltonian constraint (4.4) (Shapiro and Teukolsky 1980).

V. THE HYDRO EQUATIONS

The source of the gravitational field in our cosmological models is a perfect fluid with energy-momentum tensor $T_{\mu\nu}$ given by equation (3.1). The fluid variables are evolved using the equations of general relativistic hydrodynamics, namely, the particle conservation equation

$$\nabla_\mu (\rho u^\mu) = 0 \quad (5.1)$$

and the energy-momentum conservation equations

$$\nabla_\nu T^\mu_\nu = 0. \quad (5.2)$$

Here, ∇_μ is the covariant derivative in the full four-dimensional spacetime. We also need the normalization of the fluid four-velocity

$$u^\mu u_\mu = -1 \quad (5.3)$$

and an equation of state $P = P(\rho, \epsilon)$, which we will specify below.

We now rewrite the hydro equations (5.1) and (5.2) to make them look as much like the usual Newtonian hydro equations as possible (Wilson 1979). Then, we will be able to use differencing methods that have been developed for Newtonian flows to produce a good numerical evolution. This technique of patterning the general relativistic fluid equations after the Newtonian ones works well for low and moderate velocities $v \lesssim 0.90$; cf. the code tests presented in the second paper in this series (Centrella and Wilson 1982). More specialized methods are required to treat extreme special relativistic flows.

To this end, we define the following variables:

$$\begin{aligned}
 w &= \alpha u^t && \text{boost factor} \\
 D &= \rho w && \text{density} \\
 E &= e w && \text{energy} \\
 \sigma &= \rho + e + P && \text{relativistic enthalpy} \\
 S_\mu &= \sigma w u_\mu && \text{momentum} \\
 v^i &= u^i/u^t && \text{transport 3-velocity} \\
 \sqrt{\gamma} &= \sqrt{\det(\gamma_{ij})} && \text{3-volume.}
 \end{aligned}$$

Note that our energy variable is actually a thermal energy variable. This choice will lead to an internal energy equation (5.7) rather than a total energy conservation equation. Also, we have included the special relativistic contribution of e and P to the momentum in our definition of S_μ . Finally, we choose the fluid velocity to have $u^y = 0$ but $u^z(z,t)$ and $u^x(z,t)$ both nonzero in general.

In these variables, the particle conservation equation (5.1) becomes the density equation

$$\partial_t D = - \underbrace{\frac{D}{\sqrt{\gamma}} \partial_t \sqrt{\gamma}}_{\text{volume}} - \underbrace{\frac{1}{\sqrt{\gamma}} \partial_z (D \sqrt{\gamma} v^z)}_{\text{transport}} \quad (5.4)$$

The spacelike $\mu = 1$ components of the energy-momentum conservation equations (5.2) give the two nontrivial momentum equations

$$\partial_t S_x = - \underbrace{\frac{S_x}{\sqrt{\gamma}} \partial_t \sqrt{\gamma}}_{\text{volume}} - \underbrace{\frac{1}{\sqrt{\gamma}} \partial_z (S_x \sqrt{\gamma} V^z)}_{\text{transport}} \quad (5.5)$$

and

$$\begin{aligned} \partial_t S_z = & - \underbrace{\frac{S_z}{\sqrt{\gamma}} \partial_t \sqrt{\gamma}}_{\text{volume}} - \underbrace{\frac{1}{\sqrt{\gamma}} \partial_z (S_z \sqrt{\gamma} V^z)}_{\text{transport}} \\ & - \underbrace{\alpha \partial_z P}_{\text{pressure acceleration}} - \frac{1}{2} \partial_z \underbrace{(4) g^{\mu\nu} \frac{S_\mu S_\nu}{S^t}}_{\text{gravitational acceleration}}. \end{aligned} \quad (5.6a)$$

A gravitational acceleration term appears only in the S_z equation, and in our variables is given by

$$\begin{aligned} \partial_z (4) g^{\mu\nu} \frac{S_\mu S_\nu}{S^t} = & 2\alpha \underbrace{\sigma w^2 \frac{\partial_z \alpha}{\alpha}}_{\substack{\text{acceleration} \\ \text{of observers} \\ \text{at rest in} \\ \text{the slice}}} - \underbrace{\sigma(w^2 - 1) \frac{\partial_z A}{A}}_{\substack{\text{gradient in} \\ \text{gravitational} \\ \text{field}}} \\ & - \underbrace{(S_x \partial_z \beta^x + S_z \partial_z \beta^z)}_{\text{"coriolis" forces}}. \end{aligned} \quad (5.6b)$$

Projecting equation (5.2) along u^μ gives $u^\mu \nabla_\nu T^\nu_\mu = 0$, which becomes the energy equation

$$\partial_t E = - \underbrace{\frac{E}{\sqrt{\gamma}} \partial_t \sqrt{\gamma}}_{\text{volume}} - \underbrace{\frac{1}{\sqrt{\gamma}} \partial_z (E \sqrt{\gamma} V^z)}_{\text{transport}} - \underbrace{\frac{P}{\sqrt{\gamma}} [\partial_t (w \sqrt{\gamma}) + \partial_z (w \sqrt{\gamma} V^z)]}_{\text{"PdV" work}}. \quad (5.7)$$

Finally, the velocity normalization equation (5.3) gives an equation for w ,

$$w = \left(1 + \frac{S_x^2 + S_z^2}{(A \sigma w)^2} \right)^{1/2} \quad (5.8)$$

To complete the set of hydro equations we need an equation of state for the pressure. In this code the pressure is calculated in a separate subroutine, and then fed back into the hydro equations. For most of our tests of the code and for many of our actual inhomogeneous cosmology runs, we have used the perfect gas equation of state

$$P = \frac{(\Gamma-1)E}{w}, \quad 1 \leq \Gamma \leq 2. \quad (5.9)$$

More complicated equations of state are required for certain astrophysical problems and will be discussed, along with the resulting modifications to the hydro equations, in later papers.

VI. SUMMARY

In this paper we have set up the basic machinery needed to build dynamic planar cosmological models on a computer. Having discussed our coordinate conditions and variables, we have shown how to use our gauge to produce a good set of equations (4.1) - (4.9) for the numerical solution of the kinematical and gravitational field variables. The source of the gravitational field in our code is a perfect fluid. This cosmological fluid is evolved using the equations of general relativistic hydrodynamics written in a form, (5.4) - (5.9), which makes them look as much like the Newtonian equations as possible.

The second paper in this series (Centrella and Wilson 1982) will describe the techniques we have used to difference these equations and solve them numerically, as well as to determine the accuracy and reliability of the code. Later papers will contain the results of evolutions of strong gravitational waves, nonlinear density fluctuations and shock formation, as well as other astrophysical situations of interest in cosmology.

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