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AN ANISOTROPIC ELASTIC-PLASTIC STRESS ANALYSIS OF A THICK-WALLED GRAPHITE CYLINDER

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Submitted as a thesis to the Graduate Council of The University of Tennessee in partial fulfillment of the requirements for the degree of Master of Science.

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S.E.M.

ABSTRACT

An anisotropic theory of plasticity, proposed by Hill for orthotropic metals behavior and modified in the present study, was used in conjunction with the linear theory of elasticity for the stress and strain analysis of a thick-walled graphite cylinder loaded with an internal pressure. The problem was formulated in terms of the elastic and plastic parts of the small deformation strain components. The solution to the problem - given in terms of the plastic strain increments and the increment of the load ΔP - was evaluated numerically on a digital computer using a predictor-corrector method of successive approximation. For this purpose an algorithm based on the concept of an effective-stress effective-total-strain relationship was developed. The numerical results are presented in graphical form.

The results of the analysis were compared with the inside surface and outside surface circumferential strains measured experimentally on a series of 30 thick-walled tube specimens of AGOT reactor grade graphite which were monotonically loaded with internal pressure until failure occurred. Excellent agreement between the modified theory and experimental mean values was found for the inside surface strains. Less accurate results were calculated for the outside surface strains, indicating a need for further development in the constitutive relations for graphite. The maximum computed stresses were, however, considered acceptable for design purposes.

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NOMENCLATURE

a, b	Inside and outside radius, respectively, of the thick-walled cylinder
$A_{ij} \ (i, j = 1, 2, 3)$	The elements of a 3 by 3 matrix defined in the text
C_1, C_2	Integration constants appearing in the solution to the strain equations
dW_p	$\sigma_{ij} d\epsilon_{ij}^P$; the increment of plastic work, in.·lb/in. ³
D_p	The plastic strain dilatation, permanent volume change, in. ³ /in.
$E_i \ (i = 1, 2, 3)$	The Young's moduli of elasticity in the rectangular Cartesian coordinate system, psi
E_x, E_z	The elastic moduli for transversely isotropic material in the isotropic plane and perpendicular to the isotropic plane, respectively, psi
F_z	Axial force acting on the thick-walled cylinder
G	$\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12}$; a constant of proportionality appearing in the plasticity equations
G_{ij}	Elastic shear moduli for anisotropic materials, psi
l, m, n	Superscripts designating an iteration number in the numerical algorithm
P	Internal pressure acting on the thick-walled cylinder, psi

s	The statistical standard deviation
s_r, s_θ, s_z	Fractional part of the stress components corresponding to the elastic component of the strain, psi
$\bar{x}$	The arithmetic mean value
x_1, x_2, x_3	Coordinate directions in the rectangular Cartesian coordinate system
z, θ, r	Coordinate directions in the cylindrical coordinate system
α_c	The critical angle for strain gage positioning
$\alpha_{ij} (i, j = 1, 2, 3)$	Constants in the plasticity equations related to the nonelastic materials properties
ϵ_{ij}	The total strain tensor
ϵ_{ij}^E	The elastic component of the strain tensor
ϵ_{ij}^P	The plastic component of the strain tensor
$\bar{\epsilon}_p$	The effective plastic strain, in./in.
$\epsilon_{re}, \epsilon_{\theta e}, \epsilon_{ze}$	The normal elastic strain components in the cylindrical coordinate system, in./in.
$\epsilon_{rp}, \epsilon_{\theta p}, \epsilon_{zp}$	The normal plastic strain components in the cylindrical coordinate system, in./in.
$\lambda, C_{11}, C_{12}, C_{33}, A_1, A_2, A_3, A_4, B_1$	Elastic materials constants for transversely isotropic materials related to E_x, E_z, ν_x, ν_z defined in the text

Λ	$d\bar{\epsilon}_p/\bar{\sigma}$; proportionality term appearing in the plasticity equations
μ_{ij}	$-\epsilon_j/\epsilon_i$; ratios of the total strain components where the (j) direction is transverse to the (i) direction of applied stress
ν_{ij}	$-\epsilon_j^E/\epsilon_i^E$; Poisson's ratios of the elastic strain components where the (j) direction is transverse to the (i) direction of applied stress
ν_{ij}^P	$-\epsilon_j^P/\epsilon_i^P$; ratios of the plastic strain components where the (j) direction is transverse to the (i) direction of applied stress
ν_x, ν_z	Strain ratios for transversely isotropic materials
$\bar{\sigma}$	The effective stress, psi
σ_{ij}	The stress tensor
$\sigma_r, \sigma_\theta, \sigma_z$	The normal stress components in the cylindrical coordinate system, psi
($\wedge$)	Caret used to denote an estimate

CHAPTER I

INTRODUCTION

Graphite is not generally considered as a primary candidate for a structural material. It is brittle, has a low structural strength, is normally anisotropic, inhomogeneous, and exhibits a nonlinear stress-strain relationship. Nevertheless graphite has certain unique physical properties which make its use highly desirable for certain applications. Graphite is an excellent thermal conductor, retains its strength at temperatures above 4000°F, can be manufactured in a variety of different shapes and sizes, and is one of the better neutron moderators. Graphite was used as a structural material in the first nuclear reactor, is currently being used in nuclear power reactors throughout Europe, and is the primary structural material in the core of several experimental power reactors. Graphite is also being used for the structural material in some reactor fuel elements and plays an important role in the design of reactors for nuclear rockets. It appears that graphite will continue to be an important structural material in future nuclear reactors.

In the past, elastic methods of analysis have been adequate if not entirely accurate. As the power requirements for future reactors increase, demands on the structural material will also increase. Since graphite is inherently not a strong material, on the order of 2000 to 4000 psi ultimate strength, it becomes important to take better advantage of the strength which is available. This in turn requires the use of analytical models which can describe the physical behavior of graphite more accurately. In

addition to the anisotropy, one of the most prominent characteristics of the material is that when it is mechanically loaded the strain is not proportional to the stress and upon unloading a portion of the strain is not recovered. This implies that the deformation behavior might be better described by a theory of plasticity. The purpose of the present study is to compare the behavior of an ideal structural element with predictions based on a theory of plasticity and with predictions based on the theory of elasticity.

A thick-walled cylinder loaded with an internal pressure was selected for study since an experimental investigation of such an ideal structure would be simple enough so that an adequate amount of data could be obtained at reasonable cost; and since the analytical solution of the problem would be complex enough to approximate conditions in a nuclear reactor. Laboratory investigations of the mechanical behavior of reactor grade graphite by Greenstreet et al. (1)¹ had shown that the inhomogeneous nature of the material could be accounted for by using standard concepts from statistics. The use of statistics, however, requires sufficient data to compute the mean and variance with reasonable accuracy. Consequently, 30 thick-walled tubes were tested to failure. Strain versus pressure curves were obtained for each specimen at two locations, the inside surface and the outside surface. Strains in both the circumferential and axial directions were measured using foil strain gages. The mean curves and the standard deviations were computed and compared with the analytical predictions based on

¹Numbers in parentheses refer to similarly numbered items in the List of References.

uniaxial data. In this way one can assess the importance of using a more realistic mathematical model in design and perhaps gain a better understanding of what constitutes safe design limits for graphite structures.

The most promising plasticity theory available in the open literature is the theory proposed by R. Hill in 1948 (2) for anisotropic metals. Hill's equations were based on a generalization of the von Mises yield criterion to include orthotropic behavior, in which the planes of anisotropy are perpendicular. Hu's reformulation (3) of Hill's equations were selected for the analytical portion of the study. A brief historical account of the theory of plasticity is given in Chapter II. Chapter II also contains a complete set of field equations for small deformation theory along with the elastic and plastic constitutive relations. Methods for obtaining the plasticity constants from uniaxial test data are discussed and a numerical computation of the constants is given.

The analytical solution of the thick-walled cylinder problem is discussed in Chapter III. The solution was obtained by a method of successive approximations starting with the solution to the elastic problem. For this purpose, a numerical algorithm for anisotropic incremental plasticity was developed. The numerical computations were carried out on a Control Data Corporation 1604-A digital computer using a program written especially for this problem. Normal components of the stress and strain tensor were calculated as a function of radius and internal pressure. Other parameters related to the theory of plasticity, such as the effective stress, $\bar{\sigma}$, and the effective plastic strain $\bar{\epsilon}_p$ were also calculated. Curves of these variables are shown.

Chapter IV is devoted to a description of the experimental investigation and reduction of the data. The experiments were carried out in the Experimental Stress Analysis Laboratory, Reactor Division, Oak Ridge National Laboratory with assistance from Laboratory personnel. The graphite material used for the thick-walled tube specimens was obtained from a block of AGOT reactor-grade graphite, the same material that was used in earlier investigations (1) to determine the uniaxial mechanical properties. AGOT is an extruded graphite with an axis of anisotropic materials properties parallel to the extrusion direction. Materials properties in the plane perpendicular to the extrusion are essentially isotropic. Such a material is referred to as transversely isotropic.

Comparison between the experimental measurements and the analytical predictions is discussed in Chapter V. Modifications are made to the theoretical equations to include the effects of plastic dilation (permanent volume change) and the experimental data are compared with the new predictions in Chapter VI. A summary and discussion of the entire study are contained in Chapter VII.

A definition for the effective plastic strain increment $d\bar{\epsilon}_p$, based on the hypothesis that strain hardening is a function only of the plastic work is developed in Appendix A for Hill's anisotropic loading function. A complete set of unabridged field equations in cylindrical coordinates is given in Appendix B for reference. A list of symbols used in the text is given in the nomenclature.

CHAPTER II

THEORY OF ORTHOTROPIC PLASTICITY FOR SMALL DEFORMATION

"Theory of plasticity" is the name given to the mathematical study of time independent stress and strain in permanently deformed solid bodies. The objective of the theory is twofold; first, to construct explicit relations between the stresses and the strains which agree with observations as closely as possible, and second to develop the mathematical techniques for calculating the stresses and strains in bodies which have been permanently deformed. "Plastic" strains are by definition irrecoverable. Therefore, the stress-strain relations, or constitutive equations, for materials which exhibit both recoverable and irrecoverable strains must account for both types of behavior by separate means. Conventional practice is to relate the recoverable portion of the strain (the elastic strains) to the stresses by Hooke's law. The plastic strains are related to the stresses by a suitable stress-plastic strain relation.

Theories of plasticity have been a subject for scientific study since 1864 when Tresca (4) introduced the concept of the maximum shear stress criteria for determining the onset of plastic flow. Behavior prior to yielding is described by the theory of elasticity. In 1870, St. Venant (5) formulated a continuum description of the flow phenomenon for a rigid perfectly plastic material. He postulated that the directions of the maximum strain rate should coincide with the directions of the maximum shear stress and applied Tresca's yield criterion to the solution of a plane problem. The St. Venant theory was generalized to three dimensions

by Lévy (6) in 1870. These early studies were summarized and put into perspective by St. Venant (7) in a report to the French Academy of Science in 1871.

In 1913 von Mises (8) advanced a yield criterion based on purely mathematical considerations which was interpreted by Hencky (9) in 1924, as implying that yielding occurred when the elastic shear-strain energy reached a critical value. Both the Tresca theory and the von Mises theory imply coaxiality between the stress deviator tensor and the plastic strain-increment tensor. Plastic incompressibility is a fundamental postulate of both theories, and has been demonstrated by Bridgman (10) in 1923 and others. The Lévy-Mises equations have been shown to be valid to a first approximation by Lode (11) in 1926 and Taylor and Quinney (12) in 1931.

Prandtl (13) in 1924, incorporated the elastic strains into the stress-strain relations for the plane problem. He took the total strain as the sum of the elastic and plastic parts, using Hooke's law for the elastic stress-strain relations and St. Venant's theory for the plastic relations. Reuss (14) in 1930, developed the equations in three dimensions for the elastic, perfectly plastic body. The theory was further generalized in 1932 by Schmidt (15) and by Odqvist (16) in 1933 who showed how work hardening could be brought within the framework of plasticity theory.

The concept of a plastic potential was introduced by von Mises (17) in 1928 who proposed that the strain rates for a rigid, perfectly plastic material be given in terms of a general yield function. When the plastic potential and the yield function coincide, Bland (18) in 1957 identified the law as an "associated flow rule." Melan (19) in 1938, used the concept

of a plastic potential and developed stress-strain relations for work-hardening materials.

Anisotropic materials were first considered by von Mises (17) in 1928 who studied the flow condition for crystals and proposed a yield criterion based on a homogeneous quadratic function of stresses with 15 independent coefficients. Hill in 1948 (2) and 1950 (20) used a simplified form of this criterion to describe the yield condition for anisotropic metals. He assumed that the yield stress in tension for any direction was the same as in compression. The theory was later extended by Hu (3) in 1956 to include work hardening. Hu described work hardening in terms of the effective stress and the effective plastic strain. These are the equations which are used in the present study.

A complete set of equations for small deformation theory orthotropic plasticity is given in the following sections. These are the field equations and the constitutive equations which together with the boundary conditions must be satisfied for the solution to a particular problem. In general, the materials properties are assumed to have directional dependence which coincides with the rectangular Cartesian (x_1, x_2, x_3) coordinate system. For transversely isotropic materials such as graphite which have a plane of isotropic materials properties, two of the coordinate directions are confined to lie in the isotropic plane but are otherwise unrestricted. In the present study the 2-3 plane is taken as the isotropic plane with the x_1 direction coinciding with the direction of materials symmetry.

I. THE FIELD EQUATIONS

The field equations given here are those for small deformations of a continuum [see Ref. (21)]. They consist of the strain-displacement relations, the compatibility equations, and the equilibrium equations. They are given in the rectangular Cartesian coordinate system in order to be consistent with the constitutive equations given in the next section. The field equations are given in cylindrical coordinates in Appendix B for reference.

The variables are: (1) the displacements u_i ; $i = 1, 2, 3$ in the rectangular Cartesian coordinate directions x_i , (2) the six independent components of the symmetric strain tensor ϵ_{ij} ; $i, j = 1, 2, 3$, (3) the six independent components of the stress tensor σ_{ij} ; $i, j = 1, 2, 3$, and (4) the body forces F_j ; $j = 1, 2, 3$. When the subscripts are equal, $i = j$, the stress and strain components are called "normal"; unequal subscripts $i \neq j$ denote the shear components.

Strain-Displacement Relations

The six independent strain components are defined in terms of the displacements by the following six equations.

$$\begin{aligned} \epsilon_{11} &= \frac{\partial u_1}{\partial x_1}, & \epsilon_{12} &= \frac{1}{2} \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right), \\ \epsilon_{22} &= \frac{\partial u_2}{\partial x_2}, & \epsilon_{23} &= \frac{1}{2} \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right), \\ \epsilon_{33} &= \frac{\partial u_3}{\partial x_3}, & \epsilon_{31} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right). \end{aligned} \quad (2.1)$$

The Compatibility Equations

Six second-order partial differential equations, called the compatibility equations, are additional restrictions on the strain components which must be satisfied in order to ensure continuity in the displacements. These are:

$$\begin{aligned}
 \frac{\partial^2 \epsilon_{11}}{\partial x_2 \partial x_3} &= \frac{\partial}{\partial x_1} \left(-\frac{\partial \epsilon_{23}}{\partial x_1} + \frac{\partial \epsilon_{31}}{\partial x_2} + \frac{\partial \epsilon_{12}}{\partial x_3} \right), \\
 \frac{\partial^2 \epsilon_{22}}{\partial x_3 \partial x_1} &= \frac{\partial}{\partial x_2} \left(+\frac{\partial \epsilon_{23}}{\partial x_1} - \frac{\partial \epsilon_{31}}{\partial x_2} + \frac{\partial \epsilon_{12}}{\partial x_3} \right), \\
 \frac{\partial^2 \epsilon_{33}}{\partial x_1 \partial x_2} &= \frac{\partial}{\partial x_3} \left(+\frac{\partial \epsilon_{23}}{\partial x_1} + \frac{\partial \epsilon_{31}}{\partial x_2} - \frac{\partial \epsilon_{12}}{\partial x_3} \right), \\
 \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} &= \frac{1}{2} \left(\frac{\partial^2 \epsilon_{22}}{\partial x_1^2} + \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} \right), \\
 \frac{\partial^2 \epsilon_{23}}{\partial x_2 \partial x_3} &= \frac{1}{2} \left(\frac{\partial^2 \epsilon_{33}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_3^2} \right), \\
 \frac{\partial^2 \epsilon_{31}}{\partial x_3 \partial x_1} &= \frac{1}{2} \left(\frac{\partial^2 \epsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \epsilon_{33}}{\partial x_1^2} \right).
 \end{aligned} \tag{2.2}$$

The Equilibrium Equations

The equations of force equilibrium in terms of the stresses σ_{ij} and the body forces F_j are the following three first-order partial differential equations.

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + F_1 = 0,$$

$$\begin{aligned}
\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + F_2 &= 0, \\
\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + F_3 &= 0.
\end{aligned}
\tag{2.3}$$

II. THE CONSTITUTIVE EQUATIONS

The complete stress-strain relations describe the general relationship between the stress variables and the strain variables. For a material which undergoes both elastic and plastic deformations, the strain tensor is divided into two components according to the Prandtl-Reuss (22) relationship,

$$\epsilon_{ij} = \epsilon_{ij}^E + \epsilon_{ij}^P, \tag{2.4}$$

where ϵ_{ij}^E is the elastic component of the strain tensor and ϵ_{ij}^P is the plastic strain component.

Hooke's Law

The elastic strain components ϵ_{ij}^E are related to the stresses by Hooke's law. For orthotropic materials, the strains are defined by the following set of equations (23).

$$\begin{aligned}
\epsilon_{11}^E &= \frac{1}{E_1} \sigma_{11} - \frac{\nu_{21}}{E_2} \sigma_{22} - \frac{\nu_{31}}{E_3} \sigma_{33}, & \epsilon_{12}^E &= \frac{\sigma_{12}}{2G_{12}}, \\
\epsilon_{22}^E &= \frac{1}{E_2} \sigma_{22} - \frac{\nu_{32}}{E_3} \sigma_{33} - \frac{\nu_{12}}{E_1} \sigma_{11}, & \epsilon_{23}^E &= \frac{\sigma_{23}}{2G_{23}}, \\
\epsilon_{33}^E &= \frac{1}{E_3} \sigma_{33} - \frac{\nu_{13}}{E_1} \sigma_{11} - \frac{\nu_{23}}{E_2} \sigma_{22}, & \epsilon_{31}^E &= \frac{\sigma_{31}}{2G_{31}}.
\end{aligned}
\tag{2.5}$$

where E_i are the elastic Young's moduli in the respective coordinate directions, G_{ij} are the shear moduli, and ν_{ij} are Poisson's strain ratios determined from uniaxial tests. The first subscript (i) on ν_{ij} indicates the axial direction of the uniaxial test specimen, relative to the materials properties. The second subscript (j) indicates a direction transverse to the test specimen. Thus, ν_{ij} is the negative ratio of strain in the (j) direction to the strain in the (i) direction caused by a stress in the (i) direction; $\nu_{ij} = -\epsilon_j^E / \epsilon_i^E$.

Plasticity Equations

The plastic strain components ϵ_{ij}^P are related to the stresses through the incremental-plastic-strain equations and the integral along the loading path τ

$$\epsilon_{ij}^P = \int_{\tau} d\epsilon_{ij}^P . \quad (2.6)$$

The complete set of orthotropic incremental-plastic-strain equations, as developed by Hu (3) are given below in incremental form. These equations together with Eq. (2.6) define the plastic components of the strain in terms of the stresses.

$$\begin{aligned} d\epsilon_{11}^P &= d\lambda (\alpha_{11}\sigma_{11} - \alpha_{12}\sigma_{22} - \alpha_{31}\sigma_{33}) , \\ d\epsilon_{22}^P &= d\lambda (-\alpha_{12}\sigma_{11} + \alpha_{22}\sigma_{22} - \alpha_{23}\sigma_{33}) , \\ d\epsilon_{33}^P &= d\lambda (-\alpha_{31}\sigma_{11} - \alpha_{23}\sigma_{22} + \alpha_{33}\sigma_{33}) , \\ d\epsilon_{12}^P &= d\lambda \alpha_{44}\sigma_{12} , \end{aligned} \quad (2.7)$$

$$d\epsilon_{23}^P = d\lambda \alpha_{55} \sigma_{23} , \quad (2.7)$$

$$d\epsilon_{31}^P = d\lambda \alpha_{66} \sigma_{31} ,$$

where

$$\alpha_{11} = \alpha_{12} + \alpha_{31} ,$$

$$\alpha_{22} = \alpha_{23} + \alpha_{12} , \quad (2.8)$$

$$\alpha_{33} = \alpha_{31} + \alpha_{23} .$$

The coefficients α_{ij} are determined from test data.

It is readily verified from Eqs. (2.7) and (2.8) that the plastic dilatation $\mathcal{D}_p$, the change in volume due to plastic distortion, is zero since

$$d\mathcal{D}_p = d\epsilon_{11}^P + d\epsilon_{22}^P + d\epsilon_{33}^P \equiv 0 . \quad (2.9)$$

The zero plastic volume change characteristic is common to metals and is normally stated as one of the fundamental postulates of the mathematical theory of plasticity.

The quantity $d\lambda$ in Eq. (2.7) is a positive scalar function of proportionality which expresses the strain hardening law. It is related to two parameters, the effective-stress $\bar{\sigma}$ and the effective-plastic-strain increment $d\bar{\epsilon}_p$ by

$$d\lambda = \frac{d\bar{\epsilon}_p}{\bar{\sigma}} , \quad (2.10)$$

where $\bar{\sigma}$ and $d\bar{\epsilon}_p$ are quadratic functions given by the following equations.

$$\begin{aligned} (\bar{\sigma})^2 = & [\alpha_{12}(\sigma_{11} - \sigma_{22})^2 + \alpha_{23}(\sigma_{22} - \sigma_{33})^2 + \alpha_{31}(\sigma_{33} - \sigma_{11})^2 \\ & + 2\alpha_{44}\sigma_{12}^2 + 2\alpha_{55}\sigma_{23}^2 + 2\alpha_{66}\sigma_{31}^2] , \end{aligned} \quad (2.11)$$

and

$$\begin{aligned}
 (d\bar{\epsilon}_p)^2 = & \frac{1}{G^2} \left[\alpha_{12} (\alpha_{23} d\epsilon_{11}^P - \alpha_{31} d\epsilon_{22}^P)^2 \right. \\
 & + \alpha_{23} (\alpha_{31} d\epsilon_{22}^P - \alpha_{12} d\epsilon_{33}^P)^2 + \alpha_{31} (\alpha_{12} d\epsilon_{33}^P - \alpha_{23} d\epsilon_{11}^P)^2 \left. \right] \\
 & + 2 \left[\frac{(d\epsilon_{12}^P)^2}{\alpha_{44}} + \frac{(d\epsilon_{23}^P)^2}{\alpha_{55}} + \frac{(d\epsilon_{31}^P)^2}{\alpha_{66}} \right], \quad (2.12)
 \end{aligned}$$

where

$$G = (\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12}).$$

Equations (2.10), (2.11), and (2.12) are a consistent set of relations as shown in Appendix A.

For uniaxial loading conditions, $\bar{\sigma}$ is proportional to the nonzero stress component. For the three normal uniaxial stress conditions, the effective stress becomes

$$\begin{aligned}
 \bar{\sigma} &= \sqrt{\alpha_{11}} \sigma_{11} ; & \text{only } \sigma_{11} \neq 0 , \\
 \bar{\sigma} &= \sqrt{\alpha_{22}} \sigma_{22} ; & \text{only } \sigma_{22} \neq 0 , \\
 \bar{\sigma} &= \sqrt{\alpha_{33}} \sigma_{33} ; & \text{only } \sigma_{33} \neq 0 .
 \end{aligned} \quad (2.13)$$

Similar results can be obtained for the effective plastic strain increment $d\bar{\epsilon}_p$, from Eq. (2.7) with Eqs. (2.8) and (2.13),

$$\begin{aligned}
 d\bar{\epsilon}_p &= \frac{1}{\sqrt{\alpha_{11}}} d\epsilon_{11}^P ; & \text{only } \sigma_{11} \neq 0 , \\
 d\bar{\epsilon}_p &= \frac{1}{\sqrt{\alpha_{22}}} d\epsilon_{22}^P ; & \text{only } \sigma_{22} \neq 0 , \\
 d\bar{\epsilon}_p &= \frac{1}{\sqrt{\alpha_{33}}} d\epsilon_{33}^P ; & \text{only } \sigma_{33} \neq 0 .
 \end{aligned} \quad (2.14)$$

Strain Hardening

Strain hardening or work hardening is referred to as the phenomenon that as a material deforms plastically the resistance to deformation increases. One mathematical description of plastic strain hardening is based on the hypothesis that the degree of hardening is a function only of the total plastic work and is otherwise independent of the strain paths (22, page 61). According to this hypothesis the effective stress $\bar{\sigma}$ is a function of the total plastic work W_p ,

$$\bar{\sigma} = F(W_p) , \quad (2.15)$$

where the total plastic work is defined by

$$W_p = \int_{\tau} dW_p ,$$

and

$$dW_p = \bar{\sigma} d\bar{\epsilon}_p , \quad (2.16)$$

as shown in Appendix A.

The strain-hardening hypothesis, Eq. (2.15), can be written as

$$\bar{\sigma} = F \left(\int_{\tau} \bar{\sigma} d\bar{\epsilon}_p \right) ,$$

and it follows that the effective stress $\bar{\sigma}$ is a function only of the effective plastic strain (22, page 63),

$$\bar{\sigma} = H(\bar{\epsilon}_p) = H \left(\int_{\tau} d\bar{\epsilon}_p \right) . \quad (2.17)$$

Equation (2.17) is called the effective or generalized-stress plastic-strain relationship (24).

The relationship indicated by Eq. (2.17) can be simply obtained from uniaxial tests. Since unloading was not considered in the present study,

an effective stress-plastic strain relationship valid for monotonic loading is sufficient. When Eqs. (2.13) and (2.14) are substituted into Eq. (2.17), one obtains the following functional relations.

$$\begin{aligned}\sqrt{\alpha_{11}} \sigma_{11} &= H \left(\frac{\epsilon_{11}^P}{\sqrt{\alpha_{11}}} \right) ; & \text{only } \sigma_{11} \neq 0 , \\ \sqrt{\alpha_{22}} \sigma_{22} &= H \left(\frac{\epsilon_{22}^P}{\sqrt{\alpha_{22}}} \right) ; & \text{only } \sigma_{22} \neq 0 , \\ \sqrt{\alpha_{33}} \sigma_{33} &= H \left(\frac{\epsilon_{33}^P}{\sqrt{\alpha_{33}}} \right) ; & \text{only } \sigma_{33} \neq 0 .\end{aligned}\tag{2.18}$$

When the single coefficient in one of Eqs. (2.18) is set equal to 1, essentially normalizing Eqs. (2.7), Eq. (2.17) will coincide with the stress-plastic strain relationship measured in the corresponding uniaxial test. Therefore let

$$\alpha_{11} = 1.0 .\tag{2.19}$$

Then the corresponding relationship for monotonic loading is

$$\bar{\sigma} = H(\epsilon_{11}^P) ,$$

or alternately,

$$\bar{\epsilon}_p = G(\sigma_{11}) .\tag{2.20}$$

III. DETERMINATION OF THE PLASTICITY CONSTANTS

The three constants, α_{12} , α_{23} , and α_{31} , may be determined from uniaxial test data. The other three, α_{44} , α_{55} , and α_{66} , can be determined from shear test data.

Because of symmetry only the constants α_{12} , α_{23} , and α_{31} appear in the equations for a thick-walled cylinder under internal pressure. There are three methods for calculating these constants known to the writer, two of which were developed during the current study. All three methods will be discussed since two of the methods give independent checks on the resulting values, and thereby provide a preliminary test of Hu's equations as applied to graphite. The yield stress method given by Hu is not applicable to graphite since the yield point is taken as zero. For the sake of completeness, however, Hu's method will also be discussed.

Before the methods of computation are discussed, it will be convenient to specialize the general equations for uniaxial test conditions. The principal plastic strain increments can be obtained from Eqs. (2.7) and (2.10) as follows. When $\sigma_{11} \neq 0$ and $\sigma_{12}, \sigma_{22}, \sigma_{23}, \sigma_{31}, \sigma_{33} \equiv 0$, then

$$\begin{aligned} d\epsilon_{11}^P &= \frac{d\bar{\epsilon}_P}{\bar{\sigma}} \alpha_{11} \sigma_{11} , \\ d\epsilon_{22}^P &= - \frac{d\bar{\epsilon}_P}{\bar{\sigma}} \alpha_{12} \sigma_{11} , \\ d\epsilon_{33}^P &= - \frac{d\bar{\epsilon}_P}{\bar{\sigma}} \alpha_{31} \sigma_{11} , \end{aligned} \quad (2.21)$$

and

$$d\epsilon_{12}^P = d\epsilon_{23}^P = d\epsilon_{31}^P \equiv 0 .$$

Similar results are obtained from uniaxial tests in the other two principal directions.

When $\sigma_{22} \neq 0$ and $\sigma_{11} = \sigma_{12} = \sigma_{23} = \sigma_{31} = \sigma_{33} \equiv 0$, then

$$\begin{aligned} d\epsilon_{11}^P &= -\frac{d\bar{\epsilon}_p}{\bar{\sigma}} \alpha_{12} \sigma_{22} , \\ d\epsilon_{22}^P &= \frac{d\bar{\epsilon}_p}{\bar{\sigma}} \alpha_{22} \sigma_{22} , \\ d\epsilon_{33}^P &= -\frac{d\bar{\epsilon}_p}{\bar{\sigma}} \alpha_{23} \sigma_{22} , \end{aligned} \quad (2.22)$$

and

$$d\epsilon_{12}^P = d\epsilon_{23}^P = d\epsilon_{31}^P \equiv 0 .$$

When $\sigma_{33} \neq 0$ and $\sigma_{11} = \sigma_{12} = \sigma_{22} = \sigma_{23} = \sigma_{31} \equiv 0$, then

$$\begin{aligned} d\epsilon_{11}^P &= -\frac{d\bar{\epsilon}_p}{\bar{\sigma}} \alpha_{31} \sigma_{33} , \\ d\epsilon_{22}^P &= -\frac{d\bar{\epsilon}_p}{\bar{\sigma}} \alpha_{23} \sigma_{33} , \\ d\epsilon_{33}^P &= \frac{d\bar{\epsilon}_p}{\bar{\sigma}} \alpha_{33} \sigma_{33} , \end{aligned} \quad (2.23)$$

$$d\epsilon_{12}^P = d\epsilon_{23}^P = d\epsilon_{31}^P \equiv 0 .$$

The Plastic Strain Ratio Method

The first method for calculating the constants is based on the ratios of the plastic strain measured transverse to the direction of applied stress to the plastic strain measured in the direction of the applied stress, and the assumption that these ratios are constant with longitudinal strain. An expression for calculating the plastic strain ratios from uniaxial test data is developed below.

If, for a uniaxial test in one of the principal directions x_i , the single subscripted variable ϵ_i is used to denote the strain in the direction of applied stress, and ϵ_j is used to denote the transverse strain in the x_j direction, $j \neq i$, then ϵ_i and ϵ_j are the sum of their respective elastic and plastic parts, thus,

$$\epsilon_i = \epsilon_i^E + \epsilon_i^P,$$

$$\epsilon_j = \epsilon_j^E + \epsilon_j^P.$$

Let ν_{ij} be Poisson's ratio

$$\nu_{ij} = -\epsilon_j^E / \epsilon_i^E,$$

and let μ_{ij} and ν_{ij}^P be similarly defined in terms of the total strains and plastic strain, respectively,

$$\mu_{ij} = -\epsilon_j / \epsilon_i,$$

$$\nu_{ij}^P = -\epsilon_j^P / \epsilon_i^P,$$

then ν_{ij}^P can be expressed in terms of μ_{ij} , ν_{ij} , ϵ_i , and ϵ_i^P , as shown in the following. Repeated subscripts do not indicate summations, but (i) and (j) may take on values 1,2,3 with $i \neq j$. Thus,

$$\begin{aligned} \nu_{ij}^P &= -\frac{\epsilon_j^P}{\epsilon_i^P} = \frac{1}{\epsilon_i^P} (-\epsilon_j + \epsilon_j^E) \\ &= \frac{1}{\epsilon_i^P} \left[-\frac{\epsilon_j}{\epsilon_i} (\epsilon_i^E + \epsilon_i^P) + \frac{\epsilon_j^E}{\epsilon_i^E} \epsilon_i^E \right] \\ &= \frac{1}{\epsilon_i^P} \left[\epsilon_i^E (\mu_{ij} - \nu_{ij}) + \epsilon_i^P \mu_{ij} \right], \end{aligned}$$

and finally,

$$v_{ij}^P = \epsilon_i^E / \epsilon_i^P (\mu_{ij} - v_{ij}) + \mu_{ij} ; \quad i \neq j, \text{ not summed .} \quad (2.24)$$

Two of the constants, α_{12} and α_{31} , can be found in terms of α_{11} from the uniaxial test $\sigma_{11} \neq 0$ data and Eqs. (2.21), thus,

$$\alpha_{12} = - \alpha_{11} \frac{d\epsilon_{22}^P}{d\epsilon_{11}^P} ,$$

and

$$\alpha_{31} = - \alpha_{11} \frac{d\epsilon_{33}^P}{d\epsilon_{11}^P} .$$

If the incremental strain ratios are constant with ϵ_{11} , then with $\alpha_{11} = 1.0$,

$$\alpha_{12} = - \frac{\epsilon_{22}^P}{\epsilon_{11}^P} = v_{12}^P ,$$

and

$$\alpha_{31} = - \frac{\epsilon_{33}^P}{\epsilon_{11}^P} = v_{13}^P .$$

The remaining constant α_{23} can be calculated from either the $\sigma_{22} \neq 0$ or the $\sigma_{33} \neq 0$ test data. Using the $\sigma_{22} \neq 0$ test data and Eqs. (2.22) and (2.8), one obtains

$$\alpha_{23} = - \alpha_{22} \frac{d\epsilon_{33}^P}{d\epsilon_{22}^P} = (\alpha_{23} + \alpha_{12}) v_{23}^P ,$$

$$\alpha_{12} = - \alpha_{22} \frac{d\epsilon_{11}^P}{d\epsilon_{22}^P} = (\alpha_{23} + \alpha_{12}) v_{21}^P ,$$

or

$$\alpha_{23} = \frac{v_{12}^P v_{23}^P}{v_{21}^P} .$$

If the $\sigma_{33} \neq 0$ data are used together with Eq. (2.23), then

$$\alpha_{23} = \frac{v_{13}^P v_{32}^P}{v_{31}^P} .$$

In summary, the constants α_{12} , α_{23} , and α_{31} may be expressed in terms of the plastic strain ratios, thus,

$$\begin{aligned} \alpha_{12} &= v_{12}^P , \\ \alpha_{23} &= \frac{v_{12}^P v_{23}^P}{v_{21}^P} = \frac{v_{13}^P v_{32}^P}{v_{31}^P} , \\ \alpha_{31} &= v_{13}^P , \end{aligned} \tag{2.25}$$

where the plastic strain ratios v_{ij}^P are obtained from uniaxial test data and Eq. (2.24).

The following observations are in order for certain special cases.

If the material is isotropic in the 2-3 plane, then $v_{12}^P = v_{13}^P$ and

$\alpha_{12} = \alpha_{13} = v_{12}^P$. If the total strain ratio curve is constant with ϵ_1 , then $\mu_{ij} = v_{ij}$ and consequently, from Eq. (2.24), $v_{ij}^P = v_{ij}$.

The Yield Stress Method

A second method, given by Hu, for calculating the constants, α_{12} , α_{23} , and α_{31} , is based on the intersection of the yield surface in stress space and the three coordinate axes referenced to the orthotropic directions of

the material. When the effective stress $\bar{\sigma}$, as given by Eq. (2.11), is set equal to a constant, $\bar{\sigma} = \text{const}$, and plotted in principal stress space, where the principal stress directions are assumed to coincide with the orthotropic materials directions, the result will be a cylindrical surface with an elliptical cross section, as shown in Fig. 1, described by

$$\bar{\sigma} = \text{const} = \left[\alpha_{12}(\sigma_{11} - \sigma_{22})^2 + \alpha_{23}(\sigma_{22} - \sigma_{33})^2 + \alpha_{31}(\sigma_{33} - \sigma_{11})^2 \right]^{1/2}. \quad (2.26)$$

This surface will intersect the coordinate axes at positions which correspond with the normal uniaxial stress conditions given by Eq. (2.13). Since $\bar{\sigma}$ was set equal to a constant the three relations of Eq. (2.13) are equal; thus,

$$\bar{\sigma} = \sqrt{\alpha_{11}} \sigma_{11} = \sqrt{\alpha_{22}} \sigma_{22} = \sqrt{\alpha_{33}} \sigma_{33}. \quad (2.27)$$

If the respective stress values are taken as the yield stresses Y_{11} , Y_{22} , and Y_{33} so that the effective yield stress $\bar{\sigma}_y$ is given by

$$\bar{\sigma}_y = \sqrt{\alpha_{11}} Y_{11} = \sqrt{\alpha_{22}} Y_{22} = \sqrt{\alpha_{33}} Y_{33},$$

and $\alpha_{11} = 1.0$, then

$$\alpha_{11} = 1.0, \quad \alpha_{22} = \left(\frac{Y_{11}}{Y_{22}} \right)^2, \quad \alpha_{33} = \left(\frac{Y_{11}}{Y_{33}} \right)^2. \quad (2.28)$$

Using Eq. (2.8) and solving the resulting three simultaneous equations for α_{12} , α_{23} , and α_{31} gives

$$\alpha_{12} = \frac{1}{2} \left[1 + \left(\frac{Y_{11}}{Y_{22}} \right)^2 - \left(\frac{Y_{11}}{Y_{33}} \right)^2 \right],$$

Figure 1. Hill's Anisotropic Effective Stress Function in Principal Stress Space and Its Intersection with the Coordinate Planes.

$$\alpha_{23} = \frac{1}{2} \left[-1 + \left(\frac{Y_{11}}{Y_{22}} \right)^2 + \left(\frac{Y_{11}}{Y_{33}} \right)^2 \right] ,$$

$$\alpha_{31} = \frac{1}{2} \left[1 - \left(\frac{Y_{11}}{Y_{22}} \right)^2 + \left(\frac{Y_{11}}{Y_{33}} \right)^2 \right] .$$
(2.29)

If the material is isotropic in the 2-3 plane such that the yield stresses are equal, $Y_{22} = Y_{33}$, then $\alpha_{12} = \alpha_{31} = 1/2$. If the yield stress is equal in all three directions, then all three constants equal $1/2$.

Effective Plastic Strain Method

A third method, which is a generalization of the second method, can be developed by considering the effective plastic strain and the generalized-stress plastic-strain relation, Eq. (2.17), shown schematically in Fig. 2. For strain-hardening materials, a constant value of effective stress $\bar{\sigma}$ implies also a constant value of effective plastic strain $\bar{\epsilon}_p$. Therefore, when the effective stress values for each of the three uniaxial stress conditions are equal, Eq. (2.27),

$$\bar{\sigma} = \sqrt{\alpha_{11}} \sigma_{11} = \sqrt{\alpha_{22}} \sigma_{22} = \sqrt{\alpha_{33}} \sigma_{33} ,$$

then the effective plastic strain values from Eq. (2.14) are also equal,

$$\bar{\epsilon}_p = \frac{\epsilon_{11}^P}{\sqrt{\alpha_{11}}} = \frac{\epsilon_{22}^P}{\sqrt{\alpha_{22}}} = \frac{\epsilon_{33}^P}{\sqrt{\alpha_{33}}} .$$
(2.30)

With $\alpha_{11} = 1.0$, then

$$\alpha_{22} = \left(\frac{\sigma_{11}}{\sigma_{22}} \right)^2 = \left(\frac{\epsilon_{22}^P}{\epsilon_{11}^P} \right)^2 ,$$

$$\alpha_{33} = \left(\frac{\sigma_{11}}{\sigma_{33}} \right)^2 = \left(\frac{\epsilon_{33}^P}{\epsilon_{11}^P} \right)^2 .$$
(2.31)

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Figure 2. The Effective-Stress Effective-Plastic-Strain Relation.

The effective plastic strain $\bar{\epsilon}_p$ is also equal for all three uniaxial stress conditions when the plastic work is equal, even without strain hardening. Merkle (25) developed a graphical method for determining the constants which is also based on the concept of equivalent plastic work for constant values of the effective strain.

Equations (2.31) can be solved for the constants α_{22} and α_{33} by inserting the uniaxial stress-strain relationships determined from the test data. Studies by Greenstreet et al. (1) indicate that these relations for AGOT graphite can be adequately represented by the power law,

$$\begin{aligned}\epsilon_{11}^P &= k_1 \sigma_{11}^n, \\ \epsilon_{22}^P &= k_2 \sigma_{22}^n, \\ \epsilon_{33}^P &= k_3 \sigma_{33}^n,\end{aligned}\tag{2.32}$$

where k_1 , k_2 , k_3 , and n are constants. Then,

$$\alpha_{22} = \left(\frac{\epsilon_{22}^P}{\epsilon_{11}^P} \right)^2 = \left(\frac{k_2}{k_1} \right)^2 \left(\frac{\sigma_{22}}{\sigma_{11}} \right)^{2n} = \left(\frac{k_2}{k_1} \right)^2 \alpha_{22}^{-n},$$

and

$$\alpha_{22} = \left(\frac{k_2}{k_1} \right)^{2/(n+1)}.$$

Similarly

$$\alpha_{33} = \left(\frac{k_3}{k_1} \right)^{2/(n+1)}.$$

Using Eq. (2.8) and solving simultaneously,

$$\begin{aligned}\alpha_{12} &= \frac{1}{2} \left[1 + \left(\frac{k_2}{k_1} \right)^{2/(n+1)} - \left(\frac{k_3}{k_1} \right)^{2/(n+1)} \right], \\ \alpha_{23} &= \frac{1}{2} \left[-1 + \left(\frac{k_2}{k_1} \right)^{2/(n+1)} + \left(\frac{k_3}{k_1} \right)^{2/(n+1)} \right], \\ \alpha_{31} &= \frac{1}{2} \left[1 - \left(\frac{k_2}{k_1} \right)^{2/(n+1)} + \left(\frac{k_3}{k_1} \right)^{2/(n+1)} \right].\end{aligned}\quad (2.33)$$

Again, if the material is isotropic in the 2-3 plane, $\alpha_{12} = \alpha_{31} = 1/2$, as shown by letting $k_2 = k_3$ in Eqs. (2.33).

Discussion

In summary, three independent methods have been developed for calculating the plasticity constants of the Hill-Hu equations for anisotropic plasticity from uniaxial test data, two of which can be used for graphite. Each of the methods is consistent with the basic assumptions of the theory, one of which is that the plastic dilatation is zero. Both the strain ratio method and the effective plastic strain method should therefore give the same numerical values for the constants if the assumptions of the theory hold for this particular material.

The uniaxial data of Greenstreet et al. for this graphite show that it is transversely isotropic,² thus indicating that $\alpha_{12} = \alpha_{31} = 1/2$. However, they also show that the total strain ratios μ_{12} and μ_{31} are nearly

²The coordinate directions used by Greenstreet et al. are different from the ones used here, which must be taken into account in comparing the numerical values.

constant with values of about 0.081, indicating by the strain ratio method that $\alpha_{12} = \alpha_{31} \approx 0.08$. This shows a discrepancy of more than a factor of six. In addition, calculations of α_{23} by the two methods disagree by about one order of magnitude, $\alpha_{23} = 1.76$ by the effective plastic strain method and $\alpha_{23} = 0.143$ by the strain ratio method. There are strong indications that graphite deforms with a definite plastic dilatation and therefore the present equations may not be entirely adequate. Ways of overcoming this difficulty are developed in Chapter VI and the relative importance of the disagreement is discussed.

CHAPTER III

ANALYTICAL SOLUTION FOR THE THICK-WALLED
CYLINDER WITH AN INTERNAL PRESSURE

The infinitely long thick-walled circular cylinder loaded with an internal pressure is a classical problem in the theory of elasticity. Lamé (26) is given credit for first solving the problem in 1852 and the problem is discussed in most text books on the theory of elasticity and strength of materials. The elastic problem may be reduced to that of solving one ordinary linear differential equation. The differential equation may be written in terms of a stress function, a stress variable, a strain variable, or the radial displacement. In every case the differential equation is homogeneous and once the solution to the homogeneous equation is known, stresses, strains, and displacements throughout the body can be determined with the use of the boundary equations.

The addition of the nonlinear plastic-strain terms to the stress-strain relation, however, complicates the problem. For small deformations, the elastic-plastic behavior can be described by the same differential equation as for the elastic problem except for the addition of nonhomogeneous terms. One must thus obtain a particular solution for the plastic-strain terms.

I. THE REDUCED FIELD EQUATIONS

The general field equations are simplified considerably for the thick-walled cylinder with axial symmetry (see Fig. 3). In the present study,

Figure 3. Thick-Walled Graphite Cylinder with Internal Pressure and Axial Force.

the cylinder was oriented with respect to the material so that the plane of isotropy was perpendicular to the axis of the cylinder. The field equations can then be written in cylindrical coordinates (z, θ, r) and reduced by reason of symmetry as shown in Appendix B.

Since all shear terms are zero, single subscripts are used to denote the principal or normal stress and strain components. A lower case (p) is added to denote the plastic strain component and a lower case (e) is added for the elastic strain component, thus, $\epsilon_z = \epsilon_{ze} + \epsilon_{zp}$, etc. Numbered subscripts are retained for the elastic constants C_{11} , C_{12} , and C_{33} , and for the plasticity constants α_{12} , α_{23} , and α_{31} , since in this particular case the numerical values are the same in either the rectangular Cartesian or the cylindrical coordinate system.

Strain-Displacement Relations

The infinitely long cylinder is in a condition of generalized plane strain, thus $\partial(\dots)/\partial z = 0$. Because of axial symmetry $\partial(\dots)/\partial \theta = 0$, therefore the displacements and the strain-displacement relations of Eq. (2.1) become simply:

$$\begin{aligned} u_r &= u, & u_\theta &= 0, & u_z &= \text{constant times } z, \\ \epsilon_r &= \frac{du}{dr}, & \epsilon_\theta &= \frac{u}{r}, & \epsilon_z &= \text{constant}, \\ \epsilon_{r\theta} &= \epsilon_{\theta z} = \epsilon_{zr} \equiv 0. \end{aligned} \quad (3.1)$$

The Compatibility Equation

Five of the six compatibility equations of Eq. (2.2) are identically satisfied because of the axial symmetry, leaving the single ordinary

differential equation (see Appendix B):

$$\frac{d\epsilon_{\theta}}{dr} = \frac{1}{r} (\epsilon_r - \epsilon_{\theta}) ,$$

or written alternately

(3.2)

$$\frac{d}{dr} (r\epsilon_{\theta}) = \epsilon_r .$$

The Equilibrium Equation

The only nontrivial equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_{\theta}}{r} = 0 . \quad (3.3)$$

The Stress-Strain Relations

For transversely isotropic materials with the 2-3 plane as the plane of isotropy, the following elastic constants in Hooke's law Eq. (2.5) are equivalent:³

$$E_2 = E_3 , \quad \nu_{13} = \nu_{12} , \quad \nu_{23} = \nu_{32} ,$$

$$\nu_{21} = \nu_{31} , \quad \frac{\nu_{13}}{E_1} = \frac{\nu_{21}}{E_2} = \frac{\nu_{31}}{E_3} , \quad G_{12} = G_{31} ,$$

$$G_{23} = \frac{E_2}{2(1 + \nu_{23})} .$$

The number of independent elastic constants therefore reduces to five, denoted here by the single subscripted variables:

$$E_Z = E_1 , \quad E_X = E_2 = E_3 , \quad \nu_Z = \nu_{31} = \nu_{21} , \\ \nu_X = \nu_{23} = \nu_{32} , \quad G_Z = G_{12} = G_{31} .$$

³Complete elastic stress-strain relations are given for various types of anisotropic behavior in Reference 23.

Since the shear terms vanish, the complete set of stress-strain equations for the thick-walled cylinder problem can be written as follows:

$$\begin{aligned}\epsilon_r &= \frac{1}{E_x} (\sigma_r - \nu_x \sigma_\theta - \nu_z \sigma_z) + \epsilon_{rp}, & \epsilon_{r\theta} &= \sigma_{r\theta} = 0, \\ \epsilon_\theta &= \frac{1}{E_x} (\sigma_\theta - \nu_x \sigma_r - \nu_z \sigma_z) + \epsilon_{\theta p}, & \epsilon_{\theta z} &= \sigma_{\theta z} = 0, \\ \epsilon_z &= \frac{1}{E_z} \sigma_z - \frac{\nu_z}{E_x} (\sigma_r + \sigma_\theta) + \epsilon_{zp}, & \epsilon_{zr} &= \sigma_{zr} = 0,\end{aligned}\quad (3.4)$$

where the plastic strain terms are obtained by integrating the following strain-rate equations, Eqs. (2.7) written in cylindrical coordinates (27).

$$\begin{aligned}d\epsilon_{zp} &= \frac{d\bar{\epsilon}_p}{\bar{\sigma}} [(\alpha_{12} + \alpha_{31})\sigma_z - \alpha_{12}\sigma_\theta - \alpha_{31}\sigma_r], \\ d\epsilon_{\theta p} &= \frac{d\bar{\epsilon}_p}{\bar{\sigma}} [(\alpha_{23} + \alpha_{12})\sigma_\theta - \alpha_{23}\sigma_r - \alpha_{12}\sigma_z], \\ d\epsilon_{rp} &= \frac{d\bar{\epsilon}_p}{\bar{\sigma}} [(\alpha_{31} + \alpha_{23})\sigma_r - \alpha_{31}\sigma_z - \alpha_{23}\sigma_\theta].\end{aligned}\quad (3.5)$$

The effective stress $\bar{\sigma}$, Eq. (2.11), and the effective plastic strain increment $d\bar{\epsilon}_p$, Eq. (2.12), reduce to the following:

$$\bar{\sigma} = \left[\alpha_{12}(\sigma_z - \sigma_\theta)^2 + \alpha_{23}(\sigma_\theta - \sigma_r)^2 + \alpha_{31}(\sigma_r - \sigma_z)^2 \right]^{1/2}, \quad (3.6)$$

and

$$\begin{aligned}d\bar{\epsilon}_p &= \frac{1}{G} \left[\alpha_{12}(\alpha_{23} d\epsilon_{zp} - \alpha_{31} d\epsilon_{\theta p})^2 + \alpha_{23}(\alpha_{31} d\epsilon_{\theta p} - \alpha_{12} d\epsilon_{rp})^2 \right. \\ &\quad \left. + \alpha_{31}(\alpha_{12} d\epsilon_{rp} - \alpha_{23} d\epsilon_{zp})^2 \right]^{1/2}, \quad (3.7)\end{aligned}$$

where

$$G = (\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12}).$$

The stress-strain relations as explicit functions of the stresses are:

$$\begin{aligned}\sigma_r &= \lambda \mathcal{D}_e + C_{11}(\epsilon_r - \epsilon_{rp}) + C_{12}(\epsilon_\theta - \epsilon_{\theta p}) , \\ \sigma_\theta &= \lambda \mathcal{D}_e + C_{11}(\epsilon_\theta - \epsilon_{\theta p}) + C_{12}(\epsilon_r - \epsilon_{rp}) , \\ \sigma_z &= \lambda \mathcal{D}_e + C_{33}(\epsilon_z - \epsilon_{zp}) ,\end{aligned}\tag{3.8}$$

where the elastic dilatation $\mathcal{D}_e$ is

$$\mathcal{D}_e = \mathcal{D} - \mathcal{D}_p ,$$

or

$$\mathcal{D}_e = (\epsilon_r - \epsilon_{rp}) + (\epsilon_\theta - \epsilon_{\theta p}) + (\epsilon_z - \epsilon_{zp}) .$$

When the plastic dilatation $\mathcal{D}_p$ equals zero, $\mathcal{D}_e$ equals the sum of the total strain components. The material constants λ , C_{11} , C_{12} , and C_{33} are the analog of Lamé's constants for transversely isotropic materials.

They are related to the elastic moduli and Poisson's ratios as follows:

$$\begin{aligned}\lambda &= \frac{E_x \nu_z}{(E_x/E_z)(1 - \nu_x) - 2\nu_z^2} , \\ C_{11} &= \frac{\lambda}{(1 + \nu_x)} \left[\frac{E_x}{\nu_z E_z} - (1 + \nu_x + \nu_z) \right] , \\ C_{12} &= \frac{\lambda}{(1 + \nu_x)} \left[\left(\frac{\nu_x}{\nu_z} \right) \left(\frac{E_x}{E_z} \right) - (1 + \nu_x - \nu_z) \right] , \\ C_{33} &= \frac{\lambda}{\nu_z} (1 - \nu_x - \nu_z) .\end{aligned}\tag{3.9}$$

The Boundary Conditions

For the thick-walled cylinder loaded with an internal pressure (P) and with an axial force (F_z) on the end, the following boundary conditions

apply:

$$\begin{aligned}
 \sigma_r &= -P \text{ at } r = a, \text{ the inside radius ,} \\
 \sigma_r &= 0 \text{ at } r = b, \text{ the outside radius ,} \\
 F_z &= 2\pi \int_a^b \sigma_z r \, dr .
 \end{aligned} \tag{3.10}$$

The solution to the above sets of equations is given in the next section.

II. ANALYTICAL SOLUTION FOR THE TOTAL STRAIN COMPONENTS

The method used for solving the set of equations for the stress and strain components follows closely the method outlined by Mendelson and Manson (28). The compatibility equation, Eq. (3.2) and the equilibrium equation, Eq. (3.3) were combined, using the stress-strain relations Eqs. (3.8), to form a second order differential equation in terms of the total circumferential strain ϵ_θ and the plastic strain components ϵ_{rp} , $\epsilon_{\theta p}$, and D_p :

$$\begin{aligned}
 \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r^2 \epsilon_\theta) \right] &= A_1 \frac{d}{dr} \epsilon_{rp} + A_2 \frac{d}{dr} \epsilon_{\theta p} \\
 &+ B_1 \frac{d}{dr} D_p + (A_1 - A_2) \frac{(\epsilon_{rp} - \epsilon_{\theta p})}{r} , \tag{3.11}
 \end{aligned}$$

where

$$A_1 = \frac{C_{11}}{\lambda + C_{11}} ,$$

$$A_2 = \frac{C_{12}}{\lambda + C_{11}} ,$$

$$B_1 = \frac{\lambda}{\lambda + C_{11}} .$$

Equation (3.11) can be solved for ϵ_θ by direct integration. After rearrangement:

$$\begin{aligned} \epsilon_\theta = & \frac{A_1 + A_2}{2r^2} \int_a^r (\epsilon_{rp} + \epsilon_{\theta p}) r dr \\ & + \frac{A_1 - A_2}{2} \int_a^r \frac{(\epsilon_{rp} - \epsilon_{\theta p})}{r} dr \\ & + \frac{B_1}{r^2} \int_a^r \mathcal{D}_p r dr + C_1 + \frac{C_2}{r^2}, \quad (3.12) \end{aligned}$$

where C_1 and C_2 are integration constants.

The radial strain ϵ_r was obtained by substituting Eq. (3.12) into the compatibility equation, Eq. (3.2), and solving; thus,

$$\begin{aligned} \epsilon_r = & A_1 \epsilon_{rp} + A_2 \epsilon_{\theta p} + B_1 \left[\mathcal{D}_p - \frac{1}{r^2} \int_a^r \mathcal{D}_p r dr \right] \\ & - \frac{1}{2r^2} (A_1 + A_2) \int_a^r (\epsilon_{rp} + \epsilon_{\theta p}) r dr \\ & + \frac{1}{2} (A_1 - A_2) \int_a^r \frac{(\epsilon_{rp} - \epsilon_{\theta p})}{r} dr + C_1 - \frac{C_2}{r^2}. \quad (3.13) \end{aligned}$$

The constants C_1 and C_2 were determined by using the boundary conditions on the radial stress Eq. (3.10) at $r = a$, and $r = b$, and noting that the axial strain ϵ_z is constant across the wall, Eq. (3.1). Substituting Eqs. (3.12) and (3.13) into the first of Eqs. (3.8) gives the pair of simultaneous equations

$$\begin{aligned} A_3 C_1 - \frac{A_4}{a^2} C_2 &= V_1, \\ A_3 C_1 - \frac{A_4}{b^2} C_2 &= V_2, \end{aligned} \quad (3.14)$$

where

$$A_3 = C_{11} + C_{12} + 2\lambda ,$$

$$A_4 = C_{11} - C_{12} ,$$

$$V_1 = - (P + \lambda \epsilon_z) ,$$

$$V_2 = -\lambda \epsilon_z + \frac{A_4}{2b^2} (A_1 + A_2) \int_a^b (\epsilon_{rp} + \epsilon_{\theta p}) r dr \\ - \frac{A_3(A_1 - A_2)}{2} \int_a^b \frac{(\epsilon_{rp} - \epsilon_{\theta p})}{r} dr + \frac{A_4 B_1}{b^2} \int_a^b D_p r dr .$$

The solution to Eqs. (3.14) gives

$$C_1 = -\frac{\lambda \epsilon_z}{A_3} + \frac{a^2 P}{A_3(b^2 - a^2)} - \frac{b^2(A_1 - A_2)}{2(b^2 - a^2)} \int_a^b \frac{(\epsilon_{rp} - \epsilon_{\theta p})}{r} dr \\ + \frac{A_4(A_1 + A_2)}{2A_3(b^2 - a^2)} \int_a^b (\epsilon_{rp} + \epsilon_{\theta p}) r dr \\ + \frac{\lambda(A_1 - A_2)}{A_3(b^2 - a^2)} \int_a^b D_p r dr ,$$

and

(3.15)

$$C_2 = \frac{a^2 b^2}{A_4(b^2 - a^2)} \left[P - \frac{A_3(A_1 - A_2)}{2} \int_a^b \frac{(\epsilon_{rp} - \epsilon_{\theta p})}{r} dr \right. \\ \left. + \frac{A_4(A_1 + A_2)}{2b^2} \int_a^b (\epsilon_{rp} + \epsilon_{\theta p}) r dr \right. \\ \left. + \frac{\lambda(A_1 - A_2)}{b^2} \int_a^b D_p r dr \right] .$$

Since the end force F_z was specified rather than the axial strain as one of the boundary conditions, the axial strain ϵ_z must be determined from the relation

$$F_z = 2\pi \int_a^b \sigma_z r dr . \quad (3.16)$$

When ϵ_r and ϵ_θ from Eq. (3.12) and Eq. (3.13) are substituted into the stress-strain relation for σ_z , Eq. (3.8), the axial stress becomes

$$\begin{aligned} \sigma_z = & \epsilon_z(\lambda + C_{33}) + 2\lambda C_1 + \lambda A_1 \epsilon_{rp} + \lambda A_2 \epsilon_{\theta p} - C_{33} \epsilon_{zp} \\ & - \lambda A_1 \rho_p + \lambda(A_1 - A_2) \int_a^r \frac{(\epsilon_{rp} - \epsilon_{\theta p})}{r} dr . \end{aligned} \quad (3.17)$$

Substituting C_1 from Eq. (3.15) and σ_z from Eq. (3.17) into the end force relation, Eq. (3.16), and simplifying the results by using Eqs. (3.9) one obtains

$$\frac{F_z}{2\pi} = E_z \epsilon_z \frac{(b^2 - a^2)}{2} + \frac{\nu_z E_z}{E_x} a^2 p - E_z \int_a^b \epsilon_{zp} r dr ,$$

which can be solved for the axial strain ϵ_z ; thus,

$$\epsilon_z = \frac{2}{(b^2 - a^2)} \left[\frac{F_z}{2\pi E_z} - \frac{\nu_z a^2 p}{E_x} + \int_a^b \epsilon_{zp} r dr \right] . \quad (3.18)$$

Equations (3.12), (3.13), and (3.18) together with the integration constants C_1 and C_2 from Eq. (3.15) give the solution for the total strain components in terms of the plastic strain components. The stresses can be evaluated by use of the complete stress-strain relations, Eqs. (3.8), when the numerical values of the plastic strain components have been determined. These components were evaluated numerically by using the algorithm developed in the next section.

III. NUMERICAL DETERMINATION OF THE PLASTIC STRAIN COMPONENTS

The numerical algorithm used to determine the values of the plastic strain components is an adaptation of the Mendelson-Manson technique (28), especially constructed for use with anisotropic materials properties. The algorithm was formulated in terms of the incremental theory approach, following the work of Hanson (29), where the load, in this case the internal pressure P , is divided into k increments,

$$P_k = \sum_{n=1}^k \Delta P_n ,$$

and the entire solution is evaluated at each station

$$P_i = \sum_{n=1}^i \Delta P_n ; \quad i = 1, 2, \dots, k .$$

The algorithm can also be used to develop the finite deformation solution, where total plastic strains are considered rather than incremental plastic strains, by considering the entire load P as the first increment. In that case, k equals 1 in the above expression. The deformation theory approach is quite often satisfactory for engineering calculations, as pointed out by Budianski (30).

The algorithm is a predictor-corrector method (31) based on successive approximation. The present method contains a matrix inversion step which prevents accumulation of errors during the computation, thus leading to more rapid convergence than other methods of the same general type. A similar algorithm was developed by Havner (32) for isotropic materials. He also developed a mathematical proof for convergence which guarantees against divergence. Havner's proof does not, however, rule out the

possibility of oscillation about the true solution. If oscillation can be prevented, the algorithm will develop the correct answer.

The numerical calculation proceeds in the following manner. First, an estimate of the strain distributions is obtained by solving the elastic problem. This is done by setting the plastic strain terms, ϵ_{zp} , $\epsilon_{\theta p}$, and ϵ_{rp} in Eqs. (3.12), (3.13), (3.15), and (3.18), equal to zero. These initial estimates of the strain values are used to compute estimates for the stress components, the effective stress, and the effective total strain discussed below. Corrections to these quantities are then calculated and estimates for the plastic strain increments are computed. The estimated values for the plastic strains are then substituted into Eqs. (3.12), (3.13), (3.15), and (3.18) and better estimates of the total strains are calculated. The process is repeated until a suitable convergence criterion is satisfied.

The load is then incremented to $P_2 = P_1 + \Delta P_2$. The entire process is repeated, with the plastic strains obtained for the first load increment held constant while values for the second increment of the plastic strains are calculated. The load is successively incremented by ΔP_n until the final value $P_k = P$ has been considered.

The Effective Total Strain

The effective total strain function, introduced by Mendelson and Manson for the numerical solution of thermal stress problems in the elastic-plastic range, provides a convenient parameter for correcting estimated values of the plastic strain terms, either initially set equal to zero or obtained from the previous iteration. They reasoned that the values of

the strains obtained from an elastic solution would essentially represent the total strains correctly with only the magnitude of the elastic and plastic parts, $\epsilon_t = \epsilon_e + \epsilon_p$, being in error. A function, composed of the total strains could therefore be used as an invariant for correcting the plastic strain components. For this purpose they defined the effective strain, $\bar{\epsilon}_t$, for an isotropic von Mises material, in terms of the principal strains ϵ_i , as

$$\bar{\epsilon}_t = \sqrt{2/3} \sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2},$$

which reduces for the uniaxial case, $\sigma_1 \neq 0$, to

$$\bar{\epsilon}_t = \frac{2(1 + \nu)}{3E} \sigma_1 + \epsilon_1^P.$$

It should be noted that the above expression is not a generalized strain function in the same sense as the effective plastic strain, Eq. (2.12), since it neither reduces to the uniaxial total strain, ϵ_1 , nor will it accurately represent the work of deformation (elastic strain energy plus plastic work) when combined with an effective stress. A functional relation

$$\bar{\sigma} = H(\bar{\epsilon}_t) = H(\bar{\epsilon}_e + \bar{\epsilon}_p),$$

will, however, be universal in the sense that a unique curve will be capable of representing both the general combined stress case and any one of the uniaxial stress conditions if properly constructed. Mendelson and Manson therefore chose to represent the effective elastic strain $\bar{\epsilon}_e$ by

$$\bar{\epsilon}_e = \frac{2(1 + \nu)}{3E} \bar{\sigma},$$

and to retain the conventional expressions for the effective stress $\bar{\sigma}$, and for the effective plastic strain, $\bar{\epsilon}_p$. The effective-stress

effective-total-strain relation was therefore given by

$$\bar{\epsilon}_t = \frac{2(1 + \nu)}{3E} \bar{\sigma} + \bar{\epsilon}_p, \quad (3.19)$$

where $\bar{\epsilon}_p$ is related to $\bar{\sigma}$ by the generalized stress plastic-strain relation discussed earlier, $\bar{\sigma} = H(\bar{\epsilon}_p)$ and $\bar{\epsilon}_p = \int_{\tau} d\bar{\epsilon}_p$.

Generalization of the above development for orthotropic elastic-plastic materials behavior would be unduly complicated because of the added number of elastic and plastic constants. One would also find that the resulting expressions for the uniaxial stress conditions could not be separated into elastic and plastic components, except for the special case in which the plasticity constants α_{12} , α_{23} , and α_{31} are equal to the anisotropic Poisson's ratios. This seems to be an undue restriction, especially when one considers the utility of the effective-stress effective-total-strain relation.

The advantage of a relation such as Eq. (3.19) is that, given a numerical value of $\bar{\epsilon}_t$, one can determine a value for the effective stress $\bar{\sigma}$, and a value for the effective-plastic-strain $\bar{\epsilon}_p$ which satisfy the relation $\bar{\sigma} = H(\bar{\epsilon}_p)$. Considering incremental loading, one can determine a value for $\Delta\bar{\epsilon}_p \approx d\bar{\epsilon}_p$ for use in computing estimates of the incremental plastic strains, as shown by Hanson (29).

In light of the above, an alternate definition of the effective-total-strain is proposed. In order to preserve the condition of separability shown in Eq. (3.19), let the effective total strain be defined as the sum of a linear "elastic" part and the generalized-effective-plastic strain $\bar{\epsilon}_p$, thus,

$$\bar{\epsilon}_t = \bar{\epsilon}_e + \bar{\epsilon}_p, \quad (3.20)$$

where

$$\bar{\epsilon}_p = \int_{\tau} d\bar{\epsilon}_p \cong \sum_{n=1}^k \Delta\bar{\epsilon}_{pn}$$

is the sum of the increments along the loading path τ . Also, let the effective-stress total-strain relation be

$$\bar{\epsilon}_t = f(\bar{\sigma}) = A\bar{\sigma} + \bar{\epsilon}_p(\bar{\sigma}) , \quad (3.21)$$

where $\bar{\epsilon}_p(\bar{\sigma})$ is the generalized plastic stress-strain relationship. When Eq. (3.21) is reduced to the uniaxial case for each of the three principal stress conditions, using Eqs. (2.27) and (2.30) one finds

$$\begin{aligned} \bar{\epsilon}_t &= A\sqrt{\alpha_{11}} \sigma_1 + \epsilon_{1p}/\sqrt{\alpha_{11}} ; & \text{only } \sigma_1 \neq 0 , \\ \bar{\epsilon}_t &= A\sqrt{\alpha_{22}} \sigma_2 + \epsilon_{2p}/\sqrt{\alpha_{22}} ; & \text{only } \sigma_2 \neq 0 , \\ \bar{\epsilon}_t &= A\sqrt{\alpha_{33}} \sigma_3 + \epsilon_{3p}/\sqrt{\alpha_{33}} ; & \text{only } \sigma_3 \neq 0 . \end{aligned} \quad (3.22)$$

If the constant A is set equal to $1/E_1$, and $\alpha_{11} = 1.0$ as previously discussed, then from Eq. (3.22),

$$\bar{\epsilon}_t = \frac{\sigma_1}{E_1} + \epsilon_{1p}(\sigma_1) .$$

Thus a reference curve is established which retains the advantage of separability between the elastic and plastic parts for an anisotropic plastic material with the added feature of conforming exactly to the uniaxial stress-strain curve in a given materials direction. In the present study the axial direction of the cylinder was used as the reference materials direction,

$$\bar{\epsilon}_t = f(\bar{\sigma}) = \epsilon_z = f(\sigma_z) ,$$

with the plastic strain given by the power form of Eq. (2.32)

$$\bar{\epsilon}_t = \frac{\bar{\sigma}}{E_z} + k_z \bar{\sigma}^n . \quad (3.23)$$

Moreover, since each of the uniaxial expressions of Eq. (3.22) will give the same numerical values for the effective total strain, $\bar{\epsilon}_t$, the relation of Eq. (3.21) is in this sense a generalized expression.

The Numerical Algorithm

The algorithm is described in the following sequence of operations. Each step represents an operation on the corresponding distribution through the thickness rather than at a single point. Since the procedure is one of numerical iteration, superscripts enclosed in parentheses are used to denote the iteration number or sequence. The caret (^) above a variable is used to indicate that the numerical value is an estimate. The use of the effective-stress effective-total-strain diagram is shown in Fig. 4.

1. The plastic strain components $\epsilon_{\theta p}$, ϵ_{rp} , and ϵ_{zp} are initially set equal to zero, and the corresponding elastic solution is found from Eqs. (3.12), (3.13), (3.15), and (3.18), thus giving an estimate of the strain distributions for the first increment of load $P_1 = \Delta P_1$, as shown below.

$$\epsilon_{\theta}^{(1)} = C_1 + C_2/r^2 ,$$

$$\epsilon_r^{(1)} = C_1 - C_2/r^2 ,$$

$$\epsilon_z^{(1)} = \frac{2}{(b^2 - a^2)} \left(\frac{F_z}{2\pi E_z} - \frac{\nu_z a^2 P_1}{E_x} \right) ; \quad F_z = 0 ,$$

Figure 4. Schematic of the Numerical Algorithm Based on the Effective-Stress Effective-Total-Strain Diagram.

where initially,

$$C_1 = -\frac{\lambda \epsilon_z^{(1)}}{A_3} + \frac{a^2 P_1}{A_3(b^2 - a^2)},$$

$$C_2 = \frac{a^2 b^2 P_1}{A_4(b^2 - a^2)}.$$

2. The stress components σ_r , σ_θ , and σ_z corresponding to the elastic solution are computed from the stress-strain relations, Eq. (3.8).

3. The value of the effective stress $\bar{\sigma}$ is computed from Eq. (3.6) using values for α_{ij} determined previously.

4. The corresponding value of the total effective strain $\bar{\epsilon}_t$ is calculated using Eq. (3.21), since at this point the effective plastic strain $\bar{\epsilon}_p$ is zero.

The values of $\bar{\epsilon}_t$ and $\bar{\sigma}$ determined in steps 3 and 4 locate point (1) on Fig. 4 for every point r through the thickness of the tube.

5. Using the value of the total effective strain $\bar{\epsilon}_t$ from the previous step, find the corresponding value of $\bar{\sigma}$ from the reference curve, Eq. (3.23). This operation locates point (2) on Fig. 4. A numerical solution to Eq. (3.23) may be necessary.

6. The total effective strain $\bar{\epsilon}_t$ can now be decomposed into an elastic part, $\bar{\epsilon}_e = \bar{\sigma}/E_z$, and a plastic part, $\bar{\epsilon}_p = k_z \bar{\sigma}^n$; and an estimate of the effective plastic strain increment can be calculated from

$$\Delta \bar{\epsilon}_p = \bar{\epsilon}_p^{(m)} - \bar{\epsilon}_p^{(l-1)}. \quad (3.24)$$

The superscript (m) indicates the current calculation and (l-1) indicates

the value of $\bar{\epsilon}_p$ from the previous load $P_{\ell-1}$. For the first load increment, $\bar{\epsilon}_p^{(\ell-1)}$ equals zero.

7. Using the values for $\bar{\sigma}$ obtained in step 5 and $\hat{\Delta\epsilon}_p$ obtained in step 6, calculate the ratio,

$$\Lambda = \hat{\Delta\epsilon}_p / \bar{\sigma} \quad (3.25)$$

for use in estimating the plastic strain increments from Eq. (3.5)

$$\hat{\Delta\epsilon}_{zp} = \Lambda (\alpha_{11}\sigma_z - \alpha_{12}\sigma_\theta - \alpha_{31}\sigma_r) ,$$

$$\hat{\Delta\epsilon}_{\theta p} = \Lambda (\alpha_{22}\sigma_\theta - \alpha_{23}\sigma_r - \alpha_{12}\sigma_z) ,$$

$$\hat{\Delta\epsilon}_{rp} = \Lambda (\alpha_{33}\sigma_r - \alpha_{31}\sigma_z - \alpha_{23}\sigma_\theta) .$$

8. At this point, it is possible to compute an estimate of the plastic strain component increments. However, there is an error in the stress values determined in step 2, and consequently the rate of convergence in the computation will be influenced. These errors can be removed by using the stress-strain relations, Eq. (3.8) in connection with the plasticity equations, Eq. (3.5). Since the right-hand side of Eq. (3.8) is written in terms of the total strain components and the plastic strain components, it is possible to determine the portion of the stress " s_i ," associated with the elastic part of the strains, and the portion associated with the plastic part. Equation (3.8) may then be written as follows:

$$\sigma_r = s_r - C_{11} \epsilon_{rp} - C_{12} \epsilon_{\theta p} ,$$

$$\sigma_\theta = s_\theta - C_{12} \epsilon_{rp} - C_{11} \epsilon_{\theta p} ,$$

$$\sigma_z = s_z - C_{33} \epsilon_{zp} .$$

Substituting these relations into the plastic strain increment equations, Eq. (3.5) as written in step 7, results in expressions in terms of s_i and the total plastic strain components; thus,

$$\begin{aligned}\Delta \hat{\epsilon}_{zp} = \Lambda & \left[\alpha_{11}(s_z - C_{33}\epsilon_{zp}) \right. \\ & - \alpha_{12}(s_\theta - C_{12}\epsilon_{rp} - C_{11}\epsilon_{\theta p}) \\ & \left. - \alpha_{31}(s_r - C_{11}\epsilon_{rp} - C_{12}\epsilon_{\theta p}) \right], \\ \Delta \hat{\epsilon}_{\theta p} = \Lambda & \left[\alpha_{22}(s_\theta - C_{12}\epsilon_{rp} - C_{11}\epsilon_{\theta p}) \right. \\ & - \alpha_{23}(s_r - C_{11}\epsilon_{rp} - C_{12}\epsilon_{\theta p}) \\ & \left. - \alpha_{12}(s_z - C_{33}\epsilon_{zp}) \right], \quad (3.26)\end{aligned}$$

$$\begin{aligned}\Delta \hat{\epsilon}_{rp} = \Lambda & \left[\alpha_{33}(s_r - C_{11}\epsilon_{rp} - C_{12}\epsilon_{\theta p}) \right. \\ & - \alpha_{31}(s_z - C_{33}\epsilon_{zp}) \\ & \left. - \alpha_{23}(s_\theta - C_{12}\epsilon_{rp} - C_{11}\epsilon_{\theta p}) \right].\end{aligned}$$

The plastic strain increments can be separated from the total plastic strain components by noting that

$$\begin{aligned}\epsilon_{zp}^{(\ell)} &= \epsilon_{zp}^{(\ell-1)} + \Delta \hat{\epsilon}_{zp}^{(m)}, \\ \epsilon_{\theta p}^{(\ell)} &= \epsilon_{\theta p}^{(\ell-1)} + \Delta \hat{\epsilon}_{\theta p}^{(m)}, \\ \epsilon_{rp}^{(\ell)} &= \epsilon_{rp}^{(\ell-1)} + \Delta \hat{\epsilon}_{rp}^{(m)}.\end{aligned} \quad (3.27)$$

Estimates of the plastic-strain increments for the (m) th iteration can then be determined by solving the set of equations which result from substituting these expressions into Eqs. (3.26), written below as a single

matrix equation,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \Delta \hat{\epsilon}_{zp}^{(m)} \\ \Delta \hat{\epsilon}_{\theta p}^{(m)} \\ \Delta \hat{\epsilon}_{rp}^{(m)} \end{bmatrix} = \begin{bmatrix} V_1 + W_1 \\ V_2 + W_2 \\ V_3 + W_3 \end{bmatrix} . \quad (3.28)$$

The individual terms in the matrix equation, Eq. (3.28), are given below.

$$A_{11} = -(\Lambda^{-1} + \alpha_{11} C_{33})$$

$$A_{12} = (\alpha_{12} C_{11} + \alpha_{31} C_{12})$$

$$A_{13} = (\alpha_{31} C_{11} + \alpha_{12} C_{12})$$

$$A_{21} = \alpha_{12} C_{33}$$

$$A_{22} = -(\Lambda^{-1} + \alpha_{22} C_{11} - \alpha_{23} C_{12})$$

$$A_{23} = (\alpha_{23} C_{11} - \alpha_{22} C_{12})$$

$$A_{31} = \alpha_{31} C_{33}$$

$$A_{32} = (\alpha_{23} C_{11} - \alpha_{33} C_{12})$$

$$A_{33} = -(\Lambda^{-1} + \alpha_{33} C_{11} - \alpha_{23} C_{12})$$

$$V_1 = -(\alpha_{11} s_z - \alpha_{12} s_\theta - \alpha_{31} s_r)$$

$$V_2 = -(\alpha_{22} s_\theta - \alpha_{23} s_r - \alpha_{12} s_z)$$

$$V_3 = -(\alpha_{33} s_r - \alpha_{31} s_z - \alpha_{23} s_\theta)$$

$$\begin{aligned}
W_1 &= - \left[\epsilon_{zp}^{(\ell-1)} (-\alpha_{11} C_{33}) + \epsilon_{\theta p}^{(\ell-1)} (\alpha_{12} C_{11} + \alpha_{31} C_{12}) + \epsilon_{rp}^{(\ell-1)} (\alpha_{12} C_{12} + \alpha_{31} C_{11}) \right] \\
W_2 &= - \left[\epsilon_{zp}^{(\ell-1)} (\alpha_{12} C_{33}) + \epsilon_{\theta p}^{(\ell-1)} (-\alpha_{22} C_{11} + \alpha_{23} C_{12}) + \epsilon_{rp}^{(\ell-1)} (-\alpha_{22} C_{12} + \alpha_{23} C_{11}) \right] \\
W_3 &= - \left[\epsilon_{zp}^{(\ell-1)} (\alpha_{31} C_{33}) + \epsilon_{\theta p}^{(\ell-1)} (\alpha_{23} C_{11} - \alpha_{33} C_{12}) + \epsilon_{rp}^{(\ell-1)} (-\alpha_{33} C_{11} + \alpha_{23} C_{12}) \right]
\end{aligned}$$

When the load increment $\ell = 1$, then W_1 , W_2 , and W_3 are equal to zero. The solution of Eq. (3.28) gives the desired estimates of the plastic strain increments for the (m)th iteration without the complication of induced errors in the stress terms. These calculations take place at point ② in Fig. 4, page 44.

9. Current estimates for the plastic strain components can then be calculated from Eq. (3.27).

10. New estimates of the total strain components $\epsilon_z^{(n)}$, $\epsilon_\theta^{(n)}$, and $\epsilon_r^{(n)}$ are calculated next using Eqs. (3.12), (3.13), (3.15), and (3.18) with the numerical values of the plastic strain components determined in step 9.

11. Compute the values of the stresses σ_z , σ_θ , and σ_r from Eqs. (3.8). Also calculate the numerical value of $\bar{\sigma}$ using Eq. (3.6).

12. Check $\epsilon_z^{(n)}$, $\epsilon_\theta^{(n)}$, and $\epsilon_r^{(n)}$ against the previous values $\epsilon_z^{(n-1)}$, $\epsilon_\theta^{(n-1)}$, and $\epsilon_r^{(n-1)}$ for convergence. If the convergence criteria are not satisfied, go to step 13. Otherwise go to step 16.

13. Calculate an estimate for the effective plastic strain from the definition written in terms of the estimated plastic strains,

$$\begin{aligned}
\Delta \hat{\epsilon}^{(m)} &= \frac{1}{G} \left[\alpha_{12} \left(\alpha_{23} \Delta \hat{\epsilon}_{zp}^{(m)} - \alpha_{31} \Delta \hat{\epsilon}_{\theta p}^{(m)} \right)^2 \right. \\
&\quad \left. + \alpha_{23} \left(\alpha_{31} \Delta \hat{\epsilon}_{\theta p}^{(m)} - \alpha_{12} \Delta \hat{\epsilon}_{rp}^{(m)} \right)^2 \right. \\
&\quad \left. + \alpha_{31} \left(\alpha_{12} \Delta \hat{\epsilon}_{rp}^{(m)} - \alpha_{23} \Delta \hat{\epsilon}_{zp}^{(m)} \right)^2 \right]^{1/2}. \quad (3.29)
\end{aligned}$$

14. Calculate the next estimate for the total effective strain from

$$\hat{\epsilon}_t^{(\ell)} = \frac{\bar{\sigma}}{E_1} + \frac{\bar{\epsilon}^{(\ell-1)}}{\epsilon_p} + \Delta \hat{\epsilon}_p^{(m)}, \quad (3.30)$$

which together with $\bar{\sigma}$ from step 11 locates point (3) on Fig. 4, page 44.

15. Repeat steps 5 through 11 until convergence has taken place, thus locating point (4) on Fig. 4.

16. Add the next loading increment ΔP_ℓ , and consider the load $P_\ell = P_{\ell-1} + \Delta P_\ell$. The computations for the second and subsequent loading increments can be resumed with step 10 providing $\Delta \epsilon_{rp}$, $\Delta \epsilon_{\theta p}$, and $\Delta \epsilon_{zp}$ are reset to zero.

When the above algorithm is programmed for the computer, a certain number of indexing and bookkeeping operations must also be added for checking purposes and to ensure that the desired output is printed. It is also possible to construct a more compact logic than outlined above. The logical flow diagram for the computer program is shown in Fig. 5.

It was found that convergence to machine accuracy was normally achieved in 10 to 20 iterations. Calculations done without the addition of the step 8 matrix operation normally required from 50 to 100 iterations for convergence. In some cases, computations had a marked tendency to diverge until the matrix operation of step 8 was added.

The results of the analytical solution for the thick-walled cylinder loaded with an internal pressure is given in the next section.

Figure 5. Computer Program Logic Diagram

Figure 5. (continued)

Step 9

Compute $\epsilon_{rp}^{(\ell)}$, $\epsilon_{\theta p}^{(\ell)}$, $\epsilon_{zp}^{(\ell)}$ from Eq. (3.27)

↓

Step 10

WOUT

↓

Print out ϵ_r , ϵ_θ , ϵ_z , σ_r , σ_θ , σ_z on $a \leq r \leq b$
Also write on tape for plotting

↓

If NORY = 0

Yes

Step 4

No ↓

Step 16

If P = P_k

Yes

EXIT

No ↓

Set P = P + ΔP

Set $\Delta\epsilon_{\theta p}^{(\ell-1)} = \Delta\epsilon_{\theta p}^{(\ell)}$; $\Delta\epsilon_{\theta p}^{(\ell)} = 0$

$\Delta\epsilon_{rp}^{(\ell-1)} = \Delta\epsilon_{rp}^{(\ell)}$; $\Delta\epsilon_{rp}^{(\ell)} = 0$

$\Delta\epsilon_{zp}^{(\ell-1)} = \Delta\epsilon_{zp}^{(\ell)}$; $\Delta\epsilon_{zp}^{(\ell)} = 0$

↓

Step 10

END

Figure 5. (continued)

IV. ANALYTICAL RESULTS

The computer program described in the previous section was used to compute the stress and strain components as a function of radial position through the wall thickness and as a function of internal pressure from zero to 800 psi in loading increments of 25 psi. The axial force on the end of the cylinder F_z was zero. The mechanical properties of the graphite were taken from uniaxial tensile data discussed in Reference 1.

Since the material is transversely isotropic, the uniaxial stress-strain curves were represented analytically by the pair of equations, see Eqs. (2.32) and (3.23),

$$\epsilon_z = \frac{\sigma}{E_z} + k_z \sigma^n, \quad (3.31)$$

and

$$\epsilon_\theta = \epsilon_r = \frac{\sigma}{E_x} + k_x \sigma^n. \quad (3.32)$$

The plasticity constants (α_{12} , α_{23} , α_{31}) were computed using the method of effective plastic strain and Eqs. (2.33) with $k_1 = k_z$ and $k_2 = k_3 = k_x$. The numerical values of the mechanical properties used in this study are given in Table 1.

The physical dimensions of the tube specimens for experimental study were selected so as to give an inside-to-outside radius ratio, a/b , equal to 1/2. Since all of the equations for the stresses and strains can be nondimensionalized in terms of the radius ratio, r/b , the actual dimensions of a thick-walled cylinder loaded with an internal pressure are not important for the analytical studies.

Table 1. Mechanical Properties of AGOT Graphite

Axial Direction ^a	Transverse Direction ^b
$E_z = 1.79 \times 10^6$	$E_x = 1.0 \times 10^6$
$\nu_z = 0.045$	$\nu_x = 0.08$
$k_z = 1.05 \times 10^{-11}$	$k_x = 3.73 \times 10^{-11}$
$n = 2.1$	$n = 2.1$
$\alpha_{12} = 1/2 \quad \alpha_{23} = 1.76 \quad \alpha_{31} = 1/2$	

^aSee Eq. (3.31).^bSee Eq. (3.32).

Figures 6 through 13 show the results of the calculations. Each of the stress and strain variables is shown as a function of the radial position through the wall thickness and as a function of the internal pressure in Figs. 6 through 11. Figures 12 and 13 show the effective stress $\bar{\sigma}$ and the effective plastic strain, $\bar{\epsilon}_p$, respectively. The stresses and strains were also calculated for the corresponding anisotropic elastic case. The results of these calculations are shown for comparison, except for the radial stress, Fig. 6, where the difference was too small to show. The difference between the two theories is well illustrated; the nonlinear character of the plasticity equations is clear; and the magnitude of the difference is large enough to be easily measured.

Figure 6. Three-Dimensional Plot of the Radial Stress as a Function of the Radius and the Internal Pressure.

Figure 7. Three-Dimensional Plot of the Radial Strain as a Function of the Radius and the Internal Pressure.

Figure 8. Three-Dimensional Plot of the Circumferential Stress as a Function of the Radius and the Internal Pressure.

Figure 9. Three-Dimensional Plot of the Circumferential Strain as a Function of the Radius and the Internal Pressure.

Figure 10. Three-Dimensional Plot of the Axial Stress as a Function of the Radius and the Internal Pressure.

Figure 11. Three-Dimensional Plot of the Axial Strain as a Function of the Radius and the Internal Pressure.

Figure 12. Three-Dimensional Plot of the Effective Stress as a Function of the Radius and the Internal Pressure.

Figure 13. Three-Dimensional Plot of the Effective Plastic Strain as a Function of the Radius and the Internal Pressure.

CHAPTER IV

EXPERIMENTAL INVESTIGATION

The experimental portion of the study consisted of measuring the strain in the axial and circumferential directions, ϵ_z and ϵ_θ , respectively, on thick-walled tube specimens of AGOT reactor-grade graphite loaded with an internal pressure. Since mean values of the material properties determined from uniaxial test measurements were used for the analytical solution, and since graphite is not homogeneous, a rather large number of tube specimens were tested in order to examine the statistics of the experiment.

I. THE EXPERIMENT

Thirty thick-walled tube specimens, each 4-in. long by 1/2-in. inside diameter with a 1/4-in. wall thickness, were machined from a block of AGOT reactor grade graphite with the axes of the tubes parallel to the extrusion direction of the block, corresponding to the assumptions of the analytical solution. The block of graphite was the same material used by Greenstreet et al. (1) to measure the uniaxial mechanical properties.

The strains ϵ_z and ϵ_θ were measured with epoxy backed foil strain gages 1/16-in. long mounted at the midsection of the tubes on both the inside and outside surfaces. The strain gages were attached with Eastman 910 cement. Identical gages mounted on unstressed graphite blocks located near the test specimens completed the Wheatstone bridge. Four of these specimens are shown in Fig. 14. Additional strain gages were placed along

Figure 14. Thick-Walled Tube Specimens with Outside Strain Gages.

the length of two specimens to monitor the variation in the strain measurements with axial position. This was done to insure that the tubes were long enough to avoid end effects in the center region test section. The data from these two specimens indicated that end effects were dissipated within $1/2$ in. of either end leaving a gage section about 3-in. long in the center of the tube. One of these specimens is also shown in Fig. 14.

The lead wires from the inside surface strain gages were led out through small $1/16$ -in.-diameter holes drilled through the wall near each end of the tube. A thin latex rubber liner was inserted into the specimen to contain the pressurizing fluid (mineral oil), and they were mounted in the test rig as shown in Fig. 15. Rubber O-rings mounted on the pressure lines at either end were used to seal the ends of the specimens. In this way the specimen was free to contract in the axial direction, due to the strain-ratio effect, as the internal pressure was increased. The graphite cylinders were therefore tested in a condition of generalized plane-strain with zero net end force.

Signals from the strain-gage bridge were amplified with operational amplifiers in an Electronic Associates, Inc. (EAI) TR-10 analog computer. The polarity of the gages was checked and the strain measuring system was calibrated by use of shunt resistors placed across each gage. During the test, strain readings from each of the four gages were continuously recorded on an EAI 11 $1/2$ -in. by 17-in. X-Y plotter as a function of the internal pressure. The pressure was plotted on the ordinate (y axis) to a scale of 100 psi per inch. The circumferential strains were plotted on the abscissa to a scale of 0.02 percent per inch, and the axial strains were plotted to a scale of 0.005 percent per inch.

Figure 15. Thick-Walled Tube Specimen Mounted in the Pressure Loading Test Rig.

The pressure was developed by a motorized Black Hawk pumping unit, and the specimens were loaded at a continuous rate of 200 psi per minute. The pressure was measured by a Baldwin SR-4 pressure cell which had a guaranteed accuracy of $\pm 1/4$ percent.

Each of the 30 specimens was pressurized to failure in a continuous loading cycle. Failure occurred suddenly with a resulting loss in pressure and a sudden drop in the strain-gage readings. Failure occurred in every case by a tensile crack along the length of the tube to at least one end of the specimen as seen in Fig. 14, page 65. It was not possible to locate the point of crack initiation. A histogram of the failure pressures as read from the pressure-strain plots is shown in Fig. 16. The range of the pressures at failure was from 380 psi to 765 psi with a mean value of 608.5 psi and a standard deviation of 98.5 psi. The coefficient of variation was 16 percent.

II. DATA REDUCTION

Fourteen of the 120 pressure-strain plots obtained from the test series were discarded because of either malfunction of the recorder or a bad strain gage. From the 30 specimens tested, there were 21 completed sets of data where all four gages produced good plots. The number of plots for each of the four measured strains retained for analysis is shown below.

Inside Gages		Outside Gages		Complete Sets of Data
ϵ_θ	ϵ_z	ϵ_θ	ϵ_z	
25	22	30	29	21

Figure 16. Histogram of the Failure Pressures From the Series of Thick-Walled Tube Specimens.

Most of the bad recordings were from the inside gages, probably a result of poor surface adhesion. The working space available for placing the gages inside the tube specimens was quite limited.

Strain values were read manually from the plots at intervals of 20 psi pressure and punched on data cards for computer reduction. The range of the strain data, the arithmetic mean value, the standard deviation, and the coefficient of variation in percentage of the mean are given for each pressure interval in Tables 2 through 5. The standard deviation "s" was calculated using the equation for small sample statistics (33),

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2, \quad (4.1)$$

where the mean value is given by

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i. \quad (4.2)$$

The data for the circumferential strains at the inside surface and at the outside surface are shown in Figs. 17 and 18, respectively. The spread in the data as shown by the envelope in the figures and by the range in the tables was rather large; however, the standard deviations as shown by the vertical bars were on the same order of magnitude as the uniaxial data. These data were therefore accepted as adequately representing the true behavior of this material. Both the mean curves and the data envelopes clearly show the characteristic nonlinear behavior of this graphite.

The range of the axial strain data, as given in Tables 4 and 5, was considerably wider than expected. In addition, the majority of the data from the inside gages indicated an axial expansion, whereas the majority

Table 2. Circumferential Strain Data - Inside Surface

Pressure (psi)	Range (% strain)		Mean (% strain)	Standard Deviation (% strain)	Coefficient of Variation (%)
20	0.0030	0.0064	0.0048	0.00078	16.26
40	0.0072	0.0110	0.0090	0.00089	9.93
60	0.0110	0.0170	0.0134	0.00126	9.37
80	0.0150	0.0220	0.0178	0.00151	8.46
100	0.0186	0.0260	0.0223	0.00177	7.95
120	0.0222	0.0300	0.0267	0.00206	7.72
140	0.0258	0.0360	0.0315	0.00258	8.20
160	0.0300	0.0450	0.0366	0.00337	9.23
180	0.0336	0.0490	0.0413	0.00364	8.80
200	0.0380	0.0570	0.0465	0.00443	9.52
220	0.0414	0.0610	0.0514	0.00481	9.34
240	0.0456	0.0670	0.0566	0.00528	9.33
260	0.0496	0.0730	0.0619	0.00581	9.49
280	0.0536	0.0800	0.0671	0.00653	9.72
300	0.0578	0.0870	0.0726	0.00724	9.96
320	0.0620	0.0940	0.0779	0.00825	10.58
340	0.0658	0.1010	0.0836	0.00885	10.58
360	0.0704	0.1080	0.0894	0.00960	10.74
380	0.0742	0.1160	0.0949	0.01085	11.43
400	0.0790	0.1240	0.1009	0.01198	11.87
420	0.0834	0.1320	0.1074	0.01325	12.34
440	0.0854	0.1400	0.1125	0.01378	12.25
460	0.0866	0.1490	0.1196	0.01506	12.60
480	0.0880	0.1580	0.1260	0.01684	13.36
500	0.0892	0.1690	0.1332	0.01938	14.52
520	0.1044	0.1920	0.1438	0.02002	13.92
540	0.1088	0.1870	0.1483	0.01847	12.45
560	0.1136	0.1970	0.1553	0.02002	12.89
580	0.1178	0.2080	0.1607	0.02106	13.10
600	0.1226	0.2200	0.1683	0.02296	13.64
620	0.1268	0.2320	0.1775	0.02656	14.96
640	0.1304	0.2440	0.1797	0.02995	16.67
660	0.1330	0.2144	0.1776	0.02414	13.58
680	0.1672	0.2236	0.1916	0.02083	10.87
700	0.1772	0.2332	0.1967	0.02380	12.10
720	0.1770	0.2240	0.1956	0.02139	10.94
740	0.1814	0.2330	0.2011	0.02786	13.85
760	0.1964	0.2420	0.2192	0.03224	14.71

Table 3. Circumferential Strain Data - Outside Surface

Pressure (psi)	Range (% strain)		Mean (% strain)	Standard Deviation (% strain)	Coefficient of Variation (%)
20	0.0010	0.0030	0.0019	0.00052	27.65
40	0.0028	0.0050	0.0036	0.00050	13.96
60	0.0044	0.0074	0.0054	0.00072	13.18
80	0.0058	0.0098	0.0073	0.00090	12.28
100	0.0076	0.0124	0.0093	0.00111	11.95
120	0.0092	0.0148	0.0113	0.00133	11.78
140	0.0108	0.0176	0.0133	0.00156	11.71
160	0.0128	0.0202	0.0155	0.00178	11.49
180	0.0144	0.0228	0.0176	0.00207	11.74
200	0.0164	0.0258	0.0199	0.00236	11.84
220	0.0182	0.0286	0.0222	0.00261	11.77
240	0.0200	0.0314	0.0245	0.00288	11.78
260	0.0220	0.0344	0.0269	0.00316	11.74
280	0.0240	0.0374	0.0293	0.00352	11.98
300	0.0260	0.0404	0.0318	0.00383	12.03
320	0.0280	0.0440	0.0345	0.00429	12.42
340	0.0300	0.0490	0.0371	0.00479	12.91
360	0.0322	0.0502	0.0394	0.00458	11.63
380	0.0344	0.0538	0.0422	0.00497	11.78
400	0.0364	0.0574	0.0452	0.00539	11.94
420	0.0390	0.0614	0.0483	0.00585	12.13
440	0.0416	0.0656	0.0515	0.00643	12.47
460	0.0440	0.0700	0.0551	0.00750	13.62
480	0.0466	0.0744	0.0576	0.00749	12.99
500	0.0492	0.0788	0.0610	0.00818	13.41
520	0.0518	0.0830	0.0641	0.00821	12.80
540	0.0554	0.0884	0.0680	0.00855	12.57
560	0.0578	0.0926	0.0715	0.00941	13.15
580	0.0600	0.0914	0.0747	0.00837	11.21
600	0.0624	0.0956	0.0786	0.00889	11.32
620	0.0710	0.1000	0.0845	0.00913	10.81
640	0.0758	0.1054	0.0912	0.00993	10.88
660	0.0790	0.1122	0.0974	0.01054	10.82
680	0.0834	0.1124	0.1014	0.00968	9.54
700	0.0858	0.1174	0.1053	0.01170	11.11
720	0.0886	0.1216	0.1081	0.01728	15.99
740	0.1204	0.1260	0.1232	0.00396	3.21
760	0.1304	0.1304	0.1304	--	--

Table 4. Axial Strain Data - Inside Surface

Pressure (psi)	Range (in./in. $\times 10^5$)		Mean (in./in. $\times 10^5$)	Standard Deviation (in./in. $\times 10^5$)	Coefficient of Variation (%)
20	-0.0779	2.008	0.9726	0.5901	60.67
40	-0.1295	2.941	1.2337	0.7631	61.86
60	-0.1547	3.601	1.3273	0.8409	63.36
80	-0.1535	3.988	1.3928	0.8844	63.49
100	-0.1260	4.101	1.4950	0.9417	62.99
120	-0.1143	3.940	1.5875	0.9856	62.08
140	-0.1166	3.507	1.6815	1.0091	60.01
160	-0.2615	3.540	1.7835	1.0216	57.28
180	-0.4178	3.803	1.8802	1.0839	57.65
200	-0.5873	4.061	1.9669	1.1601	58.98
220	-0.7778	4.316	2.0347	1.2461	61.24
240	-0.9825	4.567	2.0927	1.3711	65.52
260	-1.201	4.814	2.2003	1.5469	70.30
280	-1.434	5.060	2.2935	1.6846	73.45
300	-1.660	5.321	2.3932	1.8184	75.98
320	-1.882	5.587	2.5061	1.9468	77.68
340	-2.071	5.998	2.6444	2.0750	78.46
360	-2.183	6.483	2.7239	2.2190	81.46
380	-2.243	6.832	2.9297	2.3769	81.13
400	-2.251	7.165	3.0777	2.4627	80.02
420	-2.548	7.488	3.1520	2.6826	85.11
440	-2.961	7.867	3.4667	2.9186	84.19
460	-3.479	8.270	3.5064	3.1350	89.41
480	-3.740	8.716	3.8024	3.4098	89.67
500	-3.907	9.225	3.8607	3.5580	92.16
520	-5.442	9.960	3.6680	4.3223	117.84
540	-4.065	9.051	3.7666	4.2663	113.27
560	-4.208	10.550	4.7765	3.9795	83.31
580	-4.351	12.410	5.1891	4.3242	83.33
600	-4.493	14.070	5.6275	4.865	86.45
620	-4.521	15.64	5.695	5.533	97.16
640	-6.836	12.84	7.112	4.023	56.56
660	4.384	13.72	8.969	3.484	38.85
680	6.936	14.61	10.308	3.438	33.36
700	8.488	15.50	11.933	3.507	29.39
720	8.816	12.22	10.518	2.407	22.88
740	9.202	12.67	10.936	2.452	22.42
760	12.95	12.95	12.95	--	--

Table 5. Axial Strain Data - Outside Surface

Pressure (psi)	Range (in./in. $\times 10^5$)		Mean (in./in. $\times 10^5$)	Standard Deviation (in./in. $\times 10^5$)	Coefficient of Variation (%)
20	-0.277	0.174	-0.0428	0.0982	229.3
40	-0.528	0.340	-0.0982	0.1819	185.3
60	-0.754	0.499	-0.166	0.255	153.5
80	-0.954	0.651	-0.244	0.325	132.8
100	-1.129	0.795	-0.345	0.396	118.5
120	-1.278	0.933	-0.434	0.473	109.0
140	-1.401	1.077	-0.533	0.555	104.2
160	-1.498	1.239	-0.628	0.638	101.7
180	-1.864	1.409	-0.731	0.732	100.0
200	-2.000	1.598	-0.829	0.818	98.6
220	-2.167	1.817	-0.925	0.909	98.3
240	-2.457	2.059	-1.027	1.014	98.7
260	-2.854	2.340	-1.126	1.128	100.2
280	-3.445	2.663	-1.226	1.257	102.5
300	-4.472	3.048	-1.341	1.428	106.6
320	-6.197	3.486	-1.474	1.676	113.7
340	-8.276	4.148	-1.604	2.005	124.9
360	-14.420	4.969	-1.866	2.933	157.2
380	-17.83	6.605	-2.065	3.658	177.1
400	-20.51	8.269	-2.315	4.296	185.6
420	-13.32	9.932	-1.990	3.487	175.2
440	-21.52	11.67	-2.365	4.832	204.3
460	-26.25	13.89	-2.490	6.008	241.3
480	-6.182	18.55	-1.596	4.725	296.0
500	-4.983	23.50	-1.184	5.863	495.2
520	-5.472	2.788	-2.520	2.235	88.7
540	-5.985	3.119	-2.509	2.382	94.9
560	-6.702	3.484	-2.646	2.599	98.2
580	-7.437	3.819	-2.877	2.868	99.7
600	-8.099	4.256	-3.016	3.144	104.3
620	-9.243	4.986	-2.935	3.677	125.3
640	-9.572	5.949	-3.208	3.788	118.1
660	-8.206	6.565	-2.873	3.875	134.9
680	-9.021	-3.126	-4.173	2.777	66.5
700	-9.863	0.079	-4.596	3.247	70.6
720	-10.66	0.499	-3.588	4.209	117.3
740	-11.46	0.975	-4.635	6.306	136.1
760	-3.76	-3.76	-3.76	--	--

Figure 17. Inside Surface Circumferential Strain Data as a Function of the Internal Pressure.

Figure 18. Outside Surface Circumferential Strain Data as a Function of the Internal Pressure.

of the outside surface gages indicated an axial contraction. If these data are correct, one is forced to conclude that the specimens deformed in such a way that plane cross sections did not remain plane during loading. The only possibility for applying an unknown axial force on the specimens was through the rubber O-rings used to seal the ends against a pressure loss. This possibility would, however, have violated St. Venant's principal concerning the distribution of the axial force across the wall at some distance from the ends. End effects were investigated on two of the specimens as noted earlier and were found to die out within $1/2$ in. of the ends.

The probability is quite high that the axial strain gage data, at least on the inside surface, was incorrect. Several possible causes were investigated. Since the inside gages registered an expansion rather than a contraction, it is possible that these gages were polarized in reverse. Part of the standard procedure during testing, however, was to check the polarity of every strain gage before loading the specimen. This possibility was therefore ruled out as being unlikely.

A second possibility is that the pressure acting on the inside gages somehow distorted their accuracy. A search of the literature located the results of a study on the effects of pressure acting normal to the gage surface. Milligan (34) showed that pressures on the order of 10^6 psi were required to produce measurable errors. Since the maximum pressure in the graphite tube tests was less than 800 psi, this possibility was also ruled out.

A third possibility is that the axial gages were not perfectly aligned in the direction of the tube axis. If an axial gage were out of alignment by an angle α as shown schematically in Fig. 19, then the strain, ϵ_G , measured in the direction of the gage would be a function of both the axial strain and the circumferential strain. The magnitude of ϵ_G , found by rotating coordinates through the angle α (21, page 16), is given by

$$\epsilon_G = \epsilon_z \cos^2 \alpha + \epsilon_\theta \sin^2 \alpha . \quad (4.3)$$

Since the axial strain ϵ_z should be negative and much smaller than the circumferential strain ϵ_θ , a small angle of misalignment α could cause a large relative error in the gage reading. The angle necessary for the absolute value of the two components to be equal, and therefore the critical angle " α_c " for the gage to read zero, $\epsilon_G = 0$, rather than negative is then given by

$$\alpha_c = \cot^{-1} \sqrt{-\epsilon_\theta / \epsilon_z} . \quad (4.4)$$

The critical angle for a range of strain ratios is given in Table 6.

The theoretical calculations discussed in Chapter III indicated that the strain ratio should be about 35 on the inside surface and about 10 on the outside surface. It would, therefore, have been necessary for the axial strain gages on the inside surface to be correctly aligned within about 5 degrees or better to obtain reliable data. Gage alignment within 5 degrees is not an unreasonable requirement, but the working space was quite limited. Inside gages were placed 2 inches down a 1/2 inch diameter hole. The fact that the outside surface gages registered negative values

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Figure 19. Schematic Diagram of the Error in the Axial Strain Data as a Function of Positioning Error.

Table 6. The Critical Angle α_c for
Different Strain Ratios

Strain Ratio $-\epsilon_\theta/\epsilon_z$	Critical Angle α_c
10	17° 33'
20	12° 36'
30	10° 20'
40	8° 59'
50	8° 3'

whereas the inside gages registered positive values in most cases, lends some credibility to the assumption that the errors were caused by misalignment of the gages. Errors introduced into the data in the above manner would be random and of unknown magnitude. Additional experimental work would be required to fully resolve this question.

It is of value to note, however, that the negative axial strains measured on either surface were generally less than 100 microinches. These negative values are in the proper range of magnitude according to the theoretical calculations. Further analysis of these data was not considered practical.

CHAPTER V

COMPARISON OF THEORY AND EXPERIMENT

The objectives of the present study were to develop a solution, based on the theory of plasticity, for the stress analysis of a thick-walled cylinder made of graphite, loaded with an internal pressure; and to evaluate the adequacy of the solution from the results of an experiment. A solution to the theoretical problem was developed in Chapter III using an anisotropic plasticity theory proposed for metals by R. Hill (2). The experimental study and the data were discussed in Chapter IV. The purpose of the experiment was to obtain sufficient data on the strain-pressure relationship to compare with the theoretically-predicted values using average materials properties obtained from earlier uniaxial tests.

In principal, it is possible to measure two of the three principal strains at both the inside surface and at the outside surface as a function of the loading pressure. These are the circumferential strains, $\epsilon_{\theta}(a)$ and $\epsilon_{\theta}(b)$, and the axial strains, $\epsilon_z(a)$ and $\epsilon_z(b)$. Present experimental techniques do not provide means for measuring the radial strains nor for measuring strains within the wall of the cylinder. The experiment was not completely successful since the accuracy of the axial strain data was rather poor. The majority of the axial strain data from the outside surface did, however, indicate that the tube specimens decreased in length by an amount which was consistent with the theoretical predictions.

A meaningful comparison between the theoretical solution and the experiment can be made using the circumferential strains. Figures 20 and 21

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Figure 20. Theoretical Solutions and Experimental Mean Data for the Inside Surface Circumferential Strain.

Figure 21. Theoretical Solutions and Experimental Mean Data for the Outside Surface Circumferential Strain.

show circumferential strain curves based on the theory of plasticity and the theory of elasticity for the inside surface and for the outside surface, respectively. The experimental mean data points and the range of the data spread at each 20 psi increment of internal pressure are shown for comparison. The upper and lower limits of the measured strain range are shown as short line segments which form dashed envelope lines. The uneven nature of the envelope lines is the result of erratic strain behavior as individual specimens neared their respective failure loads.

The theoretical plasticity solution curves do not match the data as well as might be desired; however, they do lie within the scatterband of the data. The measured strains were generally less than predicted for the inner surface, and generally more than predicted for the outer surface. On the other hand, agreement between the plasticity curves and the data is substantially better than between the elastic theory curves and the data. A comparative error analysis of the two theoretical solutions is given in Tables 7 and 8. A positive error indicates that the theoretical calculations were too high whereas a negative error indicates that they were too low.

As these tables and Tables 2 and 3 on pages 71 and 72 in the previous chapter show, the errors in the theoretical solution for the maximum circumferential strain which occurs at the inner surface were less than the standard error in the data over the first half of the pressure loading curve, and increased to about 20 percent for the specimens which had not failed at 700 psi. Seven specimens had not failed at this pressure, as shown in the histogram, Fig. 16, page 69. At a pressure of 600 psi, near

Table 7. Comparative Errors Between Theory and Experiment
for the Inside Surface Circumferential Strains

Pressure (psi)	Experimental Mean (in./in. $\times 10^3$)	Plasticity Theory (in./in. $\times 10^3$)	Percentage Error (%)	Elasticity Theory (in./in. $\times 10^3$)	Percentage Error (%)
100	0.223	0.198	-12.6	0.174	-27.7
200	0.465	0.448	-3.8	0.349	-24.6
300	0.727	0.753	+3.5	0.524	-38.6
400	1.009	1.114	+9.4	0.698	-44.4
500	1.332	1.532	+13.1	0.873	-52.0
600	1.683	2.008	+16.2	1.048	-60.6
700	1.967	2.544	+22.7	1.223	-60.9

Table 8. Comparative Errors Between Theory and Experiment
for the Outside Surface Circumferential Strains

Pressure (psi)	Experimental Mean (in./in. $\times 10^3$)	Plasticity Theory (in./in. $\times 10^3$)	Percentage Error (%)	Elasticity Theory (in./in. $\times 10^3$)	Percentage Error (%)
100	0.093	0.073	-27.7	0.066	-39.7
200	0.199	0.159	-24.7	0.133	-49.5
300	0.318	0.261	-22.0	0.200	-59.3
400	0.452	0.377	-19.6	0.266	-69.5
500	0.610	0.509	-19.7	0.333	-83.0
600	0.786	0.657	-19.5	0.400	-96.4
700	1.053	0.821	-28.3	0.466	-125.7

the mean failure pressure, the error in the theoretical predictions was 16 percent, which is less than the standard error of the uniaxial materials properties. The error in the outside surface strain prediction was on the order of 20 to 25 percent too low. This error is somewhat larger than can reasonably be accounted for by variations in materials properties alone, although a smaller value of the elastic modulus E_x is indicated by both figures. It still is much less than the 50 to 100 percent error given by the anisotropic elastic calculations.

Although it is not possible to measure the stresses in the thick-walled cylinder, one can deduce several important conclusions regarding the stresses. First, it was noted in Chapter III, Section IV, that the influence of a plastic strain component was to reduce the maximum circumferential stress at the inner surface and to increase the minimum circumferential stress at the outer surface. Since the inner surface strain ϵ_θ was not as large in the experiment as the theory predicted, the reduction in the stress would be less than predicted. Second, the slope of the circumferential strain curve through the wall thickness was less than predicted by either the elastic theory or the plasticity theory. A more uniform strain distribution than computed indicates that the outer portion of the cylinder wall carried a larger share of the load and hence had a higher circumferential stress than computed. Since equilibrium of stresses must be maintained, this would further depress the stresses at the inner surfaces. It is therefore reasonable to conclude that the two theoretical solutions effectively bracket the maximum stresses with the plasticity solution giving a better estimate of the maximum stress value than the elasticity solution.

An important consequence of the present investigation is related to the development of a valid failure criterion for design purposes. Uniaxial test specimens of this graphite failed at stress values near 2000 psi and maximum strain values near 0.2 percent (see Ref. 1). The cylindrical tube specimens failed at maximum circumferential stress values of less than 1000 psi with a mean value of perhaps 800 psi, and strain values of 0.17 percent at the mean failure pressure, up to about 0.21 percent for the maximum value. This is a good indication that the maximum principal stress is not a valid failure criterion. The maximum principal strain criterion for failure is, however, not ruled out.

A possible failure criterion for future study is suggested by the maximum effective stress concept, since the computed value was about 2000 psi at the mean failure pressure as shown in Fig. 12, page 62. It is reasonable to assume that failure of solid brittle materials is governed by some functional relation between all the stress and strain components acting at a point rather than simply a maximum single value. In order to determine the proper functional form, one would need a series of experiments with different combinations of the stress and strain components, as well as a good theoretical description of the deformation mechanics.

CHAPTER VI

ANISOTROPIC PLASTICITY WITH DILATATION

At the end of Chapter II, it was indicated that equations could be developed which would consider the effects of plastic dilatation on the stresses and strains. In view of the apparent disagreement between the computed circumferential strains and the experimental values, it is of some interest to develop these equations and to compare the results with those presented in the previous chapter. Accordingly, Hu's anisotropic plasticity equations are extended in the development below to include non-zero plastic dilatation. Theoretical predictions of the circumferential strains using these equations are then compared with the mean experimental values.

The following development is carried out in reduced index notation where the components of the stress and plastic strain rate tensors σ_{ij} and $d\epsilon_{ij}^P$ are replaced by the single subscripted variables (21, page 60),

$$\sigma_1 = \sigma_{11} , \quad \sigma_2 = \sigma_{22} , \quad \sigma_3 = \sigma_{33} ,$$

$$\sigma_4 = \sigma_{12} , \quad \sigma_5 = \sigma_{23} , \quad \sigma_6 = \sigma_{31} ,$$

and

$$d\epsilon_1^P = d\epsilon_{11}^P , \quad d\epsilon_2^P = d\epsilon_{22}^P , \quad d\epsilon_3^P = d\epsilon_{33}^P ,$$

$$d\epsilon_4^P = 2d\epsilon_{12}^P , \quad d\epsilon_5^P = 2d\epsilon_{23}^P , \quad d\epsilon_6^P = 2d\epsilon_{31}^P .$$

The normal summation convention of Cartesian tensors is retained.

Using this notation, Hu's equations, Eq. (2.7), for the plastic-strain increments can be written as follows:

$$d\epsilon_i^P = \frac{d\bar{\epsilon}_p}{\bar{\sigma}} \alpha_{ij} \sigma_j ; \quad i, j = 1, 2, \dots, 6 , \quad (6.1)$$

where the signs preceding the coefficient terms α_{ij} ; $i \neq j$ in that formulation have been changed from negative (-) to positive (+), and the shear term coefficients α_{44} , α_{55} , and α_{66} have been adjusted by a factor of 2 in order to conform with the engineering shear strain notation used here. The coefficient matrix $[\alpha_{ij}]$ remains symmetric, $\alpha_{ij} = \alpha_{ji}$ as before, and is subject to the restrictions of Eq. (2.8), written below with the appropriate sign changes,

$$\begin{aligned} \alpha_{11} &= -(\alpha_{12} + \alpha_{31}) , \\ \alpha_{22} &= -(\alpha_{23} + \alpha_{12}) , \\ \alpha_{33} &= -(\alpha_{31} + \alpha_{23}) . \end{aligned} \quad (6.2)$$

Recall that the above restrictions guarantee that the plastic dilatation is zero, that is,

$$d\mathcal{D}_p = d\epsilon_1^P + d\epsilon_2^P + d\epsilon_3^P \equiv 0 .$$

When the plastic dilatation is not zero, Eq. (6.2) no longer holds. It then becomes necessary to redefine both the effective stress $\bar{\sigma}$ and the effective plastic strain increment $d\bar{\epsilon}_p$ in order to conform with the strain-hardening hypothesis. The appropriate definitions of $\bar{\sigma}$ and $d\bar{\epsilon}_p$ can be taken from the incremental plastic work relation, Eq. (2.16).

$$dW_p = \sigma_i d\epsilon_i = \bar{\sigma} d\bar{\epsilon}_p \quad (6.3)$$

It is then a simple matter to develop relations between $\bar{\sigma}$ and the stress components σ_i , and between $d\bar{\epsilon}_p$ and the plastic strain increment components $d\epsilon_i^P$. If one multiplies Eq. (6.1) by σ_i and sets the results equal to Eq. (6.3), then

$$\sigma_i d\epsilon_i^P = \frac{d\bar{\epsilon}_p}{\bar{\sigma}} \alpha_{ij} \sigma_i \sigma_j = \bar{\sigma} d\bar{\epsilon}_p, \quad (6.4)$$

which reduces at once to the specific relation for the effective stress

$$\bar{\sigma}^2 = \alpha_{ij} \sigma_i \sigma_j. \quad (6.5)$$

Equation (6.5) represents a quadratic surface in stress space and will describe a proper yield function if the surface is convex (35). Quadratic surfaces which are also convex are the sphere, circular or elliptical cylinder, cone, ellipsoid, and the circular and elliptic paraboloid (36).

One can use Eq. (6.5) and Eq. (6.1) to develop a similar relation for the effective plastic strain increment $d\bar{\epsilon}_p$. The inverse of Eq. (6.1) is

$$\sigma_j = \frac{\bar{\sigma}}{d\bar{\epsilon}_p} \beta_{jk} d\epsilon_k^P, \quad (6.6)$$

where the matrix $[\beta_{jk}]$ is the inverse of $[\alpha_{ij}]$, defined by the relation

$$\alpha_{ij} \beta_{jk} = \delta_{ik}, \quad (6.7)$$

and δ_{ik} is the Kronecker delta

$$\delta_{ik} = 1; \quad i = k,$$

$$\delta_{ik} = 0; \quad i \neq k.$$

Since $[\alpha_{ij}]$ is symmetric, then $[\beta_{jk}]$ is also symmetric (37),

$$\beta_{jk} = \beta_{kj} . \quad (6.8)$$

Then substituting Eq. (6.6) into Eq. (6.5) and using Eq. (6.7),

$$\bar{\sigma}^2 = \frac{\bar{\sigma}}{d\bar{\epsilon}_p} d\epsilon_k^P \alpha_k . \quad (6.9)$$

A second substitution of Eq. (6.6) then gives

$$\bar{\sigma}^2 = \left(\frac{\bar{\sigma}}{d\bar{\epsilon}_p} \right)^2 d\epsilon_k^P \beta_{jk} d\epsilon_j^P ,$$

which reduces immediately to the required relation,

$$(d\bar{\epsilon}_p)^2 = \beta_{jk} d\epsilon_j^P d\epsilon_k^P . \quad (6.10)$$

Since the shear terms are all zero in the thick-walled cylinder problem under consideration, the range of the indices (i) and (j) is restricted to three in the remaining discussion, and as before take on the directions of the cylindrical coordinate system, that is, $(1,2,3) = (z,\theta,r)$.

The numerical values of the coefficient matrix $[\alpha_{ij}]$,

$$\alpha_{ij} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix} \quad (6.11)$$

can be determined from uniaxial test data as before, and $[\beta_{ij}]$ can be computed by matrix inversion. Since there are now six independent constants rather than three, both methods discussed in Chapter II were combined.

One of the coefficients, α_{11} , can be assigned the value of 1.0 as before, and $\alpha_{22} = \alpha_{33}$ for transversely isotropic behavior in the 2-3 plane, leaving four constants to be determined.

The effective plastic strain method, Eqs. (2.31) and (2.32) with $k_z = k_1$ and $k_x = k_2$, was used to determine α_{22} ,

$$\alpha_{22} = \left(\frac{\epsilon_2^P}{\epsilon_1^P} \right)^2 = \left(\frac{k_x}{k_z} \right)^{2/(n+1)} . \quad (6.12)$$

The remaining constants were determined by the plastic strain ratio method, taking proper account of the sign changes of α_{ij} and the isotropic behavior in the 2-3 plane ($\theta, r = x$ plane),

$$\alpha_{12} = \alpha_{13} = \frac{\epsilon_2^P}{\epsilon_1^P} = -v_{12}^P ,$$

and

(6.13)

$$\alpha_{23} = \alpha_{22} \frac{\epsilon_3^P}{\epsilon_2^P} = -\alpha_{22} v_{23}^P = -\alpha_{22} v_x^P .$$

Assuming that the plastic strain ratios v_{ij}^P are constant and equal to the elastic Poisson's ratios v_{ij} with

$$v_{12} = v_{21} \frac{E_1}{E_2} = v_z \frac{E_z}{E_x}$$

as noted in Chapter III, Section I, then

$$\alpha_{12} = -v_z \frac{E_z}{E_x} ,$$

and

(6.14)

$$\alpha_{23} = -\alpha_{22} v_x .$$

Using the materials properties given earlier, in Table 1, page 55, and Eqs. (6.12) and (6.14), the numerical values of the coefficient matrix,

Eq. (6.11), become

$$\alpha_{ij} = \begin{bmatrix} 1.0 & -0.08 & -0.08 \\ -0.08 & 2.26 & -0.181 \\ -0.08 & -0.181 & 2.26 \end{bmatrix}. \quad (6.15)$$

The surface $\bar{\sigma}^2$ described by Eq. (6.5) with the numerical values of α_{ij} given in Eq. (6.15) is an ellipsoid in principal stress space, thus meeting the requirement of convexity.

It is immediately apparent from Eqs. (6.15) and (6.1) that the plastic dilatation $\mathcal{D}_p$ is not zero since the determinant of $[\alpha_{ij}]$ is different from zero. In addition, the apparent inconsistency in the uniaxial data discussed in Chapter II, Section III, has been removed.

The effective stress function $\bar{\sigma}$ of Eq. (6.5), the effective plastic strain increment $d\bar{\epsilon}_p$ of Eq. (6.10), and the numerical values of α_{ij} from Eq. (6.15) were substituted into the analysis of Chapter III in place of the Hill-Hu counterparts. A small amount of reprogramming was necessary, but the essential features of the numerical algorithm remained unchanged. As presently written the computer program will reproduce the previous results by simply using the previous numerical values of α_{ij} with the proper sign changes.

The results of the computations for the circumferential strains at the inside and outside surface are shown in Figs. 22 and 23, respectively, along with the mean experimental data points. As shown, modifying the theory produced the desired result by reducing the computed maximum strains at the inner surface. The computed strains for the outside surface, however, were also reduced, thereby further increasing the error between the

Figure 22. Improved Theoretical Solution and Experimental Mean Data for the Inside Surface Circumferential Strain.

Figure 23. Improved Theoretical Solution and Experimental Mean Data for the Outside Surface Circumferential Strain.

analysis and the experiment. As noted previously, a reduction in the numerical value of the elastic modulus E_x is justified on the basis of the apparent tangent to the experimental pressure-strain curves. Doing this, however, would be contrary to the purpose of the study, which was to see how well one could predict the behavior of a statistical sample of thick-walled tube specimens using previously determined materials properties.

Figure 24 shows the three theoretical solutions for the circumferential stress, σ_θ , as a function of the internal pressure at the inside surface and at the outside surface. The curves labeled "E" are the elasticity results, and the curves labeled "P-1" and "P-2" are the plasticity results with zero plastic dilatation (Hu's equations) and with nonzero plastic dilatation, respectively. As expected, the stresses calculated assuming nonzero plastic dilatation, the P-2 curves, lay between the elastic results and the previous plasticity results. Since the outside circumferential strains were larger in the experiment than either of the two plasticity solutions predicted, again indicates that the circumferential stress at the outer surface was larger than computed. The necessity for satisfying equilibrium then indicates that the circumferential stress was smaller at the inner surface than computed assuming nonzero plastic dilatation, the P-2 curve. Previous arguments indicated that the inner surface circumferential stress was actually larger than indicated by the P-1 curve. It can therefore be concluded that the maximum stress in the thick-walled cylinder was between the values indicated by the P-1 and the P-2 curves with Hill's function giving a lower bound, and the present

Figure 24. Comparison of Three Different Theoretical Solutions for the Circumferential Stress; Elastic Theory (E), Elastic-Plastic Theory (P-1), and Improved Elastic-Plastic Theory (P-2).

quadratic function giving an upper bound. The difference between the two stress predictions P-1 and P-2 is about 10 percent.

CHAPTER VII

SUMMARY AND DISCUSSION

An anisotropic theory of plasticity, originally developed for metals for which the plastic dilatation is zero, has been used, in conjunction with the linear theory of elasticity, to compute the stresses and strains in a thick-walled cylinder of graphite. The graphite cylinder was assumed to be free from end restraint and loaded monotonically with an internal pressure over the range from 0 psi to 800 psi. Graphite is a transversely isotropic material with a nonzero plastic dilatation. Materials properties were assumed to be equivalent to those determined in an earlier investigation with the exception of the plastic volume change which was taken as zero to conform with the theory. It was assumed that the plane of isotropy was perpendicular to the axis of the cylinder.

A numerical algorithm based on the effective-stress effective-total-strain relationship (a predictor-corrector method) was developed for incrementally solving the elastic-plastic strain equations of the thick-walled cylinder problem. The effective-stress effective-total-strain relationship was defined so as to coincide with the uniaxial stress-total-strain curve of the material in a direction perpendicular to the plane of isotropy. The plasticity constants were evaluated in terms of uniaxial materials properties. The solution for the thick-walled cylinder problem was programmed for, and the computations were performed on an electronic digital computer. Numerical values of the stress and strain components were plotted as a function of radial position and internal pressure.

A series of 30 thick-walled tube specimens cut from the same block of graphite used earlier to determine the uniaxial materials properties was strain gaged in the circumferential and axial directions on both the inside and outside surfaces and tested to failure with an internal pressure and unrestrained ends. Each of the four sets of data was averaged at 20 psi pressure intervals to obtain a mean pressure-strain curve. The standard deviations of the data were also computed at each interval. When the data were analyzed, it was found that incompatible discrepancies existed in the axial strain data. The axial strain data were therefore not used for subsequent comparison with theoretical predictions. The circumferential strain data gave good smooth mean curves up to the mean failure pressure of all the specimens. The variance was essentially the same as for the reference uniaxial data.

Comparison of the experimental mean circumferential pressure-strain curves with the theoretical curves showed fair but not accurate agreement. Subsequently the theoretical predictions were improved by modifying the basic plasticity equations to include the effects of permanent plastic dilatation. Agreement between the new theoretical strains and the experimental mean data was very good for the inner surface. Theoretical predictions for the outer surface were low by about 25 percent. The maximum stresses and the maximum strains both occur at the inner surface in the circumferential direction. It was concluded that the maximum stresses were approximately correct as calculated by the improved theory, with the unimproved theory giving a lower bound.

The original purposes of this study have been satisfied. General techniques for solving orthotropic elastic-plastic problems were developed

and applied to a particular problem. The theoretical work was verified by experiment and shown to be acceptable for design purposes. Further theoretical work should, however, be done to improve the strain predictions at the outer surface of the thick-walled cylinder. Constitutive equations, especially for graphite, have been developed (38), but are as yet unpublished. The application of these equations to the present problem should further increase our understanding of the mechanical behavior of graphite.

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APPENDICES

APPENDIX A

DETERMINATION OF THE EFFECTIVE PLASTIC STRAIN INCREMENT

The effective plastic strain increment $d\bar{\epsilon}_p$ can be derived in terms of the plastic strain increments $d\epsilon_{ij}^p$ alone, by using the definition of the effective stress $\bar{\sigma}$ in terms of the stress components σ_{ij} , Eq. (2.11) and Hu's plastic strain increment equations, Eqs. (2.7).

The definition of the effective stress $\bar{\sigma}$, Eq. (2.11) is

$$\begin{aligned} \bar{\sigma}^2 = & \alpha_{12}(\sigma_{11} - \sigma_{22})^2 + \alpha_{23}(\sigma_{22} - \sigma_{33})^2 \\ & + \alpha_{31}(\sigma_{33} - \sigma_{11})^2 + 2\alpha_{44}\sigma_{12}^2 \\ & + 2\alpha_{55}\sigma_{23}^2 + 2\alpha_{66}\sigma_{31}^2, \end{aligned} \quad (A1)$$

and Hu's plastic strain increment equations, Eqs. (2.7) written in matrix form are:

$$\begin{bmatrix} d\epsilon_{11}^p \\ d\epsilon_{22}^p \\ d\epsilon_{33}^p \\ d\epsilon_{12}^p \\ d\epsilon_{23}^p \\ d\epsilon_{31}^p \end{bmatrix} = d\lambda \begin{bmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{31} & 0 & 0 & 0 \\ -\alpha_{12} & \alpha_{22} & -\alpha_{23} & 0 & 0 & 0 \\ -\alpha_{31} & -\alpha_{23} & \alpha_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}, \quad (A2)$$

where

$$\alpha_{11} = \alpha_{12} + \alpha_{31} ,$$

$$\alpha_{22} = \alpha_{23} + \alpha_{12} ,$$

$$\alpha_{33} = \alpha_{31} + \alpha_{23} .$$

If the stress components σ_{ij} can be determined from Eq. (A2) in terms of $d\lambda$ and $d\epsilon_{ij}^P$, direct substitution into Eq. (A1) would result in the desired relation. The normal way for doing this is to premultiply the matrix equation by the inverse of the coefficient matrix. However, in this case the matrix is singular since the determinant is zero, and consequently the inverse does not exist. This is a direct consequence of the zero plastic volume change condition

$$D_p = d\epsilon_{11}^P + d\epsilon_{22}^P + d\epsilon_{33}^P = 0 ,$$

and can be easily verified.

One can, however, accomplish the same results by a simple matrix operation designed to obtain the differences of the normal stresses $(\sigma_{11} - \sigma_{22})$, etc., and the shear stresses in terms of the plastic strain increments. Since the plastic shear-strain terms are individually related to the shear stress terms by a single coefficient, only the normal plastic strain equations need to be considered. If both sides of the matrix equation relating the normal strain terms to the normal stresses,

$$\begin{bmatrix} d\epsilon_{11}^P \\ d\epsilon_{22}^P \\ d\epsilon_{33}^P \end{bmatrix} = d\lambda \begin{bmatrix} (\alpha_{12} + \alpha_{31}) & -\alpha_{12} & -\alpha_{31} \\ -\alpha_{12} & (\alpha_{23} + \alpha_{12}) & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{23} & (\alpha_{31} + \alpha_{23}) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} , \quad (A3)$$

are premultiplied by the matrix

$$R = \begin{bmatrix} \alpha_{23} & -\alpha_{31} & 0 \\ 0 & \alpha_{31} & -\alpha_{12} \\ -\alpha_{23} & 0 & \alpha_{12} \end{bmatrix}$$

the result is

$$\begin{bmatrix} \alpha_{23} d\epsilon_{11}^P - \alpha_{31} d\epsilon_{22}^P \\ \alpha_{31} d\epsilon_{22}^P - \alpha_{12} d\epsilon_{33}^P \\ \alpha_{12} d\epsilon_{33}^P - \alpha_{23} d\epsilon_{11}^P \end{bmatrix} = d\lambda G \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix}, \quad (A4)$$

where for convenience the common factor is denoted by

$$G = (\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12}).$$

The coefficient matrix is still singular; however, each algebraic equation in the matrix equation, Eq. (A4), defines one of the stress difference terms required for substituting into Eq. (A1). For example,

$$(\sigma_{11} - \sigma_{22}) = \frac{1}{d\lambda} \frac{1}{G} (\alpha_{23} d\epsilon_{11}^P - \alpha_{31} d\epsilon_{22}^P).$$

The shear terms can be obtained directly from Eq. (A2), for example,

$$\sigma_{12} = \frac{1}{d\lambda} \frac{d\epsilon_{12}^P}{\alpha_{44}}.$$

When these results are substituted into Eq. (A1),

$$\begin{aligned} (\bar{\sigma} d\lambda)^2 = \frac{1}{G^2} & \left[\alpha_{12} (\alpha_{23} d\epsilon_{11}^P - \alpha_{31} d\epsilon_{22}^P)^2 + \dots \right] \\ & + 2 \left[\frac{(d\epsilon_{12}^P)^2}{\alpha_{44}} + \dots \right]. \quad (A5) \end{aligned}$$

The right-hand side of Eq. (A5) is entirely in terms of the plastic strain increments and thus provides a definition for the effective plastic strain increment,

$$\begin{aligned}
 (d\bar{\epsilon}_p)^2 = \frac{1}{G^2} & \left[\alpha_{12} (\alpha_{23} d\epsilon_{11}^P - \alpha_{31} d\epsilon_{22}^P)^2 \right. \\
 & + \alpha_{23} (\alpha_{31} d\epsilon_{22}^P - \alpha_{12} d\epsilon_{33}^P)^2 \\
 & + \alpha_{31} (\alpha_{12} d\epsilon_{33}^P - \alpha_{23} d\epsilon_{11}^P)^2 \left. \right] \\
 & + 2 \left[\frac{(d\epsilon_{12}^P)^2}{\alpha_{44}} + \frac{(d\epsilon_{23}^P)^2}{\alpha_{55}} + \frac{(d\epsilon_{31}^P)^2}{\alpha_{66}} \right]. \quad (A6)
 \end{aligned}$$

The quantity $d\lambda$ is found from Eqs. (A5) and (A6) to be

$$d\lambda = \frac{d\bar{\epsilon}_p}{\bar{\sigma}}, \quad (A7)$$

as noted in the text.

The increment of plastic work per unit volume,

$$\begin{aligned}
 dW_p = \sigma_{11} d\epsilon_{11}^P + \sigma_{22} d\epsilon_{22}^P + \sigma_{33} d\epsilon_{33}^P \\
 + 2\sigma_{12} d\epsilon_{12}^P + 2\sigma_{23} d\epsilon_{23}^P + 2\sigma_{31} d\epsilon_{31}^P, \quad (A8)
 \end{aligned}$$

can be expressed in terms of the effective stress $\bar{\sigma}$ and the increment of effective plastic strain $d\bar{\epsilon}_p$ by substituting the plastic strain increments in terms of the stress components from Eq. (A1) into Eq. (A8). Then collecting terms and replacing $d\lambda$ according to Eq. (A7), the increment of plastic work becomes

$$\begin{aligned}
 dW_p = \frac{d\bar{\epsilon}_p}{\bar{\sigma}} & \left[\alpha_{12} (\sigma_1 - \sigma_2)^2 + \alpha_{23} (\sigma_2 - \sigma_3)^2 + \alpha_{31} (\sigma_3 - \sigma_1)^2 \right. \\
 & + 2\alpha_{44} \sigma_{12}^2 + 2\alpha_{55} \sigma_{23}^2 + 2\alpha_{66} \sigma_{31}^2 \left. \right]. \quad (A9)
 \end{aligned}$$

Using Eq. (A1),

$$dW_p = \frac{\bar{d\epsilon}_p}{\bar{\sigma}} \bar{\sigma}^2 ,$$

and finally,

$$dW_p = \bar{\sigma} \bar{d\epsilon}_p . \quad (A10)$$

APPENDIX B

UNABRIDGED FIELD EQUATIONS IN CYLINDRICAL COORDINATES

The field equations for small deformation theory can readily be found in the literature written in rectangular Cartesian coordinates, or in more advanced texts written for a general coordinate system in terms of the general tensors. These equations are, however, not usually written out in the cylindrical coordinate system. A complete set of unabridged field equations written for the cylindrical coordinate system are therefore given below for reference. The compatibility equations in terms of the cylindrical coordinate strains were published by Odqvist (39) in 1934. The strain-displacement relations and the equilibrium equations were taken from Sokolnikoff's Mathematical Theory of Elasticity (21).

The cylindrical coordinate position, $p(z, \theta, r)$, of a point in a Cartesian coordinate system, $p(x_1, x_2, x_3)$, can be obtained from the following.

$$x_1 = z$$

$$x_2 = r \cos \theta$$

$$x_3 = r \sin \theta$$

or

(B1)

$$z = x_1$$

$$r = (x_2^2 + x_3^2)^{1/2}$$

$$\theta = \cos^{-1} \left(\frac{x_2}{\sqrt{x_2^2 + x_3^2}} \right) = \tan^{-1} \left(\frac{x_3}{x_2} \right).$$

The strain-displacement relations between the strain components ϵ_{ij} ; $i, j = r, \theta, z$ and the displacements u_j ; $j = r, \theta, z$ are:

$$\begin{aligned}
 \epsilon_{rr} &= \frac{\partial u_r}{\partial r} , \\
 \epsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} , \\
 \epsilon_{zz} &= \frac{\partial u_z}{\partial z} , \\
 \epsilon_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) , \\
 \epsilon_{rz} &= \frac{1}{2} \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) , \\
 \epsilon_{z\theta} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) .
 \end{aligned} \tag{B2}$$

The equilibrium equations in terms of the stresses σ_{ij} ; $i, j = r, \theta, z$ and the body forces F_j are:

$$\begin{aligned}
 \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r &= 0 , \\
 \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{r\theta} + F_\theta &= 0 , \\
 \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{rz} + F_z &= 0 .
 \end{aligned} \tag{B3}$$

The compatibility equations in terms of the strain components are given below.

$$\begin{aligned}
\frac{\partial^2 \epsilon_{rr}}{\partial \theta \partial z} &= \frac{1}{r} \frac{\partial}{\partial r} \left(-r^2 \frac{\partial \epsilon_{\theta z}}{\partial r} + r \frac{\partial \epsilon_{zr}}{\partial \theta} + r^2 \frac{\partial \epsilon_{r\theta}}{\partial z} \right) - \frac{2}{r} \frac{\partial \epsilon_{zr}}{\partial \theta} , \\
\frac{\partial^2 (r \epsilon_{\theta\theta})}{\partial z \partial r} - \frac{\partial \epsilon_{rr}}{\partial z} &= \frac{1}{r} \frac{\partial}{\partial \theta} \left[\frac{\partial (r \epsilon_{\theta z})}{\partial r} - \frac{\partial \epsilon_{zr}}{\partial \theta} + r \frac{\partial \epsilon_{r\theta}}{\partial z} \right] , \\
\frac{\partial^2 \epsilon_{zz}}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial \epsilon_{zz}}{\partial \theta} &= \frac{\partial}{\partial z} \left(r \frac{\partial \epsilon_{\theta z}}{\partial r} + \frac{\partial \epsilon_{zr}}{\partial \theta} - r \frac{\partial \epsilon_{r\theta}}{\partial z} \right) - \frac{\partial \epsilon_{\theta z}}{\partial z} , \\
2 \frac{\partial^2 (r \epsilon_{r\theta})}{\partial r \partial \theta} &= \frac{\partial}{\partial r} \left(r^2 \frac{\partial \epsilon_{\theta\theta}}{\partial r} \right) - r \frac{\partial \epsilon_{rr}}{\partial r} + \frac{\partial^2 \epsilon_{rr}}{\partial \theta^2} , \\
2 \left(\frac{\partial^2 \epsilon_{\theta z}}{\partial \theta \partial z} + \frac{\partial \epsilon_{zr}}{\partial z} \right) &= \frac{1}{r} \frac{\partial^2 \epsilon_{zz}}{\partial \theta^2} + r \frac{\partial^2 \epsilon_{\theta\theta}}{\partial z^2} + \frac{\partial \epsilon_{zz}}{\partial r} , \\
2 \frac{\partial^2 \epsilon_{zr}}{\partial z \partial r} &= \frac{\partial^2 \epsilon_{rr}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial r^2} .
\end{aligned} \tag{B4}$$

When u_z is constant and u_θ is zero, and $\partial(\dots)/\partial\theta$ and $\partial(\dots)/\partial z$ are zero, and the body forces are zero, as for the thick-walled cylinder problem, then Eqs. (B2) reduce to the pair of strain-displacement relations

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$

and

$$\epsilon_{\theta\theta} = \frac{u_r}{r} .$$

The equilibrium equations, Eqs. (B3), reduce to the single relation

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 . \tag{B6}$$

Dropping double subscripts the compatibility equations, Eqs. (B4), become simply

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \epsilon_{\theta}}{\partial r} \right) - r \frac{\partial \epsilon_r}{\partial r} = 0 .$$

Integrating with respect to r , using the method of integration by parts results in

$$r^2 \frac{\partial \epsilon_{\theta}}{\partial r} - \left(r \epsilon_r - \int \epsilon_r dr \right) = 0 .$$

Then using Eqs. (B5)

$$r^2 \frac{\partial \epsilon_{\theta}}{\partial r} - \left(r \epsilon_r - \int \frac{\partial u_r}{\partial r} dr \right) = 0 ,$$

$$r^2 \frac{\partial \epsilon_{\theta}}{\partial r} - (r \epsilon_r - u_r) = 0 ,$$

finally results in the compatibility equation quoted in the text, Eq. (3.2),

$$\frac{d \epsilon_{\theta}}{dr} = \frac{1}{r} (\epsilon_r - \epsilon_{\theta}) . \quad (B7)$$

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