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**ON TWO-DIMENSIONAL FLUID MOTION
THROUGH A SPOUT
COMPOSED OF FOUR PLANE WALLS**

by

Gordon C. K. Yeh

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ON TWO-DIMENSIONAL FLUID MOTION THROUGH
A SPOUT COMPOSED OF FOUR PLANE WALLS

by

Gordon C. K. Yeh

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ABSTRACT

The mapping function, the complex potential, and the equations for the streamlines and equipotential lines for the title problem are obtained. One numerical example is presented and two limiting cases are discussed.

I. INTRODUCTION

In order to study the vapor entrainment in the downcomer of a boiling liquid nuclear reactor, it is essential to know the liquid stream flow from the riser to the downcomer. In this report, the two-dimensional counterpart of an idealized case of such a stream flow is solved. The idealization consists of considering the liquid to be nonviscous and incompressible, the motion to be steady and irrotational and the free liquid surface to be a rigid boundary. The problem is then one of two-dimensional potential theory with specified boundary conditions.

The problem is precisely stated in Sect. II. In Sect. III the Schwarz-Christoffel transformation is obtained, which maps the flow region from the physical plane (z -plane) to the upper half of the ζ -plane. With a source located at the origin of the ζ -plane the complex potential throughout the fluid is obtained in Sect. IV. Then the parametric equations for plotting the stream lines and equipotential lines are derived in Sect. V.

In Sect. VI, the formulation for the special case of $k = h$ is obtained, and a numerical example for the case $s = k = h$ is carried out with the resulting streamlines and equipotential lines plotted in Fig. 4. In Sections VII and VIII, the solutions for the limiting cases of $s/h \rightarrow \infty$ and $k/h = s/h \rightarrow \infty$, respectively, are derived, and these are shown to check with known results previously reported in the literature.

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II. STATEMENT OF THE PROBLEM

Figure 1-a shows a two-dimensional spout composed of three plane walls $A_\infty B$, CD_∞ and FG_∞ , of semi-infinite lengths, and one plane wall BC of finite length. Consider the irrotational steady motion of a nonviscous and incompressible fluid from a uniform stream of velocity $-U$ at $D_\infty E_\infty$ to a uniform stream of velocity V at $A_\infty G_\infty$. Assuming that such a steady motion is possible, the problem is to determine the velocity field throughout the fluid.

The principle of reflection shows that the solution of the problem also elucidates the two-dimensional flow problem depicted in Fig. 2. The axially symmetric case corresponding to Fig. 2 represents a certain idealization for the flow in the riser and the downcomer of a boiling liquid nuclear reactor (with the uniform velocity in the riser U , the radius of the riser h , annular radius of the downcomer k , and the depth of immersion s).

III. THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

$A_\infty BCD_\infty E_\infty FG_\infty$ in Fig. 1-a may be considered as a simple closed polygon with interior angles 0 , $\pi/2$, $\pi/2$, 0 , and 2π , respectively. Let us open out the polygonal boundary into the real axis in the ζ -plane (Fig. 1-b), regarding the points D_∞ and E_∞ as coincident and making them correspond to $\zeta = 0$. Let B , C and F correspond to $\zeta = -b$, $\zeta = -c$, and $\zeta = f$, respectively, where b , c , and f are real positive numbers to be chosen or determined.

(a) z-PLANE

Fig. 1

Schwarz-Christoffel
Transformation

(b) ζ -PLANE

Fig. 2

Idealization for Flow in a Reactor

According to the theorem of Schwarz and Christoffel, the transformation

$$\frac{dz}{d\zeta} = K(\zeta + b)^{-1/2} (\zeta + c)^{-1/2} (\zeta - 0)^{-1} (\zeta - f)^1 = \frac{K(\zeta - f)}{\zeta \sqrt{(\zeta + b)(\zeta + c)}} \quad (1)$$

maps the interior of the z -plane (Fig. 1-a) on the upper half of the ζ -plane (Fig. 1-b). In Eq. (1), K is a constant to be determined and may be complex.

Integrating Eq. (1), we have

$$\begin{aligned} z &= K \int \frac{d\zeta}{\sqrt{(\zeta + b)(\zeta + c)}} - Kf \int \frac{d\zeta}{\zeta \sqrt{(\zeta + b)(\zeta + c)}} \\ &= 2K \log \left(\sqrt{\zeta + b} + \sqrt{\zeta + c} \right) + \frac{Kf}{\sqrt{bc}} \log \left(\frac{\sqrt{(\zeta + b)(\zeta + c)} + \sqrt{bc}}{\zeta} + \frac{b + c}{2\sqrt{bc}} \right) + C, \end{aligned} \quad (2)$$

where the arbitrary constant C may be complex

IV THE COMPLEX POTENTIAL

In the z -plane, the flow is from a source of output Uh at $D_\infty E_\infty$ to a sink of intake Vk at $A_\infty G_\infty$. Hence, in the ζ -plane we have a source at the origin which supplies the volume Uh per unit time over an angle π . Thus, the strength of the source is Uh/π , and therefore the complex potential is

$$w = \frac{-Uh}{\pi} \log \zeta \quad (3)$$

Furthermore, the continuity condition requires that

$$Uh = Vk \quad \text{or} \quad V = \frac{h}{k} U \quad . \quad (4)$$

From Eqs. (1) and (3) we have

$$\frac{dw}{dz} = \frac{dw}{d\xi} \frac{d\xi}{dz} = \frac{-Uh}{\pi\xi} \frac{\xi \sqrt{(\xi+b)(\xi+c)}}{K(\xi-f)} = \frac{-Uh \sqrt{(\xi+b)(\xi+c)}}{\pi K(\xi-f)} \quad . \quad (5)$$

Now, at A_∞ and G_∞ ($\xi \rightarrow \pm \infty$), $dw/dz = -V = -hU/k$, if the real axis is parallel to $A_\infty B$. Therefore,

$$\frac{-h}{k} U = \frac{-Uh}{\pi K} \quad \text{or} \quad K = \frac{k}{\pi} \quad . \quad (6)$$

Again, at D_∞ and E_∞ ($\xi = 0$), $dw/dz = U$. Therefore,

$$U = \frac{Uh \sqrt{bc}}{\pi K f} \quad \text{or} \quad f = \frac{h}{\pi K} \sqrt{bc} = \frac{h}{k} \sqrt{bc} \quad . \quad (7)$$

Let us choose $f = 1$. Then Eq. (7) becomes

$$c = \left(\frac{k}{h} \right)^2 b^{-1} \quad . \quad (8)$$

Substituting $f = 1$ and Eqs. (6) and (8) into Eq. (2), and denoting

$$\lambda = k/h \quad , \quad (9)$$

we have

$$\begin{aligned} \frac{\pi}{h} (z - C) &= 2\lambda \log \left(\sqrt{\xi+b} = \sqrt{\xi+\lambda^2 b^{-1}} \right) \\ &+ \log \left[\sqrt{\left(1 + \frac{b}{\xi}\right) \left(1 + \frac{\lambda^2}{\xi b}\right)} + \frac{\lambda}{\xi} + \frac{b^2 + \lambda^2}{2\lambda b} \right] \quad . \end{aligned} \quad (10)$$

If we take $z = 0$ to correspond to $F(\xi = f = 1)$, we have

$$-\frac{\pi}{h} C = 2\lambda \log \left(\sqrt{1+b} + \sqrt{1+\lambda^2 b^{-1}} \right) + \log \left[\sqrt{(1+b)(1+\lambda^2 b^{-1})} + \lambda + \frac{b^2 + \lambda^2}{2\lambda b} \right] \quad . \quad (11)$$

Substituting Eq. (11) into Eq. (10), we obtain

$$\begin{aligned} \frac{\pi}{h} z = & 2\lambda \log \frac{\sqrt{b(\xi+b)} + \sqrt{b\xi + \lambda^2}}{\sqrt{b(1+b)} + \sqrt{b + \lambda^2}} \\ & + \log \frac{2\lambda \sqrt{b(\xi+b)(\xi b + \lambda^2)} + 2\lambda^2 b + (b^2 + \lambda^2) \xi}{2\lambda \sqrt{b(1+b)(b + \lambda^2)} + 2\lambda^2 b + b^2 + \lambda^2} - \log \xi \end{aligned} \quad (12)$$

or

$$\frac{\pi}{h} z = \left[\frac{\sqrt{b(\xi+b)} + \sqrt{\xi b + \lambda^2}}{\sqrt{b(1+b)} + \sqrt{b + \lambda^2}} \right]^{2\lambda} \frac{2\lambda \sqrt{b(\xi+b)(\xi b + \lambda^2)} + 2\lambda^2 b + (b^2 + \lambda^2) \xi}{\xi [2\lambda \sqrt{b(1+b)(b + \lambda^2)} + 2\lambda^2 b + b^2 + \lambda^2]} \quad (13)$$

It remains to determine b from the dimensions of the spout. Figure 1-a shows that point $C(\xi = -\lambda^2 b^{-1})$ corresponds to $z = -s - hi$. Substitution in Eq. (13) gives

$$-e^{-\frac{s}{h}\pi} = \left[\frac{\sqrt{b^2 - \lambda^2}}{\sqrt{b(1+b)} + \sqrt{b + \lambda^2}} \right]^{2\lambda} \frac{\lambda^2 - b^2}{2\lambda \sqrt{b(1+b)(b + \lambda^2)} + 2\lambda^2 b + b^2 + \lambda^2} \quad (14)$$

or

$$e^{\mu} = \frac{\left[\sqrt{b(1+b)} + \sqrt{b + \lambda^2} \right]^{2\lambda} \left[2\lambda \sqrt{b(1+b)(b + \lambda^2)} + b(2\lambda^2 + b) + \lambda^2 \right]}{(b^2 - \lambda^2)^{\lambda+1}}, \quad (15)$$

where μ denotes

$$\mu = (s/h) \pi \quad (16)$$

In the present problem, λ and μ are given and b is determined by approximation from the transcendental equation (15). For example, Eq. (15) may be written as

$$\begin{aligned} \mu = \Phi(b) = & 2\lambda \log \left[\sqrt{b(1+b)} + \sqrt{b + \lambda^2} \right] \\ & + \log \left[2\lambda \sqrt{b(1+b)(\lambda^2 + b)} + b(2\lambda^2 + b) + \lambda^2 \right] - (\lambda + 1) \log (b^2 - \lambda^2) \end{aligned} \quad (17)$$

The point of intersection of the $\Phi(b)$ vs. b curve and the straight line $\Phi(b) = \mu$ gives the graphical solution for b . The $\Phi(b)$ vs. b curves for $\lambda = 0.25, 0.5, 0.75, 1, 2, 3$, and 4 are shown in Fig. 3.

Fig. 3. $\Phi(b)$ vs. b Curves for Various Values of λ

From Eq. (3), we have

$$\zeta = e^{-\alpha w} \quad , \quad (18)$$

where

$$\alpha = \pi/Uh \quad (19)$$

is the reciprocal of the source strength.

Substituting Eq. (18) into Eq. (12), we obtain

$$\begin{aligned} \alpha(Uz - w) = 2\lambda \log \frac{\sqrt{be^{-\alpha w} + b^2} + \sqrt{be^{-\alpha w} + \lambda^2}}{\sqrt{b(1+b)} + \sqrt{b + \lambda^2}} \\ + \log \frac{2\lambda \sqrt{(be^{-\alpha w} + b^2)(be^{-\alpha w} + \lambda^2)} + 2\lambda^2 b + (b^2 + \lambda^2)e^{-\alpha w}}{2\lambda \sqrt{b(1+b)}(b + \lambda^2) + 2\lambda^2 b + b^2 + \lambda^2} \quad , \end{aligned} \quad (20)$$

relating z and w .

Finally, substituting Eqs. (6), (8), and (18) into Eq. (5), we have

$$\frac{dw}{dz} = \frac{-U}{\lambda} \frac{\sqrt{(e^{-\alpha w} + b)(e^{-\alpha w} + \lambda^2 b^{-1})}}{(e^{-\alpha w} - 1)} \quad (21)$$

the negative conjugate of which represents the velocity field throughout the fluid.

V. THE STREAMLINES

By writing

$$z = x + iy \quad , \quad (22)$$

$$w = \phi + i\psi \quad , \quad (23)$$

and

$$\zeta = re^{i\theta} \quad , \quad (24)$$

Eq. (3) gives

$$\left. \begin{aligned} -\alpha\phi &= \log r \\ -\alpha\psi &= \theta \end{aligned} \right\} \quad , \quad (25)$$

and Eq. (20) gives

$$\left. \begin{aligned} \alpha Ux &= \alpha\phi - 2\lambda \log \left[\sqrt{b(1+b)} + \sqrt{b + \lambda^2} \right] \\ &\quad - \log \left[2\lambda \sqrt{b(1+b)(b + \lambda^2)} + 2\lambda^2 b + b^2 + \lambda^2 \right] \\ &\quad + 2\lambda \log R_1(\phi, \psi) + \log R_2(\phi, \psi) \\ \alpha Uy &= \alpha\psi + 2\lambda \Theta_1(\phi, \psi) + \Theta_2(\phi, \psi) \end{aligned} \right\} \quad , \quad (26)$$

where R_1 , R_2 , Θ_1 and Θ_2 are real functions of ϕ and ψ such that

$$R_1 e^{i\Theta_1} = \sqrt{be^{-\alpha w} + b^2} + \sqrt{be^{-\alpha w} + \lambda^2} \quad (27)$$

and

$$R_2 e^{i\Theta_2} = 2\lambda \sqrt{(be^{-\alpha w} + b^2)(be^{-\alpha w} + \lambda^2)} + 2\lambda^2 b + (b^2 + \lambda^2) e^{-\alpha w} \quad (28)$$

Substituting Eq. (23) into Eq. (27) and simplifying, we obtain

$$\left. \begin{aligned} R_1 &= b^{1/2} e^{-\frac{\alpha\phi}{2}} \left[A_1 + B_1 + \sqrt{(A_1 + C_1)(B_1 + D_1)} + \sqrt{(A_1 - C_1)(B_1 - D_1)} \right]^{1/2} \\ \Theta_1 &= \tan^{-1} \frac{(A_1 - C_1)^{1/2} + (B_1 - D_1)^{1/2}}{(A_1 + C_1)^{1/2} + (B_1 + D_1)^{1/2}} \left(0 \leq \Theta_1 \leq \frac{\pi}{2} \right) \end{aligned} \right\}, \quad (29)$$

where

$$\left. \begin{aligned} A_1 &= (\sin^2 \alpha\psi + C_1^2)^{1/2} \\ B_1 &= (\sin^2 \alpha\psi + D_1^2)^{1/2} \\ C_1 &= \cos \alpha\psi + be^{\alpha\phi} \\ D_1 &= \cos \alpha\psi + \lambda^2 b^{-1} e^{\alpha\phi} \end{aligned} \right\} \quad (30)$$

Substituting Eq. (23) into Eq. (28) and simplifying, we obtain

$$\left. \begin{aligned} R_2 &= \lambda be^{-\alpha\phi} (A_2^2 + B_2^2)^{1/2} \\ \Theta_2 &= \tan^{-1} (B_2/A_2) \quad (0 \leq \Theta_2 \leq \pi) \end{aligned} \right\}, \quad (31)$$

where

$$\left. \begin{aligned} A_2 &= \pm \sqrt{2A_1B_1(1+C_2)} + (\lambda^{-1}b + \lambda b^{-1}) \cos \alpha\psi + 2\lambda e^{\alpha\phi} \quad (\text{for } C_1 + D_1 \geq 0) \\ B_2 &= \sqrt{2A_1B_1(1-C_2)} - (\lambda^{-1}b + \lambda b^{-1}) \sin \alpha\psi \\ C_2 &= \frac{C_1D_1 - \sin^2 \alpha\psi}{\sqrt{C_1^2D_1^2 + (C_1^2 + D_1^2) \sin^2 \alpha\psi + \sin^4 \alpha\psi}} \end{aligned} \right\} \quad (32)$$

Comparing Eqs. (25) with Fig. 1, we see that the value of ψ varies from $-\pi/\alpha = Uh$ (ABCD) to zero (EFG), and the value of ϕ varies from $-\infty$ (AG) to ∞ (DE). From Eqs. (3) and (12), we see that $w = 0$ ($\phi = \psi = 0$) at F, where $\zeta = 1$ and $z = 0$ ($x = y = 0$).

With R_1 , Θ_1 , R_2 , and Θ_2 evaluated from Eqs. (29) and (31), Eqs. (26) then provide formulations for plotting the streamlines ($\psi = \text{constant}$) and the equipotential lines ($\phi = \text{constant}$) throughout the fluid in terms of U , h , k , and s .

In what follows, the special case of $\lambda = 1$ and the limiting cases of $\mu \rightarrow \infty$ and $\lambda = \mu \rightarrow \infty$ will be investigated.

VI. SPECIAL CASE OF $\lambda = 1$

For the special case of $\lambda = 1$ (i.e., $k = h$), Eq. (15) becomes

$$e^{\mu/2}(b-1) - (b+1) = 2\sqrt{b} \quad , \quad (33)$$

which gives

$$b = (e^{\mu/2} + 1)^2 (e^{\mu/2} - 1)^{-2} \quad . \quad (34)$$

Substituting Eq. (34) into Eq. (20), we have

$$\begin{aligned} \alpha(Uz - w) = \log & \left\{ \frac{(e^\mu - 1) \left[(e^{\mu/2} - 1)^2 e^{-\alpha w} + (e^{\mu/2} + 1)^2 \right]^{1/2} \left[(e^{\mu/2} + 1)^2 e^{-\alpha w} + (e^{\mu/2} - 1)^2 \right]^{1/2}}{4e^\mu (e^\mu + 1)} \right. \\ & \left. + \frac{(e^\mu - 1)^2 e^{-\alpha w} + [(e^\mu + 1)^2 + 4e^\mu]}{4e^\mu (e^\mu + 1)} \right\} \\ & + \log \left\{ \frac{(e^\mu - 1) \left[(e^{\mu/2} - 1)^2 e^{-\alpha w} + (e^{\mu/2} + 1)^2 \right]^{1/2} \left[(e^{\mu/2} + 1)^2 e^{-\alpha w} + (e^{\mu/2} - 1)^2 \right]^{1/2}}{4e^\mu (e^\mu + 1)} \right. \\ & \left. + \frac{(e^\mu - 1)^2 + [(e^\mu + 1)^2 + 4e^\mu] e^{-\alpha w}}{4e^\mu (e^\mu + 1)} \right\} \quad (35) \end{aligned}$$

or

$$\begin{aligned} 8e^{2\mu + \alpha Uz} = & 2(e^\mu + 1)^2 + (e^{\alpha w} + e^{-\alpha w})(e^\mu - 1)^2 \\ & + (e^\mu - 1)(e^{\alpha w} + 1) \left[2e^{-\alpha w} \left\{ (e^\mu + 1)^2 + 4e^\mu \right\} + (e^\mu - 1)^2 (1 + e^{-2\alpha w}) \right]^{1/2} . \quad (36) \end{aligned}$$

Consider a numerical example in which $\mu = \pi$ (i.e., $s = h$). From Eq. (34), we have

$$b = \left(\frac{5.810}{3.810} \right)^2 = (1.525)^2 = 2.325 \quad , \quad (37)$$

which can also be obtained graphically from Fig. 3. From Eq. (36) we have

$$e^{\alpha Uz} = 0.2721 + 0.1145 \left[(e^{\alpha w} + e^{-\alpha w}) + (e^{\alpha w} + 1)(e^{-2\alpha w} + 2.755 e^{-\alpha w} + 1)^{1/2} \right] \quad . \quad (38)$$

The parametric equations for the streamline $-\alpha\psi = \pi/2$ can be formulated as follows: From Eqs. (30) and (32), we have

$$\left. \begin{aligned} C_1 &= 2.325 e^{\alpha\phi} \\ D_1 &= 0.430 e^{\alpha\phi} \\ A_1 &= (1 + C_1^2)^{1/2} \\ B_1 &= (1 + D_1^2)^{1/2} \\ C_2 &= \frac{C_1 D_1 - 1}{\sqrt{(C_1 D_1)^2 + (C_1^2 + D_1^2) + 1}} \\ A_2 &= \left[2A_1 B_1 (1 + C_2) \right]^{1/2} + 2e^{\alpha\phi} \\ B_2 &= \left[2A_1 B_1 (1 - C_2) \right]^{1/2} + 2.755 \end{aligned} \right\} ; \quad (39)$$

from Eqs. (29) and (31) we have

$$\left. \begin{aligned} R_1^2 &= 2.325 e^{-\alpha\phi} \left[A_1 + B_1 + \sqrt{(A_1 + C_1)(B_1 + D_1)} + \sqrt{(A_1 - C_1)(B_1 - D_1)} \right] \\ \Theta_1 &= \tan^{-1} \frac{(A_1 - C_1)^{1/2} + (B_1 - D_1)^{1/2}}{(A_1 + C_1)^{1/2} + (B_1 + D_1)^{1/2}} \quad \left(0 \leq \Theta_1 \leq \frac{\pi}{2} \right) \\ R_2 &= 2.325 e^{-\alpha\phi} (A_2^2 + B_2^2)^{1/2} \\ \Theta_2 &= \tan^{-1} \frac{B_2}{A_2} \quad (0 \leq \Theta_2 \leq \pi) \end{aligned} \right\} ; \quad (40)$$

and from Eqs. (26) and (19) we have

$$\left. \begin{aligned} \frac{x}{h} &= \frac{1}{\pi} (\alpha\phi + \log R_1^2 R_2) - 1.943 \\ \frac{y}{h} &= \frac{1}{\pi} (2\Theta_1 + \Theta_2) - 0.500 \end{aligned} \right\} \quad (41)$$

The equations for other streamlines can be similarly formulated. In Fig. 4, the streamlines $-\alpha\psi = 0, \pi/4, \pi/2, 3\pi/4, \text{ and } \pi$ for this numerical example are plotted.

Fig. 4. Streamlines and Equipotential Lines for the Case of $h = k = s$

VII. LIMITING CASE OF $\mu \rightarrow \infty$

For the limiting case of $\mu \rightarrow \infty$ (i.e., $s/h \rightarrow \infty$), Eq. (15) requires

$$b = \lambda, \quad (42)$$

and Eq. (20) becomes

$$\alpha(Uz - w) = (1 + \lambda) \log \frac{e^{-\alpha w} + \lambda}{1 + \lambda}. \quad (43)$$

Substituting Eqs. (22) and (23) into Eq. (43) and using Eq. (19), we have

$$\left. \begin{aligned} \frac{x}{h} &= \frac{\alpha\phi}{\pi} + \frac{(1+\lambda)}{\pi} \log \frac{(e^{-2\alpha\phi} + 2\lambda e^{-\alpha\phi} \cos \alpha\psi + \lambda^2)^{1/2}}{1 + \lambda} \\ \frac{y}{h} &= \frac{\alpha\psi}{\pi} - \frac{(1+\lambda)}{\pi} \tan^{-1} \frac{\sin \alpha\psi}{\cos \alpha\psi + \lambda e^{\alpha\phi}} \end{aligned} \right\}, \quad (44)$$

which can also be obtained by using Eqs. (26). This is the case of an infinitely long spout with a dividing wall of semi-infinite length.

For the special case that $\lambda = 1$ and $h = U^{-1} = \pi/2$ (hence $\alpha = \pi$), with the coordinates of z and w planes transformed such that $z = \bar{z} - (\pi/2)i$ and $w = \bar{w} - i$, Eq. (43) becomes

$$e^{-2\bar{z}} = 4e^{\pi\bar{w}} (1 - e^{\pi\bar{w}})^{-2}, \quad (45)$$

which checks with the known results for the electrostatic potential about an edge of a conducting plate (see, for example, Ref. 1, p. 186).

VIII. LIMITING CASE OF $\lambda = \mu \rightarrow \infty$

If $\lambda \rightarrow \infty$ (i.e., $k/h \rightarrow \infty$) as well as $\mu \rightarrow \infty$ (i.e., $s/h \rightarrow \infty$), Eq. (42) gives

$$b \rightarrow \infty \quad (46)$$

and Eq. (43) becomes

$$\begin{aligned} \alpha(Uz - w) &= \lim_{\lambda \rightarrow \infty} \frac{\log \frac{e^{-\alpha w} + \lambda}{1 + \lambda}}{(1 + \lambda)^{-1}} = \lim_{\lambda \rightarrow \infty} \frac{\frac{1 + \lambda}{e^{-\alpha w} + \lambda} \left[\frac{1}{1 + \lambda} - \frac{e^{-\alpha w} + \lambda}{(1 + \lambda)^2} \right]}{-(1 + \lambda)^{-2}} \\ &= \lim_{\lambda \rightarrow \infty} \frac{(1 + \lambda)(e^{-\alpha w} - 1)}{e^{-\alpha w} + \lambda} = e^{-\alpha w} - 1. \end{aligned} \quad (47)$$

By taking $\alpha = -1$ and shifting the origin of the z -plane to $-h/\pi - ih$, such that $z = \bar{z} - (h/\pi) - ih$, Eq. (46) becomes

$$\bar{z} = \frac{h}{\pi} (e^w - w + i\pi) \quad (48)$$

which exactly checks with the known result for the irrotational flow out of a long channel into a large reservoir or for the parallel-plate condenser (see, for example, Ref. 2, p. 511).

IX. CONCLUSIONS

The two-dimensional steady motion of an ideal fluid in a spout shown in Fig. 1-a has been analyzed. The solution applies to various dimensions h , k , and s of the spout including the limiting cases of $s/h \rightarrow \infty$ and $k/h \rightarrow \infty$.

The results also solve the electrostatic problem and the corresponding limiting cases of the potential and field distribution due to two conducting walls, ABCD and FG (Fig. 1-a), raised to different potentials.

If the axi-symmetric flow approximates the two-dimensional counterpart, the results herein may be applied to study the problem of vapor entrainment in the downcomer of a boiling liquid nuclear reactor for various combinations of velocity U , radii h and k , and immersion depth s .

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