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THE GENERATION OF POISSON TIME DISTRIBUTED RANDOM PULSES

Ned Wilde

**PHILLIPS
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ATOMIC ENERGY DIVISION

**NATIONAL REACTOR TESTING STATION
US ATOMIC ENERGY COMMISSION**

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Ned Wilde

Instrument Development Branch

PHILLIPS
PETROLEUM
COMPANY

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ABSTRACT

In the nuclear instrumentation field, a need exists for a source of pulses randomly distributed in time according to the Poisson distribution. A theoretical analysis shows that these pulses may be generated by employing an electrical "white noise" source. Calculations indicate the feasibility of a generator capable of producing Poisson pulse at a maximum average rate of one megacycle. The results of laboratory experiments (1) support the calculations and (2) demonstrate the inadequacy of using periodic pulses to determine instrument response to random pulses. A design of a nuclear signal simulator incorporating the Poisson pulse generator is suggested.

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I. INTRODUCTION

All electronic instrumentation systems used to measure nuclear phenomena can be represented by the generalized block diagram shown in Figure 1.

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Figure 1 Nuclear Instrumentation System

Since nuclear events are discrete and random in character, these systems are fundamentally random pulse systems. Depending on the nature of the source of radiation and the detector characteristics, the electrical signal will have the following properties.

- (1) A random distribution of pulses in time (usually a Poisson distribution).
- (2) A random distribution of pulse amplitudes.
- (3) A random distribution of pulse shapes (usually a normal distribution).
- (4) An integrating effect (superposition of pulses).

If the integrating effect is severe, the pulses may lose their identity as individual pulses and become part of a continuous analog signal having random fluctuations.

The electronic system in Figure 1 must be capable of properly processing these random pulses. Experimental evaluation of the response of this system to random pulses requires an electrical source of random pulses. One common method of generating these pulses uses an actual source of radiation and a detector. This technique suffers from two major disadvantages: (1) the properties of the test signal may not be well known and are difficult to vary, and (2) a health hazard exists when using the high intensity sources required to obtain high pulse rates.

The evaluation of these electronic systems could readily be performed with an electronic random pulse generator of known and variable characteristics. This paper presents a design study of a random pulse generator which generates constant amplitude constant shape pulses randomly distributed in time according to the Poisson distribution. A method of introducing random pulse amplitudes, random pulse shapes and integrating effects is suggested.

II. THEORETICAL ANALYSIS

A. General

An electrical "white noise" source is considered for the random element in a Poisson pulse generator. Figure 2 is an illustration of a random current (or voltage) having zero mean value and a normal amplitude distribution.

Figure 2 Random Noise and Threshold Level

If an adjustable threshold level is established and divided into small time increments, the binomial probability distribution describes the probability that the threshold level will be exceeded x times in n increments. In Figure 2 the threshold level has been exceeded four times in 32 increments. Thus, the probability of having a pulse in a given increment is approximately $4/32$ or 0.125 .

Under certain conditions, the binomial distribution is an excellent approximation for the Poisson distribution. A random pulse generator design based on the concept illustrated in Figure 2 will generate binomial pulses. However, if these conditions are observed, the resulting pulses can be considered as Poisson distributed pulses.

B. The Relationship Between the Binomial and Poisson Time Interval Probability Distribution

The binomial probability distribution describes the probability of x pulses occurring in n increments when the probability of an individual pulse per increment is a constant probability p .

$$B_x = \binom{n}{x} p^x (1 - p)^{n-x} \quad (1)$$

The binomial coefficient is given by

$$\binom{n}{x} = \frac{n!}{x! (n-x)!} \quad (2)$$

If n approaches infinity and p approaches zero such that the product pn (the average value of x) remains a constant, the following approximations can be made.

$$\frac{n!}{(n-x)!} \sim n^x \quad (3)$$

$$(1-p)^{n-x} \sim e^{-pn} \quad (4)$$

Equation (1) can be rewritten as

$$P_x = \frac{(pn)^x}{x!} e^{-pn} \quad (5)$$

This is the Poisson probability distribution and when considered this way, it is the limiting distribution for the binomial probability distribution.

It is more convenient to discuss the time interval distribution (i.e., the probability distribution of the intervals between pulses) rather than the pulse distribution. Time interval distributions are expressed in time rates. For Poisson distributed pulses, the probability of x pulses in time t is

$$P_x(t) = \frac{(at)^x}{x!} e^{-at} \quad (6)$$

where a is the average pulse rate. The probability of no pulses ($x = 0$) in time t is

$$P_0(t) = e^{-at} \quad (7)$$

If the period t is to be a time interval between successive pulses, the interval must terminate with the next pulse. The probability of a pulse in dt at t is $(a dt)$. The combined probability of no pulse in time t followed by a pulse in dt at t is the product of these two probabilities.

$$dP(t) = a e^{-at} dt \quad (8)$$

The probability of a time interval less than t is obtained by integrating Equation (8).

$$P(t) = \int_0^t dP(t) = (1 - e^{-at}) \quad (9)$$

Note that Equations (8) and (9) are continuous functions.

In Figure 2 the time interval is restricted to multiples of a finite resolving time. Hence, it would be desirable to have a discrete time interval distribution describing this condition. A discrete relation analogous to Equations (8) and (9) can be derived from the binomial distribution. The probability of x pulses in r time increments of length τ is

$$B_x(r) = \binom{r}{x} (a\tau)^x (1 - a\tau)^{r-x} \quad (10)$$

where $\binom{r}{x}$ is the binomial coefficient and $r = t/\tau$ is an integer. The probability of no pulses ($x = 0$) in r increments is

$$B_0(r) = (1 - a\tau)^r \quad (11)$$

The probability of a pulse in the $(r + 1)$ increment is $a\tau$. The combined probability of no pulses in r increments followed by a pulse in the $(r + 1)$ increment (i.e., the probability of a time interval r increments long) is

$$B(r) = a\tau (1 - a\tau)^r \quad (12)$$

The probability of a time interval equal to or less than n increments long is obtained by summing Equation (12)

$$B(n) = \sum_{r=0}^n B(r) = 1 - (1 - a\tau)^{n+1} \quad (13)$$

Equations (12) and (13), the desired discrete relations, are analogs to Equations (8) and (9), respectively.

It is readily shown that Equation (9) is the limiting expression for Equation (13) just as the Poisson distribution can be considered as the limiting distribution for the binomial distribution. Writing Equation (13) in terms of t and τ gives

$$B(n) = 1 - (1 - a\tau)^{\frac{t}{\tau} + 1} \quad (14)$$

If $a\tau \ll 1$, $(1 - a\tau)$ is approximately $e^{-a\tau}$ and

$$B(n) \sim 1 - e^{-a(t + \tau)} \quad (15)$$

Letting the increment τ approach zero and n approach infinity such that $t = n\tau$ is finite, yields

$$B(n) \sim 1 - e^{-at} \quad (16)$$

$$B(n) \sim P(t)$$

Numerical values for $B(n)$ are given in Table of the Binomial Probability Distribution, National Bureau of Standards, Applied Mathematics, Series No. 6, Washington D. C., 1950. For table use, $B(n)$ is rewritten in terms of the binomial expansion as

$$B(n) = \sum_{r=1}^{n+1} \binom{n+1}{r} (a\tau)^r (1 - a\tau)^{n+1-r} \quad (17)$$

Table I presents a comparison of the binomial and Poisson time interval distributions for various values of $a\tau$. Figure 3 is a plot of the data for the case $a\tau = 0.05$. Not all the histogram bars are shown. Note that for the binomial time interval distribution, time is measured in discrete steps of τ seconds each and n is the number of discrete time steps.

Figure 3 Partial Histogram of $B(n)$ for $a\tau = 0.05$ and Curve of $P(t)$

Summarizing, the Poisson time interval distribution is the desired distribution while the binomial time interval distribution is the actual distribution describing the concept of Figure 2. If the sampling increment τ were infinitesimally small, the two distributions would coincide. However, in practice, τ must have some finite value.

It is difficult to establish a criterion to determine a maximum size for τ for a given average rate a . For this is attempting to determine at what point the binomial distribution is a good approximation for the Poisson distribution. It depends on the application. This question could be resolved for a given application by comparing the response of a system under test to the signal from a radioactive source and detector and the signal from the Poisson pulse generator. As a first approximation a value of $a\tau < 0.05$ would appear reasonable. The logical approach is to build a Poisson pulse generator with the shortest τ possible and limit the maximum average rate to a value compatible with the application. It should be noted that the binomial distribution favors the shorter time intervals as compared to the Poisson. The approximation is on the conservative side.

TABLE I
PROBABILITY OF A TIME INTERVAL EQUAL TO OR LESS THAN t

<u>Binomial</u>								<u>Poisson</u>	
n	B(n) aτ = 0.10	n	B(n) aτ = 0.05	n	B(n) aτ = 0.02	n	B(n) aτ = 0.01	at	P(t)
0	0.1000	0	0.0500	0	0.0200	0	0.0100	0	0.0000
1	0.1900	2	0.1426	5	0.1142	10	0.1047	0.1	0.0952
2	0.2710	4	0.2262	10	0.1993	20	0.1903	0.2	0.1813
3	0.3439	6	0.3017	15	0.2762	30	0.2677	0.3	0.2592
4	0.4095	8	0.3698	20	0.3457	40	0.3377	0.4	0.3297
6	0.5217	12	0.4867	30	0.4654			0.6	0.4512
8	0.6126	16	0.5819	40	0.5632			0.8	0.5507
10	0.6862	20	0.6594					1.0	0.6321
12	0.7458	24	0.7226					1.2	0.6988
14	0.7941	28	0.7741					1.4	0.7534
16	0.8332	32	0.8160					1.6	0.7981
18	0.8649	36	0.8422					1.8	0.8347
20	0.8906	40	0.8779					2.0	0.8647
25	0.9354							2.5	0.9179
30	0.9618							3.0	0.9502
35	0.9775							3.5	0.9698
40	0.9867							4.0	0.9817
45	0.9921							4.5	0.9889

$$B(n) = 1 - (1 - a\tau)^{n+1}$$

where t = nτ

$$P(t) = 1 - e^{-at}$$

C. Some Statistical Properties of a Random Current

The manner of identifying threshold crossings bears investigation. The rate of threshold crossings can be determined by counting the number of times the threshold is crossed with a positive slope signal. A second method is to count the number of maxima above the threshold. This method will give a different result than the first since it is possible to have more than one maximum above the threshold per crossing. Either method can be used but the average rates will differ.

It is also possible for the signal to cross the threshold and remain above the threshold of a period of time greater than τ . A threshold detection circuit will respond to this condition in one of two ways. First, it may remain inactive during this period. Second, the circuit could continue to generate pulses spaced τ seconds apart resulting in a burst of pulses. If this overlapping of increments occurs frequently, it reduces the probability of having threshold crossings in the increments immediately following the initial crossing in the first case, and increases the probability in the second case. Both the binomial and Poisson distributions assume an equal probability for all increments. It is apparent that the overlap effect must be minimized. An analysis of the rates of threshold crossing, maxima and overlaps, follows.

Consider a random current with zero mean value and a normal amplitude distribution. The probability of $I(t)$ being in the range I and $I + dI$ at a random time is

$$dp(I) = \frac{dI}{\sqrt{2\pi} \sigma} e^{-I^2/2\sigma^2} \quad (18)$$

where $\sigma^2 = \phi_{11}(0)$ which is the mean square ac current or variance. The positive square root of the variance is the rms ac current or standard deviation. In this discussion, the frequency spectrum of the random current will be considered as a uniform spectrum of unity power per cycle from zero frequency to a cutoff frequency of f_c and zero from f_c to infinity. The undefined power unit is a characteristic of a given random source. As an example, the power unit of a thermal noise source is the well known KT watts per cycle.

Several useful expressions are presented at this time. The power normalized autocorrelation function evaluated at zero displacement and the derivatives of this function are

$$\phi_0 = f_c \quad (19)$$

$$-\phi_0'' = \frac{4}{3} \pi^2 f_c^2 \quad (20)$$

$$\phi_0^{IV} = \frac{16}{5} \pi^4 f_c^5 \quad (21)$$

where $\phi_o^n = \left[\frac{d^n \phi_{11}(\tau)}{d\tau} \right]_{\tau=0}$. The determinant and co-factors of the moments matrix are

$$|M| = -\phi_o'' (\phi_o \phi_o^{IV} - \phi_o''^2) = \frac{256}{135} \pi^6 f_c^9 \quad (22)$$

$$M_{11} = -\phi_o'' \phi_o^{IV} = \frac{64}{15} \pi^6 f_c^8 \quad (23)$$

$$M_{13} = \phi_o''^2 = \frac{16}{9} \pi^4 f_c^6 \quad (24)$$

$$M_{33} = -\phi_o'' \phi_o'' = \frac{4}{3} \pi^2 f_c^4 \quad (25)$$

Three additional expressions which are used in the following discussion of a random current are

$$y = \sqrt{\frac{I}{\phi_o}} \quad \text{and} \quad y_1 = \sqrt{\frac{I_1}{\phi_o}} \quad (26)$$

$$\psi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \quad (27)$$

$$h(y_1) = \int_0^{y_1} \psi(y) dy \quad (28)$$

Both $\psi(y)$ and $h(y_1)$ or variations of these are tabulated in tables of the normal probability function and tables of the error function (erf).

The three average time rates of interest are:

1. The average rate at which $I(t)$ will pass through a level I_1 with positive slope. This rate is called the "positive slope rate", v_s .
2. The average rate of maxima (points of zero slope and downward concave) of $I(t)$ occurring above a level I_1 . This rate is called the "rate of maxima", v_m .
3. The average rate at which $I(t)$ will pass through a level I_1 with positive slope and have a value greater than I_1 at a later increment of time τ measured from the crossover. This rate is called the "overlap rate", v_τ .

1. Positive Slope Rate, v_s

The positive slope rate is given by the expression

$$v_s = \frac{f_c}{\sqrt{3}} e^{-y_1^2/2} = v_o \sqrt{2\pi} \psi(y_1) \quad (29)$$

The average rate at which $I(t)$ will pass through the zero level ($I_1 = 0$) is determined by letting $y_1 = 0$. Thus,

$$v_o = \frac{f_c}{\sqrt{3}} \quad (30)$$

The average rate at which $I(t)$ will pass through level I_1 with either positive or negative slope is just $2v_s$

2. Rate of Maxima, v_m

S. O. Rice¹ has shown that the probability of a maximum occurring in the rectangle $dI dt$ is

$$d^2 p(y_1 t) = dy dt \frac{\phi_o^{1/2}}{2\pi M_{33}} \left\{ |M|^{1/2} \psi \left[\left(\frac{M_{11} \phi_o}{|M|} \right)^{1/2} y \right] + 2 M_{13} \left(\frac{\pi \phi_o}{2 M_{33}} \right)^{1/2} y \psi(y) \left[\frac{1}{2} + h \left(\frac{M_{12} \phi_o^{1/2}}{(|M| M_{33})^{1/2}} y \right) \right] \right\} \quad (31)$$

Substituting Equations (19) and (22) through (25) into Equation (31) yields

$$d^2 p(y_1 t) = dy dt f_c \left\{ \frac{2}{\sqrt{15}} \psi \left(\frac{3}{2} y \right) + \frac{\sqrt{2\pi}}{\sqrt{3}} y \psi(y) \left[\frac{1}{2} + h \left(\sqrt{\frac{5}{2}} y \right) \right] \right\} \quad (32)$$

The average rate of maxima occurring above I_1 is

$$v_m = \frac{dp(y_1, t)}{dt} = \int_{y_1}^{\infty} p(y, t) dy \quad (33)$$

Substituting Equations (30) and (32) into Equation (33)

$$\begin{aligned} \frac{v_m}{v_o} = & \frac{2}{\sqrt{5}} \int_{y_1}^{\infty} \psi \left(\frac{3}{2} y \right) dy + \sqrt{\frac{\pi}{2}} \int_{y_1}^{\infty} y \psi(y) dy \\ & + \sqrt{2\pi} \int_{y_1}^{\infty} y \psi(y) h \left(\frac{\sqrt{5}}{2} y \right) dy \end{aligned} \quad (34)$$

The first two integrals are straight forward. However, the third integral is a little more difficult and is best evaluated by integrating by parts. Let

$$u = h \left(\frac{\sqrt{5}}{2} y \right) \quad \text{and} \quad dv = y \psi(y) dy$$

then

$$du = \frac{\sqrt{5}}{2} \psi \left(\frac{\sqrt{5}}{2} y \right) dy \quad \text{and} \quad v = \frac{1}{\sqrt{2\pi}} - \psi(y)$$

Hence

$$\begin{aligned} uv \Big|_{y_1}^{\infty} - \int_{y_1}^{\infty} v du &= h \left(\frac{\sqrt{5}}{2} y \right) \left[\frac{1}{\sqrt{2\pi}} - \psi(y) \right] \Big|_{y_1}^{\infty} \\ &- \frac{\sqrt{5}}{2\sqrt{2\pi}} \int_{y_1}^{\infty} \psi \left(\frac{\sqrt{5}}{2} y \right) dy + \sqrt{\frac{5}{2}} \int_{y_1}^{\infty} \psi \left(\frac{\sqrt{5}}{2} y \right) \psi(y) dy \\ &= \frac{1}{2\sqrt{2\pi}} - h \left(\frac{\sqrt{5}}{2} y_1 \right) \left[\frac{1}{\sqrt{2\pi}} - \psi(y_1) \right] \\ &- \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2} - h \left(\frac{\sqrt{5}}{2} y_1 \right) \right] + \frac{\sqrt{5}}{3\sqrt{2\pi}} \left[\frac{1}{2} - h \left(\frac{3}{2} y_1 \right) \right] \end{aligned}$$

Equation (34) may now be rewritten as

$$\begin{aligned} \frac{v_m}{v_o} = & \frac{4}{3\sqrt{5}} \left[\frac{1}{2} - h \left(\frac{3}{2} y_1 \right) \right] + \sqrt{\frac{\pi}{2}} \psi(y_1) \\ & + \sqrt{2\pi} \psi(y_1) h \left(\frac{\sqrt{5}}{2} y_1 \right) + \frac{\sqrt{5}}{3} \left[\frac{1}{2} - h \left(\frac{3}{2} y_1 \right) \right] \end{aligned} \quad (35)$$

It is more convenient to write Equation (35) in the following form.

$$v_m = v_s \left[\frac{1}{2} + h \left(\sqrt{\frac{5}{2}} y_1 \right) \right] + v_o \sqrt{\frac{3}{5}} \left[\frac{1}{2} - h \left(\frac{3}{2} y_1 \right) \right] \quad (36)$$

Two limiting cases of v_m are of interest. The first is when $x_1 = -\infty$, then

$$v_m = \sqrt{\frac{3}{5}} v_o \quad (37)$$

The second is when $x_1 \rightarrow +\infty$, then

$$v_m \rightarrow v_s \quad (38)$$

3. Overlap Rate, v_τ

The expression for the joint probability of $I(t)$ being in the range between I_a and $I_a + dI$ at a random time and at a later time τ being in the range between I_b and $I_b + dI$ is

$$d^2 p(y_a, y_b) = dy_a dy_b \frac{\phi_o}{2\pi [\phi_o^2 - \phi_\tau^2]^{1/2}} \exp \left[- \frac{\phi_o^2 (y_a^2 + y_b^2) - 2 \phi_\tau \phi_o y_a y_b}{2 (\phi_o^2 - \phi_\tau^2)} \right] \quad (39)$$

The power normalized autocorrelation function for the frequency spectrum under consideration is

$$\phi_\tau = \phi_{11}(\tau) = f_c \left(\frac{\sin \omega_c \tau}{\omega_c \tau} \right) \quad (40)$$

Substituting Equation (19) and (40) into Equation (39) yields

$$d^2 p(y_a, y_b) = dy_a dy_b \frac{1}{2\pi q} \exp \left[- \frac{y_a^2 + y_b^2}{2q^2} + \frac{y_a y_b \sin(\omega_c \tau)}{\omega_c \tau q^2} \right] \quad (41)$$

$$\text{where } q^2 = \left[1 - \frac{\sin^2 \omega_c \tau}{(\omega_c \tau)^2} \right].$$

The probability that I_a will be between I_1 and $I_1 + dI$ and at a later time τ , I_b will be greater than I_1 is obtained by integrating Equation (41).

$$\begin{aligned} \frac{dp(y_a, y_1)}{dy_a} \bigg|_{y_a = y_1} &= \psi(y_a/q) \int_{y_1}^{\infty} \frac{1}{\sqrt{2\pi} q} \\ &\exp \left[-\frac{y_b^2}{2q^2} + \frac{y_a y_b \sin(\omega_c \tau)}{\omega_c \tau q^2} \right] dy_b \end{aligned} \quad (42)$$

The integral can be evaluated by completing the square of the exponent and changing the variable of integration. The result is

$$\left[dp(y_1) \right]_{y_b > y_1} = \psi(y_1) \left[\frac{1}{2} - h(r y_1) \right] dy_1 \quad (43)$$

$$\text{where } r = \frac{1}{q} \left[1 - \frac{\sin \omega_c \tau}{\omega_c \tau} \right] = \left[\frac{\omega_c \tau - \sin \omega_c \tau}{\omega_c \tau + \sin \omega_c \tau} \right]^{1/2}.$$

In a similar manner Equation (41) can be integrated from $-\infty$ to y_1 to give the probability that I_a will be between I_1 and $I_1 + dI$ and at a latter time τ , I_b will be less than I_1 .

$$\left[dp(y_1) \right]_{y_b < y_1} = \psi(y_1) \left[\frac{1}{2} + h(r y_1) \right] dy_1 \quad (44)$$

The ratio $\frac{\left[dp(y_1) \right]_{y_1 > y_1}}{\left[dp(y_1) \right]_{y_b > y_1} + \left[dp(y_1) \right]_{y_b < y_1}}$ is the probability that

if $I(t)$ crosses I_1 with either positive or negative slope at some random time, then I_b will be greater than I_1 at a period τ later. Substituting Equations (43) and (44) into this ratio results in

$\left[\frac{1}{2} - h(r y_1)\right]$. Now, the rate at which $I(t)$ crosses I_1 with a positive slope is the positive slope rate, Equation (29). The overlap rate is the product of the positive slope rate and the ratio.

$$v_T = v_s \left[\frac{1}{2} - h(r y_1)\right] \quad (45)$$

The limiting cases are: (1) for $y_1 = 0$

$$v_T = \frac{v_s}{2} \quad (46)$$

and (2) for $y_1 = -\infty$

$$v_T = 0 \quad (47)$$

4. Numerical Evaluation

Table II presents numerical values for the threshold rates (i.e., positive slope rate and rate of maxima) normalized to the spectrum cutoff frequency, f_c . The derivative of the positive slope rate is also tabulated. This can be considered as the threshold sensitivity. It is a measure of the change in v_s with respect to y_1 . The ratio of v_m/v_s has been calculated and tabulated. This represents the average number of maxima per threshold crossing as a function of the threshold.

TABLE II

Normalized Threshold Rates

y_1	v_s/f_c	v_m/f_c	$\frac{1}{f_c} \frac{dv_s}{dy_1}$	v_m/v_s
0.0	0.577350	0.675973	0.00000	1.1708
0.5	0.509510	0.533327	0.25476	1.0467
1.0	0.350181	0.355782	0.35018	1.0160
1.5	0.187438	0.188240	0.28116	1.0041
2.0	0.078136	0.078191	0.15627	1.0007
2.5	0.025367	0.025370	0.06342	1.0001
3.0	0.006414	0.006416	0.01924	<1.0001
3.5	0.001263	0.001263	0.00442	<1.0001
4.0	0.000194	0.000194	0.00078	<1.0001
4.5	0.000023	0.000023	0.00011	<1.0001
5.0	0.000002	0.000002	0.00001	<1.0001

The overlap rate ν_τ is more difficult to tabulate and plot since the parameter r is oscillatory in nature. A sketch of r as a function of $\omega_c \tau$ is shown in Figure 4.

Figure 4 Parameter r as a Function of $\omega_c \tau$

The points where $r = 1$ correspond to the points where the autocorrelation function is zero. The shaded area embraces the range of oscillations of r . The limits of this area can be considered as the limits of oscillation. A corresponding shaded area appears in Figure 5 as the oscillatory limiting value of ν_τ . The lower limit of ν_τ occurs when $r = 0$. The portion of the curve of r not in the shaded area can be approximated by a straight line.

$$r = \frac{1}{\pi} \omega_c \tau = 2 f_c \tau \quad r < 1 \quad (48)$$

For $r > 1$, the value is between 0.88 and 1.24. Once $f_c \tau$ has achieved a value of approximately $1/2$, r is in the shaded area. This permits the establishment of a rough design criterion.

$$f_c \tau > 0.5 \quad (49)$$

The design criterion can be optimized for minimum overlap rate at $r = 1.24$. It occurs at

$$f_c \tau = 0.75 \quad (50)$$

Table III contains numerical values for the limiting conditions of v_τ as a function of both r and y_1 normalized to f_c . In addition, v_τ/v_s has been calculated and tabulated. It is a measure of the number of times overlapping occurs as compared to positive slope crossings. Figure 5 is a plot of the various normalized rates.

In the preceding derivations, the autocorrelation function was normalized to unity power per cycle in order to simplify the mathematics involved. In the final expressions, the effect of the normalization is completely contained in the parameter y_1 which is found only in the normal probability functions $\psi(y_1)$ and $h(y_1)$. Hence, the effect of normalizing is removed by substituting an unnormalized y_1 for the normalized y_1 . For this reason, the symbol y_1 is used to indicate either.

The parameter y_1 is the ratio of the threshold level to the rms value of the random noise in Figure 2. Either one may be varied to effect a change in y_1 . Tables II and III show that as y_1 is increased, (1) the number of threshold crossings is reduced, (2) the number of maxima per threshold crossings is reduced and (3) the number of overlaps per threshold crossings is reduced.

The design of a Poisson pulse generator requires y_1 to be large and τ to be small so that the resulting binomial distribution is a good approximation of the Poisson distribution. The bandwidth f_c of the random noise source must be large in order to obtain a large average rate of Poisson pulses. However, (1) an increased bandwidth increases the rms value of the noise thus decreasing y_1 , and (2) a decreased value of τ increases the overlap rate which weakens the approximation. Herein lays the problem of design compromises.

D. Characteristics of a Pulse Discriminator

The performance of a Poisson pulse generator is strongly dependent on the characteristics of the pulse discriminator employed to detect the threshold crossings. The factor τ is determined by the resolving time of the discriminator. A basic form for a Poisson pulse generator is shown in Figure 6.

The noise source has a high cutoff frequency of f_c , no dc component (zero mean value) and a standard deviation σ . The discriminator may be of a synchronous type in which the time increments are determined by an oscillator clock. Or, it may be self-triggered by the random current exceeding the threshold level. In the case of the synchronous discriminator, the time between output pulses will be an integer number of increments. In the case of the self-triggering discriminator, the time between output pulses will be an integer number of increments plus some fraction of an increment. If τ is sufficiently small, there is negligible difference between these two types of discriminators.

TABLE III

Normalized Overlap Rate

y_1	$v_{\tau}/f_c (r = 0)$	$v_{\tau}/f_c (r = 0.88)$	$v_{\tau}/v_s (r = 0.88)$	$v_{\tau}/f_c (r = 1.24)$	$v_{\tau}/v_s (r = 1.24)$
0	0.288675	0.289	0.5000	0.289	0.5000
0.5	0.254755	0.168	0.3300	0.136	0.2676
1.0	0.175091	0.0664	0.1894	0.0350	0.1003
1.5	0.093719	0.0175	0.0934	0.00562	0.0300
2.0	0.039068	0.00306	0.0392	0.000516	0.0066
2.5	0.012684	0.000352	0.0139	0.000025	0.0010
3.0	0.003707	0.000026	0.0041	<0.000010	<0.0001
3.5	0.000632	<0.000010	<0.0010	<0.000010	<0.0001
4.0	0.000097	<0.000010	<0.0010	<0.000010	<0.0001
4.5	0.000011	<0.000010	<0.0010	<0.000010	<0.0001
5.0	0.000001	<0.000010	<0.0010	<0.000010	<0.0001

Figure 5 Normalized Rates

Figure 6 Basic Poisson Pulse Generator

The pulse discriminator may be designed to trigger on a positive slope threshold crossing or on a peak above the threshold. If the time increment τ is the resolving time of the discriminator, the discriminator is incapable of producing more than one pulse per increment. Hence, multiple peaks within a given increment will go undetected. In the case of overlapping the peak detecting discriminator would produce a pulse for each peak with a spacing greater than τ from the previous peak. If the overlap rate is small, the triggering mode becomes less important.

The action of the pulse generator during overlap can (1) be inactive after the initial threshold cross, (2) generate an output pulse for each peak greater than the threshold and more widely spaced than τ during the overlap period, and (3) generate an output pulse for each increment τ during the overlap period. It is possible but difficult to statistically calculate which one of these three actions would most closely approximate the Poisson distribution. Once again, the importance of this action is diminished as the overlap rate is reduced. This effect is best determined by laboratory experimentation. If it proves to be important, the calculation must be performed.

The required stability for a given application can be determined from the derivation of the threshold crossing rate with respect to the normalized threshold level y_1 . Table II and Figure 5 present this information. The stability of the threshold level is of major importance.

Summarizing, the desired characteristics of the pulse discriminator are (1) short resolving time and (2) high degree of stability. In addition, system parameters must be selected to give a low overlap rate.

The average rate of output pulses may be varied by maintaining the standard deviation of the random current constant and varying the threshold level or the threshold level may be held constant and the standard deviation varied by amplifying the random current. The latter method is inherently more stable. Additional stability can be obtained by employing AGC feedback. Figure 7 is a suggested block diagram for the Poisson pulse generator.

Figure 7 Poisson Pulse Generator

III. PRACTICAL CONSIDERATIONS

A. Sample Design of a Poission Pulse Generator

Consider an illustrative design of a Poisson pulse generator based on Figure 7. The constant level pulse discriminator is assigned the characteristics of triggering on positive slope threshold crossings and being inactive during overlap periods. The average output pulse rate is then the threshold crossing rate as given in Equation (29)

$$a = v_s = \frac{f_c}{\sqrt{3}} e^{-y_1^2/2} \quad (51)$$

where $y_1 = I_1/\sigma$.

The overlap rate can be minimized by maximizing r (see Figure 3). Equation (50) gives this optimum relation as

$$f_c \tau = 0.75$$

The average rate is to be varied between a minimum and maximum value by varying the gain of the amplifier and hence y_1 . The greatest statistical deviation of the output from Poisson distributed pulses and the maximum overlap rate will occur at the maximum average rate. Let the average maximum rate be one megacycle and the minimum average rate be one kilocycle. In addition select

$$a\tau = 0.02$$

and

$$y_1 = 3.0$$

at the maximum average rate. This results in a resolving time of

$$\tau = 20 \text{ nanoseconds}$$

and a noise source low pass bandwidth of $f_c = 37.5$ megacycles. With the aid of Tables II and III and Figure 5, the limiting values shown in Table IV can be determined.

TABLE IV
Sample Design Conditions

	$v_s = a_{\min} = 1 \text{ kc}$	$v_s = a_{\max} = 1 \text{ mc}$
v_s/f_c	6.4×10^{-6}	6.4×10^{-3}
y_1	4.7 units	3.0 units
v_τ	$\ll 1.0 \text{ cps}$	$\sim 30 \text{ cps}$
dv_s/dy_1	1.9 kc/unit	0.72 mc/unit

The amplifier gain and stability requirements can be determined from these values. Assume that the 37.5 mc noise source has a standard deviation of one milliamperere, then after being amplified

$$y_1 = \frac{I_1}{\sigma G} \quad (52)$$

where I_1 is the threshold level. Solving for the gain

$$G = \frac{I_1}{\sigma y_1} \quad (53)$$

If the threshold level is 100 milliamperes, the required gains are

$$G_{min} = 21 \quad \text{at} \quad 1 \text{ kc}$$

and

$$G_{max} = 33 \quad \text{at} \quad 1 \text{ mc.}$$

The stability requirement is determined from the relation

$$\frac{dv_s}{dy_1} \sim \frac{\Delta a}{\Delta y_1} \quad (54)$$

Solving for Δy_1

$$\Delta y_1 = \frac{\Delta a}{(dv_s/dy_1)} \quad (55)$$

The small change Δy_1 can also be expressed as

$$\Delta y_1 = (y_1)_a - (y_1)_b \quad (56)$$

Substituting Equation (52) into Equation (56)

$$\Delta y_1 = \frac{I_1}{\sigma G_a} - \frac{I_1}{\sigma G_b} = \frac{I_1}{\sigma G_a} \left(\frac{G_b - G_a}{G_a} \right) \quad (57)$$

Let $\Delta G = G_b - G_a$ and since the change is small, G_b is approximately equal to G_a and written as G . Equation (57) is rewritten as

$$\Delta y_1 \sim \frac{I_1}{\sigma G} \left(\frac{\Delta G}{G} \right) = y_1 \left(\frac{\Delta G}{G} \right)$$

or

$$\frac{\Delta y_1}{y_1} \sim \frac{\Delta G}{G} \quad (58)$$

Equation (58) shows that a small fractional change in y_1 is approximately equal to the small fractional change in gain. Thus, from Equations (55) and (58)

$$\frac{\Delta G}{G} \sim \frac{\Delta y_1}{y_1} = \frac{a}{y_1 (dv_s/dy_1)} \rho \quad (59)$$

where $\Delta a = \rho a$ and ρ is a small percentage change in a .

The minimum average rate requires the higher degree of stability. If the frequency stability of the minimum average rate is to be within 1%, ($\rho = 1\%$) then

$$\frac{\Delta G}{G_{\min}} \sim 0.1\%$$

for this design. This number represents the maximum stability as well as the maximum gain adjustment sensitivity of the amplifier. In the actual system shown in Figure 7, this stability factor is the overall system stability of the noise source, amplifier, discriminator power supply and AGC loop. Table V lists the resulting design values.

TABLE V

Sample Design Values

<u>System Characteristics</u>	
Average Rate of Pulses	1 kc to 1 mc
Frequency Stability	1% or less at all frequencies
System Stability	0.1% of minimum gain
<u>Noise Source</u>	
Amplitude Distribution	Normal
Standard Deviation (rms value)	1 ma
Frequency Bandwidth	37.5 mc
<u>Amplifier</u>	
Minimum Gain	21 (for 1 kc rate)
Maximum Gain	33 (for 1 mc rate)
Gain-adjust Increments for 1% or less Frequency change	0.02
<u>Pulse Discriminator</u>	
Type	Constant threshold level, trigger on positive slope crossing of threshold, inactive during overlap periods
Threshold Level	100 ma
Resolving Time	20 nanoseconds
Pulse Width	< 20 nanoseconds

The design specifications in Table V will result in the conditions listed in Table IV.

B. Laboratory Model Poisson Pulse Generator

A prototype model of a Poisson pulse generator was assembled in the laboratory. Figure 8 is a diagram of the model. Its performance compared favorably with the foregoing theoretical analysis. Measurements were made of (1) equivalent noise bandwidth, (2) resolving time, (3) standard deviation, (rms voltage) and (4) average pulse rate. These data combined with data from Figure 5 is presented in Table VI and Figure 9. The pulse discriminator triggered on positive slope crossings of the threshold level and remained inactive during overlap periods.

Figure 8 Laboratory Model of a Poisson Pulse Generator

TABLE VI
Characteristics of the Laboratory Model

	Average Rate - a			
	20 cps	10 kc	50 kc	250 kc
v_s/f_c	10^{-5}	5×10^{-3}	2.5×10^{-2}	1.25×10^{-1}
y_1	4.7	3.1	2.5	1.7
v_T	SMALL	60 cps	300 cps	~ 10 kc
dv_s/dy_1	100 cps	30 kc	130 kc	500 kc
a_T	0.00002	0.01	0.10	0.25

$\tau = 1 \mu\text{sec}$	$f_c \tau = 2$
$f_c = 2 \text{ mc}$	$\omega_c \tau = 4\pi \text{ (r = 1.0)}$

Several points are worth discussion.

1. An average rate of 10 kc appears to have desirable statistical properties. The output pulses should be a good approximation of Poisson pulses (see Table I and Figure 3) and the overlap rate is low. An average rate of 50 kc is classified as a borderline case of acceptable characteristics.

2. An average rate of 250 kc and above appears unacceptable. Figure 9 shows a major departure between calculated and measured data at the higher frequencies. The major contribution to this deviation is the extremely long "dead time". An average rate of 250 kc and resolving time of one microsecond means that the discriminator is incapable of generating pulses 25% of the time. This results in a lowered average rate. The output pulse distribution probably differs greatly from a Poisson distribution.

3. At an average rate of 20 cps, good statistical properties can be expected but stability becomes a problem. Considerable drift was noticed in the rms voltmeter readings. This can account for the departure between the calculated and measured curves at low average rates.

No attempt was made to measure the pulse distribution nor to compare it to a radioactive source at this time.

Figure 9 Laboratory Model Threshold Crossing Characteristics

A test was conducted to demonstrate the usefulness of a random Poisson pulse generator. A low-speed glow tube decade scaler rated as a 2 kc scaler was selected as the subject for this test. Its response to various pulse conditions was compared to the response of a 10 mc counter. The results of the test are shown in Figure 10.

First the response of the scaler to a train of periodic pulses was obtained. The scaler successfully counted all pulses up to a pulse rate of 2 kc. At this point, it became paralyzed and indicated a zero rate for all pulse rates greater than 2 kc. Apparently, this is the origin of the rating of the scaler.

Second, the dual pulse resolution of the scaler was obtained. A train of dual pulses with a 1 kc rate (i.e., one millisecond between each set of pulses) was used as the input signal. The time between pairs of pulses was reduced until the indicated count rate of the scaler was reduced by a factor of two. This indicated that the scaler was incapable of resolving the two pulses. The spacing between the pair of pulses is the dual pulse resolving time of the scaler. The resolving time is 0.35 milliseconds.

A knowledge of the resolving time permits a calculation to be made of the response of an ideal circuit to random Poisson pulses. Two types of ideal circuits are considered, paralyzable and nonparalyzable. In a paralyzable circuit, the circuit is dead for a period equal to the resolving time of the circuit. But if a second pulse appears during this time, the dead time is extended for a resolving period starting with the second pulse. If the pulses appeared fast enough, the circuit would never recover and the apparent count rate would drop to zero. For a paralyzable circuit, the apparent count rate for random pulses is given by

$$r_p = R e^{-RT} \quad (60)$$

where

r_p - apparent count rate for a paralyzable circuit
 R - true count rate
 T - resolving time

In a nonparalyzable circuit, the action is unaffected by pulses appearing during the dead time. If the pulses appeared sufficiently fast, the circuit would respond to a pulse immediately after recovering from an earlier pulse. The apparent count rate will then be limited to a rate equal to the reciprocal of the resolving time. The apparent rate or random pulse is

$$r_{np} = \frac{R}{1 + RT} \quad (61)$$

where r_{np} is the apparent count rate for a nonparalyzable circuit. These two conditions are plotted in Figure 10. They represent the two limiting cases. The response of an actual circuit would be expected to fall between these two limits. Since the train of periodic pulses resulted in a zero count rate at higher rates, this seems to indicate that the scaler is a paralyzable type instrument. Consequently, when

Figure 10 Low Speed Scaler Response

Poisson pulses are applied to the scaler input, the response curve is expected to be close to or somewhat above the ideal curve for a paralyzable circuit with a resolving time of 0.35 milliseconds.

When pulses from the laboratory model Poisson pulse generator were applied to the scaler, the unexpected response curve shown in Figure 10 was obtained. Even at lower count rates (less than 1 kc) the scaler indicated a higher apparent count rate than that expected from the previous analysis. From approximately 2.5 kc to 10 kc the apparent count rate of the scaler exceeded the true count rate. The scaler seemed to be generating pulses of its own. Above 10 kc, a region of instability was encountered. First, there appears to be a maximum count rate that the scaler cannot exceed. Next, it was found that at even higher rates the repeatability of a count was poor. In the region of 100 kc and higher, there were periods of complete paralysis. At times, the scaler indicated zero counts for a 10 second period during which time the true count was 100 kc or more.

At the higher count rate, two other factors must be considered:

- (1) The pulse generator is a poor source of Poisson pulses and (2) the 10 mc counter is assumed to have a linear response.

The explanation for the scalers response is not available. But the important point to be considered is: "It would have been impossible to have predicted this response from the results of tests using various configurations of periodic pulses". This is an excellent illustration of the need for random pulse generators.

C. A Nuclear Signal Simulator

The usefulness of the proposed Poisson pulse generator can be extended by using it as the random time element in a system to simulate the electrical signals from a nuclear radiation detector. Two additional functions are required: (1) A means of randomly varying the amplitude of each pulse to a pre-determined pulse height amplitude probability distribution function and (2) a means of randomly varying the pulse shape (jitter) about some average pulse shape. The first function is required to simulate the effect of the nuclear radiation energy spectrum as modified by the detector. A typical example is the pulse height spectrum as found in gamma-ray scintillation spectroscopy. The second factor is required to simulate the varying geometry associated with the origin of each pulse. A typical example is the effect of the orientation of the ionizing path in an ionization chamber on the pulse shape.

A change in the average rate of pulses, as found in short half life isotope or reactor power measurements, can be simulated by varying the amplifier gain in a pre-determined manner. The integrating effect as found in certain nuclear detectors is simulated by electrically integrating the output of the simulator.

Figure 11 illustrates one possible configuration for such a simulator. The random Poisson pulses are used to trigger an amplitude sampler. If a normal pulse height amplitude distribution is desired, the white noise source can be used to provide this distribution. A sufficiently long time delay must be inserted to destroy the correlation between the time and height probability distribution. Standard deviation (amplifier gain) and mean value (dc insertion) are readily controllable. Several normal pulse height distributions can be super-imposed by using parallel channels. Each must be sufficiently displaced in time by means of time delays to prevent correlation.

Figure 11 Nuclear Signal Simulator

Pulse height distributions other than normal are obtainable by generating a repetitive waveform in the shape of the desired cumulative probability function. This waveform is then sampled by the Poisson pulses. The time relationship must be chosen in a manner to prevent correlation.

A normal distributed pulse shape is probably adequate for most applications. Once again the white noise with appropriate time delay can be used to generate this variation.

The simulator shown in Figure 11 would be useful as (1) a research tool for the development of new instrumentation techniques, and (2) a signal source for testing existing instrumentations. Since the physical size of this equipment may well prohibit moving from location to location, an analog tape recorder is included. A signal or signals meeting given requirements can be simulated and recorded. The tape can be readily taken to and used at another location. As an example, the

output of an ionization chamber instrumentating a reactor changing power level may be simulated and recorded. The tape is taken to the reactor site and played back into the actual reactor instrumentation for test purposes. Thus, the effect of unsafe operating conditions on the instrumentation and control can be analyzed in a safe manner.

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IV. CONCLUSIONS

This paper has attempted to demonstrate both the theoretical and practical feasibility of developing a Poisson pulse generator using a source of "white noise" as the random element. The calculations show that a generator having a one megacycle maximum average rate is readily obtainable with present circuit techniques.

The testing of instruments intended for use in nuclear systems requires a source of random Poisson pulses. The response to periodic pulses cannot reliably predict the instruments response to random pulses. Likewise, constant amplitude, constant shape random pulses cannot reliably predict the instruments response to pulses which are random in time, height and shape. However, the proposed Poisson pulse generator can be extended to include these factors. This would result in greater usage.

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