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The LAPLACE FORTRAN Code

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The LAPLACE FORTRAN Code

by

John Knox Hayes

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ABSTRACT

The method of use and applications of the subroutine LAPLACE and its related subroutines are described.

I. INTRODUCTION

The subroutine LAPLACE is written in FORTRAN for the CDC 6600 using the LASL system. The method used by the LAPLACE subroutine to obtain a numerical approximation to a solution of Laplace's equation is different from that of most routines used to solve the problem and makes the use of the routine slightly different.

Suppose that one wants to find a numerical approximation to a solution of Laplace's equation in a two-dimensional region, G , with boundary S . LAPLACE uses the fundamental formula

$$u(x,y) = \frac{1}{2\pi} \int_S \left[\ln \left(\frac{1}{r} \right) \frac{\partial u}{\partial \nu} - \frac{\partial}{\partial \nu} \ln \left(\frac{1}{r} \right) \right] ds \quad (x,y) \in G \quad (1)$$

to approximate the desired solution, u , in G (see Ref. 1, p. 257). Either u or $\partial u / \partial n$ is assumed known at every point of the boundary, S . The problem then is to find u where $\partial u / \partial n$ is known and $\partial u / \partial n$ where u is known. The values of u and $\partial u / \partial n$ which are not known are called the complementary boundary values. To approximate the complementary boundary values, a corresponding form of formula (1) for the boundary is used. Once the complementary boundary values have been approximated, the solution or any of its derivatives can be approximated by using formula (1). Thus the use of LAPLACE is a two-step process:

- (a) Approximation of complementary boundary values, and
- (b) Approximation of the solution or its derivatives at desired points in G .

II. GENERAL PROBLEM

The most general problem to be solved by LAPLACE is as follows. Consider a two-dimensional region G with boundary S . G can be an interior or exterior region. Suppose, further, that one wants to solve for $u(x,y)$, satisfying

$$\begin{aligned} \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} &= 0 && \text{in } G \\ u &= f && \text{on } S_D \\ \frac{\partial u}{\partial n} &= g && \text{on } S_N. \end{aligned}$$

Here $S_D \cup S_N = S$, which is the boundary of G , and $S_D \cap S_N$ is, at most, a finite number of points (for example, at corners). S_N or S_D can be empty. The notation used here for $\partial / \partial n$ and $\partial / \partial \nu$ is the same as that used in Kellogg (see Ref. 2, pp. 164 and 166 for an explanation).

It is assumed that a solution, $u(x,y)$, exists and that the boundary values u and $\partial u / \partial n$ exist and that both u and $\partial u / \partial n$ have piecewise continuous second derivatives along S . It is also assumed that S has a parametric representation $\{(x(t), y(t)) | t \in (0, d)\}$ with respect to arc length. Here d is the length of S . Moreover, it is assumed that both $x(t)$ and $y(t)$ are piecewise smooth. Finally, if the region G is exterior, it must be true that $u(x,y) \rightarrow 0$ as $(x,y) \rightarrow \infty$. The last restriction comes from the fact that Eq.(1) is not in general true in an exterior region without this restriction. LAPLACE will solve many problems which cannot be explicitly stated in terms of the general problem, but can be reduced to the general problem. For instance, see the discussions of branch cuts and superposition of solutions.

III. INPUT AND OUTPUT

This section discusses the input needed for the general problem. As stated above, allowable boundary conditions are of two types: the Dirichlet boundary condition, i.e., u given, or the Neumann condition, i.e., $\partial u / \partial n$ given. For any specific problem, both types of boundary conditions can be given. For problems in which only the Neumann condition is given, the solution to the general problem is unique only up to an additive constant. To make the solution to the interior Neumann problem unique, the approximate solution is arbitrarily set equal to zero on the first point of the boundary given as input. For exterior problems, it is assumed that $u(x,y) \rightarrow 0$ as $(x,y) \rightarrow \infty$. This assumption makes the exterior Neumann problem uniquely solvable. However, arbitrary Dirichlet boundary conditions cannot be imposed for the exterior problem because of this assumption.

A. Boundary Description. Two different types of boundaries can be given as input to LAPLACE. The first is called the regular boundary to distinguish it from the second. The regular boundary must always be a finite number of closed curves. Moreover, the boundary values given on the regular boundary always refer to one side of the boundary. For instance, the boundary value problem might be on the interior of a circle or the exterior of a square.

For some other boundaries it is convenient and even desirable to assume that the region enclosed by a portion of the boundary is infinitely thin. For instance, in electrostatics, if one is setting the potential on a piece of foil it is not unreasonable to assume that the foil is infinitely thin, if its actual thickness is very small in comparison with its other dimensions. The second type of boundary thus encloses a region which is assumed to be infinitely thin, and is, therefore, called a thin-plate boundary. For a thin-plate boundary, the values are assumed to be given on both sides. Moreover, the boundary values are assumed to be the same on both sides. For this reason, thin plates must not form a closed curve. Thin plates can be used only with the Dirichlet boundary condition.

1. Simplified boundary data. The boundary and boundary values can be given as input in two different ways. The more general method will be

discussed later. For the simpler method, the boundary S is approximated by line segments, circular sections, and complete circles. The boundary values are assumed to be constant on each section of the boundary. A line segment is described by its two endpoints. The two endpoints and any interior point are used to describe a circular arc. For a complete circle, one must give the coordinates of the center of the circle, the radius of the circle, and the orientation. The orientation determines whether G is interior or exterior to the circle. A +1 denotes a positive orientation, in which case G is interior to the circle, and -1 denotes a negative orientation, in which case G is exterior to the circle.

The orientation of the other boundary sections is implicit in the input. The region G must be to the left of the regular part of the boundary as one follows the curve S in the direction given in the input. The direction of the curve is determined by the order of the points used to describe line segments and circular sections and is from the first to the second endpoint. Thus, interior regions give the boundary a positive (counterclockwise) orientation, and exterior regions give it a negative (clockwise) orientation.

The input used to describe the simple boundary conditions is given on data cards, each of which refers to a part of the boundary. The part of the boundary referred to by a given card is called a boundary section. The format statement used to read the data cards is `FORMAT(7E10.0)`. The last field in the card, i.e., columns 61 to 70, is always used for the boundary value, be it Neumann or Dirichlet. If the card is used to describe a line segment, the first four fields are used for the x and y coordinates of the first endpoint and for the x and y coordinates of the second endpoint, in that order. The fifth and sixth fields must be left blank. If the card is used to describe a circular arc, the first six fields are used to give the x and y coordinates of the first endpoint, of the interior point, and of the second endpoint, in that order. If the input card is used to describe a complete circle, the first two fields are used for the x and y coordinates of the center of the circle, the third and sixth fields must be left blank, and the fourth and fifth fields are used to give the radius and the orienta-

tion, respectively. For data cards used to describe circular sections and line segments, one can leave the first two fields blank if the first endpoint is the same as the last endpoint of the previous card. The data cards are printed as they are read.

As an example, the input for the problem given below will be discussed.

Fig. 1. Sample problem using the three types of boundary sections.

G is the region between the circle and the three-sided section in Fig. 1. The boundary conditions are assumed to be $u = 1$ on the circle, $\partial u / \partial n = 0$ on the line segment $(2, -1.5)(0, 2)$, and $u = 0$ on the rest of the boundary. The boundary input cards for this problem could be as follows:

6	10/11	13/16	20/21	23/26	30/31	33/36	40/41	43/46	50/51	53/56	60/61	63/66	70
2		2		2		2		2		2		2	
0		0		0		0		0		0		0	
0		0		0		0		0		0		0	
0		0		0		0		0		0		0	

The first two fields of the second and third cards could be left blank if desired.

Notice that in Fig. 1 the orientation of the circle is positive and that of the three-sided section is positive. This is because G is interior to the three-sided section, but exterior to the circle. Care should be taken in deciding the correct orientation. Errors in orientation are easy to make and very difficult to detect. Orientation is not used with thin plates because the region G is always exterior to a thin-plate boundary.

As an example of a problem involving a thin-plate boundary, consider the following.

Fig. 2. Thin-plate boundary problem.

In this example, G is all of the interior of the circle except the line segment $(0, 0)(10, 0)$. The boundary conditions are $u = 1$ on the circle and $u = 0$ on the line segment $(0, 0)(10, 0)$. The boundary input cards for this problem are:

6	10/11	13/16	20/21	23/26	30/31	33/36	40/41	43/46	50/51	53/56	60/61	63/66	70
0		0		0		0		0		0		0	
5		5		5		5		5		5		5	
0		0		0		0		0		0		0	
0		0		0		0		0		0		0	

2. Input variables.

The order of the boundary input cards can be arbitrary, but must correspond with the order of the other input variables used to describe each section of the boundary. Four other input variables used in determining the approximation to the boundary S, given in the call statement to LAPLACE, are as follows.

- M is the number of boundary sections. At present, it is required that $1 \leq M \leq 60$.
- LT is an integer-subscripted variable of length M. $LT(I)$, $I = 1, 2, \dots, M$ determines the type of boundary and boundary condition on the Ith boundary section. If $LT(I) = 1$, the boundary is regular and u is given. If $LT(I) = 2$, the boundary is regular and $\partial u / \partial n$ is given. If $LT(I) = 3$, the boundary is a thin plate and u is given.
- KV is an integer-subscripted variable of length M. $KV(I)$, $I = 1, 2, \dots, M$ tells how many points are to be used in approximating u and $\partial u / \partial n$ on the Ith boundary section. $KV(I)$ must be odd, and greater than or equal to 3. For the simple type of boundary input, either u or $\partial u / \partial n$ is constant on the boundary. However, the complementary boundary value is not necessarily constant. The variation in the complementary boundary value, the shape of the boundary S, and the desired accuracy de-

termine the value of $KV(I)$; no set rule can be given. The length of the I th boundary section divided by $KV(I) - 1$ is the distance between approximation points on the I th section.

- (d) DC is a real-subscripted variable of length M. For the simple type of boundary input, $DC(I)$ must always be zero. For the more general type of boundary input, $DC(I)$ is the length of the I th boundary section.

For the example given in Fig. 1, the following input variables are determined:

$$\left\{ \begin{array}{l} M = 4 \\ LT(1) = 2 \\ LT(2) = LT(3) = LT(4) = 1 \\ DC(1) = DC(2) = DC(3) = DC(4) = 0. \end{array} \right\}.$$

For the example given in Fig. 2, the determined variables are:

$$\left\{ \begin{array}{l} M = 2 \\ LT(1) = 1 \\ LT(2) = 3 \\ DC(1) = DC(2) = 0. \end{array} \right\}.$$

Two auxiliary variables are given as input. One is an integer, NMAX, with the property that

$$NMAX \geq \sum_{I=1}^M KV(I).$$

The other is an $NMAX \times NMAX$ matrix, A, used for intermediate storage. The matrix A is used only in the subroutine LAPLACE, and is available to the user once the computer has returned from LAPLACE. The matrix A is not used by any of the other routines associated with LAPLACE. $NMAX = 192$ is about as large a problem as can be handled on a 65,536 word machine. The size of the matrix A at present determines the size of the largest allowable problem.

Suppose in the second example it was decided that 31 approximation points on the circle and 21 on the line segment would give sufficient accuracy. The variables KV and $NMAX$ would have the following values:

$$\left\{ \begin{array}{l} KV(1) = 31 \\ KV(2) = 21 \\ NMAX = 52 \end{array} \right\}.$$

Moreover, one would have to dimension the matrix A as an $NMAX \times NMAX$ matrix. Here $NMAX$ can be larger than 52, but it must be true that A is at least an $NMAX \times NMAX$ matrix.

B. Branch Cuts. The next, rather involved, input is used for approximating solutions of Laplace's equation which have branch cuts and is not used for most problems. For the problems where branch cuts are not used, one must state that there are zero branch cuts. The variable NBC in the call statement for LAPLACE is used to give the number of branch cuts. Branch cuts are used in flow problems and in magnetic field problems. Only magnetic field problems will be discussed here, but flow problems are similar.

In some problems in magnetism, one has a series of parallel current-carrying conductors which are assumed to be infinite in length. One wants to find a potential function whose gradient is the magnetic field. The potential function satisfies Laplace's equation exterior to the conductors and satisfies $\partial u / \partial n = 0$ on the boundary of the conductors. However, if C is any closed curve not intersecting the conductors, one must have

$$\oint_C \nabla u \cdot d\vec{s} = 2\pi I_C,$$

where I_C is the algebraic sum of all of the currents in all of the conductors in C. Without introduction of branch cuts into the solution, the above integral would be zero for any such curve C. To remedy the situation, directed line segments which form a path to each of the conductors, and possibly one line segment running to infinity, are introduced. The solution is required to have a discontinuity across each of the line segments. Starting at a given conductor, one forms a path joining each of the conductors. Each conductor has at most two line segments associated with it, one coming and one going. The discontinuity across any given line segment is 2π times the sum of all previous currents in the path. The sum may, of course, be zero, in which case there is no discontinuity across the segment and the segment can be omitted.

Consider the following example.

Fig. 3. Branch cut problem.

Assume that the smallest circular conductor has current I_1 , the larger circular conductor I_2 , and the square conductor I_3 . The discontinuity across the line segment L_1 will be $2\pi I_1$. The discontinuity across L_2 will be $2\pi(I_1 + I_2)$, and that across L_3 will be $2\pi(I_1 + I_2 + I_3)$. Line segment L_3 runs to infinity. The conductors could have been joined in any order, but if one conductor encloses all of the others, then the last branch cut must end on the surrounding conductor.

Two types of input are needed for the branch cuts. In the call statement to the subroutine LAPLACE, one must give the number of branch cuts, called NBC. For each branch cut there must also be a data card. On each data card one gives the endpoints of the branch cut in the order in which they occur and the sum of the previous currents in the path. If one of the endpoints of the branch cut is infinite, one can give values of x and y which are much larger than any values of interest in the calculation. The format statement used to read the data cards is `FORMAT(5E10.0)`. The first two fields, columns 1 to 10 and columns 11 to 20, are used for the x and y coordinates, respectively, of the first endpoint. The third and fourth fields are used for the x and y coordinates, respectively, of the second endpoint. The fifth field is used for the sum of the currents. The data cards are printed as they are read. If in the given example $I_1 = 1$, $I_2 = 2$, and $I_3 = -2$, then the data cards for the branch cuts could be as follows:

6	10/11	15/16	20/21	25/26	30/31	35/36	40/41	45/46	50/51	55/56	60/61	65/66	70/71	75/76	80/81	85/86	90/91	95/96	100/101	105/106	110/111	115/116	120/121	125/126	130/131	135/136	140/141	145/146	150/151	155/156	160/161	165/166	170/171	175/176	180/181	185/186	190/191	195/196	200/201	205/206	210/211	215/216	220/221	225/226	230/231	235/236	240/241	245/246	250/251	255/256	260/261	265/266	270/271	275/276	280/281	285/286	290/291	295/296	300/301	305/306	310/311	315/316	320/321	325/326	330/331	335/336	340/341	345/346	350/351	355/356	360/361	365/366	370/371	375/376	380/381	385/386	390/391	395/396	400/401	405/406	410/411	415/416	420/421	425/426	430/431	435/436	440/441	445/446	450/451	455/456	460/461	465/466	470/471	475/476	480/481	485/486	490/491	495/496	500/501	505/506	510/511	515/516	520/521	525/526	530/531	535/536	540/541	545/546	550/551	555/556	560/561	565/566	570/571	575/576	580/581	585/586	590/591	595/596	600/601	605/606	610/611	615/616	620/621	625/626	630/631	635/636	640/641	645/646	650/651	655/656	660/661	665/666	670/671	675/676	680/681	685/686	690/691	695/696	700/701	705/706	710/711	715/716	720/721	725/726	730/731	735/736	740/741	745/746	750/751	755/756	760/761	765/766	770/771	775/776	780/781	785/786	790/791	795/796	800/801	805/806	810/811	815/816	820/821	825/826	830/831	835/836	840/841	845/846	850/851	855/856	860/861	865/866	870/871	875/876	880/881	885/886	890/891	895/896	900/901	905/906	910/911	915/916	920/921	925/926	930/931	935/936	940/941	945/946	950/951	955/956	960/961	965/966	970/971	975/976	980/981	985/986	990/991	995/996	1000/1001	1005/1006	1010/1011	1015/1016	1020/1021	1025/1026	1030/1031	1035/1036	1040/1041	1045/1046	1050/1051	1055/1056	1060/1061	1065/1066	1070/1071	1075/1076	1080/1081	1085/1086	1090/1091	1095/1096	1100/1101	1105/1106	1110/1111	1115/1116	1120/1121	1125/1126	1130/1131	1135/1136	1140/1141	1145/1146	1150/1151	1155/1156	1160/1161	1165/1166	1170/1171	1175/1176	1180/1181	1185/1186	1190/1191	1195/1196	1200/1201	1205/1206	1210/1211	1215/1216	1220/1221	1225/1226	1230/1231	1235/1236	1240/1241	1245/1246	1250/1251	1255/1256	1260/1261	1265/1266	1270/1271	1275/1276	1280/1281	1285/1286	1290/1291	1295/1296	1300/1301	1305/1306	1310/1311	1315/1316	1320/1321	1325/1326	1330/1331	1335/1336	1340/1341	1345/1346	1350/1351	1355/1356	1360/1361	1365/1366	1370/1371	1375/1376	1380/1381	1385/1386	1390/1391	1395/1396	1400/1401	1405/1406	1410/1411	1415/1416	1420/1421	1425/1426	1430/1431	1435/1436	1440/1441	1445/1446	1450/1451	1455/1456	1460/1461	1465/1466	1470/1471	1475/1476	1480/1481	1485/1486	1490/1491	1495/1496	1500/1501	1505/1506	1510/1511	1515/1516	1520/1521	1525/1526	1530/1531	1535/1536	1540/1541	1545/1546	1550/1551	1555/1556	1560/1561	1565/1566	1570/1571	1575/1576	1580/1581	1585/1586	1590/1591	1595/1596	1600/1601	1605/1606	1610/1611	1615/1616	1620/1621	1625/1626	1630/1631	1635/1636	1640/1641	1645/1646	1650/1651	1655/1656	1660/1661	1665/1666	1670/1671	1675/1676	1680/1681	1685/1686	1690/1691	1695/1696	1700/1701	1705/1706	1710/1711	1715/1716	1720/1721	1725/1726	1730/1731	1735/1736	1740/1741	1745/1746	1750/1751	1755/1756	1760/1761	1765/1766	1770/1771	1775/1776	1780/1781	1785/1786	1790/1791	1795/1796	1800/1801	1805/1806	1810/1811	1815/1816	1820/1821	1825/1826	1830/1831	1835/1836	1840/1841	1845/1846	1850/1851	1855/1856	1860/1861	1865/1866	1870/1871	1875/1876	1880/1881	1885/1886	1890/1891	1895/1896	1900/1901	1905/1906	1910/1911	1915/1916	1920/1921	1925/1926	1930/1931	1935/1936	1940/1941	1945/1946	1950/1951	1955/1956	1960/1961	1965/1966	1970/1971	1975/1976	1980/1981	1985/1986	1990/1991	1995/1996	2000/2001	2005/2006	2010/2011	2015/2016	2020/2021	2025/2026	2030/2031	2035/2036	2040/2041	2045/2046	2050/2051	2055/2056	2060/2061	2065/2066	2070/2071	2075/2076	2080/2081	2085/2086	2090/2091	2095/2096	2100/2101	2105/2106	2110/2111	2115/2116	2120/2121	2125/2126	2130/2131	2135/2136	2140/2141	2145/2146	2150/2151	2155/2156	2160/2161	2165/2166	2170/2171	2175/2176	2180/2181	2185/2186	2190/2191	2195/2196	2200/2201	2205/2206	2210/2211	2215/2216	2220/2221	2225/2226	2230/2231	2235/2236	2240/2241	2245/2246	2250/2251	2255/2256	2260/2261	2265/2266	2270/2271	2275/2276	2280/2281	2285/2286	2290/2291	2295/2296	2300/2301	2305/2306	2310/2311	2315/2316	2320/2321	2325/2326	2330/2331	2335/2336	2340/2341	2345/2346	2350/2351	2355/2356	2360/2361	2365/2366	2370/2371	2375/2376	2380/2381	2385/2386	2390/2391	2395/2396	2400/2401	2405/2406	2410/2411	2415/2416	2420/2421	2425/2426	2430/2431	2435/2436	2440/2441	2445/2446	2450/2451	2455/2456	2460/2461	2465/2466	2470/2471	2475/2476	2480/2481	2485/2486	2490/2491	2495/2496	2500/2501	2505/2506	2510/2511	2515/2516	2520/2521	2525/2526	2530/2531	2535/2536	2540/2541	2545/2546	2550/2551	2555/2556	2560/2561	2565/2566	2570/2571	2575/2576	2580/2581	2585/2586	2590/2591	2595/2596	2600/2601	2605/2606	2610/2611	2615/2616	2620/2621	2625/2626	2630/2631	2635/2636	2640/2641	2645/2646	2650/2651	2655/2656	2660/2661	2665/2666	2670/2671	2675/2676	2680/2681	2685/2686	2690/2691	2695/2696	2700/2701	2705/2706	2710/2711	2715/2716	2720/2721	2725/2726	2730/2731	2735/2736	2740/2741	2745/2746	2750/2751	2755/2756	2760/2761	2765/2766	2770/2771	2775/2776	2780/2781	2785/2786	2790/2791	2795/2796	2800/2801	2805/2806	2810/2811	2815/2816	2820/2821	2825/2826	2830/2831	2835/2836	2840/2841	2845/2846	2850/2851	2855/2856	2860/2861	2865/2866	2870/2871	2875/2876	2880/2881	2885/2886	2890/2891	2895/2896	2900/2901	2905/2906	2910/2911	2915/2916	2920/2921	2925/2926	2930/2931	2935/2936	2940/2941	2945/2946	2950/2951	2955/2956	2960/2961	2965/2966	2970/2971	2975/2976	2980/2981	2985/2986	2990/2991	2995/2996	3000/3001	3005/3006	3010/3011	3015/3016	3020/3021	3025/3026	3030/3031	3035/3036	3040/3041	3045/3046	3050/3051	3055/3056	3060/3061	3065/3066	3070/3071	3075/3076	3080/3081	3085/3086	3090/3091	3095/3096	3100/3101	3105/3106	3110/3111	3115/3116	3120/3121	3125/3126	3130/3131	3135/3136	3140/3141	3145/3146	3150/3151	3155/3156	3160/3161	3165/3166	3170/3171	3175/3176	3180/3181	3185/3186	3190/3191	3195/3196	3200/3201	3205/3206	3210/3211	3215/3216	3220/3221	3225/3226	3230/3231	3235/3236	3240/3241	3245/3246	3250/3251	3255/3256	3260/3261	3265/3266	3270/3271	3275/3276	3280/3281	3285/3286	3290/3291	3295/3296	3300/3301	3305/3306	3310/3311	3315/3316	3320/3321	3325/3326	3330/3331	3335/3336	3340/3341	3345/3346	3350/3351	3355/3356	3360/3361	3365/3366	3370/3371	3375/3376	3380/3381	3385/3386	3390/3391	3395/3396	3400/3401	3405/3406	3410/3411	3415/3416	3420/3421	3425/3426	3430/3431	3435/3436	3440/3441	3445/3446	3450/3451	3455/3456	3460/3461	3465/3466	3470/3471	3475/3476	3480/3481	3485/3486	3490/3491	3495/3496	3500/3501	3505/3506	3510/3511	3515/3516	3520/3521	3525/3526	3530/3531	3535/3536	3540/3541	3545/3546	3550/3551	3555/3556	3560/3561	3565/3566	3570/3571	3575/3576	3580/3581	3585/3586	3590/3591	3595/3596	3600/3601	3605/3606	3610/3611	3615/3616	3620/3621	3625/3626	3630/3631	3635/3636	3640/3641	3645/3646	3650/3651	3655/3656	3660/3661	3665/3666	3670/3671	3675/3676	3680/3681	3685/3686	3690/3691	3695/3696	3700/3701	3705/3706	3710/3711	3715/3716	3720/3721	3725/3726	3730/3731	3735/3736	3740/3741	3745/3746	3750/3751	3755/3756	3760/3761	3765/3766	3770/3771	3775/3776	3780/3781	3785/3786	3790/3791	3795/3796	3800/3801	3805/3806	3810/3811	3815/3816	3820/3821	3825/3826	3830/3831	3835/3836	3840/3841	3845/3846	3850/3851	3855/3856	3860/3861	3865/3866	3870/3871	3875/3876	3880/3881	3885/3886	3890/3891	3895/3896	3900/3901	3905/3906	3910/3911	3915/3916	3920/3921	3925/3926	3930/3931	3935/3936	3940/3941	3945/3946	3950/3951	3955/3956	3960/3961	3965/3966	3970/3971	3975/3976	3980/3981	3985/3986	3990/3991	3995/3996	4000/4001	4005/4006	4010/4011	4015/4016	4020/4021	4025/4026	4030/4031	4035/4036	4040/4041	4045/4046	4050/4051	4055/4056	4060/4061	4065/4066	4070/4071	4075/4076	4080/4081	4085/4086	4090/4091	4095/4096	4100/4101	4105/4106	4110/4111	4115/4116	4120/4121	4125/4126	4130/4131	4135/4136	4140/4141	4145/4146	4150/4151	4155/4156	4160/4161	4165/4166	4170/4171	4175/4176	4180/4181	4185/4186	4190/4191	4195/4196	4200/4201	4205/4206	4210/4211	4215/4216	4220/4221	4225/4226	4230/4231	4235/4236	4240/4241	4245/4246	4250/4251	4255/4256	4260/4261	4265/4266	4270/4271	4275/4276	4280/4281	4285/4286	4290/4291	4295/4296	4300/4301	4305/4306	4
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3. Plot routine. The plot routine plots equipotential curves on film. The two endpoints of a line segment and an integer, called K here, are given as input to the plot routine. The plot routine plots K equipotential curves. The curves all cross the given line segment. There are two curves at the ends of the segment and (K-2) other curves crossing the segment at equal spacings. The equipotential curves are a series of connected line segments of length HH. HH is given as input. To plot equipotential curves along more than one line segment, there is a provision to dimension all of the input variables mentioned above.

A total list of input variables is as follows.

XMIN is the minimum X to be plotted.
 YMIN is the minimum Y to be plotted.
 XMAX is the maximum X to be plotted.
 YMAX is the maximum Y to be plotted.
 NP is the number of line segments along which equipotential curves are to be plotted.
 X1(I) for $I = 1, 2, \dots, NP$ is the first X coordinate of the Ith line segment.
 Y1(I) for $I = 1, 2, \dots, NP$ is the first Y coordinate of the Ith line segment.
 X2(I) for $I = 1, 2, \dots, NP$ is the second X coordinate of the Ith line segment.
 Y2(I) for $I = 1, 2, \dots, NP$ is the second Y coordinate of the Ith line segment.
 K(I) for $I = 1, 2, \dots, NP$ is the number of equipotential curves to be plotted along the Ith line segment.
 HH(I) for $I = 1, 2, \dots, NP$ is the length of step to be used in constructing equipotential curves starting from the Ith segment.

The call statement for the plot routine is as follows:

CALL LAPLOT(XMIN,YMIN,XMAX,YMAX,X1,Y1,X2,Y2,K,HH,NP).

LAPLOT plots all of the boundary and all of the branch cuts inside the rectangle formed by XMIN, YMIN, XMAX, and YMAX. The equipotential curves are also plotted inside this rectangle. By making various calls to LAPLOT, one can get enlarged views of the equipotential curves in various subregions of G.

If $NP \leq 0$, only the boundary and branch cuts will be plotted. If the subroutine LAPLACE discovers an error in the boundary input data and the problem is not run to completion, then only the boundary and

branch cuts will be plotted. If $XMIN \geq XMAX$ or $YMIN \geq YMAX$, then no equipotential curves will be plotted. It must be true that $2 \leq K(I) \leq 50$, for $I = 1, 2, \dots, NP$. If this requirement is not met, it will be imposed on K(I).

The plot routine will stop plotting an equipotential curve whenever any of the following occur: (a) the equipotential curve runs outside the rectangle formed by XMIN, YMIN, XMAX, and YMAX, (b) it runs into the boundary S, (c) it becomes a closed curve, or (d) it becomes too difficult to plot. The last condition occurs most often near corners in the boundary S.

D. Errors in Input. LAPLACE can detect and act upon several types of errors in the input. The subroutine LAPLACE will stop if any of the following errors occur: (a) M does not satisfy $1 \leq M \leq 60$, (b) $\sum_{I=1}^M KV(I) > NMAX$, (c) for some $I = 1, 2, \dots, M$ $LT(I) < 1$ or $LT(I) > 3$, (d) $DC(I) < 0$ for some $I = 1, 2, \dots, M$, or (e) NEC does not satisfy $0 \leq NEC \leq 40$. These errors get a print indicating which of them has occurred. LAPLACE stops after the first error is encountered, and no further checks are made.

The most common error seems to be in tracing the boundary. If the end of one regular boundary section is not also the start of some regular boundary section (including, possibly, itself), then the regular boundary is not closed. A print tells that the error has occurred and gives the number of the boundary section where it occurred. As soon as this error is found, LAPLACE returns. Thus one can call the plot routine after the computer returns. The plot routine will usually show any discontinuity in the boundary quickly. When this error has occurred and one calls the plot routine, it will plot the boundary and branch cuts and then stop.

If one gives Dirichlet boundary conditions on two adjoining sections of the boundary, and the boundary values are not equal where the boundaries meet, this is an error. A print tells that the error has occurred and gives the number of one of the boundary sections. LAPLACE stops if this error occurs.

As mentioned earlier, the variables $KV(I)$, $I = 1, 2, \dots, M$ must all be odd integers greater than or equal to 3. If $KV(I) < 3$, it is replaced in the calculation by 3. If $KV(I)$ is even and greater than 3, it is replaced in the calculation by $KV(I) - 1$.

Problems which are actually two or more independent problems should never be run on LAPLACE. For instance, if one is solving Laplace's equation on two disjoint interior regions, the two problems must be run separately.

E. Generalized Boundary Data. The subroutine BDRY is used for generalized boundary input. Whenever the simple boundary input is the only type used, BDRY will show up as an unsatisfied external. This is a normal procedure. The generalized boundary input can be used in conjunction with the simple boundary input, or it can be used as the only source of boundary data input. To use the generalized boundary input option, the user has to write a subroutine which must be called BDRY.

Assume that one has a boundary section, S_I , on which one wants to have variable boundary values. Let S_I have the parametric representation

$$\{(x_I(t), y_I(t)) | t \in [0, d_I]\}$$

with respect to arc length. Here d_I is the length of the boundary section S_I . To keep the orientation correct for the regular boundary sections, the parametric representation must be such that as one traverses S_I with increasing t , the region remains on the left. An equivalent condition is that $(dy_I(t)/dt, -dx_I(t)/dt)$ forms the unit exterior normal vector. One will need to be able to compute $x_I(t)$, $y_I(t)$, $dx_I(t)/dt$, $dy_I(t)/dt$,

$\frac{1}{2} \left[\frac{dx_I(t)}{dt} \frac{d^2 y_I(t)}{dt^2} - \frac{dy_I(t)}{dt} \frac{d^2 x_I(t)}{dt^2} \right]$, and the appropriate boundary value as a function of t for $t \in [0, d_I]$.

BDRY has the formal parameters (I,T,X,Y,XP,YP,F,V). I and T are input to the routine BDRY. BDRY must give as output

$$\begin{aligned} X &= X_I(T) \\ Y &= Y_I(T) \\ XP &= \frac{dx_I(T)}{dT} \\ YP &= \frac{dy_I(T)}{dT} \\ V &= \frac{1}{2} \left[\frac{dx_I(T)}{dT} \frac{d^2 y_I(T)}{dT^2} - \frac{dy_I(T)}{dT} \frac{d^2 x_I(T)}{dT^2} \right], \end{aligned}$$

and the variable F must contain the appropriate

boundary value. The boundary value can be

$$U[X_I(T), Y_I(T)] \text{ or } \frac{\partial U[X_I(T), Y_I(T)]}{\partial n}.$$

To make the explanation clearer, consider the following example. Let G be the square centered at the origin with sides of length 2.

Fig. 4. Variable boundary data problem.

Let S_1 be the line segment from (1,-1) to (1,1); S_2 , that from (1,1) to (-1,1); S_3 , that from (-1,1) to (-1,-1); and S_4 , that from (-1,-1) to (1,-1).

Assume that the boundary conditions are:

$$\begin{aligned} u &= y \text{ on } S_1 \\ \frac{\partial u}{\partial n} &= 1 \text{ on } S_2 \\ u &= y \text{ on } S_3 \\ u &= -1 \text{ on } S_4. \end{aligned}$$

For the boundary sections S_2 and S_4 , the simple type of boundary input can be used. To generate the input for the boundary sections S_1 and S_3 , the subroutine BDRY must be used. One possible such routine is as follows:

```

SUBROUTINE BDRY(I,T,X,Y,XP,YP,F,V)
GOTO(1,3,2,3),I
1 X=1.
Y=-1.+T
XP=0.
YP=1.
F=Y
V=0.
RETURN
2 X=-1.
Y=1.-T
XP=0.
YP=-1.
F=Y
V=0.
RETURN
3 STOP
END

```

For this example, the determined input variables are as follows:

$$\left\{ \begin{array}{l} M = 4. \\ LT(1) = LT(3) = LT(4) = 1 \\ LT(2) = 2 \\ DC(1) = DC(3) = 2. \\ DC(2) = DC(4) = 0. \\ NBC = 0 \end{array} \right\}.$$

The other variables depend on the need for accuracy. Moreover, boundary data cards would be needed for sections 2 and 4.

In some cases it is difficult, if not impossible, to compute $x(t)$ and $y(t)$ explicitly, such as when the boundary curve is a parabola. However, in these cases one can usually get an implicit expression and use it to compute $x(t)$, $y(t)$ and the desired derivatives. The speed of BDRY is not too important, since the routine is called, at most, 500 times. All variables computed by BDRY are saved and, hence, need to be computed only once.

F. Superposition of Solutions. Another useful option is available. Suppose one wants to find a function $u(x,y)$ which has simple boundary conditions but does not satisfy all requirements for the general problem. Suppose, further, that one has a function $v(x,y)$ such that $u(x,y) - v(x,y)$ does satisfy the requirements for the general problem. Usually it is true that $u(x,y) - v(x,y)$ has involved boundary conditions, and one prefers not to set up problems with such conditions. Using the following option, one can give the boundary conditions for u as input if the function v and its first derivatives can be given as a function of x and y . The real function BC is used by LAPLACE to generate branch cuts. In this subroutine are two comment cards. After the first card, one must insert a statement,

$$\{BC = BC - V(X,Y) \quad \},$$

where $V(X,Y)$ is the function V evaluated at X and Y . After the second comment card, one must insert a statement,

$$\{BC = BC - \frac{\partial V(X,Y)}{\partial X} * YP + \frac{\partial V(X,Y)}{\partial Y} * XP\}.$$

The subroutine LAPLACE and its related subroutines automatically subtract $V(X,Y)$ from all input boundary data so that LAPLACE can solve for $u(x,y) - v(x,y)$

and then automatically add $V(X,Y)$ to all output to give an approximate solution to $u(x,y)$.

The following examples show how this option can be used. **Example 1.** Suppose one wants to solve for $u(x,y)$ satisfying Laplace's equation exterior to the unit circle with $u(x,y) \rightarrow x + y$ as $(x,y) \rightarrow \infty$, and $\partial u / \partial n = 0$ on the exterior boundary of the unit circle. In this case, $V = X + Y$. The two statements needed for BC are

$$\left\{ \begin{array}{l} BC = BC - X - Y \\ BC = BC - YP + XP \end{array} \right\}.$$

The other input for this problem is the same as if the function $V(X,Y)$ had not been introduced. **Example 2.** Suppose one now wants to solve for $u(x,y)$ in the interior of the unit circle satisfying $\Delta u = 4$ with $u = 1$ on the boundary. In this example, many choices of V will suffice. Choose $V = 2X^2$. The statements needed for BC are

$$\left\{ \begin{array}{l} BC = BC - 2.*X*X \\ BC = BC - 4.*X*YP \end{array} \right\}.$$

The other input is just the input for the function u .

G. Limitations on the Code. The limitations on LAPLACE are usually obvious if one studies the method used for solving Laplace's equation. When one has an exterior problem and is very far from the boundary (say 10^8 times the length of the boundary), the solution tends to break down. However, other methods of approximation are usually good at these distances. The absolute error from LAPLACE for fixed input is usually a function of the distance from the boundary. Thus, the relative error can become large at points where the true solution is zero.

The most important source of error seems to occur at the boundary. Both the subroutine GRADFN and the function FN break down near the boundary except when the boundary is linear in the region of interest.

If H is the distance between approximation points on a given boundary section, then both FN and GRADFN start to break down at about $1.5H$ from the boundary. The error varies with the ratio of H to the radius of curvature of the boundary. In some cases, as much as half of the significant figures have been lost at the boundary. The same type of error causes trouble whenever two different sections of the boundary are close relative to the distance between approxi-

mation points on either of them. As in the case above, this is not true when the boundary is linear. In any case, cusps are not feasible with the present program.

It should be noted that for some regions a change of scale is needed. This is true when the integral equations, to be discussed later, do not have a unique solution, a very rare occurrence.

Finally, it should be noted that the author has been unable to show mathematically, except in special cases, that the method described here will give an approximation to the true solution of the stated general problem. However, so far the only real limitations have been due to programming.

H. Timing. Probably something should be said about timing. If

$$N_1 = \left(\sum_{i=1}^M KV_i \right) / 100,$$

then to compute the complementary boundary values requires about $2.4N_1^3 + 1.3N_1^2$ seconds. To compute the value of the approximate potential at a given point requires about $0.013N_1$ second, and to compute the approximate gradient of the potential at a given point requires about $0.015N_1$ second. The time required to compute an equipotential curve is more difficult to determine. Usually only one approximate gradient and one approximate potential are needed for each step of length HH in a given equipotential curve. Thus, if one knows the length of a given equipotential curve, he can approximate the time required to plot the curve.

IV. STATEMENT OF GENERAL PROBLEM IN THE FORM OF INTEGRAL EQUATIONS

The handling of input data will not be discussed here; we assume that the data are in usable form. Moreover, the computation of derivatives of the approximation solution is similar to the computation of the approximate solution itself since one merely differentiates Eq. (1) under the integral. Hence, the computation of derivatives will not be explained. Also, only the regular boundary will be treated here, since the treatment of thin plates is similar. Finally, the exterior problem will not be discussed because in its treatment it differs in only one respect from the interior problem, the difference being that the exterior Neumann problem is uniquely determined because of the restriction $u(x,y) \rightarrow 0$ as

$$(x,y) \rightarrow \infty.$$

Using Eq. (1), the general problem reduces to the one-dimensional problem of solving for u on S_N and $\partial u / \partial v$ on S_D . If S is assumed to be one or a finite number of smooth closed curves, then Eq. (1) becomes, on the boundary,

$$u(x,y) = \frac{1}{\pi} \int_S \left[\ln\left(\frac{1}{r}\right) \frac{\partial u}{\partial v} - u \frac{\partial}{\partial v} \ln\left(\frac{1}{r}\right) \right] ds \quad (x,y) \in S \quad (2)$$

For the case in which S is only piecewise smooth, see Section VII. We assume that Eq. (2) can be inverted to solve for $\partial u / \partial v$ on S_D and u on S_N .

To explain how Eq. (2) is to be used, some notation must be developed. Assume that S has the parametric representation $\{(x(t), y(t)) | t \in (0, d)\}$ with respect to arc length. Here d is the length of S . For the parametric representation, it must be true that $(dy(t)/dt, -dx(t)/dt)$ is the unit normal vector pointing exterior to G . Define $r(s,t) = \{[x(s) - x(t)]^2 + [y(s) - y(t)]^2\}^{1/2}$. Now set $u(s) = u[x(s), y(s)]$; $\partial u(s) / \partial v = \partial u[x(s), y(s)] / \partial v$; and $K(s,t) = -1/\pi \ln[r(s,t)]$.

Using the above notation, Eq. (2) becomes

$$u(t) = \int_S \left[\frac{\partial u(s)}{\partial v} K(s,t) - u(s) \frac{\partial}{\partial v} K(s,t) \right] ds. \quad (3)$$

Now set

$$E(t) = \begin{cases} \int_{S_N} g(s) K(s,t) ds - \int_{S_D} f(s) \frac{\partial}{\partial v} K(s,t) ds - f(t) & \text{if } t \in S_D \\ \int_{S_N} g(s) K(s,t) ds - \int_{S_D} f(s) \frac{\partial}{\partial v} K(s,t) ds & \text{if } t \in S_N. \end{cases}$$

See the general problem for an explanation of $f(s)$ and $g(s)$. Assuming that Eq. (3) can be inverted to solve for $u(t)$ on S_N and $\partial u(t) / \partial v$ on S_D , it follows that one must solve the pair of coupled integral equations:

$$-\int_{S \in S_D} \frac{\partial u(s)}{\partial v} K(s,t) ds + u(t) + \int_{S \in S_N} u(s) \frac{\partial}{\partial v} K(s,t) ds = E(t) \quad t \in S_N$$

and

$$-\int_{S \in S_D} \frac{\partial u(s)}{\partial v} K(s,t) ds + \int_{S \in S_N} u(s) \frac{\partial}{\partial v} K(s,t) ds = R(t) \quad t \in S_D. \quad (4)$$

V. APPROXIMATION OF INTEGRAL EQUATIONS BY MATRIX EQUATIONS

If there exist solutions $u(s)$ and $\partial u(s)/\partial v$ to the system (4), then Eq. (1) gives the desired solution $u(x,y)$ to the stated problem. However, the system (4) is in the general case not uniquely solvable. Moreover, in some problems, to solve system (4) one needs to solve a Fredholm integral equation of the first kind. These important mathematical problems will be ignored here.

When S_D is empty, the general problem is a Neumann problem, uniquely determined up to an additive constant. If G is an interior region and the general problem is a Neumann problem, the boundary condition, $u(s)|_{s=0} = 0$, is arbitrarily imposed to make the solution unique. On pp. 183-184 of Sternberg and Smith,³ a uniqueness proof is given for the interior Neumann problem. Under our restrictions, this proof carries over to the general problem. Thus, by assumption, the general problem has a unique solution.

Set

$$K_0(s,t) = \begin{cases} -K(s,t) & s \in S_D \\ \frac{\partial}{\partial v} K(s,t) & s \in S_D \end{cases}$$

and

$$\tau(s) = \begin{cases} \frac{\partial u(s)}{\partial v} & s \in S_D \\ u(s) & s \in S_N \end{cases}$$

The system (4) then becomes

$$\begin{aligned} \tau(t) + \int_S K_0(s,t) \tau(s) ds &= E(t) \quad t \in S_N \\ \int_S K_0(s,t) \tau(s) ds &= E(t) \quad t \in S_D \end{aligned} \quad (5)$$

$\tau(s)$ and $K_0(s,t)$ are not, in general, well-defined at certain points of S . This will be discussed later. The fact that $\tau(s)$ is not well-defined everywhere on S is a constant source of difficulty in discussing the problem. $\tau(s)$; $f(s)$, and $g(s)$ are approximated piecewise by second-degree polynomials. Hence, these functions should have at least piecewise continuous second derivatives. At present, S is assumed to con-

sist of two parts, S_D and S_N . Now the boundary S must be partitioned into sections S_j ($j=1,2,\dots,M$) such that: (a) each section S_j is a smooth curve, i.e., S_j has no corners and is connected; (b) each S_j is contained entirely in either S_D or S_N ; and (c) $f(s)$ or $g(s)$, whichever is defined on S_j , has continuous second derivatives interior to S_j . In general, each S_j should be as large as possible while still satisfying the above conditions.

At the endpoints of each section S_j ($j=1,2,\dots,M$) there is a possibility that $\tau(s)$ is not well-defined. This is so because there is a possible change in the type of boundary condition. Also, if there is a corner, the Neumann derivative need not be continuous around the corner. At the endpoints of each of the boundary sections, an upper and lower limit will be defined. Let

$$\tau(s+) = \lim_{\substack{\delta \rightarrow 0 \\ \delta > 0}} \tau(s+\delta)$$

$$\tau(s-) = \lim_{\substack{\delta \rightarrow 0 \\ \delta < 0}} \tau(s+\delta)$$

Because of the assumptions concerning the S_j 's and $\tau(s)$, both limits exist for $s \in (0,d]$. Interior to each of the S_j 's, the upper and lower limits are equal. Similar definitions could be made for $K_0(s,t)$, but because $K_0(s,t)$ is always integrated one has no difficulties with undefined points.

$\tau(s)$ is approximated by using KV_j points on the section S_j for $j = 1, 2, \dots, M$. Because of the type of approximation used, each KV_j must be odd and

greater than one. Set $n = \sum_{j=1}^M KV_j$ and let DC_j be the length of S_j for $j = 1, 2, \dots, M$. Define $h_j = DC_j/(KV_j-1)$ for $j = 1, 2, \dots, M$. Next, arguments t_k ($k=1,2,\dots,n$) for the function $\tau(s)$ are defined. Since $\tau(s)$ will be approximated at the ends of the boundary sections, upper and lower limits must be used in some cases. Set

$$t_k = \begin{cases} \left(\sum_{i=1}^j DC_i \right) - & \text{if } k = \sum_{i=1}^j KV_i \text{ for some } j=1,2,\dots,m \\ \left(\sum_{i=1}^{j-1} DC_i \right) + & \text{if } k = 1 + \sum_{i=1}^{j-1} KV_i \text{ for some } j=1,2,\dots,m \\ \left(\sum_{i=1}^{j-1} DC_i \right) + \left(k+1 - \sum_{i=1}^{j-1} KV_i \right) h_j & \text{if } k \in \left(\sum_{i=1}^{j-1} KV_i, \sum_{i=1}^j KV_i \right) \\ & \text{for some } j=1,2,\dots,m. \end{cases}$$

Thus, looking at the parametric representation for S , one sees that for each section S_j there are two values of t_k corresponding to the endpoints of S_j . One value of t_k is a lower limit, and the other is an upper limit. There are $KV_j - 2$ other values of t_k equally spaced at a distance h_j along the boundary section.

Let τ^0 be the vector $(\tau_1, \tau_2, \dots, \tau_n)^t$. Each component τ_k is defined by $\tau_k = \tau(t_k)$ for $k = 1, 2, \dots, n$.

Using the vector τ^0 , $\tau(s)$ is approximated on $(0, d]$ by a piecewise polynomial of degree two called $\tau_A(s)$. On the interval (t_1, t_3) , $\tau_A(s)$ is the polynomial of degree two having $\tau_A(t_1) = \tau_1$, $\tau_A(t_2) = \tau_2$, and $\tau_A(t_3) = \tau_3$. On (t_3, t_5) , $\tau_A(s)$ is the polynomial of degree two having $\tau_A(t_3) = \tau_3$, $\tau_A(t_4) = \tau_4$, and $\tau_A(t_5) = \tau_5$. The approximation continues in this manner until the point t_{KV_1} is reached. t_{KV_1} corresponds to the end of the first boundary section, and $\tau(s)$ is possibly discontinuous at this point. Therefore the upper limit $t_{KV_1} + 1$ is used on the next interval. This type of approximation is continued through $(0, d]$.

Define $j(k)$ to be that integer-valued function such that

$$\sum_{i=1}^{j(k)-1} KV_i < k \leq \sum_{i=1}^{j(k)} KV_i. \quad \tau_A(s) \text{ can then be defined by the relation}$$

$$\tau_A(s) = \begin{cases} \frac{[\tau_{k+2}(s-t_k)(s-t_k-h_{j(k)}) - 2\tau_{k+1}(s-t_k)(s-t_k-2h_{j(k)}) + \tau_k(s-t_k-h_{j(k)})(s-t_k-2h_{j(k)})]}{2h_{j(k)}^2} \\ \text{if } se(t_k, t_{k+2}) \text{ and } \left(k - \sum_{i=1}^{j(k)-1} KV_i \right) \text{ is odd.} \\ \tau_k \text{ if } s = t_k \text{ for some } k = 1, 2, \dots, n. \end{cases}$$

Notice that

$$\tau_A(s) = \sum_{k=1}^n \theta_k(s) \tau_k, \quad (6)$$

where each $\theta_k(s)$ is a piecewise polynomial of degree two which is nonzero only in some interval about t_k . An exact definition of $\theta_k(s)$ is as follows:

$$\theta_k(s) = \frac{1}{2h_{j(k)}^2} \begin{cases} (s-t_{k+1})(s-t_{k+2}) \text{ if } k - \sum_{i=1}^{j(k)-1} KV_i \\ \text{is odd and less than } KV_{j(k)}, \\ \text{and } se[t_k, t_{k+2}]. \\ (s-t_{k-1})(s-t_{k-2}) \text{ if } k - \sum_{i=1}^{j(k)-1} KV_i \\ \text{is odd and greater than one,} \\ \text{and } se[t_{k-2}, t_k]. \\ 2(s-t_{k-1})(s-t_{k+1}) \text{ if } k - \sum_{i=1}^{j(k)-1} KV_i \\ \text{is even, and } se[t_{k-1}, t_{k+1}], \\ 0 \text{ otherwise.} \end{cases}$$

To explain some of the above notation, consider the following example. Suppose S is the unit circle with center at the origin. Let $(x(t), y(t)) = (\cos(t), \sin(t))$, $te(0, d]$ be the parametric representation for S . Assume that $S_1 = \{(x(t), y(t)) | te(0, \pi/2]\}$ and $S_2 = \{(x(t), y(t)) | te(\pi/2, 2\pi]\}$. Obviously, $M = 2$, $DC_1 = \pi/2$, and $DC_2 = 3\pi/2$. Assume that $KV_1 = 3$ and $KV_2 = 5$. Immediately, one has $n = 8$, $h_1 = \pi/4$, and $h_2 = 3\pi/8$. The locations of the arguments t_k , $k = 1, 2, \dots, 8$ are shown in Fig. 5.

Fig. 5. Distribution of approximation points on the boundary.

Notice that if the Dirichlet (or Neumann) boundary condition were given on all of S and if $f(s)$, or $g(s)$, were continuous, then it would be true that $\tau_3 = \tau_4$ and $\tau_1 = \tau_8$. However, if the Dirichlet condition were given on one section and the Neumann condition on the other, then the two pairs would not, in general, be equal.

In what follows, the term $O(h^3)$ denotes a scalar or vector quantity. In the scalar notation, it means a term bounded by some constant times h^3 . In the vector notation, it denotes a vector with each component bounded by a constant times h^3 . By assumption, $\tau(s)$ has continuous second derivatives on each of the sections S_j . Hence,

$$\tau(s) = \tau_A(s) + O(h^3); \quad (7)$$

where $h = \max(h_1, h_2, \dots, h_M)$. Thus, using this substitution in Eq. (5), one gets

$$\begin{aligned} \tau_A(t) + O(h^3) + \int_S [\tau_A(s) + O(h^3)] K_O(s, t) ds &= E(t) \\ &\quad t \in S_N \\ \int_S [\tau_A(s) + O(h^3)] K_O(s, t) ds &= E(t) \\ &\quad t \in S_D. \end{aligned} \quad (8)$$

For $t \in S_N$, it will be shown later that $K_O(s, t)$ is a continuous function with a bound dependent only on S . For $t \in S_D$, $K_O(s, t)$ has, at most, a logarithmic singularity which is also somewhat dependent on S . Thus, one can say

$$\int_S O(h^3) K_O(s, t) ds = O(h^3) \quad t \in S.$$

Hence, it follows that

$$\begin{aligned} \tau_A(t) + \int_S \tau_A(s) K_O(s, t) ds &= E(t) + O(h^3) \quad t \in S_N \\ \int_S \tau_A(s) K_O(s, t) ds &= E(t) + O(h^3) \quad t \in S_D. \end{aligned} \quad (9)$$

Now, using the representation (6) given for $\tau_A(s)$, one gets

$$\begin{aligned} \tau_A(t) + \sum_{k=1}^n \tau_k \int_S \theta_k(s) K_O(s, t) ds &= E(t) + O(h^3) \quad t \in S_N \\ \sum_{k=1}^n \tau_k \int_S \theta_k(s) K_O(s, t) ds &= E(t) + O(h^3) \quad t \in S_D. \end{aligned} \quad (10)$$

Define $E_i = E(t_i)$ and $E = (E_1, E_2, \dots, E_n)^T$. If Eqs. (10) are required to be true only at $t = t_i$ for $i = 1, 2, \dots, n$, then one gets a system of n equations. One has for $i = 1, 2, \dots, n$

$$\begin{aligned} \tau_i + \sum_{k=1}^n \tau_k \int_S \theta_k(s) K_O(s, t_i) ds &= E_i + O(h^3) \quad t_i \in S_N \\ \sum_{k=1}^n \tau_k \int_S \theta_k(s) K_O(s, t_i) ds &= E_i + O(h^3) \quad t_i \in S_D \end{aligned} \quad (11)$$

Later we will need to refer to specific equations of system (11). For $i = 1, 2, \dots, n$, Eq. (11-i) will be the equation in system (11) corresponding to $t = t_i$. Set

$$a_{ik} = \begin{cases} 1 + \int_S \theta_k(s) K_O(s, t) ds & \text{if } t_i \in S_N \text{ and } k=i \\ \int_S \theta_k(s) K_O(s, t) ds, & \text{otherwise,} \end{cases} \quad (12)$$

and set $A = (a_{ik})$. Using these definitions, the system of Eqs. (11) can be written in the form

$$A\tau^0 = E + O(h^3). \quad (13)$$

Notice that all coefficients of A and E can be computed numerically from a knowledge of S . However, A often does not have an inverse. This problem and

its remedy will be discussed in the next section. It suffices to say here that one can modify A and E slightly to get a matrix A^0 and vector E^0 satisfying the same relation; i.e.,

$$A^0 \tau^0 = E^0 + O(h^3). \quad (14)$$

Now A^0 usually has an inverse, or else a change of scale will give a matrix A^0 which has an inverse. If $(A^0)^{-1}$ does exist, then, solving the equation

$$A^0 z = E^0$$

for $z = (z_1, z_2, \dots, z_n)^t$, one has

$$z - \tau^0 = (A^0)^{-1} O(h^3). \quad (15)$$

VI. REDUCTION OF MATRIX A TO A^0

We will show how the matrix A , which is often singular, can be reduced to a matrix A^0 which usually can be inverted.

At the endpoints of each of the boundary sections $S_j (j=1, 2, \dots, M)$, two ends of curves are joined. The curves may, or may not, join at a corner, and may, or may not, have the same boundary conditions. These conditions determine how the matrix A is reduced. For each joined pair of endpoints of boundary sections, one must determine which of the following possibilities is true.

(a) One curve has the Dirichlet boundary given, and the other has the Neumann boundary condition given.

(b) The Dirichlet boundary condition is given on both curves, and there is a corner where the curves join.

(c) The last possibility includes the complement of (a) and (b). In this case both curves have the Neumann boundary condition, or both have the Dirichlet condition, and there is no corner where the curves join. For ease of explanation, assume that curve S_j joins curve S_k and that the corresponding values of t are t_{j1} and t_{k1} . Moreover, assume that t_{j1} is the start of S_j . In what follows, assume that $u(x, y)$ is continuous on G and S , and that $\partial u / \partial n$ is continuous at all points of S which are not corners.

For case (a), the change is relatively easy. Assume that the Dirichlet condition is given on S_j . Obviously, then, $\tau(t_{k1}) = f(t_{j1})$, and, thus,

$\tau_{k1} = f(t_{j1})$. Hence Eq. (11-k1) can be replaced by

$$\tau_{k1} = f(t_{j1}).$$

This change does not have to be made. However, the equation is exact, and the one it replaces was an approximation.

In case (b) it is relatively easy to see that Eq. (11-k1) is the same as Eq. (11-j1). Hence A must certainly be singular. To correct this problem one need only replace Eqs. (11-j1) and (11-k1) by the second equation in system (10) evaluated at $t = (t_{j1} + t_{j1+1})/2$ and $t = (t_{k1} + t_{k1-1})/2$.

For case (c) it is obvious that $\tau(j1) = \tau(k1)$ which implies that $\tau_{j1} = \tau_{k1}$. If both curves have the Dirichlet condition given, then Eqs. (11-j1) and (11-k1) are the same. If both curves have the Neumann condition given, then the equations are the same except that the coefficients for τ_{j1} and τ_{k1} are interchanged. In either case Eq. (11-k1) can be replaced by

$$\tau_{k1} - \tau_{j1} = 0.$$

If both curves have the Neumann condition given, then the change does not have to be made.

VII. GENERALIZED FORM OF EQUATION (2)

In Section IV it was noted that Eq. (1) tends to Eq. (2) on the boundary whenever the boundary S is smooth. In what follows, Eq. (2) will be considered for those cases when S has corners. First, note that the proof of Eq. (2) given in Ref. 4 on pp. 316-318 carries over immediately to any point of S which is not a corner. Thus Eq. (2) is true at any such point.

Next, the proof in Ref. 4 is generalized to the case in which (x, y) is a corner point of S . See Fig. 6.

Fig. 6. Region G with corner.

Assume that S has a corner at the point $(x_0, y_0) = (x(t_0), y(t_0))$. Assume also that at the point (x_0, y_0) where the curves join together, the angle interior to G is α . Draw a circle of radius $\epsilon < 1$ about the point (x_0, y_0) . Let K_ϵ be the intersection of the circle with G , and let Σ_ϵ be the boundary of K_ϵ interior to G . Finally, let S_ϵ be the boundary of S which is common to the boundary of K_ϵ .

Now apply Green's second theorem, i.e.,

$$\iint_G (u \Delta v - v \Delta u) dG = \int_S \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS,$$

to the function $u(x, y)$ satisfying the stated general problem and $v = \ln(1/r) = -\frac{1}{2} \ln[(x-x_0)^2 + (y-y_0)^2]$. The theorem is applied to the region $G - K_\epsilon$. The double integral is zero since both u and v satisfy Laplace's equation in $G - K_\epsilon$. One gets

$$\int_{(S-S_\epsilon) \cup \Sigma_\epsilon} \left[u \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) - \ln\left(\frac{1}{r}\right) \frac{\partial u}{\partial \nu} \right] dS = 0$$

or

$$\begin{aligned} \int_{S-S_\epsilon} \left[u \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) - \ln\left(\frac{1}{r}\right) \frac{\partial u}{\partial \nu} \right] dS \\ = - \int_{\Sigma_\epsilon} \left[u \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) - \ln\left(\frac{1}{r}\right) \frac{\partial u}{\partial \nu} \right] dS. \end{aligned}$$

In the next section it is shown that $\partial/\partial \nu [\ln(1/r)]$ has a bound dependent only on S . Hence,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{S-S_\epsilon} \left[u \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) - \ln\left(\frac{1}{r}\right) \frac{\partial u}{\partial \nu} \right] dS \\ = \int_S \left[u \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) - \ln\left(\frac{1}{r}\right) \frac{\partial u}{\partial \nu} \right] dS. \end{aligned}$$

Moreover,

$$\left| \int_{\Sigma_\epsilon} \ln\left(\frac{1}{r}\right) \frac{\partial u}{\partial \nu} dS \right| = \left| \ln\left(\frac{1}{\epsilon}\right) \int_{\Sigma_\epsilon} \frac{\partial u}{\partial \nu} dS \right| \leq Z \epsilon \ln\left(\frac{1}{\epsilon}\right),$$

where Z bounds $|\partial u / \partial \nu|$. Thus it follows that

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma_\epsilon} \ln\left(\frac{1}{r}\right) \frac{\partial u}{\partial \nu} dS = 0.$$

On Σ_ϵ , it is true that $\partial/\partial \nu = -\partial/\partial r$. Hence,

$$\int_{\Sigma_\epsilon} u \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) dS = \int_{\Sigma_\epsilon} u \left(\frac{1}{r}\right) dS = \frac{1}{\epsilon} \int_{\Sigma_\epsilon} u dS.$$

Since $u(x, y)$ is continuous in G , and S is smooth except at the corner, one has

$$\int_{\Sigma_\epsilon} u dS = \alpha(\epsilon) u(\xi(\epsilon), \eta(\epsilon)),$$

where $(\xi(\epsilon), \eta(\epsilon)) \in \Sigma_\epsilon$ and $\alpha(\epsilon) \rightarrow \alpha \cdot \epsilon$ as $\epsilon \rightarrow 0$. It follows that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{\Sigma_\epsilon} u \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) dS &= - \lim_{\epsilon \rightarrow 0} \frac{\alpha(\epsilon) u(\xi(\epsilon), \eta(\epsilon))}{\epsilon} \\ &= -\alpha u(x_0, y_0). \end{aligned}$$

Hence, the generalized form of Eq. (2) turns out to be

$$\alpha u(x, y) = \int_S \left[\frac{\partial u}{\partial \nu} \ln\left(\frac{1}{r}\right) - u \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) \right] dS. \quad (2\text{-generalized})$$

Here α is the angle at the corner the curve S makes with itself inside G . A more exact definition of α is

$$\alpha = \pi - \arg \left\{ \lim_{\substack{\delta \rightarrow 0 \\ \delta > 0}} \frac{[x'(t_0 + \delta) + iy'(t_0 + \delta)]}{[y'(t_0 - \delta) + iy'(t_0 - \delta)]} \right\},$$

where $i = \sqrt{-1}$ and $-\pi < \arg(\cdot) \leq \pi$.

VIII. APPROXIMATION OF THE MATRIX A AND VECTOR E

For both the known and unknown boundary values, LAPLACE uses the approximations

$$u(s) \approx u(s)_A = \sum_{k=1}^n u(t_k) \theta_k(s)$$

and

(16)

$$\frac{\partial u(s)}{\partial \nu} \approx \left(\frac{\partial u(s)}{\partial \nu} \right)_A = \sum_{k=1}^n \frac{\partial u(t_k)}{\partial \nu} \theta_k(s),$$

where $\theta_k(s)$ is defined in Section V. The basic equations used in the definition of the matrix A and the vector E , and even in subsequent evaluations of the approximate solution, are Eq. (1) and the generalized Eq. (2). Thus, by examining these equations one sees that all integral evaluations in LAPLACE can be reduced to approximations of the following pair of integrals:

$$\int_S \theta_k(s) \ln\left(\frac{1}{r}\right) dS$$

and

$$\int_S \theta_k(s) \frac{\partial}{\partial v} \ln \left(\frac{1}{r} \right) ds. \quad (17)$$

Here $r = r(s, x, y) = \{[x(s) - x]^2 + [y(s) - y]^2\}^{\frac{1}{2}}$. To evaluate the approximate solution, the pair (x, y) must be allowed to range over all G and S . For evaluation of A and E , the pair (x, y) is restricted to the values $(x(t_k), y(t_k))$, $k = 1, 2, \dots, n$.

Three different methods are used to approximate the integrals in Eq. (17). To explain the three methods, some more notation is needed. Let $T_k = \{s | \theta_k(s) \neq 0\}$, and let $h(s) = h_j$ where

$$\sum_{i=1}^{j-1} DC_i < s \leq \sum_{i=1}^j DC_i.$$

Fix k , x , and y . The three different methods correspond approximately to the following three cases:

- (a) for $s \in T_k$ $r(s, x, y) > 7h(s)$
- (b) for $s \in T_k$ $0 < r(s, x, y) < 7h(s)$
- (c) for some $s \in T_k$ $r(s, x, y) = 0$.

For case (a), the functions $\partial/\partial v \ln[r(s, x, y)]$ and $\ln[r(s, x, y)]$ are relatively smooth when thought of as functions of the single variable, s , over the region T_k . The integrals in Eq. (17) are approximated by using a Newton-Cotes formula. Assume for ease of explanation that t_k is the start of the boundary section $S_{j(k)}$. Remember that $j(k)$ is the integer such that

$$\sum_{i=1}^{j(k)-1} KV_i < k \leq \sum_{i=1}^{j(k)} KV_i.$$

Looking at the first integral in Eq. (17), one has

$$\begin{aligned} & \int_S \theta_k(s) \ln \left[\frac{1}{r(s, x, y)} \right] ds \\ &= \int_{t_k}^{t_{k+2}} (s - t_{k+1})(s - t_{k+2}) \ln \left[\frac{1}{r(s, x, y)} \right] ds / 2h_{j(k)}^2 \\ &= \int_0^{2h_{j(k)}} (s - h_{j(k)})(s - 2h_{j(k)}) \ln \left[\frac{1}{r(s + t_k, x, y)} \right] ds / 2h_{j(k)}^2. \end{aligned} \quad (18)$$

For $i = 0, 1, 2, 3, 4$ let $\xi_i = -\ln[r[h_{j(k)} + t_k, x, y]]$. Using Eq. 3.5.13 on p. 73 of Ref. 5, one gets immediately

$$\begin{aligned} & \int_0^{2h_{j(k)}} [s - h_{j(k)}][s - 2h_{j(k)}] \ln \left[\frac{1}{r(s + t_k, x, y)} \right] ds \\ &\approx \frac{h_{j(k)}}{45} [7(1)\xi_0 + 32(\frac{3}{8})\xi_1 + 12(0)\xi_2 + 32(-\frac{1}{8})\xi_3 + 7(0)\xi_4] \\ &= \frac{h_{j(k)}}{45} (7\xi_0 + 12\xi_1 - 4\xi_3). \end{aligned}$$

The second integral in Eq. (17) is treated the same way. Setting

$$\eta_i = \frac{\partial}{\partial v} \ln \left\{ \frac{1}{r[h_{j(k)}/2 + t_k, x, y]} \right\} \text{ for } 0, 1, 2, 3, 4,$$

one has

$$\int_S \theta_k(s) \frac{\partial}{\partial v} \ln \left[\frac{1}{r(s, x, y)} \right] ds = \frac{h_{j(k)}}{45} (7\eta_0 + 12\eta_1 - 4\eta_3).$$

When $\theta_k(s)$ is more involved, one has only to sum two integrals similar to those above.

For cases (b) and (c), the approximation of the integrals in Eq. (17) is more difficult. By looking at the definition of $\theta_k(s)$ one can show that to compute the integrals in Eq. (17) it is sufficient to be able to compute

$$\begin{aligned} N(m, i) &= \int_{t_m}^{t_{m+1}} (s - t_m)^i \ln \left[\frac{1}{r(s, x, y)} \right] ds \\ D(m, i) &= \int_{t_m}^{t_{m+1}} (s - t_m)^i \frac{\partial}{\partial v} \left[\frac{1}{r(s, x, y)} \right] ds \end{aligned} \quad (19)$$

for $m = 1, 2, \dots, n-1$, and $i = 0, 1, 2$. To show how this is done, the first integral in Eq. (17) will be constructed for the case in which t_k is the start of boundary section $S_{j(k)}$. From the definition of $\theta_k(s)$, it follows that

$$\theta_k(s) = \begin{cases} [s - t_k - h_{j(k)}][s - t_k - 2h_{j(k)}] / 2h_{j(k)}^2 & \text{for } s \in [t_k, t_{k+2}] \\ 0 & \text{otherwise.} \end{cases}$$

Hence, on $[t_k, t_{k+1}]$ it is true that

$$\begin{aligned} \theta_k(s) &= [(s - t_k) - h_{j(k)}][(s - t_k) - 2h_{j(k)}] / 2h_{j(k)}^2 \\ &= 1 - \frac{3(s - t_k)}{2h_{j(k)}} + \frac{(s - t_k)^2}{2h_{j(k)}^2}. \end{aligned}$$

On $[t_{k+1}, t_{k+2}]$, one has

$$\begin{aligned}\theta_k(s) &= (s-t_{k+1})((s-t_{k+1})-h_{j(k)})/2h_{j(k)}^2 \\ &= \frac{(s-t_{k+1})^2}{2h_{j(k)}^2} - \frac{(s-t_{k+1})}{2h_{j(k)}}\end{aligned}$$

Hence

$$\begin{aligned}& \int_S \theta_k(s) \ln \left[\frac{1}{r(s,x,y)} \right] ds \\ &= \int_{t_k}^{t_{k+1}} \theta_k(s) \ln \left[\frac{1}{r(s,x,y)} \right] ds + \int_{t_{k+1}}^{t_{k+2}} \theta_k(s) \ln \left[\frac{1}{r(s,x,y)} \right] ds \\ &= \left[N(k,0) - \frac{3N(k,1)}{2h_{j(k)}^2} + \frac{N(k,2)}{2h_{j(k)}^2} \right] + \left[\frac{N(k+1,2)}{2h_{j(k)}^2} - \frac{N(k+1,1)}{2h_{j(k)}^2} \right] \\ &= N(k,0) - \frac{[3N(k,1)+N(k+1,1)]}{2h_{j(k)}^2} + \frac{[N(k,2)+N(k+1,2)]}{2h_{j(k)}^2}\end{aligned}$$

In the same way one can show that

$$\begin{aligned}& \int_S \theta_k(s) \frac{\partial}{\partial v} \ln \left[\frac{1}{r(s,x,y)} \right] ds \\ &= D(k,0) - \frac{[3D(k,1)+D(k+1,1)]}{2h_{j(k)}^2} + \frac{[D(k,2)+D(k+1,2)]}{2h_{j(k)}^2}\end{aligned}$$

Thus, only the approximation of the integrals (19) will be discussed. As a point of interest, it should be noted that $D(m,0)$ can always be computed explicitly. However, LAPLACE does not use this fact. Using the definition of $\partial/\partial v$ and $r(s,x,y)$, it follows that

$$\begin{aligned}D(m,i) &= \int_{t_m}^{t_{m+1}} (s-t_m)^i \frac{\partial r(s,x,y)}{r(s,x,y)} ds \\ &= \int_{t_m}^{t_{m+1}} (s-t_m)^i \frac{\left\{ [x(s)-x] \frac{dy(s)}{ds} - [y(s)-y] \frac{dx(s)}{ds} \right\}}{r^2(s,x,y)} ds\end{aligned}$$

In what immediately follows, the discussion is restricted to case (b). For (x,y) fixed and $s \in [t_k, t_{k+1}]$, the functions $\{[x(s)-x]y'(s) - [y(s)-y]x'(s)\}$ and $r^2(s,x,y)$ will be approximated by polynomials of degree two. Set

$$p(s-t_k) = as^2 + bs + c \approx r^2(s,x,y)$$

$$q(s-t_k) = \alpha s^2 + \beta s + \gamma \approx \{[x(s)-x]y'(s) - [y(s)-y]x'(s)\}$$

and choose a, b, c, α, β , and γ such that the above approximations are true at $s = t_k, t_{k+1}$, and $(t_{k+1}+t_k)/2$. Define $D_A(k,i)$ and $N_A(k,i)$ to be the approximations to the integrals $D(k,i)$ and $N(k,i)$ resulting from the use of the polynomials $p(s)$ and $q(s)$. One then has

$$\begin{aligned}N(k,i) &\approx N_A(k,i) = - \int_{t_k}^{t_{k+1}} (s-t_k)^i \ln[p(s-t_k)] ds \\ &= - \int_0^{h_{j(k)}} s^i \ln[p(s)] ds\end{aligned}$$

and

$$\begin{aligned}D(k,i) &\approx D_A(k,i) = \int_{t_k}^{t_{k+1}} (s-t_k)^i \frac{q(s-t_k)}{p(s-t_k)} ds \\ &= \int_0^{h_{j(k)}} s^i \frac{q(s)}{p(s)} ds. \quad (20)\end{aligned}$$

Both integrals in Eq. (20) can be computed explicitly for any given $p(s)$ and $q(s)$. However, there are problems due to roundoff in evaluation of the integrals (20) whenever $|ah_{j(k)}^2/c|$ is less than 10^{-6} or both $|ah_{j(k)}^2/c|$ and $|bh_{j(k)}/c|$ are less than 10^{-4} . These two problem cases are treated specially in LAPLACE, but they seem to occur only in very special circumstances.

For case (c) it is true that $r(s,x,y) = 0$ for some $s \in [t_k, t_{k+1}]$. Because of our assumptions on the boundary S , it can be shown that when generating the matrix A and vector E , one has $r(s,x,y) = 0$ only when $s = t_i$ for some $i = 1, 2, \dots, n$. Thus, if $r(s,x,y) = 0$ for $s \in [t_k, t_{k+1}]$, it must be true that either $r(t_k, x, y) = 0$ or $r(t_{k+1}, x, y) = 0$. For ease of explanation, assume that $r(t_k, x, y) = 0$. From this assumption it follows that $r(s,x,y) > 0$ for $s \in (t_k, t_{k+1}]$.

The first integral in Eq. (19) is treated in somewhat the same way as in case (b). The only difference

is that $p(s)$ is chosen differently. Here it is assumed that $p(s) = s^2(as+b)$. Since $r(t_k, x, y) = 0$, it is true that $d/ds r^2(s, x, y)|_{s=t_k} = 0$. Hence, $p(s-t_k) = r^2(s, x, y)$ and $d/ds p(s-t_k) = d/ds r^2(s, x, y)$ at $s = t_k$. Now choose a and b so that $p(s-t_k) = r^2(s, x, y)$ at $s = (t_k + t_{k+1})/2$ and t_{k+1} . Using the approximation

$$N(k, i) \approx N_A(k, i) = -\frac{1}{2} \int_{t_k}^{t_{k+1}} (s-t_k)^i \ln[p(s-t_k)] ds$$

$$= -\frac{1}{2} \int_0^{h_j(k)} s^i \ln[p(s)] ds,$$

it follows that $N_A(k, i)$ can be integrated explicitly. However, there is a slight problem due to roundoff if $|ah_j(k)/b|$ is less than, say, 10^{-6} . In this case $\ln(as+b)$ can be expanded in a power series which converges quickly over the range of consideration.

The second integral is treated similarly to case (a). However, the integrand,

$$\frac{\partial}{\partial v} \ln \left[\frac{1}{r(s, x, y)} \right] = - \frac{[x(s)-x]y'(s)-[y(s)-y]x'(s)}{r^2(s, x, y)}$$

is not well-defined at $s = t_k$, since both the numerator and denominator are zero. Everywhere else on S it is bounded. L'Hospital's rule can be used to determine the limit as $s \rightarrow t_k$. Differentiating the numerator and denominator, one gets

$$- \frac{[x(s)-x]y''(s)-[y(s)-y]x''(s)+[x'(s)y'(s)]-y'(s)x'(s)}{2\{[x(s)-x]x'(s)+[y(s)-y]y'(s)\}}$$

$$= - \frac{1}{2} \frac{[x(s)-x]y''(s)-[y(s)-y]x''(s)}{[x(s)-x]x'(s)+[y(s)-y]y'(s)}.$$

This expression is also undefined at $s = t_k$. Differentiating the numerator and denominator again, one gets

$$- \frac{1}{2} \frac{x'(s)y''(s)-y'(s)x''(s)+[x(s)-x]y'''(s)-[y(s)-y]x'''(s)}{x'(s)^2+y'(s)^2+[x(s)-x]x''(s)+[y(s)-y]y''(s)}$$

$$= - \frac{1}{2} \frac{x'(s)y''(s)-y'(s)x''(s)+[x(s)-x]y'''(s)-[y(s)-y]x'''(s)}{1 + [x(s)-x]x''(s)+[y(s)-y]y''(s)}.$$

The relation $x'(s)^2 + y'(s)^2 = 1$ comes from the fact that $(x(s), y(s))$ is a parametric representation with

respect to arc length. The denominator is nonzero at $t = t_k$. Evaluating, one gets

$$- \frac{1}{2} [x'(t_k)y''(t_k)-y'(t_k)x''(t_k)].$$

Define now a polynomial $q(s) = as^2 + bs + v$ of degree two such that

$$q(0) = - \frac{1}{2} [x'(t_k)y''(t_k)-y'(t_k)x''(t_k)]$$

and

$$q(s-t_k) = \frac{\partial}{\partial v} \ln \left[\frac{1}{r(s, x, y)} \right] \text{ for } s = \frac{t_k + t_{k+1}}{2} \text{ and } t_{k+1}.$$

Then approximate

$$D(k, i) \approx D_A(k, i) = \int_{t_k}^{t_{k+1}} (s-t_k)^i q(s-t_k) ds = \int_0^{h_j(k)} s^i q(s) ds.$$

Obviously, this integral can be evaluated with no difficulty.

IX. ERROR ANALYSIS

A rigorous error analysis will have to wait until later. At present only the most obvious results are covered.

Let $z = (z_1, z_2, z_3, \dots, z_n)^T$ be the vector in Eq. (15). Define

$$z_A(s) = \sum_{k=1}^n \theta_k(s) z_k, \quad f_A(s) = \sum_{\substack{k \text{ with} \\ t_k \in S_D}} \theta_k(s) f(t_k),$$

and

$$g_A(s) = \sum_{\substack{k \text{ with} \\ t_k \in S_N}} \theta_k(s) g(t_k).$$

The approximate solution, $u_1(x, y)$, to the stated general problem is

$$u_1(x, y) = \frac{1}{2\pi} \left[\int_{S_N} g_A(s) \ln\left(\frac{1}{r}\right) ds + \int_{S_D} z_A(s) \ln\left(\frac{1}{r}\right) ds \right.$$

$$\left. - \int_{S_D} f_A(s) \frac{\partial}{\partial v} \ln\left(\frac{1}{r}\right) ds - \int_{S_N} z_A(s) \frac{\partial}{\partial v} \ln\left(\frac{1}{r}\right) ds \right]. \quad (21)$$

If the boundary S consists only of line segments, then all integrals can be evaluated exactly. In this case one has

$$\begin{aligned} & |u(x,y) - u_1(x,y)| \\ & \leq \frac{1}{2\pi} \left[\left| \int_{S_N} (g - g_A) \ln\left(\frac{1}{r}\right) ds \right| + \left| \int_{S_D} (\tau - z_A) \ln\left(\frac{1}{r}\right) ds \right| \right. \\ & \quad \left. + \left| \int_{S_D} (f - f_A) \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) ds \right| + \left| \int_{S_N} (\tau - z_A) \frac{\partial}{\partial \nu} \ln\left(\frac{1}{r}\right) ds \right| \right] \\ & \leq [1 + \|(A^0)^{-1}\|] O(h^3). \end{aligned}$$

Here one uses the fact that $|g - g_A| \leq O(h^3)$, $|f - f_A| \leq O(h^3)$, and $|z_A - \tau| \leq |z_A - \tau_A| + |\tau_A - \tau| \leq [1 + \|(A^0)^{-1}\|] O(h^3)$. In all but the Neumann problem, $\|(A^0)^{-1}\|$ will undoubtedly be an increasing function of n , since one is solving an integral equation of the first kind.

When the boundary S does not consist of line segments, there is an error due to the approximation of integrals similar to those in Eq. (20). However, as long as one does not have cusps, or as long as h divided by the radius of curvature does not grow too large, this error remains negligible. Hopefully, sharp error estimates can be developed later.

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