48-28-9145 (2)

PREPARED FOR THE U.S. DEPARTMENT OF ENERGY,
UNDER CONTRACT DE-AC02-76-CHO-3073

PPPL-2792
UC-427

PPPL-2792

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AND MAGNETIC FLUX OF GYRATION STATES

BY

V. ARUNASALAM

October 1991

PPPL

PRINCETON
PLASMA PHYSICS
LABORATORY

PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY

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ANGULAR MOMENTUM, g-VALUE, AND MAGNETIC FLUX OF GYRATION STATES

V. Arunasalam

Princeton University,
Princeton Plasma Physics Laboratory,
Princeton, New Jersey, 08543

PPPL--2792

DE92 001932

ABSTRACT

Two of the world's leading (Nobel laureate) physicists disagree on the definition of the orbital angular momentum L of the Landau gyration states of a spinless charged particle in a uniform external magnetic field $B = B \hat{z}$. According to Richard P. Feynman (and also Frank Wilczek) $L = (r \times \mu v) = r \times (p - qA/c)$, while Felix Bloch (and also Kerson Huang) defines it as $L = r \times p$. We show here that Bloch's definition is the correct one since it satisfies the necessary and sufficient condition $L \times L = i\hbar L$, while Feynman's definition does not. However, as a consequence of the quantized Aharonov-Bohm magnetic flux, this canonical orbital angular momentum (surprisingly enough) takes half-odd-integral values with a *zero-point gyration state* of $L_z = \hbar/2$. Further, since the diamagnetic and the paramagnetic contributions to the magnetic moment are interdependent, the g -value of these gyration states is two and not one, again a surprising result for a spinless case. The differences between the gauge invariance in classical and quantum mechanics, Onsager's suggestion that the flux quantization might be an intrinsic property of the electromagnetic field-charged particle interaction, the *possibility* that the experimentally measured fundamental unit of the flux quantum *need not* necessarily imply the existence of "electron pairing" of the Bardeen-Cooper-Schrieffer superconductivity theory, and the relationship to the Dirac's angular momentum quantization condition for the magnetic monopole-charged particle composites (i.e., Schwinger's dyons), are also briefly examined from a pedestrian viewpoint.

MASTER

I. INTRODUCTION

In classical mechanics,¹ it is well known that in the Lagrangian formalism the fundamental variables are the generalized coordinates r or x_j and the corresponding velocities $v = \dot{r} = dr/dt$ or $v_j = \dot{x}_j = dx_j/dt$ and the Lagrangian $\mathcal{L}(r, v, t) = T(v, t) - U(r, v, t)$, where T is the kinetic energy and U is the "generalized potential". For a spinless charged particle of charge q in an electromagnetic field given by the scalar potential Φ and the vector potential $\mathbf{A}$, this generalized potential is velocity dependent and is given by $U = q\Phi - q\mathbf{A} \cdot \mathbf{v}/c$. However, in the Hamiltonian formalism the independent variables are the generalized coordinates r and the corresponding generalized canonical momenta $\mathbf{p}$ defined as $\mathbf{p} = \partial \mathcal{L}(r, v, t) / \partial \mathbf{v}$. Thus, for the charged particle of mass μ in this electromagnetic field, $\mathbf{p} = \mu \mathbf{v} + q\mathbf{A}/c$. That is, the "canonical momentum," or the "dynamical momentum" or the "p-momentum"² $\mathbf{p}$ is the sum of the "mechanical" or the "kinetic" or the " $\mu\mathbf{v}$ -momentum"³ $\mu\mathbf{v}$ and the "potential momentum" or the "Aharonov-Bohm momentum" $q\mathbf{A}/c$. In quantum mechanics, which is based on the Hamiltonian formalism,^{4,5} the operator $\mathbf{p}$ is the displacement operator $-i\hbar \nabla = -i\hbar \partial / \partial \mathbf{r}$, and the free particle wave function ψ is proportional to $\exp(i\mathbf{p} \cdot \mathbf{r} / \hbar)$. That is, for any arbitrary function $f(\mathbf{r})$, one can show that $f(\mathbf{r} + \mathbf{a}) = \sum_n [(\mathbf{a} \cdot \nabla)^n / n!] f(\mathbf{r}) = [\exp(\mathbf{a} \cdot \nabla)] f(\mathbf{r}) = [\exp(i\mathbf{a} \cdot \mathbf{p} / \hbar)] f(\mathbf{r})$. Thus, the Aharonov-Bohm phase^{2,6} due to the Aharonov-Bohm momentum is $\exp(iq\mathbf{A} \cdot \mathbf{r} / \hbar c)$.

Since for the charged particle in the electromagnetic field $\mathbf{p} = \mu \mathbf{v} + q\mathbf{A}/c$, $\mathbf{r} \times \mathbf{p} = \mathbf{r} \times \mu \mathbf{v} + q\mathbf{r} \times \mathbf{A}/c$. Obviously, $q\mathbf{r} \times \mathbf{A}/c$ represents the angular momentum contribution from the charged particle-electromagnetic field interaction. It should be noted that this field contribution to the angular momentum depends only on the finite value of the vector potential $\mathbf{A}$ and not on the magnetic field $\mathbf{B}$. For example, just outside an infinitely long solenoid

$B = 0$, but A is nonzero, and hence this field contribution to the angular momentum is finite. Note that this is a field-particle interaction angular momentum, and is not the usual electromagnetic field angular momentum which is given by the volume integral^{7,8} $K = (1/4\pi c) \int dV \mathbf{r} \times (\mathbf{E}^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*)$. The question then arises as to what is the orbital angular momentum of the particle?. Is it $\mathbf{r} \times \mathbf{p}$ or is it $\mathbf{r} \times \mu \mathbf{v}$?. In the literature two of the world's leading (Nobel laureate) physicists disagree on the definition of this orbital angular momentum L of the particle. According to Richard P. Feynman⁹ (and also Frank Wilczek,¹⁰ and possibly one group of physicists) $L = (\mathbf{r} \times \mu \mathbf{v}) = \mathbf{r} \times (\mathbf{p} - q\mathbf{A}/c)$, while Felix Bloch¹¹ (and also Kerson Huang,¹² and possibly another group of physicists) defines it as $L = \mathbf{r} \times \mathbf{p}$.

It is our aim in this paper to show that Bloch's definition is the correct one since it satisfies the necessary and sufficient condition $L \times L = i\hbar L$, (i.e., the commutation relations $[L_j, L_k] = i\hbar L_l$, j, k, l , in cyclic order); while Feynman's definition does not. We will then show that for the the Landau cyclotron motion about its guiding center,¹³ as a consequence of the quantized Aharonov-Bohm magnetic flux, this canonical orbital angular momentum, surprisingly enough, takes half-odd-integral values, with a *zero-point gyration state* of $L_z = \hbar/2$. This of course implies that this particular problem with the vector potential expressed in the cylindrically symmetric gauge (i.e., $\mathbf{A} = B\mathbf{r}/2$) does not have an exact Schrodinger's spatial representation. Then we examine the striking similarity between this zero-point Landau gyration state and the Dirac's free particle *Zitterbewegung* motion.^{14,15,16} Further, since the diamagnetic and paramagnetic contributions to the magnetic moment are interdependent, we find that the canonical g-value of these gyration states is two and not one, again a surprising result for a spinless particle case. Of course, the g value of two is entirely consistent with the half-integral L_z -values. We show that $g = 2$ both classically and then quantum

mechanically. Indeed, the rigorous solutions of the Landau gyration states using the equations of motion in the Heisenberg's matrix representation¹⁷ agree exactly with Huang's solutions¹² using the "old" quantum theory via the quantization of the action integral S using the Bohr-Sommerfeld-Wilson (BSW) quantization condition.³ We then discuss the differences between the gauge invariance of the classical dynamical variables, and the gauge invariance of the corresponding quantum mechanical Hermitian operators. Indeed, a gauge invariant dynamical variable in classical mechanics goes over as the corresponding gauge-dependent Hermitian operator. We then examine the flux quantization and the fundamental unit of magnetic flux using a specific model problem of a solid cylinder of free electron gas in a uniform external magnetic field. We will show that even though the flux $\Phi_j = (hc/q) (j + 1/2)$ enclosed by the successive Landau orbits steps in units of hc/q , at sufficiently low temperatures, the average flux $\langle \Phi_j \rangle = (hc/2q) (j + 1)$ is quantized in units of the fundamental unit of flux $hc/2q$, which is characteristic of the zero-point gyration ground state. This *may indicate* the possibility that the experimentally observed¹⁸ fundamental unit of the flux quantum as $hc/2q$ instead of as hc/q *need not* necessarily imply the existence of "electron pairs" of the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity,¹⁹ a rather controversial, speculative and outrageous result. Since flux quantization is a consequence of the angular momentum quantization, there are indications that this might be an intrinsic property of the electromagnetic field-charged particle interaction, in agreement with Onsager's suggestion.^{20,21} Finally, we give a brief discussion of Schwinger's charge-monopole composites or dyons,^{22,23,24} and Wilczek's charged particle-flux tube composites.¹⁰ Here, we find that the velocity-dependent particle-electromagnetic field interaction contribution to the total angular momentum plays the role of a spin angular momentum, possibly implying the usual connection between spin and statistics as discussed by Goldhaber.²⁵

II. PRELIMINARIES

We now consider the properties of the angular momentum operator $\mathbf{J}$. We work entirely with a representation at a particular instant of time, so that we are not concerned here with the change in time of the resulting angular momentum matrices. However, if the angular momentum commutes with the Hamiltonian, it is a constant of motion, and the matrices retain their form for all time. In the Hamiltonian quantum mechanics, following Schiff,⁴ we define the angular momentum $\mathbf{J}$ of a particle about some point in terms of its displacement $\mathbf{r}$ from that point and its canonical momentum $\mathbf{p}$ as

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} . \quad (1)$$

We do not require that $\mathbf{p}$ be expressible as differential operator, but rather that the components of $\mathbf{r}$ and $\mathbf{p}$ satisfy the fundamental commutation relations

$$[r_j, p_k] = i\hbar \delta_{jk} ; \quad [r_j, r_k] = [p_j, p_k] = 0 , \quad (2)$$

where δ_{jk} has the usual meaning of the Kronecker δ symbol; $j, k = 1, 2, \text{ and } 3$; $r_1 = x, r_2 = y, r_3 = z$; and $p_1 = p_x, p_2 = p_y, \text{ and } p_3 = p_z$. It is then possible to find commutation relations between the components of $\mathbf{J}$ that do not involve $\mathbf{r}$ and $\mathbf{p}$ as

$$\mathbf{J} \times \mathbf{J} = i\hbar \mathbf{J} \quad \text{or} \quad [J_j, J_k] = i\hbar J_l ; \quad j, k, l, \text{ in cyclic order.} \quad (3)$$

Now it turns out that the relations of Eq.(3) admit of some matrix representation for $\mathbf{J}$ that are not compatible with the original definition of Eq.(1). Specifically, in the literature it is stated as a theorem^{4,5,13} that the

matrices for the components of the orbital angular momentum of a particle $L = J$ of Eqs.(1) and (2) with $p = -i\hbar\nabla$ in the Schrodinger's spatial representation must have eigenvalues that are integer multiples of $\hbar$, and they cannot have eigenvalues that are half-odd-integer multiples of $\hbar$; while the matrices for the components of J of Eq.(3) can have eigenvalues that are either integer or half-odd-integer multiples of $\hbar$. This theorem for the orbital angular momentum is based on the fact that the particle wave function for continuity be single valued for a rotation about any axis through an angle 2π , and the angular momentum eigenfunctions are the ordinary spherical harmonics. However, in the study of rigid body rotations in terms of Cayley-Klein parameters,¹ one encounters a double-valued isomorphism between a set of three-by-three real orthogonal matrices (appropriate to rotations in the physical three-dimensional space) and the corresponding set of pairs of two-by-two complex unitary matrices (appropriate to rotations in the complex two-dimensional *spinor* space). The two complex unitary matrices that form the pair are such that one is the negative of the other. Indeed, in this spinor space with the Pauli spin matrices as the unit rotators, the occurrence of half angles and the resultant double-valued property are intimately connected with the fact that the spin is half integral. Although the wave function may be double valued under rotation, all physically observable properties remain single valued, of course. This is the reason that Eq.(3) admits some matrix representations for J that are not compatible with the original definition of Eq.(1). When such a conflict arises, there is in some cases physical reason to regard Eq.(3) as more fundamental than Eq.(1).

The angular momentum J can also be defined in terms of infinitesimal rotation R that permits of generalization to more complicated systems (many interacting particles, spins, fields). The effect of a rotation about the z-axis through an infinitesimal angle $\delta\varphi$ on an arbitrary function $f(x,y,z)$ is

$$R_z(\delta\phi)f(x,y,z) = [1 + \delta\phi\{(y\partial/\partial x) - (x\partial/\partial y)\}]f(x,y,z) = [1 + (\delta\phi/i\hbar)J_z]f(x,y,z). \quad (4)$$

Here, J is given by Eq.(1) but now with p expressible as the differential operator $-i\hbar\nabla = -i\hbar\partial/\partial r$. Since Eq.(4) is true for all arbitrary functions $f(x,y,z)$, we can write the z-axis infinitesimal rotation operator as

$$R_z(\delta\phi) = 1 + (\delta\phi/i\hbar) J_z, \quad \delta\phi \text{ infinitesimal} \quad (5)$$

From Eq.(5), one can show that for rotations about the z-axis through a finite angle ϕ , $R_z(\phi) = \exp[(i\phi/\hbar)J_z]$, since as $n \rightarrow \infty$, $(1 + a/n)^n \rightarrow \exp(a)$, where $\delta\phi = \phi/n$ and $a = (i\phi/\hbar)J_z$. Equation (5) is as equally fundamental as Eq.(3), and can be used to define J even when the r and p that appear in Eq.(1) do not exist. Thus, any angular momentum operator J must satisfy the commutation relations of Eq.(3), and this in turn will satisfy the infinitesimal rotation requirements of Eq.(5). However, it is *important to realize* that in Eq.(1) we *do require* that the components of r and p satisfy the fundamental commutation relation of Eq.(2), if J is to be an angular momentum operator.^{4,15} Then and only then will the components of J satisfy the necessary and sufficient condition of Eq.(3).

Armed with this knowledge, we are now in a position to examine the correctness of some of the definitions of the angular momentum found in the literature.^{4,9-12} It is clear that if the generalized potential U of the Lagrangian of the problem under study does not depend on the particle velocity, then there is no potential momentum and the canonical momentum p is identical to the mechanical momentum μv . Then, there is no ambiguity in the definition of the orbital angular momentum as $J = L = r \times p = r \times \mu v$. However, when U is velocity dependent, $p \neq \mu v$. Then, what is the orbital angular momentum L ? Is it $r \times p$ or $r \times \mu v$? Specifically, what is the orbital

angular momentum L of a spinless charged particle of charge q and mass μ in an electromagnetic field of vector potential A . In the literature, two of the world's leading (Nobel laureate) physicists disagree on the definition of the orbital angular momentum L of the gyration states of a spinless charged particle in a uniform external magnetic field $B = B_z$, whose vector potential is conveniently taken here in the cylindrically symmetric form $A = Bx/2$. According to Richard P. Feynman⁹ (and also Frank Wilczek¹⁰) $L = r \times \mu v = r \times (p - qA/c)$, while Felix Bloch¹¹ (and also Kerson Huang¹²) defines it as $L = r \times p$. Feynman states that "From analogy to classical physics, one might expect that the angular momentum operator is now $L = r \times (p - qA/c)$." By using this definition for the orbital angular momentum L he goes on to show from the Dirac's relativistic Hamiltonian that $dL/dt = \dot{L} \neq 0$ even with central forces. Then he shows that $d[L + \hbar\sigma/2]/dt = 0$ for central forces, where σ is the familiar Pauli spin matrix. Thus, he states that "The operator $L + \hbar\sigma/2$ may be regarded as the total angular momentum operator, where L represents orbital angular momentum and $\hbar\sigma/2$ intrinsic angular momentum for spin 1/2. Thus total angular momentum is conserved with central forces." In order to discuss this same problem, Schiff⁴ on the other hand plays it very safely by assuming at the start itself that $A = 0$ and defines this orbital angular momentum as $L = r \times p$, and shows that $dL/dt = -d(\hbar\sigma'/2)/dt \neq 0$ both for the central forces and also for a *completely free* Dirac particle. This nonphysical behavior of the *Dirac free particle* is a direct consequence of the lack of manifest Lorentz covariance of the Dirac free particle Hamiltonian as discussed by this author in earlier publications.^{15,26} Frank Wilczek¹⁰ states that "It is demonstrated that the orbital angular momentum L_z of a particle of charge q orbiting around a tube with magnetic flux Φ is quantized in units $L_z = \text{integer} - q\Phi/2\pi$." This is an interesting result and we will come back to it later. A somewhat related problem of the possibility of half-odd-integral orbital angular momentum has been discussed by this author²⁷

in an unpublished Princeton University Plasma Physics Laboratory Report (MATT-755) in February, 1970. As one method of proof Wilczek states "Rotations around the z-axis, i.e., changes in the azimuthal angle φ , are generated by the covariant angular momentum $L_z = -i\partial_\varphi - qA_\varphi$. In a nonsingular gauge, $A_\varphi = \Phi/2\pi$ and the azimuthal dependence of the electron wave function is $\psi_n \propto \exp(in\varphi)$, $n = \text{integer}$ for continuity. Then $L_z\psi_n = (n - q\Phi/2\pi)\psi_n$, in agreement with the previous conclusion." It is clear that according to Wilczek the orbital angular momentum component L_z defined in terms of infinitesimal rotations about the z-axis is given by $(i_z \cdot L) = [i_z \cdot (r \times \mu v)] = [i_z \cdot (r \times (p - qA/c))] = -i\hbar\partial/\partial\varphi - qA_\varphi/c$. Here, by covariant angular momentum Wilczek means the gauge covariance and not the relativistic Lorentz covariance. Wilczek examines the problem of a particle of charge q which orbits around, but does not penerate, a solenoid running along the z-axis. Here, at the location of the particle there is no B -field, but there is an A . The entire B -field exists only inside the solenoid. *This is not a steady state problem, but a transient one.* Before the current is turned on in the solenoid, the particle experiences no force and hence moves in a straight line and *cannot orbit around* the solenoid. If a current is slowly turned on, the charged particle will feel an electric field $E(r) = -(i_z \times r_\perp)(\dot{\Phi}/2\pi c r_\perp^2)$ according to Faraday's law, where Φ is the flux through the solenoid. This gives the transient change in the angular momentum $\dot{L}_z = [r_\perp \times (qE)]_z = (q\dot{\Phi}/2\pi c)$. However, once the current in the solenoid has reached its steady state value, $\dot{\Phi} = 0$, $E = 0$, and $\dot{L}_z = 0$, and the particle again moves in the force-free straight line motion and *cannot orbit around* the solenoid. Thus, in the transient stage, the angular momentum is quantized in units $L_z = \text{integer} - q\Phi/2\pi c$. This is however not true in the steady state where $\dot{\Phi}$ is zero, while Φ may be nonzero. Let us now apply Wilczek analysis to the classical cyclotron motion

about the guiding center of a spinless charged particle in this uniform magnetic field $\mathbf{B} = B\mathbf{i}_z$, whose vector potential in the cylindrically symmetric gauge is $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2 = (B/2) \mathbf{i}_z \times \mathbf{r}_\perp$. Here, $\perp$ refers to directions perpendicular to the $\mathbf{B}$ -field. In sharp contrast to Wilczek's problem, *this is a steady state and not a transient problem*. Further, at the location of the particle, both $\mathbf{B}$ and $\mathbf{A}$ are nonzero and are related as $\mathbf{B} = \nabla \times \mathbf{A}$.

$$\begin{aligned} L_z \mathbf{i}_z &= \mathbf{r}_\perp \times \mu \mathbf{v}_\perp = \mathbf{r}_\perp \times \mathbf{p}_\perp - (qB/2c) (\mathbf{r}_\perp \times \mathbf{i}_z \times \mathbf{r}_\perp) = \mathbf{r}_\perp \times \mathbf{p}_\perp - (qB/2c) (\mathbf{r}_\perp \cdot \mathbf{r}_\perp) \mathbf{i}_z \\ &= \mathbf{r}_\perp \times \mathbf{p}_\perp - (q/2\pi c) (\pi r_\perp^2 B) \mathbf{i}_z = [-i\hbar \partial/\partial\phi - qA_\phi/c] \mathbf{i}_z, \quad (6) \end{aligned}$$

where $A_\phi = (\pi r_\perp^2 B)/2\pi = \Phi/2\pi$, $\Phi = \pi r_\perp^2 B$ is the magnetic flux enclosed by the cyclotron orbit of radius $r_\perp$, $\mathbf{p} = -i\hbar \nabla$, and we have used the relation $(\mathbf{r}_\perp \times \mathbf{i}_z \times \mathbf{r}_\perp) = (\mathbf{r}_\perp \cdot \mathbf{r}_\perp) \mathbf{i}_z - (\mathbf{r}_\perp \cdot \mathbf{i}_z) \mathbf{r}_\perp = (\mathbf{r}_\perp \cdot \mathbf{r}_\perp) \mathbf{i}_z$, since $\mathbf{r}_\perp \cdot \mathbf{i}_z = 0$. If we now assume that the azimuthal dependence of the particle wave function is $\psi_m \propto \exp(in\phi)$, m = integer for continuity and single valuedness for rotations through 2π , then Eq.(6) yield the Wilczek's result $L_z = [\text{integer}]\hbar - q\Phi/2\pi c$.

Felix Bloch,¹¹ on the other hand, defines this orbital angular momentum of such a particle in the vector potential $\mathbf{A}$ as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and he assumes that the components of this orbital angular momentum must have eigenvalues that are integer multiples of $\hbar$. In this way he tries to interpret some of the characteristic features of superconductivity in terms of general principles of physics. Kerson Huang,¹² in his discussion of the Landau diamagnetism (at high enough temperatures) and the De Hass-Van Alphen effect (at low enough temperatures), also defines this orbital angular momentum as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, in agreement with Bloch and Schiff but in disagreement with Feynman and Wilczek.

However, Huang assumes that the components of this orbital angular momentum must have eigenvalues that are half-odd-integer multiples of $\hbar$, in disagreement with Bloch. He makes use of the "old" quantum theory and assumes that the action integral S over one complete cycle of motion is

$$S = \oint \mathbf{p}_\perp \cdot d\mathbf{r}_\perp = \oint \mathbf{p}_\perp \cdot (\mathbf{r} \times d\boldsymbol{\Omega})_\perp = -\int (\mathbf{r}_\perp \times \mathbf{p}_\perp) \cdot d\boldsymbol{\Omega}_Z \mathbf{i}_Z = -2\pi L_Z$$

$$= (n + 1/2)2\pi\hbar = (n + 1/2)h \quad (n = 0, 1, 2, \dots) \quad , \quad (7)$$

where $d\boldsymbol{\Omega}$ is the infinitesimal rotation pseudovector.¹ As pointed out by Crawford,³ using classical Hamilton-Jacobi theory, Ehrenfest proved that S is an adiabatic invariant, and postulated that S is to be quantized as

$$S = (n + n_0)2\pi\hbar = (n + n_0)h \quad , \quad (8)$$

where $n = 0, 1, 2, \dots, \infty$, and n_0 is a constant adjusted to agree with experiment so as to give the correct ground state. Equation (8) with $n_0 = 1/2$ corresponds to the Bohr-Sommerfeld-Wilson (BSW) quantization rule of the old quantum theory. In Eq.(7), the choice of $n_0 = 1/2$ by Huang is *necessary and sufficient* to ensure that the allowed perpendicular energy eigenvalues $E_n = (n + 1/2)\hbar\omega_D$ are of the harmonic oscillator type, and are in agreement with the earlier results of Landau.¹³ Here, the cyclotron frequency $\omega_D = qB/\mu c$. If Huang would have chosen $n_0 = 0$ in agreement with Bloch, he would have obtained $E_n = n\hbar\omega_D$, with no zero-point energy for the ground state, and this result will be in disagreement with that of Landau. The finite value of the ground state zero-point energy $\hbar\omega_D/2$ for a harmonic oscillator is characteristic of quantum mechanics and is related to the Heisenberg's

uncertainty principle.^{4,5} A somewhat similar view of assuming $n_0 = 1/2$ so as to give the correct Landau gyration ground state is also found in the paper by Crawford.³ He also gives an excellent discussion of the adiabatic invariants associated with this cyclotron motion about the guiding center. Thus, the Landau gyration ground state has an energy $E_0 = \hbar\omega_D/2$ and an action integral $S_0 = \pi\hbar = h/2$, so that the ratio (E_n/S_n) is the same for all Landau levels including the ground state. That is, the Landau gyration ground state is *the zero point gyration state* with energy $\hbar\omega_D/2$ and an associated z-component of the angular momentum of $\hbar/2$.

Thus, according to Huang's "old" quantum theory, the orbital angular momentum of the Landau gyration states is $L = r \times p = r(\mu v + qA/c)$ and is not given by $L = r \times \mu v$, and the z-component of this orbital angular momentum takes half-odd-integral values, i.e., $L_z = (n + 1/2)\hbar$. Also, according to this BSW quantization rule, the "allowed" orbits have radii satisfying the condition

$$(r_{\perp}^2)_n = (2\hbar c/qB) (n + 1/2) . \quad (9)$$

That is, the magnetic flux Φ_n enclosed by the n-th Landau orbit is

$$\Phi_n = \pi(r_{\perp}^2)_n B = (2\pi\hbar c/q) (n + 1/2) = (hc/q) (n + 1/2) , \quad (10)$$

and the associated z-component M_z of the magnetic moment M is

$$(M_z)_n = q\omega_D \pi(r_{\perp}^2)_n / c = (q/\mu c) (n + 1/2) \hbar . \quad (11)$$

Thus, from Eqs.(7) and (11), one finds that the g-value of these gyration states

is

$$g = M / (q/2\mu c)L = M_z/(q/2\mu c)L_z = 2 \quad (\text{for all } n). \quad (12)$$

Thus we see that the Bohr-Sommerfeld-Wilson quantization rule of the old quantum theory as used by Huang, yield a somewhat peculiar but nevertheless correct results for the Landau energy levels, the eigenvalues of the z-component of the canonical orbital angular momentum, and the g-value of these Landau gyration states. The g-value of two is of course entirely consistent with the half-odd-integral values for the angular momentum. What is really surprising is the fact that the z-component of the canonical orbital angular momentum $L_z = r \times p = r \times (\mu v + qA/c)$ takes half-odd-integral values. This is really tied to the fact that the force on the particle is purely velocity dependent and the motion is a circular motion about the guiding center in the x-y plane, but the energy eigenvalues are those of the projected motion on one of these (either x or y) axis only. That is, according to Landau, as far as the energy eigenvalues are concerned, this cyclotron motion about its guiding center is identical to that of a linear harmonic oscillator, i.e., $E_n = (n + 1/2) \hbar \omega_D$. This linear oscillator must of course have the zero-point energy of $\hbar \omega_D/2$ in order to satisfy the Heisenberg's uncertainty principle. Since the motion in the x-y plane is circular with a frequency ω_D , it must also have a z-component of the angular momentum L_z such that the ratio (E_n/L_{zn}) is equal to the frequency ω_D for all the Landau levels $n = 0, 1, 2, \dots, \infty$. Thus, on physical grounds, this half-odd-integral values for the z-component of the canonical orbital angular momentum is consistent with the Landau energy eigenvalues for this circular harmonic oscillator.

In the Heisenberg's matrix representation using the generalized abstract

Hilbert space of Dirac's bra and ket vectors, the occurrence of this half-odd-integral orbital angular momentum poses no real problem, since here we only require that the components of $\mathbf{r}$ and $\mathbf{p}$ satisfy the fundamental commutation relations of Eq.(2) and *we do not require that $\mathbf{p}$ be expressible as differential operator $-i\hbar\nabla$* . However, if we assume that all spinless problems of quantum mechanics, regardless of whether the generalized potential U is velocity dependent or not, must have a Schrodinger's spatial representation and the canonical momentum operator $\mathbf{p}$ *must be expressible as the differential operator $-i\hbar\nabla$* in this $\mathbf{r}$ -representation, then the occurrence of half-odd-integral values for the components of the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ poses a serious threat to the continuity and single valuedness for a spatial rotation through 2π of the eigenfunctions $\psi_n(\mathbf{r}, t)$. In this Schrodinger's $\mathbf{r}$ -representation, the spherical harmonics play an important role in the study of motions whose orbital angular momentum takes integral values of $\hbar$. Thus, it would appear that some linear combination of Hermite polynomials in the x and y coordinates with a phase difference of $\pi/2$ may play an important role in the study of motions whose orbital angular momentum takes half-odd-integral values of $\hbar$, since the wave functions of the stationary states of a Landau particle in the *nonsymmetric gauge* are essentially the Hermite polynomials.¹³

Alternatively, it is quite possible that this problem does not have an exactly equivalent Schrodinger's spatial representation in the cylindrically symmetric gauge. It is not at all clear that, when the generalized potential U of the problem under study is purely velocity dependent, whether one can indeed find an exact Schrodinger's spatial representation, for example, without destroying the physically obvious cylindrical symmetry of the problem in this particular case. There are of course problems in quantum mechanics that do not have a Schrodinger's spatial representation. These problems always involve half-odd-integral angular momentum which are admissible to the definitions of

Eqs.(3) and (5), but are certainly not admissible to that of Eq.(1) with $\mathbf{p} = -i\hbar\nabla$. For example, the Pauli spin Hamiltonian $H = \boldsymbol{\sigma} \cdot \nabla \times \mathbf{A}$ do not have a spatial representation in which one can write down a Schrodinger equation. Such a problem can only be handled as an eigenvalue problem in the Heisenberg's matrix representation in terms of the bra and ket spin vectors.

In order to get some more physical insight into this problem, let us take a detour and examine the similarity¹⁵ between the "Zitterbewegung" motion of the Dirac's free particle Hamiltonian theory and the nonrelativistic Landau cyclotron motion of a spinless particle in the uniform $\mathbf{B}$ -field with $\mathbf{A}(\mathbf{r}) = \mathbf{B} \times \mathbf{r}/2$. The commutation relation between the operators of the velocity components v_j (for $j = 1, 2$, and 3) of the Landau particle may be written^{5,13}

$$[v_j, v_k] = i(q\hbar/c\mu^2) B_l, \quad (j, k, l \text{ in cyclic order}) \quad (13)$$

The Dirac Hamiltonian for a relativistic free spin 1/2 particle is^{4,15}

$$H = E = -\boldsymbol{\alpha} \cdot \mathbf{p} - \beta \mu c^2, \quad (14)$$

where α_j for $j = 1, 2$, and 3 , and β denote four Hermitian operators which acts on the spin variables alone, are independent of $\mathbf{r}_j$, $\mathbf{p}_j$, t , and E , and hence, commute with all of them. Nevertheless, for this Dirac free particle

$$v_j = d\mathbf{r}_j/dt = (i\hbar)^{-1} [\mathbf{r}_j, H] = -c\boldsymbol{\alpha}_j \quad (15)$$

and since $[\alpha_j, \alpha_k] = 2i\sigma_l$ (j, k, l in cyclic order), we get

$$[v_j, v_k] = -2c^2 \sigma_l \quad (j, k, l \text{ in cyclic order}), \quad (16)$$

where the α_j , and σ_j' for $j = 1, 2$, and 3 are taken here in the standard representation.^{4,9,15} Comparing Eqs(13) and(16), we see that the Dirac's free-particle "Zitterbewegung" motion may be thought as some sort of Landau cyclotron motion in a "magnetic-type spinor field" $B_j \rightarrow (2\mu^2 c^3 / q\hbar) \sigma_j'$ for $j = 1, 2$, and 3 . Then the angular frequency of the oscillatory motion $\omega_{bj} = (qB_j / \mu c) \rightarrow (2\mu c^2 / \hbar) \sigma_j'$. That is, the eigenvalues of the oscillation frequency are $\pm(2\mu c^2 / \hbar)$. Since, $v_j = -c\alpha_j \rightarrow r\omega_{bj}$, the eigenvalues of the amplitude of the oscillation are $\pm(\hbar / 2\mu c)$. We have shown earlier that the *Landau's zero point gyration state* carries an orbital angular momentum of $\pm \hbar/2$ with a g-value of 2. Thus, we see a striking similarity between the "Zitterbewegung" motion of the Dirac spin 1/2 particle and the Landau's zero point gyration state. This similarity has been cast into a rigorous mathematical form by Barut and Zanghi.¹⁶ Further, this striking analogy between the *real Zitterbewegung* of the spin 1/2 Dirac *free* particle and the *zero point gyration state* (with the canonical orbital angular momentum $\hbar/2$) of the Landau particle may also shed some light on the *degree of freeness* of the Dirac free particle.¹⁵

III. GAUGE INVARIANCE AND THE CORRECT ORBITAL ANGULAR MOMENTUM

We have seen that when the generalized potential U in the Lagrangian $\mathcal{L}$ of any system is velocity dependent, then the canonical momentum $p = \partial \mathcal{L} / \partial v$ is not equal to the mechanical momentum μv . Then, what is the orbital angular momentum L of such a system?. Is it $r \times p$, or is it $r \times \mu v$?. For our case $U = q\Phi - q\mathbf{A} \cdot \mathbf{v} / c$, and hence $p = \mu v + q\mathbf{A} / c$. From what has been said

earlier, it is clear that any angular momentum operator J must satisfy the requirements of Eq.(3) or Eq.(5). In particular, if we satisfy Eq.(3), we will automatically satisfy Eq.(5) and vice versa. These Eqs.(3) and (5) are true in the abstract generalized Hilbert space of Dirac's bra and ket vectors of the Heisenberg's matrix representation. However, the orbital angular momentum L of any system must be definable in the Schrodinger's spatial representation with the aid of Eq.(1). But, Eq.(1) will satisfy the requirements of Eq.(3) *if and only if* the components of r and p satisfy the fundamental commutation relations of Eq.(2), even if p is not expressible as the differential operator $-i\hbar\nabla$. Of course, if $p = -i\hbar\nabla$, the components of p will automatically satisfy the requirements of Eq.(2). It is the fundamental assumption in the Hamiltonian formalisms of both the classical and quantum mechanics, the coordinate operator r and the corresponding *canonical momentum operator* p always satisfy the requirements of Eq.(2). In classical mechanics of course the Poisson bracket $\{\cdot, \cdot\}$ replaces the commutator bracket as $(1/i\hbar)[\cdot, \cdot]$ in Eq.(2). Thus, it is clear that the canonical orbital angular momentum L defined as $L = r \times p$, where p is the canonical momentum, will clearly satisfy the requirements of Eqs. (1) to (5). In the Schrodinger's spatial representation, this canonical momentum $p = -i\hbar\nabla$, and consequently the rotations around the z-axis, i.e., the changes in the azimuthal angle φ , are generated by the z-component of this canonical angular momentum $L_z = -i\hbar\partial/\partial\varphi$. This view is in complete agreement with those of Bloch,¹¹ Schiff,⁴ and Huang;¹² but is in complete disagreement with those of Feynman⁹ and Wilczek.¹⁰ Indeed, according to Wilczek : " Rotations around the z-axis, i.e., the changes in the azimuthal angle φ are generated by the covariant angular momentum $L_z = -i\partial_\varphi - qA_\varphi$."

This covariant angular momentum L_F of Wilczek and also of Feynman is related to the magnetic moment M by a g-factor of one and not two, and is

given by

$$L_F = r \times \mu v = r \times [p - qA/c] = (2\mu c/q)M \quad (17)$$

According to Eqs.(3) or (5), this Feynman's angular momentum L_F is not an acceptable angular momentum operator, since neither $L_F \times L_F = i\hbar L_F$, nor is it a rotation operator. First of all, according to Eq.(13), we see that in a magnetic field, the operators of the velocity components v_j , $j = 1, 2$, and 3 , of a charged particle do not commute. This means that the particle cannot simultaneously have definite values of the velocity components in all three directions. This violates the basic requirements of Eq.(2). Second, from Eqs.(2) and (13), one can show that the components of L_F satisfy the following commutation relations:⁵

$$[L_{Fj}, L_{Fk}] = i\hbar (L_{Fl} + (q r_l)(r \cdot B)) \quad (j, k, l \text{ in cyclic order}) \quad (18)$$

When the B -field is not zero, it is clear from Eqs.(3) and (18) that the commutation relations of L_F are completely different from those of an angular momentum operator J . The operator L_F therefore does not, *a priori*, possess the properties of an angular momenta. *Hence, we conclude that Bloch's definition of the orbital angular momentum is the correct one, and the Feynman's definition is totally incorrect.*

Let us now briefly examine the gauge invariance of the *dynamical variables* of a system in classical mechanics, and the gauge invariance of the *corresponding Hermitian operators* in quantum mechanics. In the gauge transformation of the electromagnetic field, the scalar and the vector potentials transform as

$$\Phi'(r,t) \rightarrow \Phi(r,t) - (1/c) \partial\chi(r,t)/\partial t, \text{ and}$$

$$A'(r,t) \rightarrow A(r,t) + \nabla\chi(r,t) \quad (19)$$

In both the classical and quantum mechanics, all physically meaningful results must not be modified when such a gauge transformation is performed, i.e., classical and quantum mechanics possess the property of *gauge invariance*. Both in the Lagrangian and in the Newton's equations of motion formalisms of classical mechanics, the fundamental dynamical variables are the position r and the velocity v , and these variables take on, at every instant, values independent of the gauge. These are usually referred⁵ to as *true physical quantities or true physical dynamical variables*. In this case, the Feynman and Wilczek pseudovector $L_F = r \times \mu v$, although not an angular momentum operator, is a true physical quantity independent of the gauge. Then, the canonical momentum $p = \mu v + qA/c$ and the canonical angular momentum pseudovector $L = r \times p$, although a true angular momentum operator, are gauge dependent. These gauge dependent dynamical variables in classical mechanics are usually referred⁵ to as *non-physical quantities or non-physical dynamical variables*.

However, in quantum mechanics which is based on the Hamiltonian formalism, the fundamental Hermitian operators which replace the classical dynamical variables are the position and the canonical momentum operators r and p , respectively. These operators and the associated commutation relations of Eq.(2) are the same in all the gauges. When the problem under study has a Schrodinger's spatial representation, $p = -i\hbar\nabla$ in all gauges. Thus, in quantum mechanics, the canonical angular momentum operator $L = r \times p$ is the same in all gauges. Thus, the relations of Eqs.(1) to (5) are gauge independent. But, the velocity operator $v = \mu^{-1}[p - qA/c]$ and the Feynman-Wilczek operator $L_F =$

$r \times \mu v$, on the other hand, depend on the gauge chosen. *In general, the Hermitian operator of quantum mechanics, corresponding to a true physical quantity (or a true physical dynamical variable) of classical mechanics, depends on the gauge chosen.*

In quantum mechanics, under the gauge transformation of Eq.(3), the Dirac's ket vector transforms as

$$|n\rangle' \rightarrow [\exp\{i(q/\hbar c)\chi(r,t)\}] |n\rangle \quad . \quad (20)$$

That is, for the wave function, the gauge transformation corresponds to a phase change which varies from one point to another, and is not therefore a global phase factor [as in Eqs.(45), (47), and (48), for example]. Infact, one can show that χ is not single valued but changes by

$$\Delta\chi = \oint \mathbf{A} \cdot d\mathbf{l} = \oint \nabla\chi \cdot d\mathbf{l} = \iint \mathbf{B} \cdot d\mathbf{s} = \Phi \quad (21)$$

whenever one goes around a closed loop. This phase change $\Delta\chi$ around a closed loop is proportional to the Aharonov-Bohm flux enclosed by the closed path.^{2,21} Thus, in quantum mechanics, the vector potential is a measurable quantity via the Aharonov-Bohm phase change in the wave functions; and as Feynman² says it is a *real field*. Feynman defines the "real field" as a set of numbers we specify in such a way that what happens *at a point* depends only on the numbers *at that point*.

IV. CLASSICAL CYCLOTRON MOTION ABOUT THE GUIDING CENTER

Let us now examine briefly the classical cyclotron motion about the

guiding center of a charged particle of charge q and mass μ in a uniform magnetic field $\mathbf{B} = B\mathbf{i}_z$ whose vector potential in the cylindrically symmetric gauge is $\mathbf{A}(\mathbf{r}) = \mathbf{A}_\perp(\mathbf{r}_\perp) = B\mathbf{r}/2 = (B/2)(\mathbf{i}_z \times \mathbf{r}_\perp)$, where $\perp$ refers to directions perpendicular to the uniform $\mathbf{B}$ -field. The equation of motion of such a particle is⁷

$$d\mathbf{v}_\perp/dt = \omega_b (\mathbf{v} \times \mathbf{i}_z), \quad (22)$$

where the cyclotron frequency $\omega_b = qB/\mu c$. The motion described by Eq.(22) is a circular motion in the plane perpendicular to $\mathbf{B}$. The solution for $\mathbf{v}_\perp$ is easily shown to be

$$\mathbf{v}_\perp(t) = \omega_b r_{\perp b} (\mathbf{i}_x - i \mathbf{i}_y) e^{-i\omega_b t}, \quad (23)$$

where $r_{\perp b}$ is the gyration radius. The convention is that the real part of the equation is to be taken. Then one can see that Eq.(23) represents a counterclockwise rotation for a positive charge q when viewed in the direction of $\mathbf{B}$. Another integration yields the radius vector of the particle about its guiding center as

$$\mathbf{r}_\perp(t) = i r_{\perp b} (\mathbf{i}_x - i \mathbf{i}_y) e^{-i\omega_b t}. \quad (24)$$

Thus, from Eqs.(23) and (24), we get

$$\mathbf{v}_\perp = -\omega_b (\mathbf{i}_z \times \mathbf{r}_\perp). \quad (25)$$

Hence, using Eq.(25), we get

$$\mathbf{p}_\perp = \mu \mathbf{v}_\perp + q \mathbf{A}_\perp / c = \mu \mathbf{v}_\perp + (\mu/2) (\omega_b \mathbf{i}_z \times \mathbf{r}_\perp) = \mu \mathbf{v}_\perp - \mu \mathbf{v}_\perp / 2 = \mu \mathbf{v}_\perp / 2 . \quad (26)$$

Thus, in this cylindrically symmetric gauge, the z-component of the canonical orbital angular momentum is

$$L_z \mathbf{i}_z = \mathbf{r}_\perp \times \mathbf{p}_\perp = (\mu/2) \mathbf{r}_\perp \times \mathbf{v}_\perp . \quad (27)$$

But the z-component of the magnetic moment $\mathbf{M}$ may be written

$$M_z \mathbf{i}_z = (q/2\mu c) \mathbf{r}_\perp \times \mu \mathbf{v}_\perp . \quad (28)$$

Thus, from Eqs.(27) and (28), we get the classical canonical g-value as

$$g = M_z / (q/2\mu c) L_z = 2 . \quad (29)$$

We see then, even classically, the canonical g-value of the cyclotron motion about the guiding center is two and not one. This of course is due to the peculiar relation of Eq.(26) between the canonical momentum $\mathbf{p}$ and the mechanical momentum $\mu \mathbf{v}$, which in turn is due to the peculiar velocity dependent force in the equation of motion of Eq.(22). That is, as pointed out by Crawford³ the contribution to the action integral S from the potential momentum $q\mathbf{A}/c$ has half the magnitude and opposite in sign of the contribution of the $\mu \mathbf{v}$ -momentum. Stated differently, the diamagnetic contribution to the magnetic moment which is due to the Aharonov-Bohm phase integral around the orbit is exactly equal to the paramagnetic contribution to the magnetic moment which is due to the canonical angular momentum.^{2,8}

V. LANDAU GYRATION STATES IN THE HEISENBERG'S MATRIX REPRESENTATION

We again consider the motion of a spinless particle of charge q and mass μ in a uniform external magnetic field $\mathbf{B} = B \mathbf{i}_z$. Let $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{p}$ be, respectively, the position, velocity, and canonical momentum vectors of the particle. The vector potential $\mathbf{A}(\mathbf{r})$ for such a uniform $\mathbf{B}$ -field may be written

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_\perp(\mathbf{r}_\perp) = (1/2) B \mathbf{r} \times \mathbf{i}_z = (B/2) \mathbf{i}_z \times \mathbf{r}_\perp. \quad (30)$$

The Hamiltonian describing the motion of the particle is

$$H = (\mathbf{p} - q\mathbf{A}/c)^2/2\mu = H_\perp + H_\parallel, \quad (31)$$

where

$$H_\perp = (\mathbf{p}_\perp - q\mathbf{A}/c)^2/2\mu = (p_x \mathbf{i}_x + p_y \mathbf{i}_y - q\mathbf{A}/c)^2/2\mu \quad (32)$$

and

$$H_\parallel = p_\parallel^2/2\mu = p_z^2/2\mu. \quad (33)$$

Here, $\perp$ and $\parallel$ refer to directions perpendicular and parallel, respectively, to the uniform magnetic field $\mathbf{B} = B \mathbf{i}_z$. Since $\mathbf{A} \cdot \mathbf{i}_z = 0$ for the uniform magnetic field $\mathbf{B} = B \mathbf{i}_z$, we see that the total Hamiltonian (H) separates into two parts, one describing the cyclotron motion of the charged spinless particle in a plane perpendicular to the $\mathbf{B}$ -field ($H_\perp$) and the other describing its free

motion parallel to the $\mathbf{B}$ -field. It is shown elsewhere^{5,13} that this $H_{\perp}$ can be written in a form which is analogous to that of a Hamiltonian of a *one-dimensional* harmonic oscillator. This equivalence to a one-dimensional harmonic oscillator was first shown by Landau¹³ in the Schrodinger's r -representation using the non-symmetric gauge where $A_x = -By$, $A_y = A_z = 0$. In this Landau's Schrodinger equation formalism, the problem under study does not exhibit the cylindrical symmetry it actually possess. Also one can show⁵ that, in this particular cylindrically symmetric gauge of Eq.(30), this same operator $H_{\perp}$ is simply related to the Hamiltonian of a *two-dimensional* harmonic oscillator. These two results (i.e., the simultaneous similarity to both the one-dimensional and the two-dimensional harmonic oscillators) are not contradictory; they simply correspond to two different decomposition of the same Hamiltonian, which must obviously lead to the same physical conclusions. *However, one must not lose sight of the fact that the Hamiltonian $H_{\perp}$ involves a physical problem which is completely different from that of a two-dimensional harmonic oscillator: the charged particle is subjected to a vector potential (describing a uniform magnetic field) and not a harmonic scalar potential (which would describe, for example, a non-uniform electric field). It so happens that, in the gauge chosen, the effects of the magnetic field can be likened to those of a fictitious harmonic scalar potential. Classically we know that the particle motion is a circle in a plane perpendicular to the $\mathbf{B}$ -field, and the x and y coordinates of the particle describe linear harmonic motion with a 90° phase difference between them. Although the projected motions on the x and y axes are formally identical to that of a linear harmonic oscillator, it should be emphasized that the motion of the particle under study is determined by a velocity-dependent potential ($q\mathbf{A}\cdot\mathbf{v}/c$) in its Lagrangian [or its velocity-dependent force ($q\mathbf{v}\times\mathbf{B}/c$)]. For these reasons one can show³ that the contribution to the "action integral" S of the "potential momentum" $q\mathbf{A}/c$ has half the magnitude and opposite in sign to the*

contribution of the " μv momentum". This in turn is responsible for the g-value of two and not one for the cyclotron motion about the guiding center under study, and this behavior is quite different from that of a two-dimensional harmonic oscillator.⁵

In the spirit of Dirac's suggestion,¹⁷ we will now take a detour and examine this problem (of the cyclotron motion about its guiding center) in the Heisenberg's matrix representation via its equations of motion. This method of analysis will explicitly exhibit the cylindrical symmetry about the uniform magnetic field $B = B i_z$. The Heisenberg's operator equation of motion for the perpendicular velocity operator $v_{\perp} = v_x i_x + v_y i_y$ is given by

$$\dot{v}_{\perp} = (i/\hbar) [H_{\perp}, v_{\perp}] = (i/\hbar) (H_{\perp} v_{\perp} - v_{\perp} H_{\perp}). \quad (34)$$

Since the uniform magnetic field $B = B i_z$ commutes with $v_{\perp}$, the quantum mechanical symmetric average⁴ $(v_{\perp} \times B - B \times v_{\perp})/2 = v_{\perp} \times B$, and hence the vector equation of the perpendicular motion is⁷

$$\dot{v}_{\perp} = (q/\mu c) (v_{\perp} \times B) = \omega_b (v_{\perp} \times i_z), \quad (35)$$

where $\omega_b = (qB/\mu c)$ is the cyclotron frequency. We now choose the perpendicular energy representation in which $H_{\perp}$ is diagonal. That is, $H_{\perp} m n = E_n \delta_{m n}$ [or $H_{\perp} |n\rangle = E_n |n\rangle$ and $\langle m | n \rangle = \delta_{m n}$]. Then in terms of the matrix elements, Eqs.(34) and (35) become

$$\dot{v}_{\perp m n} = (i/\hbar) (E_m - E_n) v_{\perp m n} \quad (36)$$

and

$$\dot{v}_{\perp mn} = \omega_D (v_{\perp} \times i_z)_{mn} \quad , \quad (37)$$

respectively. From Eqs.(36) and (37) we get

$$(E_m - E_n + \hbar\omega_D) v_{+mn} = 0 \quad (38)$$

$$(E_m - E_n - \hbar\omega_D) v_{-mn} = 0 \quad , \quad (39)$$

where

$$v_{\pm} = 2^{-1/2} (v_x \pm i v_y) \quad . \quad (40)$$

It can be shown that the following commutation relations are true:^{5,13}

$$[v_+ , v_-]_{mn} = -i [v_x , v_y]_{mn} = (\hbar\omega_D/\mu) \delta_{mn}$$

$$[v_x , v_z]_{mn} = [v_y , v_z]_{mn} = 0 \quad . \quad (41)$$

The perpendicular velocity operator $v_{\perp}$ is given by

$$v_{\perp} = dr_{\perp}/dt = (i/\hbar) [H_{\perp} , r_{\perp}] = (p_{\perp} - qA_{\perp}/c)/\mu \quad (42)$$

Thus on making use of Eqs.(41) and (42) in Eq.(32) we get

$$H_{\perp mn} = \mu(v_+ v_-)_{mn} - (\hbar\omega_D/2) \delta_{mn} = \mu(v_- v_+)_{mn} + (\hbar\omega_D/2) \delta_{mn} . \quad (43)$$

From Eqs.(38) to (43) we get, by exact analogy to the customary theory of a linear harmonic oscillator,^{4,5,13} the energy level spectrum and the nonzero matrix elements of v_+ and v_- as

$$E_n = H_{\perp nn} = (n + 1/2) \hbar \omega_b \quad , \quad (44)$$

where $n = 0, 1, 2, \dots, \infty$,

$$v_{+ n-1,n} = (v_{- n,n-1})^* = (\hbar \omega_b / \mu)^{1/2} n^{1/2} e^{i\varphi_n} \quad , \quad (45)$$

where φ_n is an arbitrary phase factor and the asterisk means the complex conjugate. Since

$$v_{\perp mn} = \dot{r}_{\perp mn} = (i/\hbar) [H_{\perp}, r_{\perp}]_{mn} = i \omega_b (m - n) r_{\perp mn} \quad , \quad (46)$$

from Eq.(45) we get the nonzero matrix elements of the operators r_+ and r_- as

$$r_{+ n-1,n} = (r_{- n,n-1})^* = i [(\hbar/\mu \omega_b) = (\hbar c/qB)]^{1/2} n^{1/2} e^{i\varphi_n} \quad , \quad (47)$$

where $r_{\pm} = 2^{-1/2} (x \pm iy)$. From Eqs.(45) and (47), the nonzero matrix elements of the perpendicular velocity and position operators are

$$v_{x n,n-1} = (v_{x n-1,n})^* = -iv_{y n,n-1} = (iv_{y n-1,n})^* = (\hbar \omega_b / 2\mu)^{1/2} n^{1/2} e^{-i\varphi_n} \quad ,$$

$$x_{n,n-1} = (x_{n-1,n})^* = -iy_{n,n-1} = (iy_{n-1,n})^* = -i(\hbar/2\mu \omega_b)^{1/2} n^{1/2} e^{-i\varphi_n} \quad . \quad (48)$$

From Eqs.(30) and (42), it is relatively easy to see that

$$p_{\perp} = p_x i_x + p_y i_y = \mu(v_x - y\omega_D/2)i_x + \mu(v_y + x\omega_D/2)i_y, \quad (49)$$

and

$$p_{\pm} = 2^{-1/2} (p_x \pm ip_y) = \mu v_{\pm} \pm i(\mu\omega_D/2)r_{\pm}. \quad (50)$$

Thus, on making use of Eqs.(45), (47), and (48) in Eqs.(49) and (50), one can easily obtain the nonzero matrix elements of the perpendicular canonical momentum operator $p_{\perp}$. Since, $\pm i = \exp(\pm i\pi/2)$, it is seen from Eq.(48) that the particle motion about the guiding center is a circle in a plane perpendicular to the B -field and the x and y coordinates of the particle describe linear harmonic motions with a phase difference of $\pi/2$. Thus, in sharp contrast to the Landau's Schrodinger equation analysis in the r -representation,¹³ our analysis in terms of the equations of motion in the Heisenberg matrix representation yields the expected classical picture of the particle motion about its guiding center.

For any volume distribution of current the magnetic moment M is defined in Gaussian units as

$$M = (1/2c) \int (r \times j) dV, \quad (51)$$

where dV is the volume element, and j is the current density. For a plane loop of thin wire of cross-sectional area dS in which a current I is flowing, $j dV = j dS dl = I dl$, where dl is an element of length along the direction of the current flow. Then M of Eq.(51) becomes

$$\mathbf{M} = (1/2c) \oint (\mathbf{r} \times d\mathbf{l}) = (A/c) \mathbf{n} , \quad (52)$$

where the area of the loop $A = \oint (\mathbf{r} \times d\mathbf{l})/2$, and $\mathbf{n}$ denotes a unit vector normal to the plane of the loop. For our problem, $\mathbf{n} = -\mathbf{i}_z$, $l = q\omega_D$, and $A = (\pi r_\perp^2) = \pi[(r_+ r_-) + (r_- r_+)]$. Thus, on making use of Eq.(47), we get

$$(M_z)_{nn} \mathbf{i}_z = \langle n | \mathbf{M} \cdot \mathbf{i}_z | n \rangle \mathbf{i}_z = (-\mathbf{i}_z) (q\hbar/\mu c) (n + 1/2) . \quad (53)$$

We note that this result of Eq.(53) can also be obtained trivially by making use of the Feynman's theorem, according to which²⁸

$$(M_z)_{nn} = \langle n | M_z | n \rangle = \langle n | -\partial H_\perp / \partial B | n \rangle = -\partial E_n / \partial B = -(q\hbar/\mu c) (n + 1/2) , \quad (54)$$

where we have made use of Eq.(44). Since,

$$\begin{aligned} M_z \mathbf{i}_z &= (q/2\mu c) (\mathbf{r}_\perp \times \mu \mathbf{v}_\perp) = (q/2\mu c) [\mathbf{r}_\perp \times (\mathbf{p}_\perp - q\mathbf{A}_\perp/c)] = (q/2\mu c) [\mathbf{r}_\perp \times \mathbf{p}_\perp - \\ &\quad (qB/2c) (\mathbf{r}_\perp \times \mathbf{i}_z \times \mathbf{r}_\perp)] = (q/2\mu c) L_z \mathbf{i}_z - (q/2\mu c) (qB/2c) (\mathbf{r}_\perp \cdot \mathbf{r}_\perp) \mathbf{i}_z, \end{aligned} \quad (55)$$

where we have made use of Eq.(30), the relation $\mu \mathbf{v} = \mathbf{p} - q\mathbf{A}/c$, Eq.(28), and the fact that $(\mathbf{i}_z \cdot \mathbf{r}_\perp) = 0$. The last equality of Eq.(55) exhibits the paramagnetic and the diamagnetic contributions explicitly. Thus, from Eq.(55) we get

$$\begin{aligned} (L_z)_{nn} &= (2\mu c/q) (M_z)_{nn} + (qB/2c)[(r_+ r_-) + (r_- r_+)]_{nn} \\ &= -2 (n + 1/2) \hbar + (n + 1/2) \hbar = - (n + 1/2) \hbar , \end{aligned} \quad (56)$$

where we have made use of Eqs.(47) and (54). From Eq.(47), the eigenvalues of the magnetic flux of these gyration states are

$$\Phi_{nn} = \pi(\mathbf{r}_\perp \cdot \mathbf{r}_\perp)_{nn} B = \pi B [(r_+ r_-) + (r_- r_+)]_{nn} = (hc/q) (n + 1/2). \quad (57)$$

From Eqs.(54) and (56), we see that the g-value of these Landau gyration states is $g = [M_z / (q/2\mu c)L_z] = 2$ for all n . As seen from Eqs.(53) to (56), this $g = 2$ is a consequence of the fact that the diamagnetic and the paramagnetic contributions to the magnetic moment are interdependent. Indeed, the diamagnetic contribution is exactly equal to the paramagnetic one, and is related to the Aharonov-Bohm magnetic flux. Thus we see that the rigorous analysis of the cyclotron motion about its guiding center using the cylindrically symmetric gauge in the Heisenberg matrix representation via Dirac's suggestion of using its equation of motion, yield the same results as those previously obtained by Huang^{1,2} using the "old" quantum theory with the Bohr-Sommerfeld-Wilson quantization condition for the action integral S with the choice of $n_0 = 1/2$ so as to give the correct Landau ground state. It is interesting to note that in the old quantum theory the quantization of the action integral S , or equivalently, the components of the orbital angular momentum L , is the most fundamental entity. Then the quantization of energy, magnetic moment, and the magnetic flux follow as a consequence of the S -quantization. While in the modern quantum theory the system Hamiltonian is the most fundamental. In both the nonrelativistic and in the manifestly Lorentz noncovariant but Lorentz invariant^{15,26} Dirac's relativistic theories, the Hamiltonian of the system is the total energy E of this system. Hence, in the modern quantum theory, the quantization of the energy E is the fundamental entity. Then the quantization of the angular momentum, magnetic moment and the magnetic flux follow as a consequence of the E -quantization. It is also interesting to note from Eqs.(10) and (57) that the magnetic flux

associated with these Landau gyration states is quantized in steps of (hc/q) , but with a zero-point flux of $(hc/2q)$. We also pointed out the striking similarity between the Dirac free particle *Zitterbewegung* motion and the ground state of the Landau cyclotron motion, so that spin is the orbital angular momentum of this *Landau zero-point gyration state* or the *Zitterbewegung*.^{16,15} This is suggestive of the Onsager's idea that the flux quantization in units of this zero-point gyration value of $(hc/2q)$ might be an intrinsic property of the electromagnetic field-charged particle interaction.²⁰ We know that the zero-point vibrations of the quantized radiation field determines the change in the radiative self-energy of quantum states which are responsible to the Lamb shift and the Schwinger $(g - 2)$ value.^{29,30} It is therefore not unreasonable to speculate on the link between the intrinsic flux quantization unit and the flux associated with the Landau zero-point gyration state. It is not clear whether this has any implications to the concept of "electron pairing" in the Bardeen-Cooper-Schrieffer (BCS) theory of low temperature superconductivity.¹⁹ We will come back to examine this link in some detail later.

VI. FLUX QUANTIZATION AND THE UNIT OF MAGNETIC FLUX

It has been shown by London,³¹ Onsager,²⁰ and Byers and Yang²¹ that the magnetic flux embraced by a superconducting ring (i.e., a ring of noninteracting free electron gas) ought to be quantized in units of hc/q , where q is the magnitude of the electronic charge. However, the magnetic flux through a superconducting ring has been found experimentally¹⁸ to be quantized in units of $hc/2q$. The experimental discovery of steps just this size has been claimed to provide a convincing direct proof for the pairing of electrons as predicted by the theory of Bardeen, Cooper and Schrieffer (BCS).¹⁹ This pairing is due to

the interaction of the conduction electrons in the superconductive state, and consequently the carriers of electricity in the superconductor has twice the charge of the electron.^{2,19} It is our aim here to present a somewhat controversial and speculative counter argument to show that the discovery of the fundamental unit of flux quantum as $hc/2q$ instead of hc/q need not necessarily imply the existence of "electron pairing" in the superconductive state.

For this purpose, let us consider a solid cylinder of finite radius R placed in a uniform magnetic field $B = B\hat{z}$ with its axis parallel to the B -field. Let this cylinder contain N electrons and we will assume that these N electrons form a noninteracting free electron gas, similar to that in a superconductor. We will further assume that the temperature T of this cylinder is low enough that we are *not* in the Landau diamagnetic regime, but are in the De Hass-Van Alphen regime. It is clear from Eqs.(10) and (57) that the magnetic flux associated with the n -th Landau gyration state is $\Phi_n = (hc/q)(n + 1/2)$. That is, the flux enclosed by the successive Landau orbits increases in steps of hc/q starting from the ground state value of $hc/2q$. Following Huang,¹² it is relatively easy to show that each of these Landau gyration states are continuously degenerate, and the degeneracy of the energy level E_n is the same for all n and is given by

$$G = (R^2 q B / 2 \hbar c), \quad (58)$$

since the cross-sectional area of the cylinder is πR^2 . Physically this degeneracy arises from the fact that the Landau orbit in the magnetic field has the same energy, wherever its guiding center. Since we have N free electrons in a cylinder of radius R , let j be the largest integer for which $(r_{\perp}^2)_j$ of Eq.(9) is less than or equal to R^2 , i. e.,

$$(r_{\perp}^2)_j = (2\hbar c/qB) (j + 1/2) \leq R^2. \quad (59)$$

That is, only the levels $0, 1, 2, \dots, j$ are to be occupied by the available N free electrons. It is clear from Eq.(59) that the value of the magnetic field B_j at which the j -th Landau orbit will become available for occupation by the N free electrons under study (i.e., the value of the B field at which the radius of the j -th Landau orbit will equal R) is

$$B_j = (\hbar c/qR^2) (2j + 1). \quad (60)$$

Hence for $j = 0, 1, 2$, and 3 , for example, the values of B field at which the magnetic flux Φ_j will make a quantum jump are: $B_0 = (\hbar c/qR^2)$, $B_1 = 3(\hbar c/qR^2)$, $B_2 = 5(\hbar c/qR^2)$, and $B_3 = 7(\hbar c/qR^2)$, respectively. That is, the ratio of the fields at which the magnetic flux jumps will occur are approximately $1, 3, 5$, and 7 . This ratio of the critical fields has a striking similarity to those observed experimentally for a somewhat different problem.¹⁸ These experiments on quantized flux were done with hollow superconducting cylinders, while the result of Eq.(60) is derived only for a solid cylinder of a free electron gas. Nevertheless, this accidental agreement in the ratio of critical fields is somewhat remarkable.

We have N electrons to be distributed among $(j + 1)$ Landau levels each of degeneracy G . Hence the number of electrons N_n in the n -th level is

$$N_n \propto \begin{cases} N G e^{-E_n/KT} & \text{for } n \leq j \\ 0 & \text{for } n \geq j \end{cases}, \quad (61)$$

where T is the temperature and k is the Boltzmann constant. We will assume that this temperature T is low enough so that we are in the De Hass-Van Alphen regime and not in the Landau diamagnetic regime, but the highest occupied level j is such that $E_j = (j + 1/2)\hbar\omega_D = (q\hbar B/\mu c)(j + 1/2) \ll kT$, so that $\exp(-E_n/kT) \approx 1$ for all $n \leq j$ in Eq.(61). Since, $\sum_{n=0}^j N_n = N$, Eq.(61) becomes

$$N_n \approx \begin{cases} N/(j+1) & \text{for } n \leq j \\ 0 & \text{for } n \geq j \end{cases} \quad (62)$$

When the j -th Landau orbit is the largest orbit inside the cylinder of radius R , from Eqs.(10) or (57) and Eq.(62), we get the average magnetic flux $\langle \phi_j \rangle$ inside this cylinder of radius R as

$$\begin{aligned} \langle \phi_j \rangle &\approx (1/N) \sum_{n=0}^j (N_n \phi_n) = (hc/q) (j+1)^{-1} \sum_{n=0}^j (n + 1/2) \\ &= (hc/q) (j+1)^{-1} [(j+1)(j+1/2 + 1/2)/2] = (hc/2q) (j+1) \quad (63) \end{aligned}$$

Thus, for $j = 0, 1, 2$, and 3 for example, the average magnetic flux $\langle \phi_j \rangle$ of Eq.(63) take the values $\langle \phi_0 \rangle \approx (hc/2q)$, $\langle \phi_1 \rangle \approx 2(hc/2q)$, $\langle \phi_2 \rangle \approx 3(hc/2q)$, and $\langle \phi_3 \rangle \approx 4(hc/2q)$, respectively. Since, for $B < B_0$ there is no Landau level inside the cylinder of radius R , the average magnetic flux $\langle \phi_j \rangle$ will take the values

$$\langle \Phi_j \rangle \approx \begin{cases} 0 & \text{for } 0 < B < B_0 \approx (\hbar c / q R^2) \\ (hc/2q) & \text{for } B_0 < B < B_1 \approx 3(\hbar c / q R^2) \\ 2(hc/2q) & \text{for } B_1 < B < B_2 \approx 5(\hbar c / q R^2) \\ 3(hc/2q) & \text{for } B_2 < B < B_3 \approx 7(\hbar c / q R^2) \\ \vdots & \vdots \\ (j+1)(hc/2q) & \text{for } B_{j-1} < B < B_j \approx (2j+1)(\hbar c / q R^2) \end{cases} \quad (64)$$

Here again, the result of Eq.(64) has a striking similarity to those observed experimentally for a somewhat different problem. That is, this result of Eq.(64) for a solid cylinder has a striking similarity to the experimental results obtained with hollow superconducting cylinders. Further, what is measured in the experiments is the *trapped flux*, while these calculations refer to the average flux enclosed by the *complete* Landau gyration orbits which can be accommodated inside this solid cylinder. It is not unreasonable to assume that only these *complete* Landau gyration orbit fluxes enclosed inside this solid cylinder can be frozen in or trapped when the metal cylinder makes the transition to the superconductive state as its temperature is lowered. In any case, we find that the average magnetic flux $\langle \Phi_j \rangle$ within a finite radius cylinder of a noninteracting free electron gas in an axially symmetric uniform magnetic field is quantized in units of $hc/2q$, even though the flux $\Phi_j = (hc/q) (j + 1/2)$ enclosed by the Landau orbits steps by hc/q . At low temperatures, when the energy of the system has been reduced very,very low, instead of a large number of states being involved, only a very, very small number near the ground state are involved. Under these circumstances the quantum mechanical character of that ground state can appear on a macroscopic scale. This is the reason that the average flux $\langle \Phi_j \rangle = (hc/2q) (j + 1)$ is

quantized in units of $hc/2q$, while the flux $\Phi_j = (hc/q) (j + 1/2)$ enclosed by the successive Landau orbits steps in units of hc/q . Thus the experimental discovery of the fundamental unit of flux quantum as $hc/2q$ instead of as hc/q need not necessarily imply the existence of "electron pairing" in the superconductive state, as predicted by the BCS theory. It could very well be that this unit of flux is tied to the fact that the system is not in the Landau diamagnetic regime but is in the De Hass-Van Alphen regime of temperatures, as in the case studied here. However, what is discussed here is not a theory of superconductivity, and there is nothing in our formalism that will predict the field expulsion and the start up of the supercurrent, i.e., the Meissner effect, for example.

VII. CHARGE-MONOPOLE COMPOSITES OR DYONS, SPIN AND STATISTICS

We have seen that according to Wilczek, rotations around the z-axis, i.e., changes in the azimuthal angle φ are generated by the z-component of the Feynman's "angular momentum operator" $L_F = r \times \mu v = r \times p - (q/c) r \times A$. This is really a magnetic moment operator, *and is in no sense an angular momentum operator*. In classical mechanics this pseudovector L_F is a true physical dynamical variable, independent of the gauge. But in quantum mechanics, all Hermitian operators corresponding to these true physical dynamical variables depend on the gauge chosen. Thus, in the Hamiltonian quantum mechanics, the canonical momentum $p = \mu v + qA/c$ and the canonical orbital angular momentum operator $L = r \times p$ are the same in all the gauges, and hence are gauge covariant operators. This is due to the fact that the commutation relations of Eq.(2) and (3) are the same in all gauges. Using this Feynman's "angular momentum operator" in the Schrodinger's spatial representation where

$\mathbf{p} = -\hbar\nabla$, one obtains the relation of Eq.(6). Here, of course, we have assumed *a priori* that the problem under study, regardless of the fact that the generalized potential $U = -q\mathbf{A}\cdot\mathbf{v}/c$ of this problem is entirely velocity dependent, has indeed an exact Schrodinger's spatial representation. If we further assume that the azimuthal dependence of the particle wave function is $\psi_m \propto \exp(im\phi)$, m integer for continuity and single valuedness for rotations through 2π , then Eq.(6) yield the Wilczek¹⁰ result

$$L_{FZ} = (\text{integer})\hbar - q\Phi/2\pi c. \quad (65)$$

If now we assume that a single magnetic monopole of magnetic charge q_m existed, then $\nabla\cdot\mathbf{B} = 4\pi q_m$. Then by Gauss's divergence theorem $\int \nabla\cdot\mathbf{B} dV = \int \mathbf{B}\cdot d\mathbf{S}$. Thus, the total flux $\Phi = 4\pi q_m$. Using these, Wilczek argues that for a charged particle orbiting infinitesimally above this pole the flux is $2\pi q_m$ while for a particle infinitesimally below it is $-2\pi q_m$. The corresponding shifts in Eq.(65) are $L_{FZ} = (\text{integer})\hbar - (\pm qq_m/c)$. For these two spectra of L_{FZ} values to agree one must have $(qq_m/c) = [(1/2) + \text{integer}]\hbar$. This is the Dirac quantization condition.²⁴

In a profound paper on a magnetic model of matter Schwinger²² presents a conceivable dynamical interpretation of the subnuclear world. This he erects on the basis of the speculative but theoretically well-founded hypothesis that electric and magnetic charge can reside on a single particle. He names these dual-charged particles as dyons. Here, he first shows that the generalized Maxwell's equations are invariant for rotation through an arbitrary angle θ , and the observed absence of magnetic charge is described as the coexistence of electric and magnetic charge in the universal ratio $q_m/q = \tan\theta$. Then he

considers the nonrelativistic behavior of a particle with mass μ , carrying an electric charge q_1 and magnetic charge q_{m1} , which moves with velocity v in the field of a stationary body that possesses charges q_2 and q_{m2} . From Newton's equation of motion, he recognizes the conserved angular momentum vector as

$$J = r \times \mu v - (q_1 q_{m2} - q_2 q_{m1}) r / cr \quad (66)$$

The quantization of the component of this angular momentum along the connecting line of the particles then gives the charge quantization condition

$$[(q_1 q_{m2} - q_2 q_{m1}) / c] = 2\pi n, \quad (67)$$

where n is an integer. According to Schwinger, if there are *dual-charged* particles, rather than electrically charged particles and magnetically charged particles, the quantization of $qq_m / \hbar c$ must be *even integers*. But according to Dirac and Wilczek $qq_m / \hbar c = 1/2 + \text{integer}$. It appears that Witten³² is comfortable with both these points of view, i.e., either integer or half-integer value for zero vacuum angle θ . Dirac²⁴ arrives at half-integral values simply by quantizing the uncertainty in the electrodynamic action integral according to Bohr's quantization condition of Eqs.(7) and (8) with $n_0 = 0$, or equivalently, by using the condition that the wavefunction must be single valued. Wilczek's explanation¹⁰ of the relationship between the Dirac condition and the spin shift of $1/2$ unit, in terms of the continuity in the L_{FZ} spectrum is also very appealing. However, Schwinger²² also gives a simple argument in behalf of integer quantization. This argument is based on the fact that matter is normally magnetically neutral, and when the electromagnetic angular momentum

of Eq.(66) with $q_2 = 0$ is made to vanish by a compensating magnetic charge $-q_{m2}$ brought in from infinity to the origin so as to neutralize the magnetic charge q_{m2} at the origin, the net mechanical angular momentum $\mathbf{r} \times \mu \mathbf{v}$ of Eq.(66) must take only integral values. On shifting the viewpoint between the physically equivalent two-particle system and the three-particle system with an infinitely remote compensating charge, paradoxical transitions between $(\text{integer} + 1/2)$ and integer values of the angular momentum will be avoided if $qq_m/\hbar c$ is restricted to integral values. These Schwinger's arguments rely on quantizing the action integral based on classical Newton's equations of motion. When one has a non-zero vector potential $\mathbf{A}$ with a vanishing $\mathbf{B}$, then these arguments for $\mathbf{r} \times \mu \mathbf{v}$ may be incorrect. We wish to caution the readers about such situations. Furthermore, Schwinger introduces a spin $\mathbf{S}$ by defining the canonical momentum $\mathbf{p}$ as

$$\mathbf{p} = \mu \mathbf{v} - \mathbf{S} \times \mathbf{r} / r^2, \quad (68)$$

where

$$\mathbf{S} \cdot \mathbf{r} / r = -(q_1 q_{m2} - q_2 q_{m1}) / c. \quad (69)$$

With this definition of $\mathbf{S}$, Eq.(66) yields the familiar relation $\mathbf{J} = \mathbf{r} \times \mathbf{p} + \mathbf{S} = \mathbf{L} + \mathbf{S}$. The properties of $\mathbf{p}$ and $\mathbf{S}$ are indeed those suggested by this familiar combination involving the orbital angular momentum vector $\mathbf{L}$. Thus, one sees that the electromagnetic angular momentum of Eq.(66), which is a consequence of the velocity dependent interaction between the particles, plays the role of a spin angular momentum. Here, we have used the fact that $d(\mathbf{r}/r)/dt = (\mathbf{v}/r) - \mathbf{r}(\mathbf{r} \cdot \mathbf{v})/r^3 = \mathbf{r} \times \mathbf{v} \times \mathbf{r}/r^3$. Schwinger then goes on to construct a picture in which hadronic matter is viewed as a magnetically neutral

composite of dual-charged particles or dyons that are based electrically on a new unit of charge $q_0 = q/3$ (i.e., one-third the usual electronic charge), and $q_{m0} = q_m/3$.

For the Landau cyclotron motion

$$\begin{aligned} \mathbf{p} &= \mu\mathbf{v} + q\mathbf{A}/c = \mu\mathbf{v} + (q/2c) \mathbf{B}\times\mathbf{r} = \mu\mathbf{v} + (q/2\pi c) (\pi r^2 \mathbf{B})\times\mathbf{r}/r^2 \\ &= \mu\mathbf{v} + (q\Phi/2\pi c) i_z \times \mathbf{r}/r^2 = \mu\mathbf{v} - \mathbf{S}\times\mathbf{r}/r^2, \end{aligned} \quad (70)$$

where, in analogy with Eq.(68), the spin operator for this problem is $\mathbf{S} = -(q\Phi/2\pi c)i_z$, and Φ is the magnetic flux. Then, $\mathbf{r}\times\mathbf{p} = \mathbf{r}\times\mu\mathbf{v} - \mathbf{r}\times\mathbf{S}\times\mathbf{r}/r^2 = \mathbf{r}\times\mu\mathbf{v} - \mathbf{S}(\mathbf{r}\cdot\mathbf{r})/r^2 + \mathbf{r}(\mathbf{r}\cdot\mathbf{S})/r^2$. That is the total angular momentum,

$$\mathbf{J} = \mathbf{r}\times\mathbf{p} + \mathbf{S} = \mathbf{L} + \mathbf{S} = \mathbf{r}\times\mu\mathbf{v} + \mathbf{r}(\mathbf{r}\cdot\mathbf{S})/r^2 = \mathbf{L}_F + \mathbf{r}(\mathbf{r}\cdot\mathbf{S})/r^2. \quad (71)$$

Equation (70) for the Landau cyclotron motion is equivalent to Eq.(68) of Schwinger's charge-monopole composite. Here again, $\mathbf{p}$ and $\mathbf{S}$ of Eqs.(70) and (71) are those suggested by the familiar combination involving the orbital angular momentum vector $\mathbf{L}$. Thus we see that the Wilczek's flux factor $(-q\Phi i_z/2\pi c)$ plays the role of the spin operator $\mathbf{S}$ for the charged particle-flux tube composites, just the same way as Schwinger's $\mathbf{S}$ defined by Eqs.(68) and (69) plays the role of spin for the charge-monopole composites or dyons. We note that $[\mathbf{r}\times(\mathbf{p} - \mu\mathbf{v})] = (q/c) \mathbf{r}\times\mathbf{A} = (qB/2c) \mathbf{r}\times i_z \times \mathbf{r} = (qB/2c) [r^2 i_z - \mathbf{r}(\mathbf{r}\cdot i_z)] = (q\Phi/2\pi c) i_z - (qB/2c) \mathbf{r}(\mathbf{r}\cdot i_z) = -\mathbf{S} - (q/2c) \mathbf{r}(\mathbf{r}\cdot\mathbf{B}) = -\mathbf{S} - (q/2c) \mathbf{r}(\mathbf{r}\cdot\nabla\times\mathbf{A})$. That is, for this Landau cyclotron motion, we have

$$\mathbf{S} = -\mathbf{r}\times(\mathbf{p} - \mu\mathbf{v}) - (1/2) \mathbf{r} [\mathbf{r}\cdot\nabla\times(\mathbf{p} - \mu\mathbf{v})], \quad (72)$$

where $(\mathbf{p} - \mu\mathbf{v}) = q\mathbf{A}/c$. Also from Eqs.(68) or (70), we get

$$\mathbf{S} = -\mathbf{r} \times (\mathbf{p} - \mu\mathbf{v}) - \mathbf{r} (\mathbf{r} \cdot \mathbf{S})/r^2. \quad (73)$$

We pointed out earlier that when the generalized potential U of any physical problem is velocity dependent, $\mathbf{p} \neq \mu\mathbf{v}$. In this case, following Schwinger, we can define a generalized spin operator $\mathbf{S}$ for such problems with the aid of Eqs.(72) and / or (73). Such a definition of spin might prove useful in examining the connection between the spin and statistics of composite particles, such as for example, the charge-monopole composites discussed by Goldhaber,²⁵ charged particle-flux tube composites discussed by Wilczek¹⁰. Here "statistics" is used, in the sense of Goldhaber, to mean "permutation symmetry of indistinguishable-particle wave functions." Further, if this spin $\mathbf{S}$ takes integral values, then the total angular momentum $\mathbf{J}$ is integral, and the problem will have an exact Schrodinger's spatial representation. However, if $\mathbf{S}$ is half-integral, then the problem under study has no exact spatial representation. For the Landau gyration states, $\mathbf{S} = -(q\Phi/2\pi c)\mathbf{i}_z = -i_z(n - 1/2)\hbar$, using the Φ_n of Eqs.(10) or (57). Thus, we see that these Landau gyration states do not have an exact Schrodinger's spatial representation.

We wish to conclude this section by pointing out some of the difficulties that one may encounter in the spin-statistics connection of physical problems which have velocity-dependent generalized potentials U . First and foremost, one does not know *a priori* that this problem indeed has an exact Schrodinger's spatial representation. Only in this representation, we have $\mathbf{p} = -i\hbar\nabla$, and $\mathbf{L} = -i\hbar\mathbf{r} \times \nabla$, and the ordinary spherical harmonics are the eigenfunctions of the orbital angular momentum operator. As Goldhaber²⁵ states, "The *usual* connection between spin and statistics allowed for point particles in local

field theory is equivalent to the requirement that a pair of distinguishable particles have an even $L + S$, where L is the relative orbital angular momentum and S is the total spin of the pair. This may be reduced to the common statement about symmetry or antisymmetry of wavefunctions under interchange of particle coordinates, provided certain phase conventions are established. In particular, one depends on the choice of angular momentum eigenfunctions which acquire a phase $(-1)^L$ upon inversion of relative position coordinates $\mathbf{r}$. Ordinary spherical harmonics exhibit this behavior. They are eigenfunctions of the orbital angular momentum operator $-i\hbar \mathbf{r} \times \nabla$, meaning that the relative momentum operator is $-i\hbar \nabla$. However, magnetic effects imply the existence of velocity-dependent potentials which may (and in this case do) alter the phase produced by the inversion of $\mathbf{r}$. For our case, for example, it appears that in the cylindrically symmetric gauge, the Landau cyclotron motion about its guiding center cannot have an exact Schrodinger's spatial representation. This is apparent from the fact that the z-component of its canonical angular momentum takes on half-odd-integral values, the g-value is two and not one, and has a zero-point gyration state whose behavior is analogous to that of the *Zitterbewegung* of the Dirac's free particle so that spin $1/2$ is its orbital angular momentum.^{16,15} However, Landau¹³ has shown that this problem has a Schrodinger's spatial representation in the cylindrically nonsymmetric gauge. In this representation, the eigenfunctions are the Hermite polynomials, and the motion is a one-dimensional harmonic motion instead of the actual circular harmonic motion. Hence, in this representation it is not clear whether one can choose the ordinary spherical harmonics, which are the orbital angular momentum eigenfunctions, as also the eigenfunctions for energy with the correct Landau energy eigenvalues. Alternatively, it would appear that some linear combination of the Hermite polynomials in x and y coordinates with a phase difference of $\pi/2$ may play an important role in the study of motions whose total orbital angular momentum takes half-odd-integral values of $\hbar$.

VIII. SUMMARY AND CONCLUSIONS

We have presented a rather critical but theoretically well-founded comprehensive analysis of the angular momentum, magnetic moment, g -value, and the magnetic flux associated with the Landau gyration states. This Landau problem is a particular case of a more general class of problems where the generalized potential U of their Lagrangian is velocity dependent. For such problems, the canonical momentum p is not equal to the mechanical momentum μv . In the literature, there is considerable confusion as to how one should define the orbital angular momentum L of such systems. Is it $r \times p$ or is it $r \times \mu v$? In general, in quantum mechanics, do these problems with the velocity-dependent U have an exact Schrodinger's spatial representation always? What is the g -value of such problems? Is it the classical value of one, or is it like spin with the value two, or is it something else³³? In this paper we have presented theoretically well-founded answers to these and other somewhat related problems. These answers may appear very surprising, just as many physicists were rather shocked when Bohm and Aharonov first brought up the issue that the vector potential A is a "real" field that is a measurable quantity. Indeed, all the results discussed here are in some sense connected to this Aharonov-Bohm effect. As Feynman says,² it is interesting that something like this can be around for many years but, because of certain prejudices of what is and is not significant, continues to be ignored. We also find that the possibility exists that the experimentally observed fundamental unit of flux quantum as $hc/2q$ instead of as hc/q need not necessarily imply the existence of "electron pairs" of the BCS theory of superconductivity, a rather controversial, speculative, and outrageous result. Finally, we find that, for velocity-dependent generalized potential U , the particle-field interaction contribution to the total angular momentum plays the role of a spin angular momentum S which is determined by the departure $(p - \mu v)$, in agreement with Schwinger's dyon theory of hadrons. According to Goldhaber and also

Wilczek, this spin will determine the statistics of the system.

IX. ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy Contract No. DE-AC02-76-CHO-3073. I thank Professor Arthur S. Wightman for his 1966-comments on some parts of this problem.

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