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NOTICE

ON THE CONDITION FOR A SINGLE BUNCH HIGH FREQUENCY FAST BLOW-UP

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ABSTRACT

We study the longitudinal stability of a single particle bunch in a storage ring using Vlasov equation. We show that the Vlasov equation has solutions corresponding to a fast, microwave instability if a condition on the beam current, qualitatively similar to the stability condition for a coasting beam, is satisfied. This condition can be used to define a threshold current, and to discuss its dependence on the longitudinal coupling impedance.

MASTER

1. INTRODUCTION

Single bunch longitudinal instabilities producing an increase of the bunch length and energy spread have been observed in proton and electron synchrotrons and storage rings. Signal at microwave frequencies are observed during the bunch blow-up and because of this the effect has also been called the microwave instability.

A simple and suggestive physical model has been proposed by Boussard¹⁾ to explain this instability. His starting point is the observation that at the high frequencies of the signals detected during the instability the bunch can be considered as a coasting beam having a current equal to the bunch peak current. It is then possible to apply the coasting beam instability theory²⁻⁵⁾ to the single bunch, obtaining a qualitative agreement with the observations.

Another condition that must be satisfied in order to use the coasting beam theory for the bunch is that the instability rise time must be shorter than the synchrotron frequency. Using these two conditions, high frequencies and fast rise time, many authors⁶⁻¹¹⁾ have tried to give a theoretical foundation to the Boussard conjecture. However, in our opinion, all these works use such approximations that they do not give a sufficiently firm base for understanding the single bunch fast blow-up

In this paper we study again this problem and show that the Vlasov equation describing the bunch and its electromagnetic interaction with the beam environment, has solutions corresponding to the condition of a "microwave fast instability". We derive a condition for the existence of these solutions, which defines a threshold current. This definition of threshold is conceptually different from the one used in most instability theories, as it will be discussed in section 4. The expression of this threshold current is similar to that obtained by Boussard, but gives us more information on the effect of the environment impedance and the most dangerous frequency range.

2. THE VLASOV EQUATION

We write the equations of synchrotron oscillations using the azimuthal angle ϕ , and the relative energy deviation from the synchronous particle, ϵ as¹²⁾

$$\dot{\phi}(t) = \alpha \omega_0 \epsilon(t) \quad (1)$$

$$\dot{\epsilon}(t) = -\frac{\omega_0}{\alpha} \frac{2}{\omega_0} \phi(t) + \xi(t) \quad (2)$$

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where γ is the momentum compaction factor, ν_0 the revolution frequency, ν_0 the small amplitude synchrotron tune and ξ the energy loss per revolution due to the interaction of the beam with its environment. In equation (2) we have linearized the force due to the RF system.

The quantity ξ can be written in terms of the longitudinal coupling impedance, $Z(\omega)$, as

$$\xi(t) = -\frac{e^2 \omega_0^2}{4\pi^2 E} \sum_{m=-\infty}^{+\infty} Z[(m-\nu_0)\omega_0] \times \int d\phi \lambda(\phi, t) e^{-i(m-\nu_0)[\phi - \phi(t)]} \quad (3)$$

where E is the beam energy and $\lambda(\phi, t)$ is the bunch longitudinal density distribution which is assumed to oscillate at a coherent frequency ν ,

$$\lambda(\phi, t) = \lambda(\phi) e^{i\nu\omega_0 t} \quad (4)$$

The sum over the index m represents the sum over all the frequencies which are excited by the bunch in its periodic motion around the ring, namely all the multiples of the revolution frequency plus or minus the coherent oscillation frequency ν .

It is convenient to rewrite the equations of motion (1), (2) using the action-angle variable I, θ , related to ϕ, ϵ by the transformation function

$$F = -\frac{1}{2} \nu_0 \omega_0 \phi^2 \tan \theta \quad (5)$$

or

$$\phi = \left(\frac{2I}{\nu_0 \omega_0} \right)^{1/2} \cos \theta \quad (6)$$

$$\epsilon = - \left[(2\nu_0 \omega_0 I)^{1/2} / \alpha \omega_0 \right] \sin \theta \quad (7)$$

Introducing the "potential" U such that

$$\xi = - \frac{\partial U}{\partial \phi} \quad (8)$$

the Hamiltonian of the system can be written as

$$H(I, \theta, t) = \nu_0 \omega_0 I + U(\theta, I, t) \quad (9)$$

Using the Hamiltonian (9) we can write the Vlasov equation as

$$\frac{\partial f}{\partial t} + \theta \frac{\partial f}{\partial \theta} + I \frac{\partial f}{\partial I} = 0 \quad (10)$$

$$\dot{\theta} = \frac{\partial H}{\partial I} \quad (11)$$

$$\dot{I} = - \frac{\partial H}{\partial \theta} \quad (12)$$

If $\xi=0$ a steady state solution of (10) is given by any function of I , $f=f_0(I)$. When ξ is different from zero but is still small, we can look for a solution of (10) of the form

$$f(I, \theta, t) = f_0(I) + f_{01}(I, \theta) + f_1(I, \theta) e^{i\nu\omega_0 t} \quad (13)$$

f_{01}, f_1 being of order ξ .

The function f_{01} describes the change in the equilibrium distribution due to the interaction with the environment. In this paper we are not interested in studying this change and we assume it to be small. The term f_1 describes the stability properties of the system. Neglecting terms of second order in ξ the equation for f_1 can be written as

$$i\nu\omega_0 f_1 + \nu\omega_0 \frac{\partial f_1}{\partial \theta} + \eta(I, \theta, f_1) \frac{df_0}{dI} = 0 \quad (14)$$

Using eq. (3) with

$$\lambda(I, t) = \int d\epsilon f_1(\epsilon, \theta) e^{i\nu\omega_0 t} \quad (15)$$

we obtain

$$\eta(I, \theta, f_1) = + \frac{e^2 \omega_0^2}{4\pi^2 E} \frac{1}{2} \sqrt{2\nu\omega_0 I} \sin \theta \sum_{m=-\infty}^{+\infty} Z[(m-\nu)\omega_0] e^{i\nu\omega_0 t} \times \int d\bar{I} d\bar{\theta} f_1(\bar{I}, \bar{\theta}) e^{-i(m-\nu)[\lambda(\bar{I}, \bar{\theta}) - \varphi(I, \theta)]} \quad (16)$$

with $\varphi(I, \theta)$ given by (6).

Since both the functions f_1 and η are periodic in θ , eq. (14) can be integrated to give

$$f_1(I, \theta) = - \frac{e^{-i\eta\theta}}{\epsilon^{i2\pi\Omega-1}} f'_0(I) \int_a^{\theta+2\pi} d\alpha' \eta[I, \theta', f_1(I, \alpha')] e^{i\Omega\theta'} \quad (17)$$

where

$$\Omega \equiv \nu/\nu_0 \quad (18)$$

Defining

$$\rho_m \equiv \int dI d\theta f_1(I, \theta) e^{-i(m-\nu)\varphi(I, \theta)} \quad (19)$$

and using (16), eq. (17) becomes

$$\rho_m = \sum_{n=-\infty}^{\infty} T_{m,n} \rho_n \quad (20)$$

where

$$T_{m,n} = - \frac{e^2 \omega_0^2}{8\pi^2 E} Z[(m-\nu)\omega_0] \frac{e^{i2\pi\Omega}}{\epsilon^{i2\pi\Omega-1}} U_{m,n} \quad (21)$$

$$U_{m,n} = \int d\bar{\epsilon} d\bar{I} e^{-i(n-\nu)\varphi(I, \theta)} \sqrt{\frac{2I}{\nu_0 \omega_0}} f'_0(I) \int_{-2\pi}^0 d\theta' \sin(\theta+\theta') e^{i(m-\nu)(2I/\nu_0 \omega_0)^{1/2} \cos(\theta+\theta')} e^{i\Omega\theta'} \quad (22)$$

The solutions of our problem are the values of ν for which the equation

$$\det \left\{ \delta_{m,n} - T_{m,n} \right\} = 0 \quad (23)$$

is satisfied.

We emphasize here that, contrary to the coasting beam case, in a bunched beam all frequency components $m\omega_0$ are intrinsically coupled. In other words, it is not possible to excite just on single frequency component. In this coupling, or equivalently in the fact that the matrix $T_{m,n}$ is not diagonal, lies all the difficulty of solving the bunched beam problem.

3. FAST BLOW-UP REGIME

The problem under study contains several characteristic times, which allow us to define different regimes. The characteristic times are:

- i) the revolution period, $T_0 = 2\pi/\omega_0$;
- ii) the synchrotron period, $T_s = 2\pi/(\nu_0\omega_0)$;
- iii) the period of the perturbing e.m. field, T_0/m ;
- iv) the bunch duration, $\tau = L/c$, L being the bunch length;
- v) the instability rise time, $\tau_i = -1/\text{Im}(\nu\omega_0)$.

We want to study the "microwave, fast blow-up regime" defined by

$$T_0/m < \tau_i < T_s \quad (24)$$

$$T_0/m < L/c \quad (25)$$

Using Ω and the ring radius $R = T_0 c / 2\pi$ we can also write the definition of the microwave fast blow-up regime as

$$\frac{m}{\nu_0} > \text{Im}(\nu) > 1 \quad (26)$$

$$m \frac{L}{2\pi R} > 1 \quad (27)$$

The meaning of these two conditions is (i) the instability rise time is short compared to the synchrotron oscillation period and long compared to the period of the perturbing electromagnetic field; (ii) the wavelength of the perturbation is short compared to the bunch length.

We want now to show that the system (20) has an eigenvalue satisfying (26), (27), if a condition of the form

$$e I_0 \left| \frac{Z(n)}{n} \right|_{\text{eff}} \geq \frac{\pi^{3/2}}{2^{1/2}} E \alpha \langle \epsilon^2 \rangle \quad (28)$$

is fulfilled. In (28) I_0 is the average bunch current, $|Z(n)/n|_{\text{eff}}$ is a function of the bunch length and of the longitudinal impedance to be defined later and $\langle \epsilon^2 \rangle$ is the mean square particle energy distribution. Hence eq. (28) can be used as a condition for the fast blow-up, microwave instability to occur.

To establish (28), let's simplify the matrix $T_{m,n}$ as given by (21) and (22) in the short wave fast blow-up limit (26), (27). The factor $e^{iQ\theta}$ in the integrand of (22) is very small for θ' away from and smaller than zero. As a consequence, we can approximate the functions $\sin(\theta+\theta')$ and $\cos(\theta+\theta')$ in the same integrand by $\sin(\theta+\theta') \approx \sin \theta$ and $\cos(\theta+\theta') \approx \cos \theta - \theta' \sin \theta$. (The term $\theta' \cos \theta$ from $\sin(\theta+\theta')$ makes a contribution which is small by a factor $1/m$, and so we shall neglect it.) Also we can take the limits of

integration in θ' to be from $-\infty$ to 0. The result is

$$U_{m,n} = -i \int d\delta dI e^{i(m-n)\phi(I,\theta)} f'_0(I) \sqrt{\frac{2I}{v_0 \omega_0}} \sin \theta \frac{1}{\Omega - \sin(m-n) \sqrt{\frac{2I}{v_0 \omega_0}} \sin \theta} \quad (29)$$

Changing the variables of integration in (29) into ϕ and ϵ , (29) becomes

$$U_{m,n} = i \frac{1}{v_0 \omega_0} \int_{-\infty}^{\infty} d\epsilon \frac{1}{\Omega + (m-n) \frac{\alpha}{v_0} \epsilon} \int d\phi e^{i(m-n)\phi} \frac{\partial f_0}{\partial \epsilon} \quad (30)$$

Let's now illustrate the solution of (20), (21) and (30) with the special case of a Gaussian bunch:

$$f_0(\phi, \epsilon) = f_0(I) = N \lambda_0(\phi) G(\epsilon) \quad (31)$$

$$\lambda_0(\phi) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\phi^2/2\sigma^2} \quad (32)$$

and

$$G(\epsilon) = \frac{1}{\sqrt{2\pi} \sigma} \left(\frac{\alpha}{v_0} \right) e^{-\left(\frac{\alpha \epsilon}{v_0} \right)^2 / 2\sigma^2} \quad (33)$$

$\lambda_0(\cdot)$ and $G(\epsilon)$ are normalized to one.

Defining

$$\lambda_n \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-in\phi} \lambda_0(\phi) = \frac{1}{2\pi} e^{-\frac{1}{2} \sigma^2 n^2} \quad (34)$$

Eq. (21) becomes

$$T_{mn} = -i \frac{e^2 \omega_0^2}{4\pi^2 E} \frac{N}{v_0 \omega_0} \int d\epsilon \frac{G'(\epsilon)}{\Omega + m \frac{\alpha}{v_0} \epsilon} 2\pi \lambda_{n-m} Z(m\omega_0) \quad (35)$$

From now on we use (26) to approximate $m-n \approx m$ (as was already done in (35)), wherever possible.

It is interesting to compare the equations (20) and (35) with the corresponding result²⁾ for a coasting beam.

$$1 = -i e \frac{N \omega_0^2}{4\pi^2 E} \int d\epsilon \frac{G'(\epsilon)}{\omega + n \alpha \omega_0 \epsilon} Z(n\omega_0) \quad (36)$$

We note that if $2\pi \lambda_{n-m} = \delta_{n,m}$ then (20) and (35) reduce to (36).

Equation (35) can be rewritten as¹³⁾

$$T_{mn} = -i \frac{e^2 \alpha}{4\pi^2 E} \frac{N}{(v_0 \sigma)^2} h\left(\frac{\alpha}{v_0 \sigma}\right) e^{-\frac{1}{2} \sigma^2 (n-m)^2} \frac{Z(m\omega_0)}{m} \quad (37)$$

$$h(K) = \int_0^{\infty} dx x e^{-i K x - \frac{1}{2} x^2} \quad (38)$$

It can be seen easily from (38) that

$$\left. \begin{aligned} h(K) &= 1, & \text{if } K &= 0 \\ |h(K)| &< 1, & \text{if } K \neq 0 \text{ and } K_I \leq 0 \end{aligned} \right\} \quad (39)$$

where K_I is the imaginary part of K .

4. CONDITIONS FOR FAST BLOW-UP

Before deriving the threshold condition for the microwave fast blow-up in a bunched beam, it is useful to illustrate our method by showing how one can obtain the well known threshold condition for a coasting beam³⁻⁵⁾ from eq. (36).

Taking,

$$G(\epsilon) = \frac{N}{\sqrt{2\pi\langle\epsilon^2\rangle}} e^{-\epsilon^2/2\langle\epsilon^2\rangle}$$

in (36), one obtains

$$1 = ie \frac{I_0}{2\pi E \langle\epsilon^2\rangle \alpha} \frac{Z}{n} h\left(\frac{w}{|n| \alpha \omega_0 \langle\epsilon^2\rangle^{\frac{1}{2}}}\right), \quad (40)$$

where $I_0 \equiv eN\omega_0/2\pi$.

The usual threshold condition is nothing but a sufficient condition that eq. (40) cannot be satisfied for $w_I < 0$, and therefore an instability cannot occur. Thus, if we define I_{th} , the threshold current, by

$$1 = \frac{I_{th} e}{2\pi E \langle\epsilon^2\rangle \alpha} \left| \frac{Z}{n} \right| \quad (41)$$

then, because of (39), eq. (40) cannot be satisfied for $w_I \leq 0$ and $I_0 < I_{th}$.

We apply here a similar method for the microwave fast blow-up of a bunched beam, i.e. eqs. (23) and (37).

In order to draw conclusions from (23) and (37), we have to know the longitudinal impedance $Z(n\omega_0)$.

It has been pointed out¹⁴⁾ that, in the high frequency domain of interest for the microwave instability, the impedance function $Z(\omega)$ for many existing machines including AGS, ISR, SPEAR etc. can be roughly approximated by a giant resonance with unity Q-factor. The resonance frequency is taken to be the cut off frequency ω_c of the ring,*

$$z(n) \approx -1 \frac{\omega_c R_s}{i n \omega_c + (\omega^2 - \omega_c^2)} \quad (42)$$

The parameter R_s varies from one accelerator to another. We shall base our discussions on the approximate expression (42).

We take the microwave region to be that satisfying

$$w = \Omega + n\omega_0 \approx n\omega_0 \gg \omega_c$$

* Defined as $\omega_c \approx c/d$, d being the vacuum chamber dimension.

Suppose we wish to find a coherent frequency near $\omega = n_0 \omega_0$; $n_0 \omega_0 \gg \omega_c$. We see that $Z(n\omega)/n$ as given by (42) is approximately constant in the interval $n_0 - \sqrt{2T}\sigma \leq n \leq n_0 + \sqrt{2T}\sigma$, provided $1/\sqrt{2}\sigma \omega_0/\omega_c \ll 1$. We assume the last inequality to hold.

So we approximate (37) by

$$T_{m,n} = -i \frac{e}{2\pi E} \frac{I_0}{\alpha \langle \epsilon^2 \rangle} h\left(\frac{\Omega}{\sigma |n_0|}\right) e^{-\frac{1}{2}\sigma^2 (n-m)^2} \frac{Z(n_0 \omega_0)}{n_0} \quad (43)$$

where $\langle \epsilon^2 \rangle = (\sigma v_0/\alpha)^2$ and $I_0 = \frac{N \omega_0}{2\pi}$ have been used.

If the matrix T has an eigenvalue which is equal to unity for some Ω with $-I_0 \Omega \gg 1$, there will be a coherent instability with that frequency.

As an approximation we shall extend the range of n and m in (43) to be $-\infty < n, m < +\infty$. The most likely fast instability corresponds to the biggest eigenvalue of the matrix $T_{m,n}$. It is easy to see that the maximum eigenvalue of a matrix with $\exp[-(n-m)^2 \sigma^2/2]$ as its elements is given by

$$g(\sigma) = \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}n^2 \sigma^2} = \frac{\sqrt{2\pi}}{\sigma} \sum_{n=-\infty}^{\infty} e^{-2n^2 \pi^2/\sigma^2} \quad (44)$$

Therefore the condition for fast blow up becomes

$$-i \frac{e^2}{2\pi E} \frac{1}{\alpha \langle \epsilon^2 \rangle} h\left(\frac{\Omega}{\sigma |n_0|}\right) \frac{Z(n_0 \omega_0)}{n_0} I_0 g(\sigma) = 1 \quad (45)$$

Let us now define the threshold current I_{ch} by

$$\frac{e^2}{2\pi E} \frac{I_{ch}}{\alpha \langle \epsilon^2 \rangle} \left| \frac{Z(n_0 \omega_0)}{n_0} \right| = 1 \quad (46)$$

If $I_0 g(\sigma) < I_{ch}$, then we see from (39) that (45) does not have a solution with $\Omega_I < 0$. That means under this condition there is no microwave fast blow up. (We use the notation, $\Omega = \Omega_R + i\Omega_I$).

If $I_0 g(\sigma) = I_{ch}$, then (45) has a solution only for

$$\frac{\Omega_I}{n_0 \sigma} = 0^- \quad (47)$$

when the function $h(\Omega/\sigma |n_0|) = 1$. (Remember that $|\Omega_R| \ll -\Omega_I$.)

In this sense, (47) together with (24) and (25) define the threshold conditions for microwave fast blow up in analogy with the conventional slow blow-up threshold condition for bunched beams,

$$\frac{\Omega_I}{\omega_0} = 0^- \quad (48)$$

Note that the perturbing wavelength λ (in radians) is related to n_0 by $\lambda = 2\pi/n_0$. The quantity $a = \Omega_I/2\pi\sigma = \Omega_I/|n_0|\sigma$ determines the current needed to sustain a fast blow-up with

growth rate Ω_I and wavelength λ . The greater the quantity a , the bigger the current needed for that instability. I_{ch} measures the minimum current needed for any microwave fast blow-up to occur.

The quantity a_I is the product of Ω_I , which gives the ratio of the synchrotron oscillation period to the instability rise time, and λ/σ whose inverse gives the number of waves of the perturbing electromagnetic field in the bunch length. For microwave perturbation, λ/σ is small, and it is possible to have small a even if Ω_I is large.

Let us look into the properties of $I_0 g(\sigma)$. If σ is large, we can just keep the leading term in the first equality of (44), and obtain

$$I_0 g(\sigma) \approx I_0$$

On the other hand, if σ is small then from the second equality of (44) we have

$$I_0 g(\sigma) \approx I_{peak}$$

(Remember that for a Gaussian bunch the peak current I_{peak} and the average current I_0 are related by $I_{peak} = \sqrt{2\pi}/\sigma I_0$).

We have reached the conclusions that (47) together with (24) and (25) define the threshold conditions for microwave fast blow-up with the corresponding threshold current given by (46).

It is worth emphasizing that even if the current is below the microwave fast blow-up threshold, we still have to worry about the slow blow-up as defined by $-\Omega_I \leq 1$

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