ANALYTIC SOLUTION FOR THE PROBLEM OF GRIDDED GAP-ELECTRON FLOW INTERACTION*

S. KHEIFETS, S. YU AND J. JAEGER

Stanford Linear Accelerator Center

Stanford University, Stanford, California 94305

1. Introduction

The present study is motivated by the modelling of high-power klystrons. The two basic components of a Klystron are the resonant cavities and the drift spaces. This paper addresses only the first of these two components. Modelling of the drift spaces is deferred to future work. The formulation is not restricted to Klystron modelling, but is applicable to any problem involving the interaction of an electron beam with a resonant cavity.

While the theory of Klystrons has been worked out in detail in the small signal limit, the problem remains largely unsolved when the signals are large. In particular, the hydrodynamic models of electron beams used to derive the small signal theories fail when particle trajectories cross each other. In this paper, we employ a Vlasov description of the electron beam to study the Klystron problem. In the Vlasov formulation we follow the evolution of the electron distribution function in phase-space. The general framework can naturally accommodate particle crossing, and the beam dynamics is accurately described even when the signals are large.

While the Vlasov formulation is equivalent in principle to a particle simulation, the mathematical structure of the Vlasov equations makes it relatively easy to build in the steady-state condition. Since in many Klystron problems we are interested mostly in the steady-state solution, the Vlasov description is very convenient. This is an advantage that a particle simulation does not share.

* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

The self-consistent solution of the system of Vlasov equations is found under the following assumptions:

- a) One dimensional (longitudinally) nonrelativistic particle flow.
- b) Electric field uniform in the longitudinal coordinate (gridded gap).

The solution is valid a) for an arbitrary particle distribution of the flow entering the gap, b) for any gap size, c) for all beam intensities, and d) for a broad class of time dependences of the electric field in the gap, although we will be studying in detail the special case of a resonant cavity with a single dominant frequency.

In section 2 the problem is formulated in terms of the Vlasov equations. In section 3 we present the solution of the Liouville equations for the initial value problem. Then in the next section a boundary value solution of the Liouville equation is studied. In section 5 the Maxwell equations are solved for a given current density. The results of sections 4 and 5 are combined in section 6 to produce a self-consistent solution for the Vlasov equations. In the limit of a small beam intensity and/or a small electric field the solution gives the same results as the small signal theory based on the hydrodynamic beam models. (Section 7.) We have also derived a general solution in the limit of small gap size. (Section 8.) The last sections contain a numerical example, comparison with known approximations and some conclusions as well as a discussion of possible applications of the suggested solution to the klystron problem. Some of the results presented in this paper are not new. The derivation has been included here for completeness.

2. The Vlasov Equations

The most general and exact description of the electromagnetic interaction of a particle flux with environment is given by a system of equations describing the evolution of the particle phase space distribution function and the electromagnetic field produced by particle charges. This system of equations is referred to usually as the Vlasov equations.^{1,2,3} For the nonrelativistic one dimensional problem considered here, the Vlasov equations are as follows:

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

ELB

MASTER

Consider the motion of an electron in the gap in z direction with the velocity $v = dz/dt$:

$$\frac{dv}{dt} = \frac{e}{m} E(t) \quad (2.1)$$

Here $E(t)$ is the z -component of the electric field assumed to depend on time only. The physical realization of such a field takes place in a gridded gap, for example.

The first two integrals of this equation are

$$v(t) = \frac{e}{m} \int_{t_0}^t E(t') dt' + v_0 \quad (2.2)$$

and

$$z(t) = z_0 + v(t - t_0) - \frac{e}{m} \int_{t_0}^t (t' - t_0) E(t') dt' \quad (2.3)$$

The evolution of a flux of electrons inside the gap can be described by a distribution function ψ of time t , coordinate z and velocity v : $\psi = \psi(z, v, t)$. The continuity equation in the phase space z, v is called the Liouville equation. In our case it looks like ($\mathcal{L}$ is an operator):

$$\mathcal{L}\psi \equiv \frac{\partial \psi}{\partial t} + v \frac{\partial \psi}{\partial z} + \frac{e}{m} E(t) \frac{\partial \psi}{\partial v} = 0 \quad (2.4)$$

Notice that v and z in this equation are considered as independent variables.

The electric field $E(t)$ in general can be produced by the charges and currents of the flux taking into account the environment as well as by external sources. Introducing the vector and scalar potentials $\vec{A}(\vec{r}, t)$ and $\phi(\vec{r}, t)$ we can describe the electromagnetic field by the following Maxwell equations:

$$\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}(\vec{r}, t) \quad (2.5)$$

$$\nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \quad (2.6)$$

$$\vec{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad (2.7)$$

The current density $\vec{j}(\vec{r}, t)$ in (2.5) in turn can be expressed as a sum of external current density $\vec{j}_{ext}$ (produced for example by an external RF generator) and the current density of the flux itself. In our case, we assume for simplicity that the electron current is concentrated on axis, and write

$$\vec{j}(\vec{r}, t) = \vec{j}_{ext} + \frac{\delta(\vec{r})}{2\pi r} I(z, t) \quad (2.8)$$

$$I(z, t) = e \int_{-\infty}^{\infty} dv \cdot v \psi(z, v, t) \quad (2.9)$$

The system of equations (2.4) through (2.9) are the Vlasov equations. The solution of this system satisfying all the necessary initial and boundary conditions is the self consistent solution of the problem. The search for such a solution is the subject of the present paper.

3. Initial Value Solution Of The Liouville Equation

We solve first the Liouville equation (2.4) assuming for the time being $E(t)$ as a given function of time. It is known, that any function of the integrals of motion is the solution of the Liouville equation. Hence the function

$$\psi(z, v, t) = \tilde{\psi}_0 \left(z - v \cdot (t - t_0) + \frac{e}{m} \int_{t_0}^t (t' - t_0) E(t') dt', \quad v - \frac{e}{m} \int_{t_0}^t E(t') dt', \right) \quad (3.1)$$

is a solution of equation (2.4). $\tilde{\psi}_0(z, v)$ corresponds to the initial distribution at $t = t_0$. It is easy to check by direct substitution, that (3.1) indeed satisfies (2.4). We will not do this here since we are interested in the solution of the boundary value problem rather than the initial value problem.

4. Boundary Value Solution Of The Liouville Equation

Suppose now that at $z = z_0$ the distribution function is given for all times and velocities:

$$\psi_0 = \psi_0(v, t) \quad (4.1)$$

We are interested now in finding a solution $\psi(z - z_0, v, t)$ of (2.4) which goes into (4.1) for $z \rightarrow z_0$. This solution will describe the evolution of ψ_0 in z , v and t . In particular, it will give us the distribution function $\psi(d, v, t)$ at the exit of the gap $z = z_0 + d$.

The aim is achieved in the following way. Introduce first the implicit function $\Theta(z - z_0, v, t)$ as a solution of the equation

$$F(z - z_0, v, t, \Theta) \equiv z - z_0 - v \cdot (t - \Theta) + \frac{e}{m} \int_{\Theta}^t (t' - \Theta) E(t') dt' = 0 \quad (4.2)$$

which satisfies the condition:

$$\Theta(0, v, t) = t \quad (4.3)$$

Introduce next the function

$$V(z - z_0, v, t) = v - \frac{e}{m} \int_{\Theta(z - z_0, v, t)}^t E(t') dt' \quad (4.4)$$

From (4.3) it follows immediately

$$V(0, v, t) = v \quad (4.5)$$

Then

$$\psi(z, v, t) = \psi_0 \left(V(z - z_0, v, t), \Theta(z - z_0, v, t) \right) \quad (4.6)$$

is such a solution of the Liouville equation (2.4) which goes into the boundary value (4.1) when $z \rightarrow z_0$. To prove this, note first of all that

$$\mathcal{L}\psi = \frac{\partial \psi}{\partial V} \cdot \mathcal{L}V + \frac{\partial \psi}{\partial \Theta} \mathcal{L}\Theta$$

Then it is easy to see that

$$\mathcal{L}V = \frac{e}{m} E(\Theta) \cdot \mathcal{L}\Theta ,$$

therefore the only thing we have to show is that

$$\mathcal{L}\Theta = 0 \quad (4.7)$$

Indeed, if (4.7) is true then $\mathcal{L}\psi = 0$ and (4.3) and (4.5) provide that $\psi|_{z=z_0} = \psi_0(v, t)$. To prove (4.7) find $\mathcal{L}F$ (which is 0 since $F = 0$)

$$\mathcal{L}F = V \cdot \mathcal{L}\Theta = 0$$

Hence (4.7) is true. In the particular case of a harmonic electrical field:

$$E_h(t) = E_0 \cos(\omega t + \varphi) \quad (4.8)$$

formula (4.4) gives:

$$V_h = v - \frac{eE_0}{m\omega} \left[\sin(\omega t + \varphi) - \sin(\omega \Theta_h + \varphi) \right] , \quad (4.9)$$

Θ_h satisfies the following equation (from formula (4.2)):

$$\begin{aligned} z - z_0 - v(t - \Theta_h) + \frac{eE_0}{m\omega^2} \left[\cos(\omega t + \varphi) - \cos(\omega \Theta_h + \varphi) \right] \\ + \frac{eE_0}{m\omega} (t - \Theta_h) \sin(\omega t + \varphi) = 0 \end{aligned} \quad (4.10)$$

The solution (4.8) possesses an important feature of periodicity. Namely, if $\psi_0(v, t)$ and $E(t)$ are both periodic in time (T is the period):

$$\psi_0(v, t + T) = \psi_0(v, t) , \quad (4.11)$$

$$E(t + T) = E(t) \quad (4.12)$$

then

$$\begin{aligned}\psi(z, v, t + T) &= \psi_0(V(t + T), \Theta(t + T) + T) \\ &= \psi_0(V(t), \Theta(t)) = \psi(z, v, t),\end{aligned}\tag{4.13}$$

i.e. it is also periodic.

The correctness of this statement is very easy to see in the simple case of the harmonic electrical field with $\omega = 2\pi/T$. In this case (4.10) is invariant under transformation $t \rightarrow t + T, \Theta \rightarrow \Theta + T$ and so is (4.9). The proof for more general periodic function

$$E(t) = \sum_n E_n \cos(n\omega t + \varphi_n)\tag{4.14}$$

is more elaborate and we will not give it here.

The constants E_n, φ_n are to be found self-consistently from the solution of the Maxwell equation with the current density as a source of the field defined in (2.8) and (2.9).

5. Solution Of The Maxwell Equations

In this section we consider the current density as a given quantity. The solution of the Maxwell equation for an arbitrary cavity is well known and is given here only for reader's convenience.

Any cavity with the volume Ω and the metallic internal surface Σ can be characterized by a set of its eigen vector-potential functions $A_n(\vec{r})$ satisfying the following system of uniform equations

$$\nabla^2 \bar{A}_n + k_n^2 \bar{A}_n = 0\tag{5.1}$$

and the boundary conditions

$$\bar{A}_n|_{\text{on } \Sigma} = 0\tag{5.2}$$

The eigenvalues k_n are defined by the cavity geometry.

The eigenfunctions $\vec{A}_n$ corresponding to different field modes are orthogonal to each other. Since (5.1) and (5.2) are linear and uniform, the eigenfunctions can be multiplied by an arbitrary factor. It is convenient to normalize them in the following way

$$\int_{\Omega} \vec{A}_n \cdot \vec{A}_m d^3r = \delta_{nm} \quad , \quad (5.3)$$

where δ_{nm} is the Kronecker symbol

$$\delta_{nm} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad (5.4)$$

Given the current density, the electromagnetic field of the cavity can be found in terms of the eigenfunctions $\vec{A}_n(\vec{r})$. Let us represent first the vector potential $\vec{A}(\vec{r}, t)$ as an expansion in the eigenfunctions $\vec{A}_n$:

$$\vec{A}(\vec{r}, t) = \sum_n a_n(t) \vec{A}_n(\vec{r}) \quad (5.5)$$

Here $a_n(t)$ are time dependent mode "amplitudes". Due to (5.2) this function also satisfies the boundary conditions. Substitute now (5.5) into (2.5), multiply by $\vec{A}_m$ and integrate over the cavity volume. Using (5.3) one gets separate equation for each of the amplitudes a_n :

$$\frac{d^2 a_n}{dt^2} + \omega_n^2 a_n = 4\pi c \int_{\Omega} d^3r \vec{j} \cdot \vec{A}_n \quad , \quad (5.6)$$

where $\omega_n = k_n \cdot c$.

The field description presented here does not include the energy dissipation in the cavity walls. There are several ways to take into account losses. One effective way to do that is to modify (5.6) slightly:

$$\frac{d^2 a_n}{dt^2} + 2\alpha_n \frac{da_n}{dt} + \omega_n^2 a_n = 4\pi c \int_{\Omega} d^3r \vec{j} \cdot \vec{A}_n \quad , \quad (5.7)$$

where α_n is a constant describing the rate of the decay of the n th mode due to the energy loss (including ohmic loss).

Solution of (5.7) is easy to obtain by the Fourier transformation of both sides. For any function f periodic in time with period $T = 2\pi/\omega$:

$$f(t) = \sum_m f_m e^{+im\omega t} \quad (5.8)$$

$$f_m = \frac{1}{T} \int_0^T dt e^{-im\omega t} f(t) \quad (5.9)$$

We are not interested in transient processes here. Then, for the equilibrium solution of (5.7) we get

$$a_{nm} = \frac{4\pi c \int_{\Omega} d^3r \vec{j}_m \cdot \vec{A}_n}{\omega_n^2 - m^2\omega^2 + 2i\alpha_n m\omega} \quad (5.10)$$

where $\vec{j}_m$ is the m th Fourier component of the current density. For the m th harmonic of the vector potential $\vec{A}_m$ one finds from (5.5):

$$\vec{A}_m(\vec{r}) = \sum_n \frac{4\pi c \vec{A}_n(\vec{r}) \int_{\Omega} d^3r \vec{j}_m \cdot \vec{A}_n}{\omega_n^2 - m^2\omega^2 + 2i\alpha_n m\omega} \quad (5.11)$$

The integrals on the right hand side of this expression are the coupling coefficients of the m th current harmonic to the n th cavity mode.

The m th harmonic of the electrical field $\vec{E}_m(\vec{r})$ can be found from $\vec{A}_m(\vec{r})$:

$$\vec{E}_m(\vec{r}) = \frac{-im\omega}{c} \vec{A}_m(\vec{r}) = - \sum_n \frac{4\pi i m\omega \cdot \vec{A}_n(\vec{r}) \int_{\Omega} d^3r \vec{j}_m \cdot \vec{A}_n}{\omega_n^2 - m^2\omega^2 + 2i\alpha_n m\omega} \quad (5.12)$$

As the simplest example of the general formulae (5.11) and (5.12) let us consider a pillbox cavity with the radius R and length d . For cylindrically symmetric current flowing on the cavity axis the azimuthal eigennumber is equal zero. In this case the vector potential has only one nontrivial longitudinal component A_n . Since we restrict ourselves to fields uniform in z only, the longitudinal eigennumber is also equal to zero. The radial eigennumber n is related to different radial modes:

$$A_n = B_n J_0(\gamma_n r/R) \quad (5.13)$$

where γ_n is n th root of the zeroth order Bessel function $J_0(z)$. The normalization constants B_n are;

$$B_n = \frac{1}{\sqrt{\pi d R} \cdot J_1(\gamma_n)} \quad (5.14)$$

The eigenfrequencies ω_n in this case are defined by

$$\omega_n = \gamma_n c / R \quad (5.15)$$

with the first three values of γ_n being 2.405, 5.520 and 8.654. Let us now assume for the current density the uniform dependence on the radial distance from the axis in the interval $0 < r < b$:

$$j(z, r, t) = g(r)I(z, t) \quad (5.16)$$

$$g(r) = \begin{cases} 1/\pi b^2 & r < b \\ 0 & r \geq b \end{cases} \quad (5.17)$$

Then, from (5.12) we get for the longitudinal component of the electric field:

$$E_m(r) = - \sum \frac{8im\omega J_0(\gamma_n r/R) \int_0^d dz I_m(z) \int_0^b r dr J_0(\gamma_n r/R)}{b^2(\omega_n^2 - m^2\omega^2 + 2i\alpha_n m\omega) dR^2 J_1^2(\gamma_n)} \quad (5.18)$$

where

$$I_m(z) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt e^{-im\omega t} I(z, t) \quad (5.19)$$

is the m th Fourier harmonic of the current $I(z, t)$. Define now the gap voltage harmonic

$$V_m = -d\langle E_m(r) \rangle \quad (5.20)$$

where $\langle E_m(r) \rangle$ is the average electric field over the beam cross section $\langle E_m(r) \rangle = \frac{1}{\pi b^2} \int_0^b r dr d\varphi E_m(r)$ and the gap impedance

$$Z_m = \frac{V_m}{\langle I_m \rangle} \quad (5.21)$$

$$\langle I_m \rangle = \frac{1}{d} \int_0^d dz I_m(z) \quad (5.22)$$

For the gap impedance Z_m we get from (5.18)

$$Z_m = \sum_n \frac{16 i m \omega d J_1^2(\gamma_n b/R)}{\gamma_n^2 J_1^2(\gamma_n) (\omega_n^2 - m^2 \omega^2 + 2i \alpha_n m \omega) b^2} \quad (5.23)$$

In a particular case when $m\omega$ is close to one of the eigenfrequencies ω_n , only one mode contributes to Z_m . In such a case:

$$Z_m = \frac{16 i d J_1^2(\gamma_n b/R)}{\omega_n \gamma_n^2 J_1^2(\gamma_n) b^2 \left(\frac{\omega_n}{m\omega} - \frac{m\omega}{\omega_n} + \frac{2i\alpha_n}{\omega_n} \right)}$$

or substituting $\omega_n/m\omega = f_0/f$ and defining $Q = \omega_0/2\alpha_0$

$$Z_m = \frac{8 d J_1^2(\gamma_n b/R)}{\pi f_n \gamma_n^2 J_1^2(\gamma_n) b^2} \frac{Q}{1 + iQ \left(\frac{f}{f_0} - \frac{f_0}{f} \right)} \quad (5.24)$$

If one represents Z_m in the form

$$Z_m = \frac{(R/Q)_{sh} \cdot Q}{1 + iQ \left(\frac{f}{f_n} - \frac{f_n}{f} \right)} \quad (5.25)$$

then

$$(R/Q)_{sh} = \frac{8 d J_1^2(\gamma_n b/R)}{\pi f_n \gamma_n^2 J_1^2(\gamma_n) b^2} \quad (5.26)$$

For $b \rightarrow 0$

$$(R/Q)_{sh} \rightarrow (R/Q)_{b \rightarrow 0} = \frac{2 d}{\pi f_n R^2 J_1^2(\gamma_n)} \quad (5.27)$$

The dependence of $(R/Q)_{sh}$ on the cavity length d is contained solely in the factor $(R/Q)_{b \rightarrow 0}$. Rewrite (5.26) in the form

$$(R/Q)_{sh} = (R/Q)_{b \rightarrow 0} \cdot m^2(b/R) \quad (5.28)$$

where $(R/Q)_{b \rightarrow 0}$ is defined in (5.27) and $m(x) = 2 J_1(x)/x$

Consider a numerical example of a cavity with the dimensions $R \approx 1.43$ cm, $d = 0.5$ cm with a ground mode ($\gamma = 2.405$) on the frequency $f = 2.856$ GHz. From (5.27) one gets $(R/Q)_{b \rightarrow 0} = 180 \text{ ohm}$.

Dependence of $m^2(x)$ is illustrated in Table 5.1

TABLE 5.1

x	$m(x)$	$m^2(x)$
0.0	1.00	1.00
0.2	0.97	0.94
0.4	0.89	0.79
0.6	0.76	0.58
0.8	0.60	0.36
1.0	0.43	0.18

For the uniform beam with the radius $b = 1.0$ cm we get $(R/Q)_{sh} = 86 \text{ ohm}$.

6. Self-consistent Solution Of The Vlasov Equations

Introducing the notion of the impedance Z_m we can rewrite (5.21) in the following way:

$$E_m = -\frac{Z_m}{d^2} \int_0^d dz I_m(z) \quad , \quad (6.1)$$

where $E_m = -V_m/d$ and I_m are the m th harmonics of the electric field and of the full current flowing through the gap, respectively. It is more convenient to consider the current due to electron flow separately from other possible currents, e.g. the current arising from the external generator.

Consider for example the first klystron cavity. Assume that the cold cavity is excited by an external rf generator. Then it is convenient to rewrite (6.1) in the following form:

$$E_m = E_{m \text{ ext}} - \frac{Z_m}{d^2} \int_0^d dz I_m(z) \quad , \quad (6.2)$$

where now I_m is the m th harmonic of the electron flow current which in turn depends on E_m , $E_{m \text{ ext}}$ is the m th harmonic of the field excited in the cold cavity by an external generator. It can be equal zero in particular case of not excited cavity (for example the second klystron cavity).

Suppose now that resonance denominators in (5.18) are small for modes which result in the uniform field distribution along the longitudinal coordinate z and large for all other modes. One such an example provides the pillbox cavity excited on the ground mode. The gridded gap might be another example. In such a case the electric field is fully determined by its harmonic amplitudes $|E_m|$ and phases $\arg(E_m)$ of (6.2). The integral of the right hand side of (6.2) in turn depends on electric field through the current harmonic (5.19).

Consider the simplest case when only one mode (say the zeroth one) and one harmonic (assume the first one) contributes to the sum (5.18). Then (6.2) constitutes two transcendental equations for the amplitude E_0 and the phase φ (or for the real and imaginary parts) of the first harmonic of the field. Solution of these equations provide the self-consistent field $E_h = E_0 \cos(\omega t + \varphi)$ (4.8). Substitute this field back into the solution (4.6) for the distribution function. One gets now the selfconsistent solution of the Vlasov equation which satisfies the boundary value at $z = z_0$.

As an example, let us assume for the initial electron flow a dc current with no velocity spread:

$$\psi_0(v, t)|_{z=z_0} = \frac{I_0}{ev_0} \delta(v - v_0) \quad , \quad (6.3)$$

where I_0 is the dc electron current, and $v_0 = \sqrt{2eV_0/m}$ is the initial velocity due to the dc gun voltage V_0 . At this point it is convenient to introduce the following dimensionless variables

$$x = (z - z_0)/d \quad 0 \leq x \leq 1 \quad (6.4)$$

$$u = v/\omega d \quad -\infty \leq u \leq \infty \quad (6.5)$$

$$\tau = \omega t + \varphi \quad (6.6)$$

$$\tau_0 = \omega \Theta + \varphi \quad (6.7)$$

$$k = eE_0/m\omega^2 d \quad (6.8)$$

According to (4.6) the distribution function for any later coordinate and time is in the new variables

$$\psi(x, u, \tau) = \frac{I_0}{cu_0} \delta(u - k \sin \tau + k \sin \tau_0 - u_0) \quad (6.9)$$

where $u_0 = v_0/\omega d$ and the function $\tau_0 = \tau_0(x, u, \tau)$ is defined by equation:

$$x - u(\tau - \tau_0) + k[\cos \tau - \cos \tau_0 + (\tau - \tau_0)\sin \tau] = 0. \quad (6.10)$$

From the distribution function (6.9) one can find the beam density current

$$\begin{aligned} I(x, \tau) &= \frac{I_0}{u_0} \int_{-\infty}^{\infty} du u \delta(u - k \sin \tau + k \sin \tau_0 - u_0) \\ &= I_0 \frac{\bar{u}(\tau)}{|u_0 + k[(\tau - \tau_0(\bar{u})) \cos \tau_0(\bar{u})]|} \end{aligned} \quad (6.11)$$

where $\bar{u} = \bar{u}(\tau)$ is the solution of equation

$$\bar{u} + k \sin \tau_0(\bar{u}) = u_0 + k \sin \tau \quad (6.12)$$

The first harmonic of $I(x, \tau)$ is

$$I_1(x) = \frac{I_0}{2\pi} e^{i\varphi} \int_0^{2\pi} \frac{d\tau \bar{u}(\tau) e^{-i\tau}}{|u_0 + k[(\tau - \tau_0(\bar{u})) \cos \tau_0(\bar{u})]|} \quad (6.13)$$

Calculate now the average over x of this current and substitute into (6.2). Note that $E_1 = E_0 e^{i\varphi}/2$ and $E_{1ext} = E_{0ext}/2$.

$$k = k_{ext} e^{-i\varphi} - \frac{2Z_1}{d^2} \frac{eI_0}{2\pi m\omega^2} \int_0^1 dx \int_0^{2\pi} \frac{d\tau \bar{u}(\tau) e^{-i\tau}}{|u_0 + k[(\tau - \tau_0)] \cos \tau_0|} \quad (6.14)$$

where $k_{ext} = eE_{0ext}/m\omega^2 d$. The complex equation (6.14) is equivalent to two transcendental equations which define the amplitude E_0 and the phase φ (in respect to the external field) of the field in the gap.

Below we provide numerical results, which illustrate the application of the derived formulae.

7. Small Signal Approximation

It is instructive to study the obtained results in the limit of a small electric field and to compare them with the known results from the small signal theory.

In the small signal limit the lowest power of the parameter k should be retained in all expansions in power series.

In variables (6.4) - (6.8), equations (4.9), (4.10) for $u = V_h/\omega d$ and τ_0 look like:

$$u = u - k \sin \tau + k \sin \tau_0 \quad (7.1)$$

$$x - u(\tau - \tau_0) + k(\cos \tau - \cos \tau_0 + \tau \sin \tau - \tau_0 \sin \tau) = 0 \quad (7.2)$$

The solutions of (7.1) and (7.2) to the first order in k are

$$u = u - k \sin \tau + k \sin\left(\tau - \frac{x}{u}\right) \quad (7.3)$$

$$\tau_0 = \tau - \frac{x}{u} + \frac{k}{u} \left[\cos\left(\tau - \frac{x}{u}\right) - \cos \tau - \frac{x}{u} \sin \tau \right] \quad (7.4)$$

The terms independent of k here give the ballistic approximation. The last terms in (7.3) and (7.4) represent the influence of the electric field.

Let us assume for simplicity that the distribution function of the electron flow on the entrance of the gap is (8.3).

7.1 COUPLING COEFFICIENT

Let us first of all find the expression for the coupling coefficient μ as it follows from our solution. One can define μ as the ratio of the average kinetic energy change to the maximum of the energy gain in the gap.⁴ Calculate first the average $\langle u^2 \rangle$ as the function of x :

$$\langle u^2 \rangle = \frac{\int_{-\infty}^{\infty} du u^2 \delta\left[u - k \sin \tau + k \sin\left(\tau - \frac{x}{u}\right) - u_0\right]}{\int_{-\infty}^{\infty} du \delta\left[u - k \sin \tau + k \sin\left(\tau - \frac{x}{u}\right) - u_0\right]} \quad (7.5)$$

To the first order in k for $x = 1$

$$\langle v^2 \rangle_1 = v_0^2 + 2kv_0 \left[\sin \tau - \sin \left(\tau - \frac{1}{u_0} \right) \right] \quad (7.6)$$

From here we get (ϕ denotes $\tau + 1/2u_0$)

$$\mu = \frac{\langle v^2 \rangle_1 - v_0^2}{2k \cos \phi} = \frac{\sin \theta}{\theta} \quad (7.7)$$

where $\theta = \frac{1}{2u_0} = \frac{wd}{2v_0}$ is the half of the gap transit angle.

7.2 BEAM LOADING

Now, let us consider the beam loading by the electron flow as it follows from our solution. We need now the expression for the beam current density in the small signal approximation. The charge and the current densities both can found by integration of (6.9)

$$\rho(x, \tau) = \frac{I_0}{u_0 \omega d} \int_{-\infty}^{\infty} du \delta \left[u - k \sin \tau + k \sin \left(\tau - \frac{x}{u} \right) - u_0 \right] \quad (7.8)$$

$$I(x, \tau) = \frac{I_0}{u_0} \int_{-\infty}^{\infty} du u \delta \left[u - k \sin \tau + k \sin \left(\tau - \frac{x}{u} \right) - u_0 \right] \quad (7.9)$$

Performing the integrations we find to the first order in k :

$$I(x, \tau) = I_0 \left\{ 1 + \frac{k}{u_0} \left[\sin \tau - \sin \left(\tau - \frac{x}{u_0} \right) \right] - \frac{xk}{u_0^2} \cos \left(\tau - \frac{x}{u_0} \right) \right\} \quad (7.10)$$

$$\rho(x, \tau) = \frac{I_0}{u_0 \omega d} \left\{ 1 - \frac{kx}{u_0^2} \cos \left(\tau - \frac{x}{u_0} \right) \right\} \quad (7.11)$$

It is easy to see that (7.10) and (7.11) satisfy the continuity equation

$$\frac{\partial \rho}{\partial \tau} + \frac{1}{\omega d} \frac{\partial I}{\partial x} = 0 \quad (7.12)$$

As one sees from (7.10) in the small signal approximation the current density besides the dc component contains only the first harmonic. Expression (7.10) can be obtained conversely by expanding expression (6.11) in the power series in parameter k .

Using the definition (5.10) the first harmonic of the beam current density is

$$I_1(x) = I_0 e^{i\varphi} \frac{k}{2u_0} \left(\sin \frac{x}{u_0} - \frac{x}{u_0} \cos \frac{x}{u_0} - i + i \cos \frac{x}{u_0} + i \frac{x}{u_0} \sin \frac{x}{u_0} \right) \quad (7.13)$$

This expression is the same as one obtained from formula (2.1) of the paper⁴ assuming $E_x = \text{const}$. It also coincides with the corresponding expression for i_v in the book.⁵

Let us rewrite equation (6.2) in variables (6.4) - (6.7):

$$E_1 = \frac{Z_1}{d} \left[I_{1\text{ext}} - \int_0^1 dx I_1(x) \right] \quad (7.14)$$

Here $I_{1\text{ext}}$ is the first harmonic of the external current exciting the cavity. Substitute now (7.13) into (7.14) and take into account that $E_1 = E_0 e^{i\varphi}/2$, $I_{1\text{ext}} = I_{0\text{ext}}/2$. Then:

$$E_0 = \frac{Z_1}{d} I_{0\text{ext}} e^{-i\varphi} - \frac{Z_1}{d} I_0 k (B + iA) \quad (7.15)$$

where

$$B = \int_0^{1/u_0} d\sigma (\sin \sigma - \sigma \cos \sigma) = 4\theta \sin \theta \left(\frac{\sin \theta}{\theta} - \cos \theta \right) \quad (7.16)$$

$$A = - \int_0^{1/u_0} d\sigma (1 - \cos \sigma - \sigma \sin \sigma) = 4\theta \cos \theta \left(\frac{\sin \theta}{\theta} - \cos \theta \right) \quad (7.17)$$

Here $\theta = 1/2 u_0 = \omega d / 2 v_0$ is half of the transit angle for the gap. Solving (7.15) in respect to E_0 one finds:

$$E_0 = \frac{I_{0\text{ext}} e^{-i\varphi}}{d \left[\frac{1}{Z_1} + \frac{e I_0 (B + iA)}{m \omega^2 d^2} \right]} \quad (7.18)$$

Substitute now expression (5.24) for Z_1 :

$$E_0 = \frac{(R/Q)_{sh} I_{0\text{ext}} e^{-i\varphi}}{d \left[\frac{1}{Q} + i \left(\frac{f}{f_0} - \frac{f_0}{f} \right) + \frac{I_0 (R/Q)_{sh} (B + iA)}{2 V_0 4\theta^2} \right]} \quad (7.19)$$

From formula (7.19) immediately follow the usual expressions for the loaded quality Q_L and the shifted frequency f_L of the gap:

$$\frac{1}{Q_L} = \frac{1}{Q} + \frac{I_0(R/Q)_{sh}}{2V_0} \frac{\sin\theta}{\theta} \left(\frac{\sin\theta}{\theta} - \cos\theta \right) \quad (7.20)$$

$$f_L = f_0 \left[1 - \frac{I_0(R/Q)_{sh}}{4V_0} \frac{\cos\theta}{\theta} \left(\frac{\sin\theta}{\theta} - \cos\theta \right) \right] \quad (7.21)$$

8. Narrow Gap Approximation

In this section, we derive an analytic solution to the Vlasov equation in the limit of narrow gaps. The expansion is not restricted to small signals, but the result is consistent with small signal theory in the proper limit.

The assumption of narrow gaps allows us to expand the distribution function in a Taylor series. We have in general

$$\psi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \psi}{\partial z^n} \right|_{z_0} (z - z_0)^n \quad (8.1)$$

For narrow gaps, the solution is given by the first few terms of the series. We will work out the example of an initial cold distribution, with

$$\psi_0(v, t) = \frac{I_0}{ev_0} \delta(v - v_0) \quad (8.2)$$

the general solution is given by

$$\psi(z, v, t) = \frac{I_0}{ev_0} \delta \left(v - \int_{\Theta}^t a(t') dt' - v_0 \right) \quad (8.3)$$

where Θ is given by

$$z - z_0 = v(t - \Theta) - \int_{\Theta}^t (t' - \Theta) a(t') dt' \quad (8.4)$$

We will perform the expansion of (8.3) up to $n = 3$. The first term in the series ($n = 0$) is given of course by $\psi_0(v, t)$ as defined in Eq. (8.2). To obtain the $n = 1$ term, we need to evaluate

$$\frac{\partial \psi}{\partial z} = \frac{\partial \psi_0}{\partial v} a(\Theta) \frac{\partial \Theta}{\partial z} \quad (8.5)$$

Differentiating Eq. (8.4), we obtain

$$\frac{\partial z}{\partial \Theta} = -v + \int_{\Theta}^t a(t') dt' \quad (8.6)$$

The coefficient of the $n = 1$ term is then given by

$$\left. \frac{\partial \psi}{\partial z} \right|_{z_0} = -\frac{I_0}{ev_0} \delta'(v - v_0) a(t)/v, \quad n = 1 \quad (8.7)$$

The coefficient of $n = 2$ term is proportional to the second derivative of ψ and is evaluated to be

$$\left(\frac{ev_0}{I_0} \right) \left. \frac{\partial^2 \psi}{\partial z^2} \right|_{z_0} = \delta''(v - v_0) \frac{a^2(t)}{v^2} + \delta'(v - v_0) \left[\frac{a'(t)}{v^2} - \frac{a^2(t)}{v^3} \right], \quad n = 2 \quad (8.8)$$

Note that δ' refers to the derivative of the delta function with respect to velocity while $a'(t)$ is a derivative of the acceleration with respect to time. A superscript with n primes refers to the n th derivative. Finally, the $n = 3$ coefficient is evaluated to be

$$\begin{aligned} \left(\frac{ev_0}{I_0} \right) \left. \frac{\partial^3 \psi}{\partial z^3} \right|_{z_0} = & -\delta'''(v - v_0) \frac{a^3}{v^3} + 3\delta''(v - v_0) \left[\frac{aa'}{v^3} + \frac{a^3}{v^4} \right] \\ & -\delta'(v - v_0) \left[\frac{a''}{v^3} - \frac{4a'a}{v^4} + \frac{3a^3}{v^5} \right], \quad n = 3 \end{aligned} \quad (8.9)$$

The current is related to the first moment of the distribution function

$$I(z, t) = e \int dv v \psi(z, v, t) \quad (8.10)$$

In the Taylor series expansion of ψ , the velocity integrals may be evaluated term by term. The $n = 0$ term gives rise to the d.c. component of the current since

$$e \int dv v \psi_0 = \frac{I_0}{v_0} \int dv v \delta(v - v_0) = I_0 \quad (8.11)$$

The $n = 1$ component gives no contribution since

$$e \int dv v \left. \frac{\partial \psi_0}{\partial z} \right|_{z_0} = -\frac{I_0}{v_0} a(t) \int dv \delta'(v - v_0) = 0 \quad (8.12)$$

The $n = 2$ component of the distribution function has terms which are proportional to $a^2(t)$. However, these two terms cancel exactly when we take the velocity moment of the distribution function. We are then left with a contribution to the rf current

$$e \int dv v \left. \frac{1}{2} \frac{\partial^2 \psi_0}{\partial z^2} \right|_{z_0} (z - z_0)^2 = \frac{I_0}{2v_0^3} a'(t) (z - z_0)^2 \quad (8.13)$$

To evaluate the $n = 3$ component of the current, we take the velocity moment of Eq. (8.9). Again, the terms proportional to $a^3(t)$ vanish and we obtain

$$e \int dv v \left. \frac{1}{6} \frac{\partial^3 \psi_0}{\partial z^3} \right|_{z_0} (z - z_0)^3 = I_0 \left(\frac{5aa'}{v_0^5} - \frac{a''}{3v_0^4} \right) (z - z_0)^3 \quad (8.14)$$

In performing the velocity integrals, we have made use of the delta function identity

$$\int g(v) \delta^{(n)}(v - v_0) dv = (-1)^n \left. \frac{\partial^n g}{\partial v^n} \right|_{v_0} \quad (8.15)$$

Combining these results, we have that to $n = 3$ in the Taylor series expansion,

$$K(z, t) = I_0 \left\{ 1 + \frac{a'(t)}{2v_0^3} (z - z_0)^2 - \left[\frac{a''}{3v_0^4} - \frac{5aa'}{v_0^5} \right] (z - z_0)^3 \right\} \quad (8.16)$$

The term which is proportional to aa' represents our first explicit nonlinear contribution to the current. However, it is clear that if $a(t)$ is a pure first harmonic, the quadratic term in a can contribute only to the zeroth and second harmonic. Hence, to the order considered, there is no higher order contribution to the rf component of the current.

We now turn to examine more carefully the z -dependence of I_1 . The first harmonic of the current is related only to the linear terms in $a(t)$.

For

$$a(t) = \frac{eV}{md} \cos(\omega t + \varphi) \quad (8.17)$$

we have

$$\begin{aligned}
 I_1(z, t) &= \frac{I_0}{2v_0^3} \left[a'(t)(z-z_0)^2 - \frac{2a''(t)}{3v_0}(z-z_0)^3 \right] \\
 &= -\frac{I_0\omega}{2v_0^3} \frac{eV}{md} (z-z_0)^2 \left[\sin(\omega t + \varphi) - \frac{2\omega(z-z_0)}{3v_0} \cos(\omega t + \varphi) \right] \\
 &\approx -\frac{I_0\omega}{2v_0^3} \frac{eV}{md} (z-z_0)^2 \sin\left(\omega t + \varphi - \frac{2\omega(z-z_0)}{3v_0}\right) \\
 &= \frac{I_0\omega}{2v_0^3} \frac{eV}{md} (z-z_0)^2 \cos\left(\omega t + \varphi + \frac{\pi}{2} - \frac{2\omega(z-z_0)}{3v_0}\right)
 \end{aligned} \tag{8.18}$$

The analytic formula predicts a quadratic z -dependence on the amplitude and a linear z -dependence on the phase of I_1 . These features are consistent with the numerical results from Section 9. The magnitude of the amplitude and rate of phase change are also in agreement.

9. Numerical Results Comparison With The Small Signal Approximation

Here we apply formulae to the input klystron cavity. In general, the small signal approximation gives correct results for that cavity. This is true due to small values of both the input power and the length of the gap. In addition, the initial distribution of electrons at the cavity entrance in velocities and time is very simple and is easy to simulate. Hence, in this case the results obtained by using the self-consistent solution should agree with the small signal approximation.

Table I shows that is indeed the case. The first column contains the amplitude of the external voltage V_{ext} changed in steps from 1.05 kV to 2.00 kV. Next column contains, the gap voltage V_{SC} in kV as found from selfconsistent solution (6.2) for the Gaussian distribution in initial electron velocities. The third column gives the small signal approximation gap voltage V_{ssa} (in kV). The last column gives the phase shift of the gap voltage φ_{sc} in respect to the applied external electric field. These values should be compared to the small signal approximation phase shift φ_{ssa} which does not depend on the external amplitude. Table 2 summarizes the parameters of the cavity and the beam, used in these calculations. V_0 is the dc gun voltage, I_0 is the dc beam

current and σ is the rms velocity spread of the incoming beam. Q_L and f_L are the beam loaded parameters of the cavity found from equations (7.20) and (7.21).

Table 3 illustrates the dependence of the gap voltage V_{ac} and its phase φ_{ac} on σ of the initial Gaussian distribution. For σ smaller than 10^6 m/sec the result is the same as for zero spread velocity beam ($\sim \delta(v - v_0)$).

Figure 1 presents the normalized velocity distribution of the beam as function of the dimensionless velocity (8.5) at the cavity exit for different values of the rf phase τ (8.8).

Figures 2-4 are the phase plots of u/k versus x/k for $k = 0.2, 0.25$ and 0.33 respectively.

Figure 5 represents the amplitude and the phase of the first harmonic of the beam current as functions of the dimensionless distance x (8.13) inside the gap. Both the amplitude and the phase agree with the result (8.18) obtained in the narrow gap approximation.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Table 1

$$\varphi_{acc} = -0.122429D + 00$$

V_{ext}	V_{sc}	V_{acc}	φ_{acc}
0.105000D + 04	0.851658D + 03	0.851664D + 03	-0.122433D + 00
0.110000D + 04	0.892212D + 03	0.892220D + 03	-0.122433D + 00
0.115000D + 04	0.932766D + 03	0.932775D + 03	-0.122433D + 00
0.120000D + 04	0.973321D + 03	0.973331D + 03	-0.122434D + 00
0.125000D + 04	0.101387D + 04	0.101389D + 03	-0.122434D + 00
0.130000D + 04	0.105443D + 04	0.105444D + 04	-0.122434D + 00
0.135000D + 04	0.109498D + 04	0.109500D + 04	-0.122435D + 00
0.140000D + 04	0.113554D + 04	0.113555D + 04	-0.122435D + 00
0.145000D + 04	0.117609D + 04	0.117611D + 04	-0.122436D + 00
0.150000D + 04	0.121664D + 04	0.121666D + 04	-0.122436D + 00
0.155000D + 04	0.125720D + 04	0.125722D + 04	-0.122436D + 00
0.160000D + 04	0.129775D + 04	0.129777D + 04	-0.122437D + 00
0.165000D + 04	0.133830D + 04	0.133833D + 04	-0.122437D + 00
0.170000D + 04	0.137886D + 04	0.137889D + 04	-0.122438D + 00
0.175000D + 04	0.141941D + 04	0.141944D + 04	-0.122438D + 00
0.180000D + 04	0.145996D + 04	0.146000D + 04	-0.122439D + 00
0.185000D + 04	0.150052D + 04	0.150055D + 04	-0.122439D + 00
0.190000D + 04	0.154107D + 04	0.154111D + 04	-0.122440D + 00
0.195000D + 04	0.158162D + 04	0.158166D + 04	-0.122440D + 00
0.200000D + 04	0.162217D + 04	0.162222D + 04	-0.122441D + 00

Table 2

$$\text{length} = 0.016000 \text{ m}$$

$$Q = 1000.0$$

$$R/Q = 85.0 \text{ Ohm}$$

$$f = 0.3532D + 09 \text{ Hz}$$

$$\Delta f = 0.20D + 08 \text{ Hz}$$

$$eV_0 = 0.620000D + 05 \text{ eV}$$

$$I_0 = 11.50 \text{ A}$$

$$\sigma = 0.1300D + 06 \text{ m/sec}$$

$$Q_L = 949.49001$$

$$f_L = 0.35313D + 09 \text{ Hz}$$

Table 3

σ [m/sec]	V_{sc} [kV]	φ_{sc} [rad]
10^2	1.62217	-0.122441
10^3	1.62217	-0.122441
10^4	1.62217	-0.122441
10^5	1.62217	-0.122441
10^6	1.62212	-0.122453
10^7	1.61656	-0.123646
$1.4 \cdot 10^7$	1.61088	-0.124835
$1.8 \cdot 10^7$	1.60279	-0.126466

10. Conclusions

The approach suggested in this work proves to be correct. The results obtained agree to a great accuracy with the small signal approximation. In the limit of narrow gap the solution gives reasonable results both for the amplitude and phase of the resonant harmonic of the beam current.

The next questions which should be addressed are how useful and how convenient is the Vlasov approach in general and with respect to the klystron problem in particular. The calculation of the particle distribution along the klystron tube seems to be straight forward although substantial work need to be done.

Nevertheless, the approach looks promising. One can attempt to develop a one-dimensional model of a klystron which will include all important physics of the beam dynamics in multicavity system, including the interaction with the output cavity and crossover of the electron trajectories. The model takes into account the space charge effects in the cavities. The debunching effect of the space charge in drift sections of the klystron can be evaluated in perturbative manner using the ballistic approximation as the unperturbed solution. Such a model might be usefull as a fast and convenient tool for the klystron design. It can also provide information (at least as the first guess) on the amplitudes and the phases of the gap voltages for klystron cavities. That might be useful as the input for numerical models of a klystron such as MASK.

Further work is needed to extend the present formulation into the region of relativistic velocities.

Acknowledgements

The authors are grateful to all members of the Numerical Analysis Group for the interest and stimulating discussions of the problem. Our special gratitude goes to B. Herrmannsfeld and P. Wilson for the encouragement and help in our work.

References

1. Vlasov, A. A., "On the Kinetic Theory of an Assembly of Particles with Collective Interaction." *J. Phys. (U.S.S.R.)* 9, 25 (1945).
2. See, for example, N. A. Krall and A. W. Trivelpiece, "Principles of Plasma Physics," McGraw-Hill (1973).
3. T. H. Stix, "The Theory of Plasma Waves," McGraw-Hill (1962).
4. G. M. Branch, Jr., *IRE Trans. on Electron Devices*, ED-8, # 3, May 1961, p. 193.
5. M. Chodorow and C. Susskind, "Fundamentals of Microwave Electronics," McGraw-Hill Company.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5