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**One Dimensional Multi-Phase
Moving Boundary Problems with
Phases of Different Densities**

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ONE DIMENSIONAL MULTI-PHASE MOVING BOUNDARY PROBLEMS
WITH PHASES OF DIFFERENT DENSITIES*

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ABSTRACT

Density changes occur naturally in phase change problems. It is customary in analyzing such problems to ignore the bulk movement of material thus introduced and assume the densities to be equal. However, for one dimensional problems the complexity introduced by this bulk movement is more apparent than real. This can be demonstrated by posing the problem in local coordinates fixed in each phase. In this report we show how to define suitable moving coordinates and, using them, pose and solve a one dimensional, multi-phase Stefan problem with phases of distinct densities. This explicit solution is essentially a similarity solution in the local coordinates. However, utility of these local coordinates is not limited to problems for which similarity solutions exist. They could be used with finite element or finite difference schemes to analyze more complex problems.

I. Introduction

In the classical Stefan problem the densities of the two phases are considered to be equal. To do otherwise forces one to consider the movement of bulk material. If different densities are permitted in multidimensional problems, then heat conduction by convection must be considered. But in one dimension one merely has $T_t + vT_r$ in place of T_t and the change of variable $\xi = r - \int_t^t v dt'$ immediately suggests itself. When v is the same throughout a phase we may say that ξ is at rest with respect to that phase.

A few authors have analyzed one dimensional, two phase, moving boundary problems with phases of different densities. Some have used this change of variable, but others have not.

Chambre [2] considered a solidification problem for a two phase system of unequal densities in which the equations governing the distribution of velocities and temperatures are one dimensional. They are posed along the ray $r > 0$, where r may however be in any one of plane, cylindrical or spherical coordinates. He did not make a change of variable. He assumed the solid to be unmoving and of infinite thermal conductivity, so no equations were posed for the solid phase. Explicit representations were obtained for temperature and velocity profiles in the liquid phase in terms of error functions and exponential integrals. The temperature distribution is a function of the similarity variable $x/\sqrt{t}$ and the velocity distribution is proportional to a function of $x/\sqrt{t}$ multiplied by $(1/\sqrt{t})$.

In [4], Horvay considered problems similar to those posed by Chambre. He asserted that Chambre failed to take into account the conservation of mass in the liquid, and "so his results cannot be

regarded as valid." The critical item appears to have been that Chambré considered the liquid to be an incompressible fluid in a constant pressure field, but Horvay found that the absolute value of the pressure became essentially infinite everywhere in the fluid which "indicates that compressibility and the effect of pressure on freezing temperature cannot be neglected in plane front freezing."

Horvay also obtained expressions for the temperature distribution in the liquid in terms of the similarity variable; and, like Chambré, found that the speed of advance of the freezing front was proportional to $\sqrt{t}$.

In a subsequent paper [5], Horvay abandoned the simplifying assumption of the solid growing from an initial point. (This assumption had permitted the solution of the problem in essentially closed form, but had resulted in infinite negative pressures at small times.) He proposed instead that "a couple dozen liquid molecules conglomerate into a solid-like mass ('embryo') and then, suddenly, the freezing phenomenon takes over." He considered a system of equations involving temperature, velocity and pressure in the liquid phase and further included the change in freezing temperature which accompanies pressure changes. This model is much more complicated than what we discuss.

Tao [7] considered a two phase solidification problem in one Cartesian coordinate with arbitrary initial and boundary conditions. The region considered was the half line $x > 0$. The phase densities were constant but distinct. He made the change of variable $y(t, \lambda) = (x + \epsilon S(t))/(1 + \epsilon)$, where $S(t)$ is the location of the phase

front, $\epsilon = (\rho_2 - \rho_1)/\rho_2$, and ρ_i is the density of phase i , and rewrote the heat transfer problem in the moving liquid using this coordinate.

This is similar in spirit to what we propose except that, because of the factor $(1+\epsilon)^{-1}$, the y coordinate system is not at rest with respect to the liquid phase. Tao's idea has the advantage that $y(t, S(t)) = S(t)$, which simplifies the statement of the Stefan condition at the moving boundary; and, since the liquid extends to $+\infty$, no difficulty arises at another boundary to the right. However, this would not be the case for a multi-phase problem.

Tao then analyzed the transformed problem using infinite series of terms involving repeated integrals of the error function.

In a very brief treatment indeed, Carslaw and Jaeger [1, pg.291] gave what is essentially a similarity solution for a two phase Stefan problem with phases of distinct densities. They made no change of variable.

Dankwerts [3] considered several two phase problems with phases of distinct densities. These problems were posed on the entire real line so no fixed boundary condition was imposed. He considered the problems posed ab initio in coordinates at rest in the two phases. The analysis is limited to what can be accomplished with similarity solutions in the local variables.

In more recent work Gelder and Guy [6] gave an analysis of a practical problem involving the melting of glass. Their emphasis was on meeting practical requirements rather than in resolving mathematical difficulties. This paper is of interest here because, as they put it, "Some care is needed in formulating the equations as

there is a substantial change in density on melting, and the derivative $\frac{dT}{dt}$ is that following a batch particle." This provides an example of a problem whose analysis would have been facilitated by use of the local coordinates which we propose.

Philosophically we follow Dankwerts in that we consider the problem in terms of coordinates which are at rest with respect to the several phases. However, we go farther in two directions. First, we point out that the local coordinates are in a sense "natural" for these problems and may be used for whatever analyses and with whatever methods the reader may prefer. They are not in any way connected with or limited to the use of similarity transformations. Second, we push the use of the similarity variable a little further and obtain an explicit solution of a multi-phase Stefan problem with phases of distinct constant densities.

The sample problem we analyze is a modest generalization, to the case of distinct phase densities, of a problem first considered by Weiner [8]. It represents a model of the solidification and subsequent cooling of a semi-infinite slab of a material that undergoes several successive phase changes. A constant temperature is maintained at the fixed boundary. Weiner obtained a similarity solution whose existence and uniqueness we verified in [9].

In the next section we define the local coordinate systems and state some of their properties. The last section is devoted to analysis of the sample problem.

II. The Local Coordinate System

In what follows we denote by S_i for $i = 1, \dots, n-1$ the free boundaries separating phases 1, $\dots$, n . We denote by ρ_i for $i = 1, \dots, n$, the densities of the n phases. We take $S_0 \equiv 0$. S_n , the rightmost boundary, may be $+\infty$ but it need not be.

If S_n is not $+\infty$, then, although it is not a free boundary, it may move. Let M denote the mass of the slab. Conservation of mass gives

$$M = \sum_{j=1}^n \rho_j (S_j(t) - S_{j-1}(t)),$$

and thus

$$S_n(t) = M/\rho_n + \sum_{j=1}^{n-1} S_j(t)(\rho_{j+1} - \rho_j)/\rho_n, \quad (2.1)$$

which holds for as long as $S_{n-1}(t)$, determined by the free boundary problem, is not greater than $S_n(t)$, determined from equation 1.

As soon as $S_{n-1}(t)$, determined by the free boundary problem, becomes equal to $S_n(t)$, determined from equation 1, phase number n has disappeared. At this time S_{n-1} takes the place of S_n and we may think of a new $(n-1)$ phase, free boundary problem commencing. Equation 1 would then give the location of $S_{n-1}(t)$ in terms of the free boundaries S_j for $j < n-1$ until such time as the $(n-1)$ th phase disappears. And so on until only one phase remained.

It is conceivable of course that phases could also appear. But the foregoing discussion should suffice to indicate how to proceed in such a case.

We turn now to the definition of the local coordinate systems. In a body moving with only linear velocity $v(t)$, the local coordinate ξ is at rest with respect to the body if and only if $\partial_t \xi = v(t)$. We define our local coordinates so that this condition is satisfied.

For $x \in [0, S_1(t)]$, we define $\xi_1(t, x)$ by $\xi_1(t, x) = x$, and $\Xi_1(t)$, the location of the first moving boundary in the first local coordinate system, by $\Xi_1(t) = \xi_1(t, S_1(t)) = S_1(t)$.

For compactness of notation we define $\mu_i \equiv \rho_i / \rho_{i+1}$ for $i = 1, \dots, n-1$. Then for $x \in [S_1(t), S_2(t)]$ we define $\xi_2(t, x)$ and $\Xi_2(t)$ by

$$\xi_2(t, x) = x - (1 - \mu_1)\Xi_1(t), \quad (2.2a)$$

and

$$\Xi_2(t) = \xi_2(t, S_2(t)) = S_2(t) - (1 - \mu_1)\Xi_1(t). \quad (2.2b)$$

Proceeding recursively, we define $\xi_i(t, x)$ and $\Xi_i(t)$, for $i = 3, \dots, n$, for $x \in [S_{i-1}(t), S_i(t)]$ by

$$\xi_i(t, x) = x - \sum_{j=1}^{i-1} (1 - \mu_j)\Xi_j(t), \quad (2.2c)$$

and

$$\Xi_i(t) = \xi_i(t, S_i(t)) = S_i(t) - \sum_{j=1}^{i-1} (1 - \mu_j)\Xi_j(t). \quad (2.2d)$$

It is easily verified that, for each i , ξ_i is at rest in the i th phase; $\partial_t \xi_i = \sum_{j=1}^{i-1} (1 - \mu_j)\dot{\Xi}_j$, which is the velocity of the i th phase in the global coordinate system.

The latter statement follows from conservation of mass at the several interfaces. Specifically, the relations

$$\rho_{i-1} \dot{\Xi}_{i-1}(t) = \rho_i (\dot{\Xi}_{i-1}(t) - v_i(t)), \quad \text{for } i = 2, \dots, n,$$

where v_i is the velocity of the i th phase with respect to the $(i-1)$ th phase, express conservation of mass at the several interfaces.

Solving for the relative velocities, we have $v_i(t) = (1 - \mu_{i-1})\dot{\Xi}_{i-1}$.

The velocity of the i th phase with respect to the global coordinate system is just the sum of these relative velocities.

This completes the definition, and justification, of our local coordinates. But before proceeding, we discuss some consequences of our definitions and a minor modification of potential interest.

For future reference we evaluate $\xi_i(t, S_{i-1}(t))$, the location of the $(i-1)$ th interface in the coordinate system at rest with respect to the i th phase. For $i = 2, \dots, n$ we have

$$\begin{aligned}\xi_i(t, S_{i-1}(t)) &= S_{i-1}(t) - \sum_{j=1}^{i-1} (1-\mu_j) \Xi_j(t) \\ &= \Xi_{i-1}(t) + \sum_{j=1}^{i-2} (1-\mu_j) \Xi_j - \sum_{j=1}^{i-1} (1-\mu_j) \Xi_i(t) \\ &= \mu_{i-1} \Xi_{i-1}(t).\end{aligned}\quad (2.3)$$

By defining the dummy $\Xi_0 \equiv 0$, we logically extend this definition to $i = 1$.

Since we propose analyzing problems in the local coordinates, determining the $\Xi_i(t)$ and functions of t and ξ_i and transforming results to a global coordinate system, it seems reasonable to give the definitions of x and the S_i in terms of the ξ_i and Ξ_i . They are:

$$x = \xi_1(t, x) + \sum_{j=1}^{i-1} (1-\mu_j) \Xi_j(t), \quad \text{for } \xi_i \in (\mu_{i-1} \Xi_{i-1}, \Xi_i) \quad (2.4)$$

and

$$S_i(t) = \Xi_i(t) + \sum_{j=1}^{i-1} (1-\mu_j) \Xi_j(t), \quad (2.5)$$

for $i = 1, \dots, n$ where again we have taken $\Xi_0 = 0$.

Equation 5 shows that if each Ξ_i is proportional to t^γ for some γ , then each S_i will be also. Furthermore, this equation may be viewed as a linear system for the Ξ_i 's in terms of the S_i 's. This system is easily inverted with the result

$$\Xi_i = \rho_i^{-1} \sum_{j=1}^i \rho_j (S_j - S_{j-1}), \quad \text{for } i = 1, \dots, n-1. \quad (2.6)$$

This is very appealing on physical grounds, for it says that Ξ_i is

located in the i th coordinate system so that if the total mass to the left of Ξ_i were of density ρ_i , then it would equal the total mass to the left of S_i in the fixed coordinate system.

There is a certain aesthetic shortcoming in our definitions of the local coordinates. If the S_i do not all coincide at the origin when $t = 0$, then for $i > 1$ the local coordinate systems do not coincide with the global system initially. As a consequence $\Xi_i(0)$ will not be equal to $S_i(0)$ for $i > 1$.

This does not seem to be a serious shortcoming since equation 6, evaluated when $t = 0$, gives the initial values for the Ξ_i . Furthermore, this defect could easily be overcome; for the definitions of the ξ_i could easily be modified to achieve $\xi_i(0, x) = x$, and hence $\Xi_i(0) = S_i(0) = 0$. In fact the definition could even be changed so that $\Xi_i(0) = 0$ for each i . All of this can be done by adding constants to the definitions of the ξ_i , which of course maintains the condition that each ξ_i moves with its corresponding phase.

Alternative definitions, which must be introduced recursively as before, are:

$$\xi_i(t, x) = x - \sum_{j=1}^{i-1} (\Xi_j(t) - \Xi_j(0)), \quad (2.7a)$$

$$\Xi_i(t) = \xi_i(t, S_i(t)) = S_i(t) - \sum_{j=1}^{i-1} (\Xi_j(t) - \Xi_j(0)), \quad (2.7b)$$

which achieves $\Xi_i(0) = S_i(0)$; and

$$\xi_i(t, x) = x - S_i(0) - \sum_{j=1}^{i-1} (\Xi_j(0)), \quad (2.8a)$$

$$\Xi_i(t) = \xi_i(t, S_i(t)) = S_i(t) - S_i(0) - \sum_{j=1}^{i-1} (\Xi_j(t) - \Xi_j(0)), \quad (2.8b)$$

which achieves $\Xi_i(0) = 0$.

All three sets of definitions agree when $S_i(0) = 0$ for each i .

III. A One Dimensional Multi-Phase Stefan Problem with Phases of Distinct Constant Densities

The verbal description of the problem to be considered is very similar to that given in [8] and [9]. The diffusion equation is posed in a sequence of contiguous regions on the half line $x > 0$. These regions are bounded by free boundaries whose locations are to be determined. Between each pair of adjacent boundaries separate phases of constant, but possibly distinct, densities exist. We suppose that as the process continues and multiple phase changes occur the interspersed regions slide along the x -axis so that contact is maintained. We do not consider the energy expended in maintaining this contact. This undeniably introduces an error into the analysis. For, a complete energy balance would include terms associated with accelerating mass and doing work against friction, gravity or some other restraint. Hopefully, these terms are small.

At each free boundary the dependent variable (temperature, concentration, enthalpy) has a specified constant limiting value from each side. These values may differ by a finite jump. The dependent variable is also constant at the fixed boundary $x = 0$. The locations of the free boundaries are determined by flux conditions namely "the jump in flux at a boundary makes the boundary move." Initially, all free boundaries coincide at $x = 0$ and the dependent variable takes a constant value, U_n , for $x > 0$. In addition, U_n is the limiting value of the dependent variable for large x at each time. We denote the several boundaries by $S_i(t)$ with S_0 and S_n denoting 0 and $+\infty$ respectively.

The boundary condition at the fixed boundary, $x = 0$, determines the solution. If, as we suppose, the dependent variable is a constant there, then under certain reasonable assumptions a solution exists which is essentially a similarity solution.

We consider the diffusion equation in each phase in terms of a coordinate system at rest with respect to that phase. The correspondence between the local coordinate systems and a global coordinate system is established by equations 2.2 through 2.5. The i th local problem is partly given by:

$$\begin{aligned} u_t &= D_i u_{\xi\xi} \quad \text{for } t > 0 \text{ and } \xi \in (L_i(t), R_i(t)), \\ u(t, L_i^+(t)) &= u_{i-1}, \\ u(t, R_i^+(t)) &= u_i + \Delta_i. \end{aligned} \quad (3.1)$$

This specification is incomplete since we have not said how L_i and R_i , the left and right boundaries respectively of the i th phase, are to be determined. Of course our intention is to juxtapose the regions and impose the Stefan conditions that define the L_i and R_i .

Since these free boundary conditions involve limiting values of u_ξ from both sides of the interfaces, it is necessary to subscript the functions u . For u_i defined on $(0, \infty) \times (L_i(t), R_i(t))$ and satisfying equations 1 above, the interface condition at R_i is

$$\rho_i H_i \dot{R}_i(t) = -k_i u_{i\xi}(t, R_i^-(t)) + k_{i+1} u_{i+1,\xi}(t, L_{i+1}^+(t)), \quad (3.2)$$

where ρ_i and k_i are the density and conductivity respectively for the i th phase and H_i is the "latent heat per unit mass" of the i th phase of the transition between phases i and $i + 1$. (In Dankwert's terminology, these are problems of class "A".)

In what follows we replace $R_i(t)$ by $\Xi_i(t)$ and, from equation 2.3, $L_i(t)$ by $\mu_{i-1}\Xi_{i-1}(t)$ where as before $\mu_i = \rho_i/\rho_{i+1}$, for $i = 1, \dots, n-1$. The resulting mathematical problem is:

$$u_{it} = D_i u_{i\xi\xi} \text{ for } \xi \in (\mu_{i-1}\Xi_{i-1}(t), \Xi_i(t)), \quad t > 0,$$

$$\text{and } i = 1, \dots, n;$$

$$u_i(t, \mu_{i-1}\Xi_{i-1}^+(t)) = U_{i-1}, \text{ for each } t > 0, \text{ and } i = 1, \dots, n;$$

$$u_i(t, \Xi_i^-(t)) = U_i + \Delta_i, \quad \text{for each } t > 0, \text{ and } i = 1, \dots, n-1;$$

$$\lim_{\xi \rightarrow +\infty} u_n(t, \xi) = U_n, \text{ for each } t > 0;$$

$$\rho_i H_i \dot{\Xi}_i = -k_i u_{i\xi}(t, \Xi_i^-) + k_{i+1} u_{i+1, \xi}(t, \mu_i \Xi_i^+), \text{ for } i = 1, \dots, n-1;$$

$$\Xi_0(t) = 0, \quad \text{for all } t \geq 0;$$

$$\Xi_i(0^+) = 0, \quad \text{for } i = 1, \dots, n-1;$$

$$\Xi_n(t) = +\infty, \quad \text{for all } t \geq 0;$$

$$u_n(0^+, \xi) = U_n, \text{ for each } \xi > 0.$$

Provided that the data satisfy certain reasonable assumptions, which we shall exhibit in a moment, we are assured of the existence and uniqueness of a similarity solution for this problem. The form of this solution, essentially the solution given by Weiner but expressed in the local coordinates, is

$$u_i(t, \xi) = (U_{i-1} + \alpha_i) \{ \operatorname{erfc}[\xi / \sqrt{(4D_i t)}] / \operatorname{erfc}[a_{i-1} \mu_{i-1} \sqrt{(D_{i-1}/D_i)}] \} - \alpha_i$$

for $t > 0, \xi \in (\mu_{i-1}\Xi_{i-1}(t), \Xi_i(t)), i = 1, \dots, n-1;$ (3.3a)

$$\Xi_i(t) = a_i \sqrt{(4D_i t)}, \quad \text{for } i = 1, \dots, n-1; \quad (3.3b)$$

and for $t > 0, \xi > \Xi_{n-1}(t)$

$$u_n(t, \xi) = \begin{cases} U_n & \text{if } D_n = 0, \\ U_n + \frac{(U_{n-1} - U_n) \operatorname{erfc}[\xi / \sqrt{(4D_n t)}]}{\operatorname{erfc}[a_{n-1} \mu_{n-1} \sqrt{(D_{n-1}/D_n)}]} & \text{if } D_n > 0; \end{cases} \quad (3.3c)$$

where for $i = 1, \dots, n-1$

$$\alpha_i = -U_{i-1} + (U_{i-1} - U_i - \Delta) \operatorname{erfc}[a_{i-1} \mu_{i-1} \sqrt{(D_{i-1}/D_i)}] / \phi_i;$$

$$\phi_i = \operatorname{erf}(a_i) - \operatorname{erf}[a_{i-1} \mu_{i-1} \sqrt{(D_{i-1}/D_i)}];$$

and the $\{a_i\}$ $i = 1, \dots, n-1$ are the unique solution of the nonlinear equations:

$$\begin{aligned} -\rho_i H_i D_i a_i \sqrt{\pi} &= -K_i (U_{i-1} - U_i - \Delta_i) \exp(-a_i^2) / (\phi_i \sqrt{D_i}) \\ &\quad + K_{i+1} (U_i - U_{i+1} - \Delta_{i+1}) \exp(-a_{i+1}^2 \mu_i D_i / D_{i+1}) / (\phi_{i+1} \sqrt{D_{i+1}}), \\ &\quad \text{for } i = 1, \dots, n-2; \end{aligned} \quad (3.4a)$$

and

$$\begin{aligned} \rho_{n-1} H_{n-1} D_{n-1} a_{n-1} \sqrt{\pi} &+ K_{n-1} (U_{n-2} - U_{n-1} - \Delta_{n-1}) \exp(-a_{n-1}^2) / (\phi_{n-1} \sqrt{D_{n-1}}) \\ &= 0, \text{ if } D_n = 0. \\ &= +K_n (U_{n-1} - U_n) \exp(-a_{n-1}^2 \mu_{n-1}^2 D_{n-1} / D_n) / (\phi_n \sqrt{D_n}), \text{ if } D_n \neq 0. \end{aligned} \quad (3.4b)$$

In equation 4b we take $\phi_n = 1 - \operatorname{erf}[a_{n-1} \mu_{n-1} \sqrt{(D_{n-1}/D_n)}]$, which is consistent with the previous definition if we make the identification $\operatorname{erf}(a_n) \equiv 1$. In equation 4a the dummy parameters a_0 and D_0 , which appear in the definition of ϕ_1 , are to be taken to be zero.

The relations among the data which assure the existence and uniqueness of a solution to equations 4, and hence of the similarity solution, are $D_i > 0$, $K_i > 0$ for $i = 1, \dots, n-1$ and either $D_n > 0$, $K_n > 0$ or $D_n = K_n = 0$; and either $U_{i-1} > U_i + \Delta_i$, $H_i \geq 0$ for $i = 1, \dots, n-1$ and $U_{n-1} \geq U_n$; or $U_{i-1} < U_i + \Delta_i$, $H_i \leq 0$ for $i = 1, \dots, n-1$ and $U_{n-1} \leq U_n$; with the proviso that if $U_{n-1} = U_n$, then $H_{n-1} \neq 0$; and if $D_n = K_n = 0$, then $U_{n-1} = U_n$.

This assertion follows from results established in ref [9]. For if we proceed as we did in [9] and combine the several constants of the

problem into fewer, and incidentally nondimensional parameters, and then rearrange the equations 4 so that each $\text{erf}(a_{i+1})$ is given explicitly in terms of a_i and a_{i-1} , then the resulting equations are precisely those analyzed in [9].

The alternate parameters are:

$$\beta_i = \mu_i / (D_i / D_{i+1}), \quad \text{for } i = 1, \dots, n-2; \quad (3.5)$$

$$\beta_{n-1} = \begin{cases} \mu_{n-1} / (D_{n-1} / D_n), & \text{if } D_n > 0, \\ 1, & \text{if } D_n = 0; \end{cases}$$

$$A_i = (\beta_i \mu_i) (K_{i+1} / K_i) (U_i - U_{i+1} - \Delta_{i+1}) / (U_{i+1} - U_i - \Delta_i), \\ \text{for } i = 1, \dots, n-2;$$

$$A_{n-1} = (\beta_{n-1} / \mu_{n-1}) (K_n / K_{n-1}) (U_{n-1} - U_n) / (U_{n-2} - U_{n-1} - \Delta_{n-1});$$

$$B_i = \rho_i H_i D_i / \pi [K_i (U_{i-1} - U_i - \Delta_i)] \text{ for } i = 1, \dots, n-1;$$

and the rearranged equations for the $\{a_i\}$ are

$$\text{erf}(a_{i+1}) = \text{erf}(\beta_i a_i) \\ + A_i \exp[a_i^2 (1 - \beta_i^2)] \phi_i / (1 - B_i a_i \exp(a_i^2) \phi_i), \quad (3.6a) \\ \text{for } i = 1, \dots, n-2;$$

where as before $\phi_i = \text{erf}(a_i) - \text{erf}(a_{i-1} \beta_{i-1})$; and

$$\text{erf}(a_{n-2} \beta_{n-2}) = \text{erf}(a_{n-1}) \\ - \{B_{n-1} a_{n-1} \exp(a_{n-1}^2) \\ + A_{n-1} \exp[a_{n-1}^2 (1 - \beta_{n-1}^2)] / \text{erfc}(a_{n-1} \beta_{n-1})\}^{-1}. \quad (3.6b)$$

These are the equations analyzed in detail in [9]. For $n > 2$ our assumed ordering of the boundary values guarantees $A_i > 0$, $B_i \geq 0$ for $i = 1, \dots, n-2$, and, provided $B_i A_i \exp(a_i^2) \phi_i < 1$, this implies $a_{i+1} > a_i \beta_i$ and hence $\Xi_{i+1}(t) > \mu_i \Xi_i(t)$ for $i = 1, \dots, n-2$. This is the desired result.

As we have emphasized before, the variable ξ represents, for each i , a coordinate value in a coordinate system at rest with respect to the i th phase. Thus in equation 3b the i th moving interface is represented by the function $\Xi_i(t)$ whose values are coordinate values in that same coordinate system at rest with respect to the i th phase.

The locations of the several moving interfaces in one fixed coordinate system may be obtained by substituting equations 3b into equation 2.5. The result is

$$S_i(t) = 2(a_i\sqrt{D_i} + \sum_{j=1}^{i-1}(1-\mu_j)a_j\sqrt{D_j})\sqrt{t} \quad \text{for } i = 2, \dots, n. \quad (3.7)$$

Recall that $S_1(t) = \Xi_1(t) = 2a_1\sqrt{D_1t}$, since the first local coordinate system is not moving.

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