Plasma transport in stochastic magnetic fields.**III: Kinetics of test-particle diffusion**

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Abstract

A discussion is given of test particle transport in the presence of specified stochastic magnetic fields, with particular emphasis on the collisional limit. *Certain paradoxes and inconsistencies in the literature regarding the form of the scaling laws are resolved by carefully distinguishing a number of physically distinct correlation lengths, and thus by identifying several collisional subregimes.* The common procedure of averaging the conventional fluid equations over the statistics of a random field is shown to fail in some important cases because of breakdown of the Chapman-Enskog ordering in the presence of a stochastic field component with short autocorrelation length. A modified perturbation theory is introduced which leads to a Kubo-like formula valid in all collisionality regimes. The direct-interaction approximation is shown to fail in the interesting limit in which the orbit exponentiation length L_K appears explicitly. A higher order renormalized kinetic theory in which L_K appears naturally is discussed and used to rederive more systematically the results of the heuristic scaling arguments.

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Fig 1

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1. Introduction

The problem of cross-field plasma transport in the presence of magnetic fields which are assumed to be stochastic has received considerable attention in the literature because of its importance in providing a potential explanation of the anomalous electron thermal conductivity κ_e in tokamaks. In terms of a magnetic fluctuation spectrum which is assumed to be known — e.g., from experiment — formulas have been given for κ_e in essentially all parameter regimes of possible physical relevance (Stix 1973; Rechester & Rosenbluth 1978; Stix 1978; Krommes, Kleva, & Oberman 1978; Krommes 1978; Kadomtsev & Pogutse 1979). While it is, of course, difficult to develop a quantitative theory universally valid in all regimes, the scaling laws should asymptotically match across the appropriate boundaries in parameter space. However, scrutiny of several recent reports leads one to conclude that certain problems, paradoxes, and confusions concerned with that matching still exist. In some cases, there are outright disagreements between two or more authors on the form of κ_e in certain regimes. It is our purpose here to attempt to resolve some of these problems, although we caution that certain aspects of our work — in particular, the formal kinetic theory — require further study and justification.

One apparent paradox which arises in the collisional limit (which for present purposes may be very loosely defined as the regime where a fluid description is appropriate, and will be more rigorously defined later) is concerned with the plausible procedure of averaging a classical fluid equation, including transport parallel to a strong magnetic field, over the slightly random direction of that field in order to deduce an effective radial transport equation with "anomalous" coefficient. For example, in the presence of a very strong magnetic field which varies slowly in space, a good approximation to the temperature transport equation is (Braginskii 1965)

$$\frac{\partial T}{\partial t} = \kappa_{\parallel} (\hat{n} \cdot \nabla)^2 T, \quad (1)$$

where $\hat{n} \equiv \mathbf{B}/B$ and $\kappa_{\parallel} \sim v_t^2/\nu$. Here $v_t \equiv (T/m)^{1/2}$ is the thermal velocity, ν is the classical collision frequency, and for brevity we shall not continue to indicate species dependence explicitly. (In writing (1), we ignore the possibility that $\kappa_{\parallel}$ depends on space, as this is not essential to our arguments.) If the field is now assumed to have a small, static, but random component, $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{B}_0 \equiv B_0 \hat{n}_0$, $\mathbf{b} \equiv \delta \mathbf{B}/|\mathbf{B}|$, in the x direction, then upon averaging (1) over $\mathbf{b}$, y , and z we are led to the mean transport law

$$\frac{\partial}{\partial t} \langle T \rangle = \kappa^{(f)} \frac{\partial^2}{\partial x^2} \langle T \rangle + \kappa_{\parallel} \langle \mathbf{b} (\hat{n}_0 \cdot \nabla) \delta T \rangle, \quad (2)$$

where

$$\kappa^{(f)} \equiv \kappa_{\parallel} \langle b^2 \rangle. \quad (3)$$

In (2), the term in $\kappa^{(f)}$ is the result recently found by Rose (1982) by different means, and has a plausible interpretation in terms of a projection of the parallel collisional motion onto the radial direction. However, it is known that when the temperature fluctuation δT is evaluated from (1) in the quasilinear approximation and in the low frequency limit, the second term of (2) precisely cancels the first term, so that in the limit where $\kappa_{\parallel}$ dominates there is no anomalous transport at all (Kadomtsev & Pogutse 1979; Haas, Thyagaraja, & Cook 1981). Therefore, it is essential to include a classical perpendicular transport term $\propto \kappa_{\perp}$ in (1). Kadomtsev & Pogutse (1979) evaluated (2) in this case and found a nonzero anomalous coefficient; it is interesting that in the limit of sufficiently large $\kappa_{\perp}$ their formula reduces to (3). However, neither the general formula of Kadomtsev & Pogutse [their Eq. (21)] nor the special limit (3) agrees with the Rechester-Rosenbluth (1978) result for the collisional limit,

$$\kappa^{(R)} \sim D_m \kappa_{\parallel} / L_K, \quad (4)$$

where L_K is the Kolmogorov-Lyapunov length (Chirikov 1969; Krommes, Kleva, & Oberman 1978; Krommes 1978; Rechester, Rosenbluth, & White 1979) which describes the exponential divergence of neighboring field lines (see App. A) and is a measure of the degree of stochasticity of the magnetic field, and D_m is the diffusion coefficient for the lines (Rosenbluth et al. 1966) and scales as (see App. A)

$$D_m \sim \langle b^2 \rangle L_{ac}, \quad (5)$$

L_{ac} being the (parallel) correlation length for the field. Now it has recently been asserted (Horton 1981) that $\kappa^{(R)}$ agrees with the result (3) of a proper fluid description (to within a logarithmic factor involving a perpendicular transport coefficient). Of course, such a logarithmic factor would be interesting in its own right; either fluid equations such as (1) are correct, or they are not. More importantly, however, the ratio of (4) to (3) is

$$\frac{\kappa^{(R)}}{\kappa^{(f)}} \sim \frac{L_{ac}}{L_K}. \quad (6)$$

Because L_{ac} and L_K are quite distinct physical lengths, the ratio L_{ac}/L_K is not, in general, equal to one. In the quasilinear limit ($\langle b^2 \rangle$ not too

large), L_{ac} is a measure of the (parallel) spectral width ($L_{ac} = L_0 \sim \Delta k^{-1}$; see App. A), is small, and is independent of the fluctuation intensity. The quantity L_K , however, depends explicitly on the fluctuation level (see App. A) and is large for small $\langle b^2 \rangle$. Thus, in the quasilinear limit of weak perturbed magnetic fields (Rosenbluth et al. 1966), $\kappa^{(R)}/\kappa^{(f)} < 1$. In fact, a number of results for various regimes of collisionality and fluctuation intensity have been given. $\kappa^{(R)}$ agrees with none of them, although its heuristic derivation is compelling.

[We use the notation L_{ac} rather than L_c because Rechester & Rosenbluth use L_c for the quantity we call L_K . It is important to keep in mind that the autocorrelation length L_{ac} of the magnetic field is nowhere discussed in the works by Rosenbluth et al. (1966) or Rechester & Rosenbluth (1978). (Those authors implicitly set L_{ac} to 0.)]

Most features of the above problems can be resolved by a careful elucidation and study of the various lengths and orderings which pertain to the problem of stochastic magnetic fields. In our early work (Krommes, Kleva, & Oberman 1978), we presented many aspects of such a study, including the effects of collisions from a Langevin point of view. Unfortunately, although that work was widely distributed and discussed (Kleva, Krommes, & Oberman 1977), the details were never published because of lingering doubts about certain approximations in the nonlinear kinetic theory (an issue separate from the above) which was employed. [A review of the principal results has been published (Krommes 1978).] Because the detailed arguments presented in our 1978 report contain the resolution of the above problems, we have rewritten and considerably augmented the discussion and included it in Sec. 2 and App. A. By carefully distinguishing the quasilinear autocorrelation length L_0 from the various nonlinear lengths, we are able to proceed directly from a single particle picture to show in Sec. 2 that the fluid result $\kappa^{(f)}$ is correct only in a certain extremely collisional limit when the mean free path λ_v is much less than L_{ac} . (There are additional important restrictions; see Sec. 2.2.) When the autocorrelation length is short, procedures which may appear to give rise to $\kappa^{(f)}$ fail by invoking an incorrect factorization of a certain four-point correlation function, and thus by neglecting an interesting and subtle long-time correlation between the fluctuating magnetic field and the parallel velocity of the particle — the double-diffusion effect discussed in more heuristic, less detailed terms by Rechester & Rosenbluth. By a careful discussion of the double-diffusion process, we can show that there is no contradiction between the collisional results of Rechester & Rosenbluth and of Kadomtsev & Pogutse; they are valid in different collisional subregimes. In particular, the results of Kadomtsev & Pogutse emerge by using a decorrelation time of particles from lines different than the one involving L_K used by

Rechester and Rosenbluth. In fact, we are able to exhibit smooth transitions of the scaling laws between all regimes of collisionality. Furthermore, we have identified a new "double-streaming" regime.

The issue as to whether the exponentiation length L_K should or should not enter the formula for κ explicitly in any regime appears to be somewhat controversial. It will never emerge from a fluid theory based on classical transport equations. Furthermore, in our early work we observed that the well-known direct-interaction approximation does not lead to the result of Rechester & Rosenbluth. [Rose (1982) has recently arrived at the same conclusion, using a somewhat different point of view.] This is very interesting and perhaps confusing, because arguments have been advanced that the approximation should be adequate in the stochastic regime. However, the efficacy of any approximation depends on what quantities it is asked to compute. The direct-interaction approximation also arises by a certain factorization of a four-point function (different than the one discussed previously), and that factorization can be expected to fail at very small space and time scales where correlations are significantly non-Gaussian. In fact, by introducing in our 1978 report a joint probability distribution for test particles and stochastic field, we were able to argue that the small scales played an essential role in determining the (Lagrangian) correlation function whose time integral is the diffusion coefficient. Although certain of the approximations used there were very crude, we believe the qualitative conclusions we drew remain correct. In Sec. 4 we give a revised and more complete version of our earlier calculations.

In this paper we shall follow Rechester & Rosenbluth and explicitly study the transport of test particles inserted into a specified stochastic magnetic field. It is conventionally believed that the scaling laws for the test particle density diffusion coefficient of species s agree with those of the thermal conduction coefficient of species s for the plasma fluid, and it is this presumed connection which makes our work relevant to the problems just described. However, there are certain subtleties associated with this relationship, which we shall address in a future paper. Although we shall have this relationship in mind as we proceed, to avoid any possible confusion we stress that the mathematics to follow is concerned exclusively with the transport of test particles.

We shall of necessity make many approximations. Fundamentally, we shall use the same assumptions used by Rechester & Rosenbluth, including in particular the assumption that the perturbed magnetic field is static. This may be appropriate for low frequency modes which are predominantly electromagnetically polarized. It does, however, neglect the fluctuating $\vec{E} \times \vec{B}$ velocity

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and the motion of the magnetic lines. These effects are very important in some cases and should be included in a complete theory; we plan to do so in future work. However, in view of the attention which the Rechester & Rosenbluth model has received and because of the confusion which remains in the literature, we feel it is entirely in order to give a careful discussion of the restricted model. There is much to be learned from this approximation, and that insight will be very useful in developing more complete theories of transport in stochastic magnetic fields.

The remainder of the paper, then, is organized as follows. In Sec. 2 we present a heuristic discussion of the scaling laws for transport in a specified static stochastic magnetic field which considerably generalizes the earlier arguments of Rechester & Rosenbluth (1978) and Stix (1978). This discussion begins with the so-called double diffusion theory (of test particle diffusion in the limit where the particles are tied to the stochastic field lines), which we discuss in detail in Sec. 2.1. However, the long-time radial diffusion coefficient vanishes in that limit, so in Sec. 2.2 we show how to combine the double diffusion theory with a heuristic theory of perpendicular decorrelation times (of the particle from its initial line) in order to obtain the scaling law for the radial transport coefficient in any collisionality regime.

Secs. 3 and 4 are devoted to more formal aspects of the statistical dynamics and kinetic theory of test particle transport. In Sec. 3.1 we identify two general procedures which give rise to the transport coefficient: (1) computation of a certain random or conditional (on the stochastic field) hydrodynamic eigenvalue, followed by averaging over the field; (2) finding the hydrodynamic eigenvalue of a renormalized kinetic equation. In Sec. 3.2 we discuss a generalized Chapman-Enskog perturbation theory for the random hydrodynamic eigenvalue. In Sec. 4 we develop the renormalized kinetic theory. We summarize and discuss our results in Sec. 5. In App. A we review the theory of the underlying stochastic magnetic field and the magnetic diffusion coefficient. In App. B we indicate some aspects of renormalized perturbation theory. Finally, in App. C we briefly review the general theory of renormalization and state the statistical closure which we explore in the text.

2. Heuristic theory of test particle diffusion in specified stochastic magnetic fields

2.1: Theory of double diffusion

Conceptually, the theory of test particle transport in stochastic magnetic fields may be split naturally into two parts: (1) a description of the spatial diffusion of magnetic lines, characterized by the magnetic diffusion

coefficient D_m (App. A) at a given time; and (2) a description of the motion of particles along, and decorrelation of particles from, a given line as a function of time. It is thus appropriate to introduce (Krommes, Kleva, & Oberman 1978; Krommes 1978) the "radial" (perpendicular to the unperturbed flux surfaces) particle dispersion $\langle \delta x^2(t) \rangle$ and to write its derivative as

$$D(t) \equiv \frac{1}{2} \frac{d}{dt} \langle \delta x^2(t) \rangle \equiv D_m v_t d(t). \quad (7a)$$

The average $\langle \dots \rangle$ is considered to be taken over both the specified statistics of the random magnetic field, the initial spatial and velocity distribution of the test particles, and over the statistics of the microfields responsible for classical collisional scattering. Later, we shall also consider averages conditional on the initial velocity $v_{||}$ of the test particle along the line, and we will write the conditional diffusion function $D(t|v_{||})$ as

$$D(t|v_{||}) \equiv D_m |v_{||}| d(t|v_{||}). \quad (7b)$$

(We shall also consider more general conditional averages, for which we will use the conventional notation $\langle a|b \rangle$ to denote the average of a conditional on b .)

If in the limit $t \rightarrow \infty$ $D(t)$ approaches a constant value D , then D is the radial diffusion coefficient of the test particle:

$$D = D_m v_t d(\infty). \quad (8)$$

If a particle becomes decorrelated from its initial line in a time τ_c , then $d(\infty) = d(\tau_c)$ (Fig. 1). Decorrelation mechanisms include temporal fluctuations of the random field, which we shall not consider in this paper, and cross-field diffusion due to classical collisions (Rechester & Rosenbluth 1978). We will discuss the latter mechanism at some length in Sec. 2.2. Of course, if several decorrelation mechanisms are at hand, the shortest τ_c should be used. This observation will be of great help in resolving some of the confusion in the literature.

The simplest form of $d(t)$ in the collisional limit $vt > 1$, where v is the collision frequency, follows from arguments similar to those of Rechester & Rosenbluth and of Stix, on which we elaborate. Let l describe length along a reference field line. By assumption, the lines diffuse in x for parallel distances much greater than a correlation length L_{ac} . Thus, we imagine coarse-graining the l axis into arbitrary but fixed units Δl satisfying $\Delta l \gg L_{ac}$ (Fig. 2). For distances $l \equiv N\Delta l$ (N fixed; > 1), the radial dispersion conditional on Δl will diffuse,

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$$\langle \delta x^2(N) | \Delta l \rangle \sim D_m l = D_m N \Delta l. \quad (9)$$

However, particles do not always move a specified distance Δl in a fixed time Δt ; rather, they diffuse along the lines because of classical collisions. We must consider two possible regimes for the parallel mean free path $\lambda_v \equiv v_t/v$ with respect to L_{ac} . (A deficiency of the early papers was that these regimes were not identified.) In the first, a very collisional limit, we assume $\lambda_v < L_{ac}$. In this case, we imagine subdividing L_{ac} into units δl satisfying $L_{ac} > \delta l > \lambda_v$ (Fig. 3). Parallel spatial diffusion ensues for times corresponding to distances greater than δl , so since $\Delta l > \delta l$ we may consider Δl to be a Gaussian random variable whose distribution $P(\Delta l)$ has zero mean and mean square $\langle \Delta l^2 \rangle \sim D_{||} \Delta t$. From (9), then,

$$\langle \delta x^2(N) \rangle = \int d(\Delta l) \langle \delta x^2(N) | \Delta l \rangle P(\Delta l) \approx 0. \quad (10)$$

This approximation for the second moment is valid at long times. It implies the important fact that (for particles tied to lines) the long-time radial diffusion coefficient vanishes, or that the radial transport is slower than diffusive. We can estimate the time dependence of the even x moments at finite times by noting that $\Delta t = t/N^2$ and considering the fourth moment. From (9),

$$\langle \delta x^4(N) \rangle \sim (D_m N)^2 \langle \Delta l^2 \rangle \sim D_m^2 D_{||} t, \quad (11)$$

or

$$\langle \delta x^4 \rangle^{1/2} / t \sim D_m (D_{||}/t)^{1/2}. \quad (12)$$

If we use the classical expression $D_{||} \sim v^2/v$ and the definition (7a), and ignore the numerical difference between $\langle (\delta x^2)^2 \rangle^{1/2}$ and $\langle \delta x^2 \rangle$, we estimate

$$d(t) \rightarrow d_0(t) \sim (vt)^{-1/2}, \quad (13)$$

where we will use the notation d_0 to indicate that we are (at the moment) studying only the consequences of parallel collisional diffusion, with particles tied to lines.

In the second limit, we assume that λ_v satisfies $L_{ac} < \lambda_v$. (This is the regime implicitly considered by Rechester & Rosenbluth.) In this case, we may choose for the coarse-graining space steps Δl (for radial diffusion of the lines) and δl (for parallel diffusion of the particles) the ordering

$$L_{ac} < \Delta l < \lambda_v < \delta l \quad (14)$$

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(Fig. 4). Here the first (as a function of λ or t) process to set in is radial diffusion, according to

$$\langle \delta x^2(\lambda) \rangle \sim D_m \lambda \quad (15)$$

for any $\lambda > \Delta \lambda$. For $\lambda > \delta \lambda$, parallel diffusion ensues according to

$$\langle \lambda^2 | \delta \lambda \rangle \sim D_{\parallel} t. \quad (16)$$

In order to express (15) in terms of the elapsed time, we will combine (16) with (15) by referring both λ and t to the arbitrary [subject to (14)] but fixed step $\delta \lambda$. Thus, rewriting (16),

$$\delta \lambda \sim (D_{\parallel} t)^{1/2} \left(\frac{\delta \lambda}{\langle \lambda^2 | \delta \lambda \rangle^{1/2}} \right) \quad (17)$$

and then from (15)

$$\begin{aligned} \langle \delta x^2(\lambda) \rangle &= \left(\frac{\langle \lambda^2 | \delta \lambda \rangle^{1/2}}{\delta \lambda} \right) \langle \delta x^2(\delta \lambda) \rangle \\ &\sim D_m (D_{\parallel} t)^{1/2}, \end{aligned} \quad (18)$$

which leads us again to the result (13). Thus, for arbitrary λ_v/L_{ac} , there is no radial diffusion in the long-time limit if the particles remain tied to the lines (Rechester & Rosenbluth 1973).

Although $d_o(t) \rightarrow 0$ for $vt > 1$ and predicts no long-time diffusion, the function $d_o(t)$ plays a very important role in the heuristic scaling theory. This is so because for times less than a perpendicular correlation time, $d_o(t)$ approximates the true function $d(t)$. Thus, we have the important relations

$$d(\infty) = d(\tau_c) = d_o(\tau_c), \quad (19)$$

so that knowledge of $d_o(t)$ together with τ_c is sufficient to specify the scaling. This procedure is illustrated in Fig. 5 for the case $vt_c > 1$, although the opposite limit is possible and will be briefly discussed for completeness.

It is very useful and instructive to rederive the above results for $d_o(t)$ more systematically. To this end, we consider the following time correlation function expression for the time-dependent "diffusion" function:

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$$D(t) = \int_0^t d\tau \langle v_x(\tau) v_x(0) \rangle \\ = \int_0^t d\tau \langle v_{\parallel}(\tau) b(\tau) v_{\parallel} b(0) \rangle, \quad (20)$$

where $b(\tau)$ indicates the Lagrangian time dependence $b(\tau) \equiv b(\underline{x}(\tau), \tau)$, $\underline{x}(\tau)$ being the (random) particle trajectory, and where $v_{\parallel}$ stands for the initial velocity $v_{\parallel}(0)$. (Note that expression (20) differs from the definition (7a) by a term of $O(t/\tau_c)$ for $t/\tau_c < 1$, but is correct in the limit $t/\tau_c \gg 1$, which defines the time-asymptotic diffusion coefficient. Also note that (20) ignores the $\underline{E} \times \underline{B}$ velocity; recall our discussion in the Introduction.) The average is taken over the statistics of the random magnetic field and random $v_{\parallel}(\tau)$, and, in general, over a thermal distribution of $v_{\parallel}$. In Secs. 3 and 4 we shall see how the thermal average enters in a natural way when one passes to the appropriate hydrodynamic limit. However, in this section it will generally be convenient to work with averages conditional on $v_{\parallel}$.

We assume throughout that the Fourier amplitudes of the stochastic field are not affected by the particle motion. [This defines the so-called non-selfconsistent or (generalized) stochastic acceleration problem, and is adequate for the discussion of test particle (but not fluid) transport.] Of course, this does not mean that the test particle orbits are statistically independent of the field; in general, there are important correlations, because the particles move along the actual field lines, and furthermore move from line to line in a way which may depend in detail on the statistical structure of the field. However, to study $d(t)$, we retain only motion along the original line (thus setting the particle-line correlation time τ_c to ∞). Now the motion along the lines is, by assumption, independent of the statistics of the field. Then, illustrating for the important quasilinear limit $L_{ac} < L_K$, we can write (20) for the special case of a static, sheared slab magnetic geometry (App. A) in the form

$$D_o(t) = \int_0^t d\tau \int \frac{dk_{\parallel}}{2\pi} \langle b^2 \rangle_{k_{\parallel}} \langle v_{\parallel}(\tau) v_{\parallel} \exp(ik_{\parallel} \lambda(\tau)) \rangle, \quad (21)$$

where $\lambda(0) = 0$ and $\langle b^2 \rangle_{k_{\parallel}}$ is the spectral intensity of the stochastic field in parallel wavenumber $k_{\parallel}$.

Before giving a general discussion of (21), let us comment on an important pitfall of (21) with the aid of a particularly simple and illustrative special case. If the spectral intensity is absolutely uniform in $k_{\parallel}$ with a width $\Delta k_{\parallel} \rightarrow \infty$, so that the autocorrelation length (App. A) $L_{ac} = L_o \sim \Delta k_{\parallel}^{-1}$ is essentially zero (and thus $\lambda_v > L_{ac}$), then we can perform the $k_{\parallel}$ integral to find

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$$D_o(t) = 2D_m \int_0^t d\tau \langle v_{\parallel}(\tau) v_{\parallel} \delta(l(\tau)) \rangle, \quad (22)$$

where D_m is the conventional quasilinear diffusion coefficient (Rosenbluth et al. 1966; App. A),

$$D_m \equiv \langle b^2 \rangle_{L_o}, \quad (23)$$

$$L_o \equiv \pi / \Delta k_{\parallel}, \quad (24)$$

and

$$\langle b^2 \rangle = \int \frac{dk_{\parallel}}{2\pi} \langle b^2 \rangle_{k_{\parallel}} = \left(\frac{\Delta k_{\parallel}}{2\pi} \right) \langle b^2 \rangle_{\bar{k}_{\parallel}}, \quad (25)$$

where $\bar{k}_{\parallel}$ is a typical wavenumber. Now suppose the correlation function in (22) is factored according to an (in general, unjustified, as we shall see) independence hypothesis of the form

$$D_o(t|v_{\parallel}) = 2D_m \int_0^t d\tau \langle v_{\parallel}(\tau) | v_{\parallel} \rangle \delta(l(\tau)) \quad (26)$$

To evaluate qualitatively the indicated time-dependent expectations, we shall use the well-known Langevin theory of one-dimensional (in the parallel direction) Brownian motion (Uhlenbeck & Ornstein 1930; Wang & Uhlenbeck 1945), assuming a drag term of the form $-v_{\parallel}$. (Although this form does not correctly model the velocity dependence of the polarization drag except in special limits, our results will be qualitatively correct.) Since according to that theory

$$\langle v_{\parallel}(\tau) | v_{\parallel} \rangle = e^{-v_{\parallel} \tau / v_t^2}, \quad (27)$$

the integrand of (26) is small for $\tau > v^{-1}$ and the short time (streaming) limit $l(\tau) = v_{\parallel} \tau$ is adequate. Then

$$\begin{aligned} D_o(t|v_{\parallel}) &= 2D_m \int_0^t d\tau e^{-v_{\parallel} \tau / v_t^2} |v_{\parallel}| \delta(\tau) \\ &= D_m |v_{\parallel}|, \end{aligned} \quad (28a)$$

or

$$D_o(t) = \left(\frac{2}{\pi} \right)^{1/2} D_m v_t. \quad (28b)$$

This is the well-known collisionless result (Jokipii & Parker 1969), and is thus in disagreement with the double-diffusion arguments for $t \gtrsim v^{-1}$. The error is in the factorization (26), and is easy to understand on the basis of

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the Langevin theory. A density pulse released with velocity $v_{\parallel}$ at $\tau = 0$ quickly streams out of the correlation distance L_0 in a time $L_0/|v_{\parallel}|$, and $d_0(t|v_{\parallel})$ becomes stationary with the collisionless value 1. However, the streaming proceeds only for $t < \tau^{-1}$. By roughly a collision time, directed momentum of the pulse is lost, the mean position becomes stationary at the mean free path, and spatial diffusion along the line ensues. In particular, some particles diffuse backwards into the correlation distance of the field, re-establishing the correlation which was lost when the pulse streamed out of the range L_0 at short times; $d_0(t)$ begins to decrease. Mathematically, this effect is described by the term $\langle v_{\parallel}(\tau) \delta(l(\tau)) \rangle$, which cannot be factored for times longer than a collision time.

Because the conditional probability $P(v_{\parallel}, l, \tau | v_0, l_0)$ is known in the Langevin approximation, it is possible to give a quite general expression for $D_0(t)$ which embraces the double-diffusion scaling. Returning to (21) and using (7b), we see that

$$d_0(t|v_{\parallel}) = \text{sgn}(v_{\parallel}) \int_0^t d\tau \left(\frac{\int dk_{\parallel} \langle b^2 \rangle_{k_{\parallel}} \langle v_{\parallel}(\tau) \exp(ik_{\parallel} l(\tau)) \rangle}{L_0 \int dk_{\parallel} \langle b^2 \rangle_{k_{\parallel}}} \right) \\ = \text{sgn}(v_{\parallel}) L_0^{-1} \int_0^t d\tau \int \frac{dk_{\parallel}}{2\pi} C_{k_{\parallel}} \left\langle \frac{dl}{d\tau} \exp(ik_{\parallel} l(\tau)) \right\rangle, \quad (29)$$

where

$$C(l) \equiv \langle b(l)b(0) \rangle / \langle b^2(0) \rangle \quad (30a)$$

is the Eulerian spatial correlation function of the static field. (In terms of $C(l)$, the autocorrelation distance L_{ac} is defined by

$$L_{ac} \equiv \int_0^{\infty} dl C(l). \quad (30b)$$

The time integral in (29) can be performed, because the term inside the brackets is $(ik_{\parallel})^{-1}$ times the derivative of the exponential. Upon performing the resulting average by the method of cumulants (the Langevin displacement is a Gaussian random variable), we find

$$\bar{d}_0(t|v_{\parallel}) = \text{sgn}(v_{\parallel}) L_0^{-1} \int \frac{dk_{\parallel}}{2\pi} C_{k_{\parallel}} (ik_{\parallel})^{-1} \left(\exp(ik_{\parallel} \langle l(t) \rangle - \frac{1}{2} k_{\parallel}^2 \sigma^2(t)) - 1 \right), \quad (31)$$

where the results

$$\langle l(t) \rangle = (1 - e^{-v\tau}) \lambda_0, \quad (32a)$$

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$$\lambda_o \equiv v_{||}/v, \quad (32b)$$

and

$$\sigma^2(t) \equiv \langle \delta \lambda^2(t) \rangle = (2vt - 3 + 4e^{-vt} - e^{-2vt}) \lambda_v^2, \quad (33a)$$

$$\lambda_v \equiv v_t/v, \quad (33b)$$

are well-known (Uhlenbeck & Ornstein 1930). In the collisional limit $vt > 1$,

$$\langle \lambda(t) \rangle \rightarrow \lambda_o, \quad (34a)$$

$$\sigma^2(t) \rightarrow 2D_{||}t, \quad (D_{||} \equiv \lambda_v^2 v = v_t^2/v). \quad (34b,c)$$

By employing the convolution theorem for Fourier transformations, one can put (31) into an alternative, suggestive form:

$$d_o(t|v_{||}) = \text{sgn}(v_{||}) L_o^{-1} \int_0^{\langle \lambda(t) \rangle} d\zeta \langle C \rangle_{\zeta}, \quad (35)$$

where

$$\langle C \rangle_{\zeta} \equiv \int d\lambda C(\zeta - \lambda) P(\lambda, t), \quad (36a)$$

$$P(\lambda, t) \equiv [2\pi\sigma^2(t)]^{-1/2} \exp\{-\lambda^2/2\sigma^2(t)\}. \quad (36b)$$

The function $\langle C \rangle_{\zeta}$ describes the fractional particle density which has become decorrelated from the magnetic field. For times $L_o/|v_{||}| < t < v^{-1}$, this fraction is obviously unity, because the pulse has streamed through the autocorrelation length:

$$P(\lambda, t) \approx \delta(\lambda), \quad (37a)$$

$$\langle C \rangle_{\zeta} = C(\zeta), \quad (37b)$$

$$\begin{aligned} d_o(t|v_{||}) &= \text{sgn}(v_{||}) L_o^{-1} \int_0^{\langle \lambda(t) \rangle} d\zeta C(\zeta) \\ &= [\text{sgn}(v_{||}) L_o^{-1}] [\text{sgn}(v_{||}) L_o] = 1. \end{aligned} \quad (37c)$$

(For shorter times, the pre-magnetic diffusion limit,

$$d_o(t|v_{||}) = L_o^{-1} |v_{||}| t;$$

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this result will be of use later.) For collisional times, fewer and fewer particles remain uncorrelated as more and more particles diffuse backwards into the correlation region. Because particles are conserved, the rate at which correlations re-establish themselves is determined by the value of P at the spatial origin, which is just the normalization $P \sim \sigma^{-1} \sim (vt)^{-1/2}$. This is precisely the double-diffusion scaling law.

This behavior may be demonstrated explicitly in the extreme quasilinear limit $L_0 \rightarrow 0$, where the correlation function may be modelled by $C(\zeta) = \delta(\zeta/L_0)$. In this case

$$d_0(t|v_{\parallel}) = \text{sgn}(v_{\parallel}) \int_0^{\langle \lambda(t) \rangle} d\zeta P(\zeta, t) \\ = \text{erf}(|\langle \lambda(t) \rangle| / \sqrt{2\sigma(t)}) \quad (38a)$$

$$= \begin{cases} 1 & (vt < 1), \end{cases} \quad (38b)$$

$$= \begin{cases} [|v_{\parallel}|/v_t] (\pi vt)^{-1/2} & (vt > 1). \end{cases} \quad (38c)$$

The significant deficiency of the example discussed in the last paragraph is that it precludes study of the collisional regime $\lambda_v < L_0$. In order to study this regime qualitatively, we may consider the box function

$$C(\lambda) = [H(\lambda - L_0) - H(\lambda + L_0)] / 2L_0. \quad (39)$$

Upon straightforward integration, we are led from (35) to the result

$$d_0(t|v_{\parallel}) = \text{sgn}(v_{\parallel}) \frac{\sqrt{2\sigma(t)}}{2L_0} \left[\text{erfc}_1\left(\frac{\langle \lambda(t) \rangle + L_0}{\sqrt{2\sigma(t)}}\right) - \text{erfc}_1\left(\frac{\langle \lambda(t) \rangle - L_0}{\sqrt{2\sigma(t)}}\right) \right. \\ \left. - \text{erfc}_1\left(\frac{L_0}{\sqrt{2\sigma(t)}}\right) + \text{erfc}_1\left(\frac{-L_0}{\sqrt{2\sigma(t)}}\right) \right], \quad (40)$$

where $\text{erfc}_1(z) \equiv \int_z^{\infty} du \text{erfc}(u)$. For $L_0 \rightarrow 0$, this reduces to the quasilinear model (38a). However, in the limit $|\langle \lambda(t) \rangle| < L_0$ (which need not imply $vt < 1$ if $\lambda_v < L_0$), we have

$$d_0(t|v_{\parallel}) = \left(\frac{|\langle \lambda(t) \rangle|}{L_0} \right) \text{erf}\left(\frac{L_0}{\sqrt{2\sigma(t)}}\right) \quad (|\langle \lambda(t) \rangle| < L_0) \quad (41a)$$

$$+ \begin{cases} |\langle \lambda(t) \rangle| / L_0 & (\sigma(t) < L_0), \end{cases} \quad (41b)$$

$$+ \begin{cases} |\langle \lambda(t) \rangle| / [2\pi\sigma^2(t)]^{1/2} & (\sigma(t) > L_0). \end{cases} \quad (41c)$$

Thus, for $\lambda_v < L_0$, times $t < v^{-1}$ correspond to a linear regime $d_0(t|v_{\parallel}) \sim |v_{\parallel}|t/L_0$. (As we noted following Eq. (37c), this result holds quite

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generally for sufficiently short times.) For times $1 < vt < (L_0/\lambda_v)^2$, $d(t) \approx \lambda_v/L_0$ and

$$\begin{aligned} D_o(t) &\approx D_{mt} \lambda_v / L_o \\ &= \langle b^2 \rangle_{D_{||}} \equiv D^{(f)}. \end{aligned} \quad (42)$$

However, from the point of view of the parallel dynamics (all we have retained so far), this apparently time-independent result is transient. According to (41c), for times satisfying $vt > (L_0/\lambda_v)^2$, so that a significant number of particles have diffused out of the correlation regime, the long-time tail of the double-diffusion law emerges once again and $D_o(t) \rightarrow 0$. To determine the irreversible, time-asymptotic diffusion coefficient, one must introduce a theory of the decorrelation time τ_c . Of course, if τ_c falls in the range $1 < v\tau_c < (L_0/\lambda_v)^2$, $D^{(f)}$ will be the true result. We will return to this observation in the next section.

2.2 Perpendicular diffusion and decorrelation times

As we have seen, the double-diffusion theory for $d_o(t)$ predicts a vanishing time-asymptotic diffusion coefficient, because it assumes that particles never leave the field lines, and transport along the lines is slow for collisional times. According to the discussion around Eq. (19), a theory of the correlation time τ_c of the particles with their initial line is needed to determine the long-time radial diffusion coefficient: $D = D_{mt} d_o(\tau_c)$. This formula holds regardless of the size of $v\tau_c$.

Let us first discuss the collisional cases $v\tau_c > 1$. Since we do not wish to commit ourselves here to a particular normal mode, we will not explicitly discuss the possibly important case where decorrelation occurs because the background fields fluctuate in time. (Formula (8) would still apply, with $\tau_c \sim \omega^{-1}$.) In a static field, the most obvious decorrelation mechanism is that particles leave their initial line because of perpendicular collisional diffusion (Rechester & Rosenbluth 1978). There are several times associated with this process. First, we may attempt to ignore the stochasticity of the field and define a decorrelation time $\tau_{\perp}$ by the time it takes to collisionally diffuse one perpendicular wavelength $\bar{k}_{\perp}^{-1} \equiv \delta_{\perp}$ across the field lines of the nonstochastic (unperturbed) background field:

$$\tau_{\perp} \equiv (\bar{k}_{\perp}^2 D_{\perp})^{-1} = \delta_{\perp}^2 / D_{\perp}. \quad (43)$$

(The inequality $v\tau_{\perp} \gg 1$ is well-satisfied for electrons when $\delta_{\perp}$ is scaled to the ion gyroradius and the classical expression $D_{\perp} \sim \rho^2 v^{\perp}$ is used.)

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Rechester & Rosenbluth rejected this time (without discussion) in favor of a time τ_K , defined when $v\tau_K > 1$ by

$$\tau_K \sim L_{K\delta}^2 / D_{\parallel}, \quad (44)$$

where $L_{K\delta}$ scales as the exponentiation length L_K times a logarithmic factor involving $D_{\perp}$ and $\bar{k}_{\perp}$. They argued that even a very small amount of cross-field collisional diffusion was sufficient to move the particle to a neighboring line. Because the field is stochastic, the new line diverges exponentially from the old (reference) line, as a function of distance along the reference line. Because the particle now moves along the new line (by collisional diffusion for $vt > 1$), it will be carried a perpendicular distance $\delta_{\perp}$ from the reference line in a time τ_K . The associated diffusion coefficient is

$$\begin{aligned} D^{(R)} &\sim D_m v_t (v\tau_K)^{-1/2} \\ &\sim D_m v_t (\lambda_v / L_{K\delta}) \sim D_m D_{\parallel} / L_{K\delta}. \end{aligned} \quad (45)$$

However, Rechester & Rosenbluth failed to remark that the shorter of $\tau_{\perp}$ and τ_K should be used as the proper correlation time τ_c . We have

$$\frac{\tau_{\perp}}{\tau_K} \sim \left(\frac{\delta_{\perp}}{L_K}\right)^2 \left(\frac{D_{\parallel}}{D_{\perp}}\right). \quad (46)$$

Thus, when the mean free path becomes short enough so that

$$D_{\parallel} / D_{\perp} < (L_K / \delta_{\perp})^2, \quad (47)$$

we have $\tau_{\perp} < \tau_K$. In this limit,

$$\begin{aligned} D &= D_m v_t (v\delta_{\perp}^2 / D_{\perp})^{-1/2} \\ &= D_m (D_{\parallel} D_{\perp})^{1/2} / \delta_{\perp}, \end{aligned} \quad (48)$$

which is just the result of the fluid calculation of Kadomtsev & Pogutse (1979) in the limit where $D_{\perp}$ is not too large. (When $D_{\perp}$ is classical, $D_{\perp} \sim \rho^2 v$, the transition to the fluid regime occurs when $\lambda_v / L_K < \rho / \delta_{\perp}$.) For purposes of distinction, we will call (48) the "Kadomtsev-Pogutse" result, although the "fluid" result discussed in the next paragraph also follows from their formula (21) in the limit of $D_{\perp}$ very large.

If λ_v is decreased still further, one can in principle achieve $v\tau_{\perp} < (L_o / \lambda_v)^2$ [classically, $\lambda_v / L_o < \rho / \delta_{\perp}$, or $\lambda_v / L_K < (L_o / L_K)(\rho / \delta_{\perp})$]

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and, according to the discussion at the end of Sec. 2.1, we recover the extremely collisional result $D^{(f)}$, which we shall call the "fluid" result. This appears to exhaust all possibilities for the collisional ($v\tau_c > 1$), quasilinear ($L_0 < L_K$) regimes.

For completeness, we will also discuss the collisionless regime $v\tau_c < 1$. For times so short that

$$vt < \min(L_0/\lambda_v, 1), \quad (49)$$

particles simply stream along the perturbed lines and one can easily estimate

$$d_0(t) \approx \left(\frac{\lambda_v}{L_0}\right) vt. \quad (50)$$

Because for such times particles are neither decorrelated due to parallel collisions nor decorrelated from the magnetic field, we may call this regime the double streaming limit. If $v\tau_c$ satisfies the inequality (49), we have

$$D \sim D_{mt} \left[\left(\frac{\lambda_v}{L_0}\right) (v\tau_c) \right] \sim \langle b^2 \rangle v^2 \tau_c \quad (51a)$$

$$\sim D^{(f)}(v\tau_c) < D^{(f)}. \quad (51b)$$

To our knowledge, this result is new. Of course, it requires a very short correlation time which may not arise in practice.

Finally, if $L_0 < \lambda_v$, we have the possibility $1 > v\tau_c > L_0/\lambda_v$, for which $d_0(\tau_c) = 1$ and we recover the well-known collisionless result $D \sim D_{mt}$ (Rechester & Rosenbluth 1978; Stix 1978; Krommes, Kleva, & Oberman 1978; Krommes 1978).

The principle results of this discussion are summarized in Table 1. The various regimes are delineated by inequalities involving the three dimensionless parameters $\alpha \equiv (v\tau_c)^{-1}$, λ_v/L_{ac} , and λ_v/L_K . Although the three-dimensional nature of the parameter space precludes a straightforward graphical representation, it is useful to imagine selecting a regime by first fixing L_{ac} and L_K , then choosing the correlation time τ_c , finally picking the mean free path λ_v (although all these quantities may be interrelated). The table is then arranged in a way that demonstrates how the form of the scaling laws changes continuously as each boundary in parameter space is crossed by varying λ_v . In particular, there are three contiguous but distinct collisional regimes: The Rechester-Rosenbluth regime, the Kadomtsev-Pogutse regime, and the fluid regime. The double-streaming regime is entered into the table twice, because it is independent of the size of λ_v/L_{ac} and is entered by

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reversing one of the inequalities which defines either the collisionless or the fluid regime.

Table 1 is explicitly correct for the quasilinear limit $L_{ac} = L_o < L_K$, where, according to App. A, L_o is determined by the parallel spectral width [cf. Eq. (24)] so that $L_o \sim L_s$, and

$$L_K \sim L_s^{2/3} L_\perp^{1/3}, \quad (52)$$

where L_s is the shear length and $L_\perp \equiv (k_\perp^2 D_m)^{-1}$. Thus,

$$L_o : L_K : L_\perp \sim 1 : \left(\frac{L_\perp}{L_s}\right)^{1/3} : \left(\frac{L_\perp}{L_s}\right), \quad (53)$$

so in the quasilinear regime we have

$$L_o < L_K < L_\perp \quad (\text{quasilinear regime}). \quad (54)$$

When the fluctuations grow so large that $L_K \sim L_o$, all three lengths become equal, and for still larger fluctuations (strong turbulence) the inequalities (54) are reversed:

$$L_\perp < L_K < L_o \quad (\text{strong turbulence}). \quad (55)$$

Thus, according to (54) and (55), the true autocorrelation length for the magnetic field is

$$L_{ac} = \min(L_o, L_\perp); \quad (56)$$

in contrast to the assertion of Rechester & Rosenbluth, the exponentiation length L_K is never the true autocorrelation length. Brief reflection then leads us to conclude that Table 1 remains correct even for the strong turbulence regime provided only that for that case L_K must be replaced by L_{ac} . The consequences of that change are (1) the Kadomtsev-Pogutse collisional regime disappears, and (2) the Rechester-Rosenbluth regime becomes identical to the fluid regime. Of course, the value of D_m is now determined nonlinearly; viz.

$$D_m \sim \langle b^2 \rangle (k_\perp^2 D_m)^{-1}, \quad (57)$$

or

$$D_m \sim \langle b^2 \rangle^{1/2} \delta_\perp; \quad (58)$$

however, the forms of the particle transport coefficients in terms of D_m as recorded in Table 1 remain unchanged.

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This concludes our heuristic discussion of the scaling laws for test particle transport in given stochastic magnetic fields. In the next two sections, we discuss how these results can be derived more systematically from kinetic theory.

3. Hydrodynamics and test particle transport

3.1: Introduction and general theory

The goal of a kinetic theory is to derive, from the fundamental dynamical equations of motion, low order statistical information about the system. As usual, the statistical mean is interpreted in the ensemble sense and is performed over all random variables in the problem (initial conditions, random particle forces in the Langevin theory, stochastic magnetic fields, etc.) Often, one attempts to derive a kinetic equation — that is, a closed equation for the mean distribution function (of the test particles, in this case). From the kinetic equation, transport coefficients can be deduced by standard (e.g., Chapman-Enskog) techniques. Although we shall study here only the problem of statistically specified magnetic fields, a major advantage of this method is that it generalizes naturally to the problem of self-consistent field evolution. The procedure, based on a renormalized stochastic Liouville equation (Kubo 1963) for the test particles, allows us to make close contact with modern advances and insights of turbulence theory.

If one is interested only in the long time, long wavelength behavior of the system, the intermediate derivation of the kinetic equation is not necessary, and alternative techniques based on direct computation of the hydrodynamic eigenvalues are available. We discuss this further shortly.

We shall work in the guiding center approximation, thus implicitly considering electrons. In addition to assuming that the particles move along the actual magnetic field lines, we also assume that they suffer Gaussian white noise accelerations (from the electric microfields of the discrete particles) in such a way that for short times they diffuse in the parallel direction in velocity space and in the perpendicular direction in real space. We ignore collective electrostatic fluctuations, and hence the $\mathbf{E} \times \mathbf{B}$ drift (although this term can be easily reinstated), and assume that the field is static. These are the same assumptions used by Rechester & Rosenbluth (1978). (Furthermore, we shall ignore ∇B and curvature drifts in the background field. In reality, those effects are important (Mynick & Krommes 1979) and introduce significant energy-dependent form factors into the scaling laws (Mynick & Krommes 1980) which are very necessary in attempts to properly interpret the experimental data (Mynick & Strachan 1981). However, here we are concerned more with elucidating the fundamental properties of the kinetic theory than

with quantitative accuracy, from which point of view the form factors are a technical complication of importance secondary to the already very complicated problem which exists for homogeneous background.) The standard theory of Markov processes allows us to perform the averages over the white noise immediately, and we thus consider the stochastic Liouville equation (Kubo 1963; Van Kampen 1976) for the guiding centers

$$0 = \frac{\partial}{\partial t} \tilde{P}(\tilde{x}, V, t) + \tilde{V}_n \cdot \tilde{\nabla} \tilde{P} - D_{\perp} \nabla_{\perp}^2 \tilde{P} + \hat{C}(V) \tilde{P}, \quad (59)$$

where $\tilde{\nabla}_{\perp} \equiv (\tilde{I} - \hat{n}\hat{n}) \cdot \tilde{\nabla}$ and where the (linear) Fokker-Planck collision operator $\hat{C}$ equivalent to the assumed Langevin dynamics along the lines is

$$\hat{C}(V) = - \frac{\partial}{\partial V} (v V + D_v \frac{\partial}{\partial V}) , \quad (60)$$

where D_v is related to the collision frequency ν and the background temperature T by the Einstein relation $D_v = (T/m)\nu$. In (59) and elsewhere, the super-tilde notation signifies a random variable. Although it is quite possible to proceed in a general magnetic geometry, this does not appear to be crucial for the general problems we wish to discuss, and we will pass to the usual sheared slab approximation (App. A),

$$0 = \frac{\partial \tilde{P}}{\partial t} + v \left(\frac{\partial}{\partial z} + \frac{x}{L_s} \frac{\partial}{\partial y} \right) \tilde{P} - D_{\perp} \frac{\partial^2}{\partial y^2} \tilde{P} + \hat{C} \tilde{P} + v \tilde{b} \frac{\partial}{\partial x} \tilde{P}, \quad (61)$$

where we have approximated $\tilde{\nabla}_{\perp}^2$ by $\partial^2 / \partial y^2$. (We neglect a term in $D_{\perp} \partial^2 / \partial x^2$ by assuming its effect to be small compared to that of the radial diffusion which will emerge from the term in $\tilde{b}$.)

We shall treat the magnetic field perturbation $\tilde{b}$ as a Gaussian random variable. This assumption will enable us ultimately to close the equations with just the spectral quantity $\langle \tilde{b} \tilde{b} \rangle$ left as an unknown function to be specified. Fields which may arise in practice are often expected to be nearly Gaussian, and we do not believe that the Gaussian assumption is a severe restriction. We may thus write Eq. (61) in the generic form

$$\left(\frac{\partial}{\partial t} + i\tilde{L} \right) \tilde{P} + i\tilde{\Omega} \tilde{P} = 0, \quad (62)$$

where

$$i\tilde{\Omega} \equiv v \tilde{b} \frac{\partial}{\partial x} \quad (63)$$

is a random linear operator with known (Gaussian) statistics and where $\tilde{L}$ is a nonrandom linear operator whose form is obvious from (61). Equation (62) is an operator version of the well-known stochastic oscillator (Kraichnan 1961;

Krommes 1981). Although it is linear in the dynamical variable $\tilde{\Psi}$, it is quadratically nonlinear in random variables ($\tilde{\Phi}\tilde{\Psi}$). As is well-known, this stochastic nonlinearity makes a straightforward statistical average of Eq. (62) of limited utility, because closure in terms of $\langle \tilde{\Psi} \rangle \equiv \tilde{P}$ cannot be immediately effected. Instead, at least two general techniques are available for extracting the radial diffusion coefficient from (62): (1) first compute the random or conditional (on a given realization of the random field) hydrodynamic (low frequency, long wavelength) eigenvalue of the operator $i\tilde{L} + i\tilde{Q}$ by Chapman-Enskog-like analysis, then average over the statistics of $\tilde{\Psi}$ to find the true (mean) eigenvalue; (2) first effect statistical closure by renormalizing Eq. (62), then find the true hydrodynamic eigenvalue of the renormalized kinetic equation. A related distinction has often been made in the general transport theory of many-body systems. There the choice was between (1') the one-step process of computing the hydrodynamic eigenvalue directly from the Liouville equation (for the distribution function of the ensemble), and (2') the two-step process whereby one first derives the kinetic equation for the mean one-body distribution, then finds the eigenvalue of that equation. Our method (1) differs from (1') in that we first compute a random eigenvalue, then average. The distinction is significant. Method (1) is the approach followed, however implicitly or, in some cases, incorrectly, by those authors who attempt to average the classical Braginskii equations over a random magnetic field. We describe it now in detail.

3.2 The random hydrodynamic eigenvalue and test particle transport

In this section we shall describe a perturbation theory analysis of Eq. (61). In so doing, we must pay close attention to the length scales associated with the stochastic magnetic field, and also to the possibility that the classical perpendicular transport may be large. By appropriately taking these possibilities into account, we will be led to a time-correlation function formula identical to that discussed in the previous section, but substantially different in some cases from the standard result associated with the classical Chapman-Enskog transport equations.

According to linear response theory and the fluctuation-dissipation theorem [a good review of these topics is given by Reichl (1980)], the transport coefficients of a slightly nonuniform system are related to the low frequency, long wavelength eigenvalues of the infinitesimal equilibrium response function. In the usual formulations, all statistical averaging is performed before the eigenvalues are computed. Here, however, we will temporarily eschew averaging over the random field and, thus, search for a random eigenvalue $\tilde{\omega}$. That is, we study a linear response theory conditional on $\tilde{\Psi}$. We consider, then, small perturbations about the thermal equilibrium solution

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$$P_{eq}(V) = \phi_M(V) \equiv (2\pi v_t^2)^{-1/2} \exp(-V^2/2v_t^2), \quad (64)$$

which is the unique normalizable solution of $\hat{C}P_{eq} = 0$. The random eigenvalue equation for the perturbed test particle distribution $\delta\tilde{P}$ is

$$(-i\tilde{\omega} + i\tilde{L})\delta\tilde{P} = 0, \quad (65)$$

where

$$\tilde{L} \equiv L_0 + \tilde{Q} + L_\perp - i\hat{C}, \quad (66a)$$

$$L_0 \equiv -iVn_0 \cdot \nabla, \quad (66b)$$

and

$$L_\perp \equiv iD_\perp \frac{\partial^2}{\partial y^2}. \quad (66c)$$

We may assume that $\hat{C} = O(1)$. To isolate the hydrodynamic eigenvalue, it is assumed in the conventional Chapman-Enskog ordering (cf. Uhlenbeck & Ford 1963) that spatial gradients are $O(\epsilon)$. Here, we must be more careful. More precisely, the hydrodynamic limit states that differential operators in space are small when they act on perturbations averaged over all randomness, including that of the magnetic field; they are not necessarily small, however, when they act on any functional of the stochastic field. Since the characteristic scale of $\tilde{b}$ is L_{ac} in the parallel direction and $\delta_\perp$ in the perpendicular direction, and since $\delta\tilde{P}$ inherits the spatial structure of $\tilde{b}$, the scales of $\delta\tilde{P}$ are not hydrodynamic unless $\lambda_v < L_{ac}$ and $v\tau_\perp > 1$ ($\alpha < 1$). In all other regimes, the ordering must be modified. Even when the hydrodynamic criteria are satisfied, care must be exercised in view of our previous discussion of the double-diffusion process. These matters are most conveniently handled by introducing a multiple-length scale expansion,

$$x_0 \equiv \underline{x}, \quad x_1 \equiv \epsilon \underline{X}, \quad \dots, \quad (67a)$$

$$\underline{v} = \underline{v}_0 + \epsilon \underline{v}_1 + \dots, \quad (67b)$$

and assuming that $\tilde{b} = \tilde{b}(x_0)$ whereas mean hydrodynamic gradients depend on x_1 . By assuming that $\tilde{b}$ does not depend on x_1 , we are essentially making a WKB assumption that diffusion develops on spatial scales short compared to the hydrodynamic scales. This implies that Fourier transformation with respect to x_1 is appropriate, and upon performing this operation we find

$$L_0 = -iV(\hat{n}_0 \cdot \vec{v}_0 + \varepsilon i k_y) , \quad (68a)$$

$$\tilde{\omega} = -iV\tilde{b}(\frac{\partial}{\partial x_0} + \varepsilon i k_x) , \quad (68b)$$

$$L_1 = iD_1[\frac{\partial^2}{\partial y_0^2} + \varepsilon(2ik_y \frac{\partial}{\partial y_0}) - \varepsilon^2 k_y^2] . \quad (68c)$$

Notice from (68c) that we make no further ordering on D_1 at this time. The presence of a dissipative term in real space at all at this point in the calculation is a major difference from the classical Chapman-Enskog theory, where the fundamental kinetic equation contains an irreversible term only in velocity space. By allowing D_1 to enter at $O(1)$ in (68c), we use a maximal ordering from which any regime in which D_1 plays a role can be extracted by a subsidiary expansion. This also allows the $O(1)$ streaming term in (68a) to either dominate or be dominated by the term in D_1 . This allows the Kadomtsev & Pogutse and fluid regimes to be distinguished.

It is convenient to proceed by writing the perturbed distribution in the form

$$\delta\tilde{P} \equiv \phi_M \chi , \quad (69)$$

and by introducing a scalar product such that, in Dirac notation,

$$|\chi\rangle \equiv \phi_M \chi , \quad (70a)$$

$$\langle\phi|\chi\rangle \equiv W^{-1} \int dx_0 dV \phi_M^* \chi , \quad (70b)$$

where $W \equiv \int dx_0$. With respect to this scalar product, $\hat{C}$ is Hermitean, while $\partial/\partial x_0$ is anti-Hermitean. Equation (65) then takes the form

$$(-i\tilde{\omega} + i\tilde{L})|\chi\rangle = 0 . \quad (71)$$

This may be solved by expanding the eigenvalue $\tilde{\omega}$ in the form

$$\tilde{\omega} = \tilde{\omega}^{(0)} + \varepsilon \tilde{\omega}^{(1)} + \varepsilon^2 \tilde{\omega}^{(2)} + \dots , \quad (72)$$

where we choose $\tilde{\omega}^{(0)}$ to vanish because we are seeking a hydrodynamic eigenvalue; $\tilde{\omega}^{(1)}$ and $\tilde{\omega}^{(2)}$ must be determined. An expansion similar to (72) holds for the eigenfunction χ .

The lowest order equation is

$$(-i\tilde{\omega}^{(0)} + iL^{(0)})|\chi^{(0)}\rangle = 0, \quad (73)$$

where

$$L^{(0)} \equiv -iVn \cdot \nabla_0 + iD_{\perp} \frac{\partial^2}{\partial y_0^2} - i\hat{C}. \quad (74)$$

We may choose the lowest order perturbation to be constant in space, in which case we must solve

$$\hat{C}|\chi^{(0)}\rangle = 0, \quad (75)$$

the unique normalizable solution of which is a constant which we may choose to be one since the size of the eigenfunction is irrelevant in linear theory:

$$|\chi^{(0)}\rangle = |1\rangle. \quad (76)$$

To satisfy higher order solubility conditions, we will also have need for the adjoint eigenfunction $\langle\chi^{(0)}|$. Now in general $L^{(0)}$ has no definite symmetry,

$$(L^{(0)})^{\dagger} = -iVn \cdot \nabla_0 - iD_{\perp} \frac{\partial^2}{\partial y_0^2} + i\hat{C}, \quad (77)$$

so that for arbitrary $\omega^{(0)}$ the left and right eigenfunctions will differ. However, for the hydrodynamic eigenvalue it is consistent with our result (76) and the form of (75) to choose

$$\langle\chi^{(0)}| = \langle 1|. \quad (78)$$

Proceeding to first order, we have

$$(-i\tilde{\omega}^{(0)} + iL^{(0)})|\chi^{(1)}\rangle = -(-i\tilde{\omega}^{(1)} + iL^{(1)})|\chi^{(0)}\rangle. \quad (79)$$

The left-hand side of (79) is annihilated by $\langle\chi^{(0)}|$, and the solubility condition then determines $\tilde{\omega}^{(1)}$ to be

$$\begin{aligned} \tilde{\omega}^{(1)} &= \langle 1|L^{(1)}|1\rangle \\ &= \langle 1|k_{\parallel}V + k_{\perp}Vb - 2D_{\perp}k_y \frac{\partial}{\partial y_0}|1\rangle \\ &= 0 \end{aligned} \quad (80)$$

by symmetry. Then formally

$$|\chi^{(1)}\rangle = -(L^{(0)})^{-1}(k_{\parallel}v + k_x v\tilde{b})|1\rangle. \quad (81)$$

To second order,

$$(-i\tilde{\omega}^{(0)} + iL^{(0)})|\chi^{(2)}\rangle = -(-i\tilde{\omega}^{(1)} + iL^{(1)})|\chi^{(1)}\rangle - (-i\tilde{\omega}^{(2)} + iL^{(2)})|\chi^{(0)}\rangle, \quad (82)$$

the solubility condition for which is

$$\begin{aligned} \tilde{\omega}^{(2)} &= \langle 1|L^{(2)}|1\rangle + \langle 1|\tilde{L}^{(1)}|\chi^{(1)}\rangle \\ &= -ik_x^2 D_{\perp} - i\langle k_{\parallel}v + k_x v\tilde{b} | (iL^{(0)})^{-1} | (k_{\parallel}v + k_x v\tilde{b}) \rangle. \end{aligned} \quad (83)$$

The terms in k_y and $k_{\parallel}$ obviously describe the classical collisional diffusion. [For example, if $\hat{C}$ dominates in $L^{(0)}$, the term quadratic in $k_{\parallel}$ is $-ik_{\parallel}^2 D_{\parallel}$, where

$$\begin{aligned} D_{\parallel} &\equiv \langle v | \hat{C}^{-1} | v \rangle \\ &= v_t^2 / \nu. \end{aligned} \quad (84)$$

(The solution of $\hat{C}|\psi\rangle = |v\rangle$ is $\psi = v/\nu$.) Of course, this result is in complete agreement with the direct long-time solution of the Langevin equations which underly (60).] To extract the radial diffusion coefficient, we may set k_y and $k_{\parallel}$ to 0, obtaining the random eigenvalue $\tilde{\omega}_x^{(2)}$ associated with radial transport in the form

$$\tilde{\omega}_x^{(2)} = -ik_x^2 \tilde{D}, \quad (85a)$$

$$\tilde{D} \equiv -ik_x^2 \langle v\tilde{b} | (iL^{(0)})^{-1} | v\tilde{b} \rangle. \quad (85b)$$

If we use the formal operator identity

$$\hat{A}^{-1} = \int_0^{\infty} d\tau \exp(-\hat{A}\tau) \quad (86)$$

($\hat{A}$ linear) and average over $\tilde{b}$ to find the true eigenvalue and diffusion coefficient, we find

$$\begin{aligned} \omega_x^{(2)} &= -ik_x^2 D, \\ D &\equiv \int_0^{\infty} d\tau \langle v\tilde{b} \exp[(\hat{V}\hat{n} \cdot \hat{V}_0 - D_{\perp} \frac{\partial^2}{\partial y_0^2} - \hat{C})\tau] v\tilde{b} \rangle, \end{aligned} \quad (87)$$

where the angular brackets once again mean the statistical average, including

both the average over $\tilde{b}$ and over a Maxwellian distribution of initial velocities. Finally, if we recall that formally

$$\exp\left(\alpha \frac{\partial}{\partial x}\right) f(x) = f(x+\alpha) \quad (88)$$

and undo the average over Gaussian white noise which gave rise to $D_{\perp}$ and $\hat{C}$, we find that (87) becomes

$$D = \int_0^{\infty} d\tau \langle V \tilde{b} \tilde{V}(-\tau) \tilde{b}(-\tau) \rangle, \quad (89)$$

where the brackets mean an average over all random processes. Formula (89) is essentially just the long-time limit of (20).

We may now discuss how the result would have differed had we failed to recognize that $\tilde{b}$ contained intrinsic length scales distinct from the hydrodynamic scales. In that case, all terms involving derivatives with respect to x_0 would have been absent from the formalism, the lowest order operator $L^{(0)}$ would have been just $-i\hat{C}$, and (83) would have been replaced by

$$\tilde{\omega}^{(2)} = -ik_y^2 D_{\perp} - ik_{\parallel}^2 D_{\parallel} - ik_x^2 \langle V \tilde{b} | \hat{C}^{-1} | V \tilde{b} \rangle. \quad (90)$$

Upon averaging (90) over $\tilde{b}$ and using (84), we would have found

$$\omega^{(2)} = -i(k_y^2 D_{\perp} + k_{\parallel}^2 D_{\parallel} + k_x^2 D^{(f)}) \quad (91a)$$

$$D^{(f)} \equiv \langle V | \hat{C}^{-1} | V \rangle \langle \tilde{b}^2 \rangle = D_{\parallel} \langle \tilde{b}^2 \rangle. \quad (91b)$$

Now it may appear that the last term of (91a) is just the fluid result we have described previously. It is indeed the result found by Rose. However, it is clear from Eq. (90) that $D_{\perp}$ has entirely disappeared (by being taken small) from the computation of $D^{(f)}$, as has the streaming term. Therefore, according to our discussion in Sec. II, the result (91) is appropriate only to the case of an infinite autocorrelation length (consistent with the neglect of the streaming term). For the physical case of a finite autocorrelation length, in the absence of $D_{\perp}$ particles would eventually diffuse out of the autocorrelation region and, because of the double-diffusion, the time-asymptotic diffusion coefficient would vanish. If we wish to describe the fluid regime properly, we must retain $D_{\perp}$ and analyze formulas such as (85b) or (89) in the appropriate limit (see Table 1)

Thus, we may conclude that the classical Chapman-Enskog procedure is incorrect for the random eigenvalue problem when a stochastic magnetic field is present, even when that field is very small. A modified perturbation

theory must be introduced to take account of the intrinsic scales associated with the random component of the field. When this is done, one recovers a Kubo-like formula in complete agreement with the heuristic one discussed in Sec. 1. Of course, we are left with the nontrivial task of performing the statistical average required in (89). In Sec. 2 we discussed that heuristic-ally. In the next section, we discuss more formal aspects of this question.

Recently, Haas et al. (1981) considered a two-fluid description of transport in random magnetic fields, but, with a formula such as (3) in mind, argued that no one has properly computed $\kappa_{\parallel}$ in the regime of long mean free path. In order to force their formulas into reasonable agreement with experiment, they replaced, in an ad hoc fashion, the mean free path by the connection length when the latter was short compared with the former. We disagree with their emphasis. It is not that the proper $\kappa_{\parallel}$ has not been computed in the long mean free path regime, it is that equations of the form (1) are not appropriate in that case. Rather, the arguments of Rechester & Rosenbluth and others may be applied directly. Alternatively, if one insists on casting all regimes into a hydrodynamic framework, then we may say that our calculations in this section have effected the computation of the effective $\kappa_{\parallel}$ which Haas et al. called for. From our procedure, which is systematic, not ad hoc, we of course recover the collisionless formula $D \sim d_{\perp} v_t$. If one wishes, he can obtain this result from $D^{(f)}$ by replacing $\kappa_{\parallel}/v_t \sim \lambda_v$ by L_{ac} (not, fundamentally, the connection length). In the quasilinear case when L_{ac} does scale as the connection length, we see that the procedure of Haas et al. is a very ad hoc way of recovering the collisionless result of Rechester & Rosenbluth. We do not feel that this procedure is the appropriate way of viewing the problem, and we argue for the general framework of double diffusion which we have discussed.

4. Renormalized kinetic theory of test particle transport

4.1: The Dyson equation and the test particle diffusion coefficient

In this section we develop a renormalized kinetic equation for the probability distribution of the test particles, then extract the radial diffusion coefficient from the long time, long wavelength behavior of that equation. It is convenient to develop the formal theory in terms of a Green's function approach. Suppose we subject the plasma to an arbitrary infinitesimal perturbation at time $t = t'$. The hydrodynamic behavior of the system will manifest itself in the long, long wavelength mean response of the system to that perturbation (Kadanoff & Martin 1963). However, the precise form of the perturbation is not important; it is sufficient to consider the mean Green's function (infinitesimal response function) $R \equiv \langle \hat{R} \rangle$, where $\hat{R}$ is the stochastic Green's function defined from Eqs. (59) or (62):

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$$\left(\frac{\partial}{\partial t} + i\bar{L} + i\bar{\Omega}\right)\tilde{R}(x,y,z,V,t;x',y',z',V',t') \\ = \delta(t-t')\delta(x-x')\delta(y-y')\delta(z-z')\delta(V-V') . \quad (92)$$

We face again the closure problem because of the quadratic term in $\tilde{R}$. Formally, we can write for the mean response function the Dyson equation (Martin, Siggia, & Rose 1973)

$$\left(\frac{\partial}{\partial t} + i\bar{L}\right)R(1;1') + \int d\bar{1} \Sigma(1;\bar{1})R(\bar{1};1') = \delta(1-1') , \quad (93)$$

where the "mass operator" Σ is causal [$\Sigma(t;t') \propto H(t-t')$], a functional of $\langle b^2 \rangle$, and must be determined through a renormalization program. [A review with many references has been given by Krommes (1981).] Both perturbation theoretic (Frisch 1968) and powerful functional techniques (Martin, Siggia, & Rose 1973; Krommes 1978; Jensen 1981) have been used to provide a formally exact equation — the Bethe-Salpeter equation — for Σ . Of course, approximations must be made before that equation becomes tractable. In App. B and C, we discuss aspects of these techniques, and in App. C we advance the approximation which we will study later in this section. First, however, we discuss how to extract the diffusion coefficient from the Dyson equation.

The response function R can be interpreted as the probability distribution for a unit pulse inserted at time t' and phase space point (x',y',z',V') . Thus, we expect that the radial diffusion coefficient can be determined from the long-time limit of the second x moment of R . However, such an analysis is somewhat complicated by the fact that R is not translationally invariant in x because of the shear, so the diffusion coefficient depends on macroscopic position. This complication may be approached through a WKB analysis. Define

$$\underline{\rho} \equiv \underline{x} - \underline{x}' , \quad (94a)$$

$$\tau \equiv t - t' , \quad (94b)$$

and

$$\underline{x} \equiv \frac{1}{2}(\underline{x} + \underline{x}') \quad (94c)$$

(since $\tilde{b}$ is static, there is no need to introduce a mean time variable), and write two-point functions such as R in the form

$$R(\underline{x},V,t;\underline{x}',V',t') \equiv R(\underline{\rho},V,\tau|\underline{x},V') . \quad (95)$$

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The nonlinear term in (93) then becomes

$$\int d\bar{l} \Sigma(1;\bar{l}) R(\bar{l};1') = \int_0^{\bar{\tau}} d\bar{\tau} \int d\bar{\rho} d\bar{V} \Sigma(\bar{\rho}, \bar{V}, \bar{\tau} | \bar{x} + \frac{1}{2}(\bar{x} - \bar{\rho}), \bar{V}) \\ \times R(\bar{\rho} - \bar{\rho}, \bar{V}, \bar{\tau} - \bar{\tau} | \bar{x} - \frac{1}{2}(\bar{x} - \bar{\rho}), \bar{V}') . \quad (96)$$

It is assumed that the correlation range of Σ in $\bar{\rho}$ and $\bar{\tau}$ is finite, so that diffusion develops on length scales less than the macroscopic scale X . [We use the notation (ρ, X) for the radial components of the vectors $(\bar{\rho}, \bar{x})$.] Furthermore, the hydrodynamic limit, defined by passing to the limit of large ρ , or more precisely to the $k \rightarrow 0$ limit of the Fourier transform with respect to ρ , can then be taken in the approximation $\rho < X$, or $k > L_n^{-1}$. (Here L_n is the radial scale length of the mean probability density of the test particle.) Then upon expanding (96) in small $\bar{\rho}$ and $\bar{\tau}$ we arrive at the Markovian approximation

$$\int \Sigma R = \int_0^{\bar{\tau}} d\bar{\tau} \int d\bar{\rho} d\bar{V} \Sigma(\bar{\rho}, \bar{\tau}, \bar{V}, |X, \bar{V}) R(\bar{\rho}, \bar{V}, \tau | X, V') , \quad (97)$$

where we noted that Σ and R depend only on X rather than $\bar{x}$. Now the operator Σ contains two factors of

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial X} + \frac{\partial}{\partial \rho} \right) . \quad (98)$$

(Cf. Apps. B, C.) However, in the WKB ordering, we must approximate

$$\frac{\partial}{\partial \rho} \gg \frac{\partial}{\partial X} . \quad (99)$$

Therefore, Eq. (93) takes the form

$$\frac{\partial}{\partial \tau} R(\rho, V, \tau | X, V') + V \left(\frac{\partial}{\partial z} + L_s^{-1} X \frac{\partial}{\partial y} \right) R - D_{\perp} \frac{\partial^2}{\partial y^2} R + \hat{C} R \\ - \frac{\partial}{\partial \rho} \int_0^{\bar{\tau}} d\bar{\tau} \int d\bar{\rho} d\bar{V} V \hat{\Sigma}(\bar{\rho}, \bar{V}, \bar{\tau} | X, \bar{V}) \bar{V} \frac{\partial}{\partial \rho} R(\bar{\rho}, \bar{V}, \tau | X, V') = \delta(\tau) \delta(\rho) \delta(V - V') , \quad (100)$$

where $\hat{\Sigma}$ is defined by

$$\Sigma \equiv V \frac{\partial}{\partial \rho} \hat{\Sigma} \bar{V} \frac{\partial}{\partial \rho} . \quad (101)$$

By integrating Eq. (100) over all ρ and V , we find that the local number density N obeys

$$N(\tau | X, V') \equiv \int d\rho dV R(\rho, V, \tau | X, V') = H(\tau) . \quad (102)$$

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Thus, number density is appropriately conserved, and $R(\rho, V, \tau | X, V')$ can be interpreted as a local (X -dependent) probability distribution in ρ and V , conditional on V' . Conditional moments can be defined according to

$$\langle (\rho^m V^n)(\tau) \rangle \equiv \langle \rho^m V^n, \tau | X, V' \rangle \equiv \int d\rho dV \rho^m V^n R(\rho, V, \tau | X, V') . \quad (103)$$

In particular,

$$\frac{\partial}{\partial \tau} \langle \rho \rangle = 0 , \text{ or } \langle \rho \rangle = 0 , \quad (104a)$$

$$\frac{\partial}{\partial \tau} \langle \rho^2 \rangle = \frac{\partial}{\partial \tau} \langle (\rho - \langle \rho \rangle)^2 \rangle = 2D(\tau) , \quad (104b)$$

where

$$D(\tau) \equiv \int_0^\tau d\bar{\tau} \int d\rho d\bar{\rho} dV d\bar{V} V \hat{\Sigma}(\bar{\rho}, V, \bar{\tau} | X, \bar{V}) \bar{V} R(\rho, \bar{V}, \tau | X, V') . \quad (105)$$

The quantity $\int d\rho R(\rho, V, \tau | X, V') \equiv R_{\tilde{k}=0}$ which appears in (105) is just the $\tilde{k} = 0$ component of the Fourier transform with respect to ρ at constant X . It obeys

$$\frac{\partial}{\partial \tau} R_{\tilde{k}=0}(V, \tau | X, V') + \hat{C}R_{\tilde{k}=0} = \delta(\tau) \delta(V - V') , \quad (106)$$

the solution of which is just

$$R_{\tilde{k}=0} = P(V, \tau | V') , \quad (107)$$

P being the conditional Langevin velocity distribution. In the hydrodynamic limit we have $\nu\tau > 1$, in which case

$$P(V, \tau | V') \rightarrow \phi_M(V) , \quad (108)$$

ϕ_M being defined by (64). (Equation (108) describes complete thermalization of the test particle with the plasma bath.) Then the long-time radial diffusion coefficient can be written

$$D = D(\tau \rightarrow \infty) , \quad (109a)$$

$$D(\tau) = \int d\bar{V} D(\tau | \bar{V}) \phi_M(\bar{V}) , \quad (109b)$$

$$D(\tau | \bar{V}) \equiv \int_0^\tau d\bar{\tau} \left[\int dV V \hat{\Sigma}_{\tilde{k}=0}(\bar{\tau}, V | X, \bar{V}) \bar{V} \right] . \quad (109c)$$

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Obviously, the quantity in brackets in (109c) is to be identified with the four-point correlation function introduced in Sec. 2.1 [cf. Eq. (20)], with $\bar{V}$ playing the role of $V_1(0)$. Within the framework of the WKB analysis and Markovian approximation appropriate to the hydrodynamic limit, Eqs. (109) are exact.

4.2: The Bethe-Salpeter equation and the nonlinear collision operator

The quantity $\hat{\Sigma}(\rho, V, \tau | X, V')$ is the diagonal component of a quantity $K(1, 2; 1', 2')$, which is related to the joint probability distribution for test particles and stochastic field, and is the solution of the Bethe-Salpeter equation (App. C). To expedite the WKB analysis of that equation, it is convenient to define

$$\rho_1 \equiv x_1 - x'_1, \quad \rho_2 \equiv x_2 - x'_2, \quad (110a, b)$$

$$\tau_1 \equiv t_1 - t'_1, \quad \tau_2 \equiv t_2 - t'_2, \quad (110c, d)$$

$$\tilde{x}_{12} \equiv 1/2 (x_1 + x_2), \quad \tilde{x}'_{12} \equiv 1/2 (x'_1 + x'_2), \quad (110e, f)$$

to introduce the variables

$$\delta\rho \equiv \rho_2 - \rho_1, \quad \bar{\rho} \equiv 1/2 (\rho_1 + \rho_2), \quad (111a, b)$$

$$\delta\tau \equiv \tau_2 - \tau_1, \quad \bar{\tau} \equiv 1/2 (\tau_1 + \tau_2), \quad (111c, d)$$

and

$$\delta\tilde{x}' \equiv \tilde{x}'_2 - \tilde{x}'_1, \quad \bar{\tilde{x}} \equiv 1/2 (\tilde{x}_{12} + \tilde{x}'_{12}), \quad (111e, f)$$

and to write

$$K(1, 2; 1', 2') \equiv K(\delta\rho, \delta\tau, \bar{\rho}, \bar{\tau}, V_2 | \delta\tilde{x}', \bar{\tilde{x}}, V'_2). \quad (112)$$

For reference, the inverse of the transformation defined by Eqs. (111) is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \bar{x} + 1/2 \bar{\rho} \mp 1/2 (\delta\rho + \delta\tilde{x}'), \quad (113a, b)$$

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \bar{x} - 1/2 \bar{\rho} \mp 1/2 \delta\tilde{x}'. \quad (113c, d)$$

Dependence of K on $\bar{x}$ and $\delta\tilde{x}'$ will be weak. Furthermore,

$$\hat{\Sigma}(\rho, V, \tau | X, \bar{V}) = K(\delta\rho=0, \delta\tau=0, \bar{\rho}=\rho, \bar{\tau}=\tau, V | \delta\tilde{x}'=0, \bar{\tilde{x}}=X, \bar{V}). \quad (114)$$

It is noteworthy that K is required on the diagonal $\delta \underline{\rho} = 0, \delta \tau = 0$. The kinetic theory of such small scales is nontrivial. For example, as we will show, the direct-interaction approximation does not properly describe the regime of small $(\delta \underline{\rho}, \delta \tau)$.

According to App. C, the source (initial condition) term in the Bethe-Salpeter equation is

$$F(1,1') \delta(\tau_2) \delta(\underline{\rho}_2) \delta(\underline{v}_2 - \underline{v}_{2'}), \quad (115)$$

where

$$F(1,1') \equiv \langle b(1)b(1') \rangle \quad (116)$$

is the specified Eulerian correlation function of the stochastic field. F depends neither on velocity nor, for static fields, on time. It is thus convenient to introduce in place of K a field line - test particle Green's function $G^{(2)}$ such that in general

$$K(1,2;1',2') = \int d\bar{1} G^{(2)}(1,2;\bar{1},2') F(\bar{1},1'), \quad (117a)$$

or, using the special form of F to restrict the argument dependence of $G^{(2)}$,

$$K(\underline{x}_1, 2; \underline{x}'_1, 2') = \int d\bar{\underline{x}}_1 G^{(2)}(\underline{x}_1, 2; \bar{\underline{x}}_1, 2') F(\bar{\underline{x}}_1, \underline{x}'_1). \quad (117b)$$

The quantity $G^{(2)}$ defined by Eq. (117b) will obey an equation of the form

$$\frac{\partial G^{(2)}}{\partial \tau_2} + \dots = \delta(\tau_2) \delta(\underline{\rho}) \delta(\delta \underline{\rho}) \delta(\underline{v}_2 - \underline{v}_{2'}), \quad (118)$$

which shows explicitly that dependence of $G^{(2)}$ on $\bar{\underline{X}}$ and $\delta \underline{x}'$ arises only from the non-translationally invariant shear term in $\underline{x}/L_s$. From here on we consider only $\delta \underline{x}' = 0$, since that is what is required for $\hat{\Sigma}$ according to (114). Also, we will often not explicitly indicate the slow $\underline{X}$ dependence. Furthermore, for static fields neither $G^{(2)}$ nor K will depend on τ_1 , and it is simpler, although less symmetrical, to consider K to be a function of τ_2 rather than $\delta \tau$ and $\bar{\tau}$; according to (114), $\tau_2 = \tau$. Then, in the WKB approximation Eq. (117b) can be simplified:

$$K(\delta \underline{\rho}, \underline{\rho}, \underline{v}, \tau_2 | \underline{v}') = \int \frac{d\underline{\ell}}{(2\pi)^3} \frac{d\underline{k}}{(2\pi)^3} \frac{d\omega_2}{2\pi} \exp(i\underline{\ell} \cdot \delta \underline{\rho} + i\underline{k} \cdot \underline{\rho} - i\omega_2 \tau_2) \\ \times G^{(2)}_{\underline{\ell}, \underline{k}, \omega_2}(\underline{v}_2 | \underline{v}') F_{-\underline{\ell} + \frac{1}{2}\underline{k}}. \quad (119)$$

Noting Eqs. (109) and using (119), we find

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$$\int_0^\infty d\tau \hat{\Sigma}_{\underline{k}=0}(V, \tau|V') = \int_0^\infty d\tau \int d\underline{\rho} K(\delta\underline{\rho}=0, \bar{\underline{\rho}}=\underline{\rho}, V, \tau_2=\tau|V') \quad (120a)$$

$$= \int_0^\infty d\tau \int \frac{d\underline{\rho}}{(2\pi)^3} G_{\underline{k}, \underline{k}=0}^{(2)}(V, \tau|V') F_{-\underline{k}} \quad (120b)$$

$$= \int \frac{d\underline{\rho}}{(2\pi)^3} G_{\underline{k}, \underline{k}=0, \omega=0}^{(2)}(V|V') F_{-\underline{k}}. \quad (120c)$$

(The result (120c) was obtained by extending the τ integral to $-\infty$, which is permissible because $\hat{\Sigma}$ and $G^{(2)}$ are causal.) Within the framework of the WKB and Markovian approximations, formulas (120) are exact.

4.3: The direct-interaction approximation to the Bethe-Salpeter equation

Insight may be gained into the form of Eqs. (120b) or (120c) by comparing them to the predictions of the direct-interaction approximation. (See App. C for the definition and references.) For problems of statistically specified fields, this approximation is the natural renormalized generalization of quasilinear theory. Although we will conclude that the approximation is inadequate in certain regimes, it is still very instructive and useful.

According to App. C, the direct-interaction approximation is represented by the factorization

$$K(1, 2; 1', 2') \approx F(1, 1') R(2; 2'), \quad (121)$$

or

$$\begin{aligned} G_{\underline{k}, 2; \underline{x}_1, 2'}^{(2)} &= \delta(\underline{\rho}_1) R(2; 2') \\ &= \delta(\bar{\underline{\rho}} - 1/2 \delta \underline{\rho}) R(\bar{\underline{\rho}} + 1/2 \delta \underline{\rho}, V_2, \tau_2 | \bar{\underline{x}} + 1/2 \delta \underline{x}' + 1/4 \delta \underline{\rho}, V_2, \tau_2) \\ &= \delta(\bar{\underline{\rho}} - 1/2 \delta \underline{\rho}) R(\delta \underline{\rho}, V_2, \tau_2 | \bar{\underline{x}}, V_2, \tau_2). \end{aligned} \quad (122)$$

Therefore, in the direct-interaction approximation

$$G_{\underline{k}, 0}^{(2)}(V, \tau|V') = R_{\underline{k}}(V, \tau | \bar{\underline{x}}, V'), \quad (123)$$

and (120b) reduces to

$$\int_0^\infty d\tau \hat{\Sigma}_{\underline{k}=0}(V, \tau|V') = \int_0^\infty d\tau \int \frac{d\underline{\rho}}{(2\pi)^3} R_{\underline{k}}(V, \tau|V') F_{-\underline{k}}. \quad (124)$$

If R were replaced by the function $R^{(0)}$ obtained by solving Eq. (93) with Σ neglected, (124) would be precisely the quasilinear approximation.

We must now ask what decorrelation mechanisms the direct-interaction

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approximation (124) contains. Consider the Dyson equation for R in the form

$$\begin{aligned} \frac{\partial}{\partial \tau} R(\rho, v, \tau | X, v') + v \left(\frac{\partial}{\partial z} + \frac{X}{L_s} \frac{\partial}{\partial y} \right) R - D_{\perp} \frac{\partial^2}{\partial y^2} R + \hat{C}R \\ - \frac{\partial}{\partial \rho} \int d\bar{v} D(\tau | \bar{v}) \frac{\partial}{\partial \rho} R(\bar{v} | v') = \delta(\tau) \delta(\rho) \delta(v - v') . \end{aligned} \quad (125)$$

To understand the qualitative features of this equation, we will Fourier transform with respect to ρ and write the term in D very approximately as $k_x^2 D$. The terms in $D_{\perp}$ and D may then be entirely factored from the solution by defining a new function χ_k according to

$$R_k \equiv \exp(-k_y^2 D_{\perp} \tau - k_x^2 D \tau) \chi_k, \quad (126)$$

where χ obeys

$$\frac{\partial}{\partial \tau} \chi_k(v, \tau | v') + ik_{\parallel} v \chi + \hat{C} \chi = \delta(\tau) \delta(v - v'), \quad (127)$$

with the local parallel wavenumber being defined by

$$k_{\parallel} \equiv k_{\parallel}(X) \equiv k_z + \frac{X}{L_s} k_y. \quad (128)$$

The solution of Eq. (127) is

$$\chi_k = \int d\ell \exp(-ik_{\parallel} \ell) P(\ell, v, \tau | v'), \quad (129)$$

where P is the conditional Langevin distribution in both (parallel) position and velocity. Upon combining (129) with (128), (124), and (109b), we find that the direct-interaction approximation to the radial diffusion coefficient is

$$\begin{aligned} D(\tau) = \int_0^{\tau} d\bar{\tau} \int \frac{dk_{\parallel}}{2\pi} \sum_{k_{\parallel}, k_x, k_y} \langle b^2 \rangle_{k_{\parallel}, k_x, k_y} \exp(-k_x^2 D \bar{\tau} - k_y^2 D_{\perp} \bar{\tau}) \\ \times \int dV d\bar{v} d\ell V \exp(-ik_{\parallel} \ell) P(\ell, v, \bar{\tau} | \bar{v}) \phi_M(\bar{v}). \end{aligned} \quad (130)$$

If the terms in D and $D_{\perp}$ are ignored, this is just an alternative but equivalent way of writing Eq. (21); we recover the double-diffusion theory. The term in $k_y^2 D_{\perp}$ cuts off the time integral in a time $\tau_{\perp}$, so if $k_x^2 D$ is negligible and $v\tau > 1$, we recover either the Kadomtsev-Pogutse or fluid results. When $k_x^2 D$ dominates, we recover the appropriate strong turbulence limit according to the discussion in Sec. 2.2. In fact, we may conclude that the direct-interaction approximation describes every regime in Table 1 except the quasilinear collisionless and Rechester-Rosenbluth regimes where $\tau_c = \tau_K$.

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The correlation time τ_K is absent from the direct-interaction approximation, in which there is no hint of exponential orbit divergence.

To understand this deficiency, let us note that the relative coordinate δp can be interpreted as the distance of the test particle from a particular reference magnetic field line. The direct-interaction approximation states that the test particle diffuses in space (for times greater than an autocorrelation time) independent of the microstructure of the lines. However, this is physically reasonable only when the test particle is farther from its initial line than a perpendicular correlation length of the stochastic field. For shorter perpendicular scales, and thus times shorter than τ_c , the test particle "understands" that a particular line was specified and that neighboring lines may exponentially diverge from that line; significant non-Gaussian correlations may become important.

4.4: Beyond the direct-interaction approximation; a renormalized theory of the correlation time τ_K

We now attempt to quantify the above arguments. That the direct-interaction approximation fails at small scales can be demonstrated in perturbation theory (App. B). However, our goal is a renormalized equation for the joint distribution $G^{(2)}$. To this end, we find it both conceptually and technically most useful to apply the functional methods of Martin, Siggia, and Rose (App. C). In particular, we shall begin with the first self-consistent vertex-renormalized equation (Kraichnan 1961; Martin, Siggia, & Rose 1973; Krommes 1981). However, rather than evaluating the interaction kernel I which describes the non-Gaussian corrections to K self-consistently in terms of K itself, we shall simplify the equation by applying the direct-interaction to I . We are then led to the equation

$$\begin{aligned} & \frac{\partial}{\partial \tau_2} G^{(2)}(\underline{1}, \tau_1, \underline{2}, \tau_2; \underline{1}', \underline{2}') + iL(2)G \\ & - \int_0^{\tau_2} d\bar{\tau} \int d\bar{\underline{2}} F(\underline{2}, \bar{\underline{2}}, \bar{\tau}) \frac{\partial}{\partial x_2} V_2 R(\underline{2}, \bar{\tau}; \bar{\underline{2}}) V_{\bar{2}} \frac{\partial}{\partial x_{\bar{2}}} G^{(2)}(\underline{1}, \tau_1, \bar{\underline{2}}, \tau_2 - \bar{\tau}; \underline{1}', \underline{2}') \\ & - \int_0^{\tau_2} d\bar{\tau} \int d\bar{\underline{1}} F(\underline{1}, \bar{\underline{1}}, \bar{\tau} - \delta\tau) \left[\frac{\partial}{\partial x_2} V_2 R(\underline{2}, \bar{\tau}; \bar{\underline{1}}) \right] V_{\bar{1}} \frac{\partial}{\partial x_{\bar{1}}} G^{(2)}(\underline{2}, \tau_2, \bar{\underline{1}}, \tau_2 - \bar{\tau}; \underline{1}', \underline{2}') \\ & = \delta(1-1')\delta(2-2'), \end{aligned} \quad (131)$$

where we have temporarily allowed for dependence on all variables in order to better display the symmetries of the equation. The operator on $G^{(2)}$ in the first term in F is the direct-interaction approximation to Σ . (In a more

refined approximation, FR would be selfconsistently replaced by K.) In the second term in F, which was absent in the direct-interaction approximation described in Sec. 4.3, note how the first arguments of F and G are permuted from the corresponding ones in Σ . This is how the mathematics ensures non-trivial joint correlations between the test particle and the reference field line.

The approximate Eq. (131) is still much too complicated to be solved exactly. We can, however, discuss its qualitative features with the aid of a series of approximations, which we describe in the following paragraphs. We believe that these approximations are not unreasonable and capture the essence of the physics: they do no obvious violence to the mathematics, and they lead us to a kinetic theory which has considerable intuitive appeal and agrees with the heuristic arguments of Rechester & Rosenbluth in the relevant regimes. However, as is quite common in complex theories of this kind, we cannot say that our approximations are entirely systematically derived by expansion in a small parameter. Further study is very desirable.

It is obvious from the form of (131) that if F is independent of velocity or time, the left-hand side does not introduce such dependence, so (131) may be replaced by

$$\begin{aligned} & \frac{\partial}{\partial \tau} G^{(2)}(\bar{x}_1, \bar{x}_2, v, \tau; \bar{x}_1', \bar{x}_2', v') + iL(2)G^{(2)} \\ & - \int_0^\tau d\bar{\tau} \int d\bar{x}_2 d\bar{v} F(\bar{x}_2, \bar{x}_2) \frac{\partial}{\partial x_2} v R(\bar{x}_2, v, \bar{\tau}; \bar{x}_2, \bar{v}) \bar{v} \frac{\partial}{\partial x_2} G^{(2)}(\bar{x}_1, \bar{x}_2, \tau - \bar{\tau}, \bar{v}; \bar{x}_1', \bar{x}_2', v') \\ & - \int_0^\tau d\bar{\tau} \int d\bar{x}_1 \int d\bar{v} F(\bar{x}_1, \bar{x}_1) \left[\frac{\partial}{\partial x_2} v R(\bar{x}_2, v, \bar{\tau}; \bar{x}_1, \bar{v}) \bar{v} \right] \frac{\partial}{\partial x_1} G^{(2)}(\bar{x}_2, \bar{x}_1, \tau - \bar{\tau}, \bar{v}; \bar{x}_1', \bar{x}_2', v') \\ & = \delta(\tau) \delta(\bar{x}_1 - \bar{x}_1') \delta(\bar{x}_2 - \bar{x}_2') \delta(v - v') . \end{aligned} \quad (132)$$

Next, we propose the following Markovian approximation:

$$\begin{aligned} & \left(\frac{\partial}{\partial \tau} + iL(2) \right) G^{(2)}(v, \tau; v') - \frac{\partial}{\partial x_2} \int d\bar{v} \hat{D}(0, v, \tau; \bar{v}) \frac{\partial}{\partial x_2} G^{(2)}(\bar{1}, \bar{2}, \bar{v}, \tau; \bar{1}', \bar{2}', v') \\ & - \frac{\partial}{\partial x_2} \int d\bar{v} \hat{D}(\delta y, v, \tau; \bar{v}) \frac{\partial}{\partial x_1} G^{(2)}(\bar{2}, \bar{1}, \bar{v}, \tau; \bar{1}', \bar{2}', v') \\ & = \delta(\tau) \delta(\bar{1} - \bar{1}') \delta(\bar{2} - \bar{2}') \delta(v - v') , \end{aligned} \quad (133)$$

where (see App. A for notation relating to the sheared geometry)

$$\hat{D}(\delta y, v, \tau; \bar{v}) = \int_0^\tau d\bar{\tau} \int d\bar{x}_1 d\bar{v} F(\bar{x}_1, \bar{x}_1) v R(\bar{x}_2, v, \bar{\tau}; \bar{x}_1, \bar{v}) \bar{v}$$

$$= \int_0^{\tau} d\bar{\tau} \sum_{\mu} \int \frac{dk_{\parallel}}{2\pi} F_{-\mu} \exp(-ik_y \delta y^{\dagger}) V R_{\mu}(\bar{\rho}, \bar{v}, \bar{\tau}; \bar{V}) \bar{V}, \quad (134)$$

where $\delta y^{\dagger}$ is measured with respect to the parallel direction and in the frame of a particle streaming or diffusing along the reference line. (This latter motion is described by Eq. (127); in that equation, $\delta y^{\dagger} = 0$.) This approximate form of the second term in $\hat{D}$ is derived by consideration of the properties of that term in perturbation theory, together with our physical expectations about the form of the solution. It is perhaps better called a model rather than a systematic approximation; as we have said, further study is desirable.

Clearly, for $\delta y^{\dagger}$ greater than, or of order, a characteristic perpendicular scale of the stochastic field, $\hat{D}(\delta y^{\dagger}) \rightarrow 0$ and the direct-interaction approximation is recovered. However, since the initial condition for $\delta y^{\dagger}$ is zero, that approximation is not valid at short times. To examine the evolution of the small scales, it is convenient to introduce the relative coordinates and, noting the arguments of $G^{(2)}$ in the second term in $\hat{D}$, to define two new functions:

$$(\bar{G}_{\Delta G})(\delta \bar{\rho}, \bar{v}, \tau) \equiv 1/2 [G^{(2)}(\delta \bar{\rho}, \bar{v}, \tau) \pm G^{(2)}(-\delta \bar{\rho}, \bar{v}, \tau)], \quad (135)$$

so that $G^{(2)} = \bar{G} + \Delta G$. In the equations for these functions, we may drop all derivatives with respect to $\bar{x}$ according to the WKB procedure. Furthermore, we will assume that the effects of motion along the reference line can be approximately factored from the cross-field decorrelation effects. We implement this factorization very approximately by temporarily dropping the terms which give rise to the streaming term $ik_{\parallel}(X)V$ in (127); at the end of the calculation, we will insert a factor corresponding to the solution (129) of (127). (Effectively, we are introducing a kind of interaction representation.) Also dropping the dagger for notational simplicity, we are then left with

$$\begin{aligned} & \frac{\partial}{\partial \tau} (\bar{G}_{\Delta G}) + \left(\frac{v}{2L_s}\right) \delta \rho \frac{\partial}{\partial \delta y} (\bar{G}_{\Delta G}) + \left(\frac{v}{2L_s}\right) 1/2 \delta \rho \frac{\partial}{\partial y} \left(\frac{\Delta G}{\bar{G}}\right) \\ & - D_{\perp} \left[1/4 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \delta y^2} \right] (\bar{G}_{\Delta G}) - D_{\perp} \frac{\partial}{\partial y} \frac{\partial}{\partial \delta y} \left(\frac{\Delta G}{\bar{G}}\right) + \hat{C} \left(\frac{\bar{G}}{\Delta G}\right) \\ & - [\hat{D} \mp \hat{D}(\delta y)] \frac{\partial^2}{\partial \delta \rho^2} (\bar{G}_{\Delta G}) - 1/4 [\hat{D} \pm \hat{D}(\delta y)] \frac{\partial^2}{\partial \bar{\rho}^2} \left(\frac{\bar{G}}{\Delta G}\right) - \hat{D} \frac{\partial}{\partial \bar{\rho}} \frac{\partial}{\partial \delta \rho} \left(\frac{\Delta G}{\bar{G}}\right) \\ & = \delta(\tau) \delta(\bar{\rho}) \delta(\delta \rho) \delta(v-v') \left(\frac{1}{0}\right), \end{aligned} \quad (136)$$

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where the terms in $\hat{D}$ include an implied velocity integration.

Since we are concerned here only with the small scales, we may approximate

$$\hat{D} \pm \hat{D}(\delta y) = \left(\frac{2\hat{D}}{1/2 \delta y^2} \hat{D}'' \right), \quad (137)$$

where

$$\hat{D}''(V, \tau; \bar{V}) \equiv - \frac{\partial^2}{\partial \delta y^2} \hat{D}(\delta y, V, \tau; \bar{V}) \Big|_{\delta y=0}. \quad (138)$$

We then find

$$\begin{aligned} & \frac{\partial}{\partial \tau} \left(\frac{\bar{G}}{\Delta G} \right) + \left(\frac{V}{2L_s} \right) \delta \rho \frac{\partial}{\partial \delta y} \left(\frac{\bar{G}}{\Delta G} \right) + \left(\frac{V}{2L_s} \right) \left(\frac{1}{2} \delta \rho \frac{\partial}{\partial y} \right) \left(\frac{\Delta G}{\bar{G}} \right) \\ & - D_1 \left(\frac{1}{4} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \delta y^2} \right) \left(\frac{\bar{G}}{\Delta G} \right) - D_1 \frac{\partial}{\partial y} \frac{\partial}{\partial \delta y} \left(\frac{\Delta G}{\bar{G}} \right) - \frac{\partial}{\partial \bar{V}} (V V + D_V \frac{\partial}{\partial \bar{V}}) \left(\frac{\bar{G}}{\Delta G} \right) \\ & - \left(\frac{1/2 \delta y^2 \hat{D}''(\tau)}{2D(\tau)} \right) \frac{\partial^2}{\partial \delta \rho^2} \left(\frac{\bar{G}}{\Delta G} \right) - \left(\frac{2D(\tau)}{1/2 \delta y^2 \hat{D}''(\tau)} \right) \frac{\partial^2}{\partial \rho^2} \left(\frac{\bar{G}}{\Delta G} \right) \\ & - \hat{D}(\tau) \frac{\partial}{\partial \bar{\rho}} \frac{\partial}{\partial \delta \rho} \left(\frac{\Delta G}{\bar{G}} \right) = \delta(\tau) \delta(\bar{\rho}) \delta(\bar{y}) \delta(\delta \rho) \delta(\delta y) \delta(V - V') \left(\frac{1}{0} \right). \quad (139) \end{aligned}$$

Now note that

$$\Delta G_{\bar{\rho}} = 1/2 (G_{\bar{\rho}} - G_{-\bar{\rho}}) = i \operatorname{Im}(G_{\bar{\rho}}), \quad (140)$$

so ΔG does not contribute under the $\bar{\rho}$ integral required in (120b). Thus, we need determine only $\bar{G}$. If we integrate Eq. (139) over all $\bar{\rho}$, $\delta \rho$, and V , we find that $\bar{G}$ is normalized to 1 (ΔG is antinormalized to 0). Thus, $\bar{G}$ can be interpreted as a probability density in the $(\bar{\rho}, \delta \rho, V)$ space.

We are unable to solve Eq. (139) to find a closed form expression for $\bar{G}$. In lieu of an exact solution, a useful alternative procedure is to represent $\bar{G}$ in terms of its cumulant expansion, and determine the low-order cumulants by taking moments of Eq. (139). [The cumulant expansion for a one-dimensional distribution $P(x)$ is conveniently expressed in terms of its Fourier transform $\langle \exp(-ikx) \rangle$:

$$\langle \exp(-ikx) \rangle = \exp \left\{ \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle \right\}, \quad (141)$$

where the n -th cumulant $\langle\langle x^n \rangle\rangle$ is expressible in terms of moments of order n or less. For example,

$$\langle\langle x^2 \rangle\rangle = \langle x^2 \rangle - \langle x \rangle^2. \quad (142)$$

The multi-dimensional generalization is straightforward (Kubo 1962).] We will write

$$\int d\bar{\rho} \, d(\delta\rho) \, dV \, \bar{\rho}^{\lambda} \delta\rho^m V^n \bar{G} \equiv \langle \bar{\rho}^{\lambda} \delta\rho^m V^n \rangle. \quad (143)$$

Recall that we need only the $k = 0$ component of $G^{(2)}$. Thus, we need not determine any cumulants involving $\bar{\rho}$ unless they appear as intermediate quantities in the course of the calculation. Furthermore, we will not need to compute integrals of ΔG .

We readily determine that

$$\langle \delta\rho \rangle = 0. \quad (144)$$

We also find

$$\frac{\partial}{\partial \tau} \langle \delta y \rangle - (2L_s)^{-1} \langle V \delta\rho \rangle = 0. \quad (145)$$

However, since

$$\frac{\partial}{\partial \tau} \langle V \delta\rho \rangle + 2\nu \langle V \delta\rho \rangle = 0 \quad (146)$$

and we are taking into account initial conditions through terms proportional to $\delta(\tau)$ on the right-hand side, we have

$$\langle V \delta\rho \rangle = 0 \quad (147)$$

and thus also

$$\langle \delta y \rangle = 0. \quad (148)$$

The second cumulants of δy and $\delta\rho$ are thus identical to their second moments. Turning to these, we have

$$\frac{\partial}{\partial \tau} \langle \delta y^2 \rangle - L_s^{-1} \langle V \delta y \delta\rho \rangle - 2D_1 = 0. \quad (149)$$

Also,

$$\frac{\partial}{\partial \tau} \langle v \delta y \delta \rho \rangle - (2L_s)^{-1} \langle v^2 \delta \rho^2 \rangle + v \langle v \delta y \delta \rho \rangle = 0. \quad (150)$$

Neglecting a fourth order cumulant, we will write

$$\langle v^2 \delta \rho^2 \rangle \approx \langle v^2 \rangle \langle \delta \rho^2 \rangle. \quad (151)$$

This is certainly accurate for times $v\tau \ll 1$, when the collisional term is negligible and $\bar{G} \propto \delta(v-v')$. However, such quasi-Gaussian factorizations appear to be reasonable in general when even moments are concerned.

Finally, we have

$$\frac{\partial}{\partial \tau} \langle \delta \rho^2 \rangle = \int d\bar{\rho} d(\delta \rho) dv d\bar{v} (\delta y^2 \hat{D}''(v, \tau; \bar{v})) \bar{G}(\bar{v}; v'). \quad (152)$$

For times $v\tau > 1$, we will assume that

$$\bar{G} \sim \phi_M(v) G^\dagger(v; v'), \quad (153)$$

where the dependence of $G^\dagger$ on v is weak and negligible when even velocity moments are concerned. Then

$$\frac{\partial}{\partial \tau} \langle \delta \rho^2 \rangle \approx D''(\tau) \langle \delta y^2 \rangle, \quad (154)$$

where

$$D'' \equiv \int dv \int d\bar{v} \hat{D}''(v, \tau; \bar{v}) \phi_M(\bar{v}) \quad (155a)$$

$$\sim D''_{\bar{m} \bar{t}} A_{\bar{v}}(\tau) \sim D''_{\bar{m} \bar{t}} (v\tau)^{-1/2}, \quad (155b)$$

where $D''_{\bar{m}}$ has the form of $D_{\bar{m}}$ [cf. Eq. (A10)], except with an additional factor of k_y^2 in the integrand. For times long compared to a collision time, the first term of (150) is small. Equations (149), (150), and (154) can then be combined to yield a single equation for $\langle \delta y^2 \rangle$. If we introduce a characteristic wavenumber $\bar{k}_y$, we may define the dimensionless variables

$$Y(\tau) \equiv \bar{k}_y^2 \langle \delta y^2(\tau) \rangle, \quad (156)$$

$$\bar{\tau} \equiv 2^{-5/3} (z_c(\tau) / \bar{L}_K)^2, \quad (157)$$

where

$$z_c(\tau) \equiv (2D_1 \tau)^{1/2}, \quad (158a)$$

$$D_{\parallel} \equiv v^2/\nu, \quad (158b)$$

and

$$\bar{L}_K \equiv L_s (\bar{k}_y^2 \bar{D}_m)^{-1/3}. \quad (159a)$$

$$\bar{D}_m \equiv \bar{k}_y^{-2} D_m. \quad (159b)$$

Then we obtain

$$\frac{d^2 Y}{d\tau^2} = \tau^{-1/2} Y, \quad (160)$$

with

$$Y(0) = 0, \quad \dot{Y}(0) = 2\bar{\tau}_\perp^{-1}, \quad (161a, b)$$

where

$$\bar{\tau}_\perp \equiv 2^{-5/3} (z_c(\tau_\perp)/\bar{L}_K)^2 \sim [(\bar{k}_y^2 D_\perp)(\bar{L}_K^2/D_\parallel)]^{-1}. \quad (162)$$

Although Eq. (160) can be solved exactly in terms of modified Bessel functions, it is more revealing to examine the asymptotic limits. For $\bar{\tau} > 1$,

$$Y \sim \left(\frac{\bar{\tau}}{\bar{\tau}_\perp}\right)^{-7/8} \exp(4\bar{\tau}^{3/4}/3) \quad (\bar{\tau} > 1). \quad (163)$$

We define the correlation time by the time to achieve $Y \sim 1$:

$$\bar{\tau}_c \rightarrow \bar{\tau}_K \sim \ln^{4/3}(\bar{\tau}_\perp) \quad (\bar{\tau}_\perp > 1). \quad (164)$$

Except for the exponent of $4/3$, this is the same scaling as the correlation time of Rechester & Rosenbluth. However, for $\bar{\tau}_\perp < 1$, this approximate solution becomes meaningless, and the short time solution of (160) becomes appropriate:

$$Y \sim 2\bar{\tau}/\bar{\tau}_\perp \quad (\bar{\tau} < 1) \quad (165)$$

and

$$\bar{\tau}_c \rightarrow \bar{\tau}_\perp \quad (\bar{\tau}_\perp < 1). \quad (166)$$

4. Renormalized kinetic theory

A similar computation of the cumulants in the interaction representation can be performed for the case $v\tau_c < 1$. We have discussed that case in our earlier work (Krommes, Kleva, & Oberman 1978; Krommes 1978) and will not discuss it further here except to say that one obtains an equation of the general form (A23) with z replaced by Vt , V being the initial streaming velocity.

With the cumulants in hand, we can now use the definition (143) and the multidimensional generalization of (141) to find the form of $\bar{G}$. Because of the importance of shear, it can easily be estimated that the $\langle \delta y^2 \rangle$ cumulant dominates. Upon recalling the discussion of factorization between Eqs. (135) and (136) and the discussion of the direct-interaction approximation in Sec. 4.3, we are finally led to the approximate result

$$D = \int_0^\infty d\tau \int \frac{dk_{\parallel}}{2\pi} \sum_{k_x, k_y} \langle b^2 \rangle_{k_{\parallel}, k_x, k_y} \exp(-k_x^2 D \tau - k_y^2 D_{\perp} \tau - \frac{1}{2} k_y^2 \langle \delta y^2(\tau) \rangle) \times \int dV d\bar{V} d\ell V \exp(-ik_{\parallel} \ell) P(\ell, V, \tau | \bar{V}) \phi_M(\bar{V}), \quad (167)$$

which is identical to the direct-interaction approximation (130) except for the additional term in $\langle \delta y^2(\tau) \rangle$. This term introduces τ_K , defined when $v\tau_K > 1$ by Eq. (164), as a possible correlation time. With this ingredient we can recover from (167) all regimes discussed heuristically in Sec. 2. When $\tau_K < \tau_{\perp}$, the higher order kinetic theory discussed in this section must be invoked in some form; then, the results of Rechester & Rosenbluth emerge. When $\tau_{\perp} > \tau_K$, the direct-interaction approximation is adequate and the Kadomtsev-Pogutse or fluid results are obtained.

5. Summary and discussion

In this paper we have studied the kinetic theory of test particle transport in specified stochastic magnetic fields, both heuristically and, to some extent, formally. Our principal results were as follows:

- (1) We discussed scaling laws valid in all collisionality regimes, and demonstrated a smooth transition between each regime.
- (2) We showed that the procedure of averaging a classical fluid equation over a random magnetic field is in general incorrect, as is the standard Chapman-Enskog ordering in the presence of a stochastic field. We introduced a modified perturbation theory which took account of the intrinsic scales of the stochastic field and which led to a Kubo formula valid in all regimes.

- (3) We showed that the direct-interaction approximation fails to describe regimes in which the stochastic instability of the field lines plays an explicit role in determining the correlation time. We described a higher order renormalization based on the Bethe-Salpeter equation which seemed to be general enough to embrace all regimes.

All of the present work relied on the foundations laid in our earlier papers, where we identified the fundamental lengths for the problem of stochastic magnetic fields and carefully distinguished the autocorrelation length L_{ac} ($\sim L_0$ quasilinearly) from the exponentiation length L_K . Unfortunately, we believe that there remains confusion in the literature about L_{ac} , perhaps because many authors envision a quasilinear analysis in which L_{ac} is often implicitly taken to vanish. We hope that App. A will be of some help in this respect.

With regard to result (1), our discussion was based on a detailed analysis of the theory of double diffusion (of the test particles tied to the stochastic lines) combined with a heuristic theory of the correlation time of particles with lines. The basic ingredients of such an analysis were presented by us previously (Krommes, Kleva, & Oberman 1978; Krommes 1978). However, in that work we restricted our attention to the quasilinear Rechester-Rosenbluth regimes, as we considered those to be the regimes of most physical relevance.

The result (2) is perhaps the most interesting and potentially important aspect of our work. The common desire to perform statistical analysis using a fluid description is quite understandable and has many antecedents. [Although more complicated, an analogous attempt occurs when one attempts to develop a turbulence theory by averaging the Vlasov equation, even though that equation describes a mean distribution function, being a (severe) truncation of the BBGKY moment hierarchy based on the Liouville equation.] It has often been warned that there may be inherent contradictions and problems with such double averaging procedures (Grad 1981), and our work in Sec. 3 shows an explicit example where this is true. As we have shown, there are also (hydrodynamic) regimes where the double averaging procedure is permissible. However, we believe that our work is the first in which the validity constraints have been displayed explicitly, and also in which the collisionless results have been reconciled with a modified Chapman-Enskog procedure.

With regard to result (3), it is not, in itself, surprising that the direct-interaction approximation may sometimes fail. It is, after all, an approximation, however sophisticated, and its deficiencies have been well

documented in its original field of application, fluid turbulence. What is more interesting is the fact that it is quite adequate to describe the magnetic diffusion coefficient D_m , yet it fails in some cases to properly describe the associated particle diffusion. (We did not give a separate detailed discussion of the renormalized theory of D_m (although see App. A), but the success of the direct-interaction approximation in this context can be inferred from the features and properties of renormalized quasilinear theory in other contexts.) The point is that the test particles serve as probes which are able to "open up" the microscopic correlations. In fact, we believe that the failure of the direct-interaction approximation to properly describe the orbit exponentiation is intimately related to its failure to preserve random Galilean invariance (Kraichnan 1964b), although we have not attempted to make the relationship mathematically explicit. In both cases, problems arise because the direct-interaction approximation drops important high order Eulerian correlations. As we have suggested previously (Krommes, Kleva, & Oberman 1978), this implies that it may be more appropriate to develop the theory of test particle transport in a Lagrangian framework.

The distinction between Lagrangian vs. Eulerian formulations brings us to a possible paradox. Kraichnan (1964b) has argued quite generally that no systematic Eulerian-based renormalization will be invariant to random Galilean transformations, and a similar result may be expected for the present problem. On the other hand, we argued in Sec. 4.4 that we could capture the essential features of the stochastic instability by proceeding to higher order in the Eulerian-based renormalization scheme of Martin, Siggia, & Rose. The resolution is simply that we made (many) further approximations, such as the Markovian approximation, to the fundamental MSR equations. It is well-known that such procedures can turn Eulerian theories into effectively Lagrangian ones. The fluid direct-interaction approximation can be made random Galilean invariant by the simple artifice of inserting a long wavelength cutoff (Kraichnan 1964b). Dupree's (1974) theory of two-dimensional turbulence is random Galilean invariant, but is derived by Markovianizing Eulerian-based equations.

Returning to more applications-oriented questions, we follow many other authors in remarking that the theory of test particle diffusion is very special in that there is no constraint of ambipolarity. That the plasma fluid density transport must obey that constraint is perhaps the principal new feature which must be accounted for when one attempts to extend the present calculations. Progress has been made in this direction (Kotschenreuther 1982). As we have stated, it may be expected that many features of the scaling laws for the thermal conductivity of the plasma fluid may be deduced from the test particle density diffusion coefficient. Of course, if the fluc-

tuation intensity of the electromagnetic fields is unknown, a separate nonlinear calculation must be performed, and this is generally extremely difficult. However, we believe that the ideas and formalism we have introduced in the present paper provide the appropriate framework for further research.

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Acknowledgments

Appendix A: The magnetic diffusion coefficient, and characteristic lengths for the problem of stochastic magnetic fields

Here we define and carefully distinguish between several characteristic lengths of the stochastic magnetic field: L_0 , the quasilinear autocorrelation length; L_K , the typical Kolmogorov or "exponentiation" length of adjacent field lines; and $L_\perp$, the decorrelation length in the strong turbulence regime. All of these emerge in the heuristic theory of the diffusion coefficient of magnetic lines, which we shall now review. [The following discussion is a substantial elaboration on, and addition to, the work of Rosenbluth et al. (1966).]

Although we shall work in a sheared slab geometry, it is instructive to begin by considering a standard model equilibrium field $\tilde{B}_0$ with shear, appropriate to the cylindrical limit of a tokamak. In (r, θ, ϕ) coordinates,

$$\tilde{B}_0 = (0, \varepsilon(r)/q(r), 1) \tilde{B}_0, \quad (A1)$$

where $\varepsilon \equiv r/R \ll 1$ and B_0 and R are constant. Topologically, this field consists of nested concentric circular flux surfaces, and is not stochastic. To this field we now add a small perturbation $\delta \tilde{B} \equiv \tilde{b}(r, \theta, \phi) \tilde{B}_0$ such that $|\tilde{b}| \equiv b \ll 1$, $b_\theta < \varepsilon/q$. If we assume harmonic dependence of the form $b \sim \exp(im\theta - in\phi)$, the total field becomes stochastic when multiple harmonics $\{m, n\} \equiv \mu$ are present with incommensurate helicities, $m/n \neq m'/n'$, such that the Chirikov resonance overlap criterion (Chirikov 1969, 1979) is well-satisfied. The equations of the field lines, $d\tilde{x} \times \tilde{B} = 0$, are

$$\frac{dr}{Rd\phi} = b_r = \sum_{\mu} b_{\mu} \exp(im\theta - in\phi), \quad (A2a)$$

$$\frac{rd\theta}{Rd\phi} = \frac{\varepsilon(r)}{q(r)} + b_{\theta} \approx \frac{r}{Rq(r)}. \quad (A2b)$$

To pass to the sheared slab approximation, we expand around a typical rational surface specified by μ_0 : $m_0 - n_0 q(r_0) = 0$. In terms of the variables

$$x \equiv r - r_0, \quad (A3a)$$

$$y \equiv r_0 (\theta - \phi/q_0), \quad (A3b)$$

$$z \equiv R\phi, \quad (A3c)$$

we find

$$\frac{dx}{dz} = \sum_{\mu} b_{\mu} \exp(ik_y y + ik_z z), \quad (A4a)$$

$$\frac{dy}{dz} = -\frac{x}{L_s}, \quad (\text{A4b})$$

where

$$L_s \equiv Rq_0/s, \quad (\text{A5a})$$

$$s \equiv d(\ln q)/d(\ln r)|_{r_0}, \quad (\text{A5b})$$

$$k_y \equiv m/r_0, \quad (\text{A5c})$$

$$k_{||} \equiv k_{||}(r_0), \quad (\text{A5d})$$

$$k_{||}(r) \equiv [m-nq(r)]/Rq(r), \quad (\text{A5e})$$

and the μ summation must be interpreted as an appropriate sum over k_y and $k_{||}$. The difficulties of such summation subject to the periodicity constraints are well-known, and are most efficiently handled by WKB techniques (Dewar 1982). However, such analysis is not essential for the simple arguments which follow. It is adequate to pass to the strict slab approximation in which $k_{||}$ is treated as a continuous variable, and we write

$$\sum_{\mu} = \sum_{k_y} R \int dk_{||}. \quad (\text{A6})$$

[Passage to the continuum in $k_{||}$ is justified in the stochastic limit because of the resonance-broadening associated with the nonlinear diffusion; cf. Galeev & Sagdeev (1981) or Krommes (1981).]

It is important to note that although $k_{||}(\mu_0) = 0$, $k_{||}(\mu) \neq 0$ in general. If a typical eigenfunction has half-width δr around r_0 , then the maximum $k_{||}$, called $\delta k_{||}$, which contributes to field line motion in the vicinity of r_0 can be estimated from

$$k_{||}(r_0 + \delta r) = 0, \quad (\text{A7})$$

or

$$\delta k_{||} \sim \left| \delta r \frac{\partial k_{||}}{\partial r_0} \right| \quad (\text{A8a})$$

$$= k_y \delta r / L_s. \quad (\text{A8b})$$

When the excited wavenumbers are such that $k_y \delta r \sim 1$, this gives rise to the often-used estimate $\delta k_{||} \sim L_s^{-1}$. However, other limits are possible.

The magnetic diffusion coefficient is defined, at a fixed time, by

$$D_m \equiv \lim_{z \rightarrow \infty} \frac{1}{2} \frac{d}{dz} \langle x^2(z) \rangle, \quad (A9)$$

where the angular brackets mean an average over initial conditions y_0 and z_0 , and over b if b is random. We can write

$$D_m = \int_0^\infty dz \sum_\mu \langle |b_\mu|^2 \rangle \exp(ik_\parallel z) N(z), \quad (A10)$$

where $N(z)$ describes the nonlinear envelope of the correlation function and arises from stochastic perturbations to the zeroth order lines. We write

$$\langle |b_\mu|^2 \rangle = (2\pi R)^{-1} \langle b^2 \rangle_{k_y, k_\parallel}, \quad (A11)$$

normalized in such a way that

$$\langle b^2 \rangle = \sum_\mu \langle |b_\mu|^2 \rangle = \sum_{k_y} \int \frac{dk_\parallel}{2\pi} \langle b^2 \rangle_{k_y, k_\parallel}. \quad (A12)$$

Furthermore, introduce a function Δ (the quasilinear autocorrelation function for specified k_y) such that

$$\Delta(z; L_0) \equiv \left(\frac{\int dk_\parallel \langle b^2 \rangle_{k_y, k_\parallel} \exp(ik_\parallel z)}{\int dk_\parallel \langle b^2 \rangle_{k_y, k_\parallel}} \right), \quad (A13a)$$

$$L_0 \equiv \pi / \Delta k_\parallel, \quad (A13b)$$

$$\Delta k_\parallel \equiv \int dk_\parallel \left(\frac{\langle b^2 \rangle_{k_y, k_\parallel}}{\langle b^2 \rangle_{k_y, 0}} \right). \quad (A13c)$$

$\Delta k_\parallel$ is the (parallel) spectral width, and L_0 is the quasilinear autocorrelation length. Then

$$D_m = \sum_{k_y} \int \frac{dk_\parallel}{2\pi} \langle b^2 \rangle_{k_y, k_\parallel} \int_0^\infty dz \Delta(z; L_0) N_{k_y}(z). \quad (A14)$$

Because

$$\int_{-\infty}^\infty dz \Delta(z; L_0) = 2L_0, \quad (A15)$$

it can be seen that

$$\lim_{L_0 \rightarrow 0} \Delta(z; L_0) = 2\delta(z/L_0). \quad (A16)$$

Thus, if L_0 is less than the characteristic scale of the envelope function $N(z)$, we have

$$\begin{aligned} D_m &\approx \left(\sum_{k_y} \int \frac{dk_{\parallel}}{2\pi} \langle b^2 \rangle_{k_y, k_{\parallel}} \right) \left(\int_0^{\infty} dz \Delta(z; L_0) \right) N(0) \\ &= \langle b^2 \rangle_{L_0}, \end{aligned} \quad (A17)$$

which is the quasilinear result.

Intuitively, it may be expected that the scale of $N(z)$ is related to the characteristic scale for divergence of two neighboring lines. The latter may be estimated by considering the statistical behavior of infinitesimal deviations of a second line from a reference line:

$$\frac{d}{dz} \Delta y = -L_s^{-1} \Delta x, \quad (A18a)$$

$$\frac{d}{dz} \Delta x = \bar{k}_y \bar{b}'(z) \Delta y, \quad (A18b)$$

where $\bar{k}_y$ is a characteristic wavenumber and

$$\bar{b}'(z) \equiv \bar{k}_y^{-1} \frac{\partial}{\partial y} b(y, z) \Big|_{y=y(z)}. \quad (A19)$$

Upon averaging over an ensemble of initial conditions for the second line, we will have $\langle \Delta y \rangle = \langle \Delta x \rangle = 0$. The second cumulants obey exactly

$$\frac{d}{dz} \langle \Delta y^2 \rangle = -2L_s^{-1} \langle \Delta y \Delta x \rangle, \quad (A20a)$$

$$\frac{d}{dz} \langle \Delta y \Delta x \rangle = -L_s^{-1} \langle \Delta x^2 \rangle + \bar{k}_y \langle \bar{b}'(z) \Delta y^2 \rangle, \quad (A20b)$$

$$\frac{d}{dz} \langle \Delta x^2 \rangle = 2\bar{k}_y \langle \bar{b}'(z) \Delta y \Delta x \rangle. \quad (A20c)$$

A quasilinear analysis of Eqs. (A20) leads us to

$$\frac{d}{dz} \langle \Delta y^2 \rangle = -2L_s^{-1} \langle \Delta y \Delta x \rangle, \quad (A21a)$$

$$\frac{d}{dz} \langle \Delta y \Delta x \rangle = -L_s^{-1} \langle \Delta x^2 \rangle, \quad (A21b)$$

$$\frac{d}{dz} \langle \Delta x^2 \rangle = 2\bar{k}_y^2 \bar{D}_m'' \langle \Delta y^2 \rangle, \quad (A21c)$$

where

$$\bar{D}_m'' \equiv \int_0^{\infty} dz \langle \bar{b}'(z) \bar{b}'(0) \rangle \quad (A22)$$

is of order D_m . Equations (A21) can be combined to give

$$\frac{d^3}{dz^3} \langle \Delta y^2 \rangle = (2/L_K)^3 \langle \Delta y^2 \rangle, \quad (A23)$$

where

$$L_K \equiv 2^{1/3} \bar{L}_K, \quad (A24a)$$

$$\bar{L}_K \equiv L_s (k_y^2 \bar{D}_m'' L_s)^{-1/3}. \quad (A24b)$$

The characteristic wavenumbers of (A23) are

$$k = -2iL_K^{-1} \{1, -1/2 (1 \pm \sqrt{3})\}, \quad (A25)$$

so one solution grows as $\exp(2z/L_K)$ and two solutions decay as $\exp(-z/L_K)$. Thus, we may identify $\bar{L}_K$ with the characteristic exponentiation length. The quasilinear analysis is valid for $L_0 \ll L_K$. Some remarks on the opposite regime $L_0 > L_K$ are given at the end of Sec. 2.2.

The preceding quasilinear estimate for the exponentiation length L_K was first given by us in our 1978 paper, where we stressed two important qualitative features of (A24b). The principle feature is the importance of shear. (A24b) can be written

$$\bar{L}_K = (k_y^2 \bar{D}_m'')^{-1/3} L_s^{2/3}, \quad (A26)$$

so $\bar{L}_K$ is composed more of shear than of radial diffusion. Hirshman & Molvig (1979) used a related fact in their theory of nonlinear saturation of the universal instability. A secondary feature of (A24b) is that it is not D_m itself which enters, but rather the quantity $\bar{D}_m''$, which scales as D_m but involves a different spectral weight. Subsequently, Rechester, Rosenbluth, & White (1979) refined the quantitative aspects of this calculation with the aid of a detailed solution of the relevant Fokker-Planck equation. They found compelling agreement with numerical computations.

Appendix B: Aspects of renormalized perturbation theory

We give here a cursory discussion of the first few terms of the renormalized perturbation theory of stochastic differential equations of the form of Eqs. (62) or (92). The procedure discussed here, consolidation of infinite diagrammatic series, affords a more primitive, but perhaps more concrete, alternative to the powerful functional procedures discussed in App. C. It is useful in that it provides an independent check of certain signs, for example.

App. B: Renormalized perturbation theory

Let us define the unperturbed (nonrandom) response function $R^{(0)}(1,1')$ by the solution of

$$\left(\frac{\partial}{\partial t_1} + i\bar{L}_1\right)R^{(0)}(1,1') = \delta(1-1') . \quad (B1)$$

The random infinitesimal response function associated with Eq. (62) can then be rewritten as the integral equation

$$\tilde{R}(1,1') = R^{(0)}(1,1') - R^{(0)}(1,\bar{1})i\tilde{Q}(\bar{1})\tilde{R}(\bar{1},1') . \quad (B2)$$

Suppose that the stochastically nonlinear term proportional to $\tilde{Q}$ were small. [It is not small, in general, especially near zeros of $\partial_t + i\bar{L}$ (resonances).] The iterative solution of Eq. (B2) then yields in n-th order

$$R^{(n)} = \langle \tilde{R}^{(n)} \rangle = \langle (-R^{(0)}i\tilde{Q})^n R^{(0)} \rangle . \quad (B3)$$

We have defined the averaging operator so that $i\tilde{Q} \propto \tilde{b}$ is Gaussian; all moments $\langle i\tilde{Q}(1) \dots i\tilde{Q}(n) \rangle$ then vanish for n odd. (This procedure is possible in the present problem where b is specified; it must be modified in the self-consistent problem where the nonlinear coupling of the internal plasma fluctuations drive in nonvanishing cumulants of all orders even if the cumulants are initially Gaussian.) Thus, for example,

$$R^{(2)} = R^{(0)} \langle (-i\tilde{Q})R^{(0)}(-i\tilde{Q})R^{(0)} \rangle \quad (B4)$$

(Fig. 6a),

$$R^{(4)} = R^{(0)} \langle (-i\tilde{Q})R^{(0)}(-i\tilde{Q})R^{(0)} \langle (-i\tilde{Q})R^{(0)}(-i\tilde{Q})R^{(0)} \rangle \rangle + (\text{two more terms}) \quad (B5)$$

(Fig. 6b), where in Fig. 6 the curly lines denote correlations, or averaging, dots denote the bare interaction vertex $-i\tilde{Q}$, and the horizontal lines denote the unperturbed response function $R^{(0)}$. The object $\Sigma^{(0)}$ defined by $-\langle i\tilde{Q}R^{(0)}i\tilde{Q} \rangle$ is the non-Markovian version of the quasilinear operator: for arbitrary H ,

$$\Sigma^{(0)}H \equiv \int_0^t d\tau \frac{\partial}{\partial x} V R^{(0)}(\underline{1}, \tau; \bar{1}) \bar{V} \langle i\tilde{Q}(\underline{1})i\tilde{Q}(\bar{1}) \rangle \frac{\partial}{\partial x} H(\bar{1}, t-\tau) , \quad (B6)$$

though note that the equation for R correct through second order,

$$(R^{(0)})^{-1}R + \Sigma^{(0)}R^{(0)} = 1 , \quad (B7)$$

is not a true quasilinear equation because of the presence of $R^{(0)}$ rather than R following $\Sigma^{(0)}$. However, it is not hard to see that the higher order disconnected terms (with no crossed lines) — e.g., the first two terms of Fig. 6b or Fig. 7a — sum to replace $R^{(0)}$ by R everywhere on the right-hand side of Eq. (B7), giving the renormalized quasilinear equation

$$(R^{(0)})^{-1}R + \Sigma R = 1. \quad (B8)$$

More generally, the sum of all disconnected diagrams (including crossed lines) yields the mass operator or turbulent collision operator Σ and the Dyson equation

$$(R^{(0)})^{-1}R + \Sigma R = 1. \quad (B9)$$

Terms involving crossed lines describe intrinsic two-body correlations. These are defined as those parts of the joint distribution $K_{o,+;o,-}^{(1,2;3,4)}$ (which describes the probability of finding the field ("o") at 1 and a particle ("+" at 2, and also the field at 3 and a perturbation ("-") in the particle probability at 4) which do not factor into the product $\langle b(1)b(3) \rangle R(2,4)$ of the singlet probabilities. To see this more clearly, consider the last term of Fig. (6b) and all similar higher order terms such as Fig. (7b), each connected factor of which contains at most one crossed line. These crossed terms sum to

$$R(1,1') = R^{(0)}(1,2) V_2 \frac{\partial}{\partial x_2} \hat{K}(2',2;\bar{1},\bar{1})|_{2'=2} V_{\bar{1}} \frac{\partial}{\partial x_{\bar{1}}} R(\bar{1},1'), \quad (B10)$$

where

$$\hat{K}(1,2;3,4) \equiv F(1,3)R(2,4) + R(2,\bar{2}) \left(V_2 \frac{\partial}{\partial x_2} R(\bar{2},\bar{4}) F(1,\bar{4}) + V_{\bar{4}} \frac{\partial}{\partial x_{\bar{4}}} R(\bar{4},4) F(\bar{2},3) \right). \quad (B11)$$

The last term in (B11) arises from the crossed graphs and is clearly nonfactorable [note the term $F(1,\bar{4})R(\bar{4},4)$]. The form of (B11) strongly suggests that $\hat{K}$ represents the first two terms of an expansion of K in powers of $\langle \tilde{Q}^2 \rangle$. It does not elucidate the nature of the neglected terms. The technique of Martin, Siggia, & Rose, which we discuss in App. (C), permits a much more complete and systematic discussion.

Appendix C: The MSR procedure and the Bethe-Salpeter equation

Here we derive the renormalized approximation we shall study in the main text from the renormalized equations of Martin, Siggia, & Rose (1973), as extended to include the case of a statistically specified random coefficient. Because one of us has recently reviewed the general procedure (Krommes 1981), we may be brief. Consider an evolution equation, for the random variable ψ , with quadratic nonlinearity:

$$\frac{\partial}{\partial t_1} \psi(1) = U(1,2)\psi(2) + U'(1,2,3)i\tilde{\Omega}(2)\psi(3) + \frac{1}{2}\bar{U}(1,2,3)\psi(2)\psi(3) . \quad (C1)$$

Here the bare interaction vertices U , U' , and $\bar{U}$ are specified and nonrandom, and the usual integration convention over repeated indices is assumed. Martin, Siggia, & Rose considered the dynamically nonlinear case where U' was absent; Dekker & Haake (1975) extended the formalism to the case of present interest by including the stochastically nonlinear term in U' . Recently, Jensen (1981) discussed substantial generalizations of the form of Eq. (C1).

It is convenient to introduce the symmetric two-point correlation matrix

$$G(1,1') \equiv \begin{vmatrix} C(1,1') & R(1;1') & M(1,1') \\ R^T(1;1') & 0 & 0 \\ M^T(1,1') & 0 & F(1,1') \end{vmatrix} , \quad (C2)$$

where

$$C(1,1') \equiv \langle \delta\psi(1)\delta\psi(1') \rangle , \quad (C3a)$$

$$M(1,1') \equiv \langle \delta\psi(1)\delta[i\tilde{\Omega}(1')] \rangle , \quad (C3b)$$

$$F(1,1') \equiv \langle \delta[i\tilde{\Omega}(1)]\delta[i\tilde{\Omega}(1')] \rangle , \quad (C3c)$$

$$R(1,1') \equiv \left. \frac{\delta\langle\psi(1)\rangle}{\delta\eta(1')} \right|_{\eta=0} , \quad (C3d)$$

and "t" denotes transpose. By considering the functional variations with respect to η of the two-point cumulants at finite η , it is possible to derive closed functional equations relating $\langle\psi\rangle$, G , and certain other three- or four-point correlation functions. For the interesting case of Gaussian initial conditions and random forces, one can write a matrix Dyson equation

$$(G^{(0)})^{-1}G - \gamma\langle\psi\rangle G + \Sigma G = 1 , \quad (C4)$$

where γ is an appropriately symmetrized matrix containing U' and $\bar{U}$. Equation (C4) is represented diagrammatically in Fig. (8a). The "mass" or "collision" operator Σ is defined in terms of the two-body scattering matrix $K(1,2;3,4)$,

$$\Sigma(1, \bar{1}) = -\frac{1}{2} \gamma(1,2,3) K(2,3; \bar{2}, \bar{3}) \gamma(\bar{2}, \bar{3}, \bar{1}) \quad (C5)$$

(Fig. 8b), where K is the solution of the Bethe-Salpeter equation (Frisch 1968; Martin, Siggia, & Rose 1973; Krommes 1978, 1981)

$$K(1,2;3,4) = \frac{1}{2} [G(1,3)G(2,4) + G(1,4)G(2,3)] \\ + G(1, \bar{1}) G(2, \bar{2}) I(\bar{1}, \bar{2}; \bar{3}, \bar{4}) K(\bar{3}, \bar{4}; 3, 4) \quad (C6)$$

(Fig. 9a). The interaction kernel I is defined formally as

$$I(1,2;3,4) \equiv \frac{\delta \Sigma(1,2)}{\delta G(3,4)} \quad (C7)$$

and is the formal expression of the intrinsic two-body correlations.

The appearance of the two-body function K in Eq. (C5) is closely related to the fact that diffusion coefficients are most conveniently couched as integrals of Lagrangian (taken along the orbit) correlation functions (Taylor 1921). Kraichnan (1964a) was one of the first to attempt a renormalized description which related a Lagrangian function to Eulerian ones.

To date, reduction of the functional derivative (C7) to practical form has been accomplished only through some sort of perturbation theory. Direct expansion in powers of the bare interaction γ defeats the purpose of renormalization and leads to unphysical divergences (Kraichnan 1961; Martin, Siggia, & Rose 1973). A better procedure (Martin, Siggia, & Rose 1973) is to expand I in powers of the renormalized vertex $\Gamma \equiv G^{-2} K \gamma$ (Fig. 8c); more precisely, in powers of $\bar{\Gamma} \equiv G^{3/2} \Gamma$, yielding to lowest order

$$I(1,2;3,4) \approx \Gamma(1,5,3) G(5, \bar{5}) \Gamma(2, \bar{5}, 4) \quad (C8)$$

(Fig. 9b). This procedure is based on the fact that $\bar{\Gamma} \sim \langle \delta \psi^3 \rangle / \langle \delta \psi^2 \rangle^{3/2}$ is a generalized skewness parameter (really, an operator). For systems where the nonlinearities imply a high degree of stochasticity, it can be hoped (Martin, Siggia, & Rose 1973) that the true (measurable) skewness is small (less than one) even though the bare skewness $G^{3/2} \gamma$ need not be small. This must, of course, be verified, either experimentally or analytically. Either task is formidable, in general. It is especially important to note that $\bar{\Gamma}$, being

in general a differential, time-dependent operator, need not be uniformly small over all ranges of its arguments.

If $\bar{\Gamma}$ were uniformly small, it would be justified to neglect the interaction term $GGIK$ in Eq. (C6). The resulting approximation,

$$\Sigma(1, \bar{1}) \approx -\frac{1}{2} \gamma(1, 2, 3) G(2, \bar{2}) G(3, \bar{3}) \gamma(\bar{3}, \bar{2}, \bar{1}) \quad (C9)$$

(Fig. 8e), is called the direct-interaction approximation after Kraichnan's original derivation for the Navier-Stokes equation (Kraichnan 1959). For linear stochastic differential equations, this reduces to renormalized quasilinear theory. When, in addition, the generalized Reynolds number is uniformly small, the theory reduces further to the well-known Bourret (1962) approximation.

In the present problem $\bar{\Gamma}$ is not uniformly small, first because γ is a differential operator which acts on radial structures of all scales, second because of the stochastic instability with which are associated very specific, non-Gaussian correlations responsible for the exponential divergence of lines. The first vertex renormalization (C8) is a kind of renormalized "Boltzmann" equation, to use the terminology of many-body physics, and if particle discreteness effects were included, the classical Boltzmann operator would follow from this order [but not from the direct-interaction approximation, which leads to a renormalized Balescu-Lenard operator (Krommes & Oberman 1976)].

We will base our theory of stochastic magnetic fields on a generalized diffusion approximation,

$$I \approx \gamma G \gamma \quad (C10)$$

(Fig. 9c). This does not provide a fully self-consistent determination of K (essentially, the diffusion operators in the Bethe-Salpeter equation are given by their direct-interaction approximation). However, we believe that it provides an adequate description, especially in the quasilinear regime $\lambda < L_K$.

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Name of regime	Inequality which defines regime	$D/D_m v_t$ ($D_m \equiv \langle b^2 \rangle L_{ac}$)	D	τ_c which defines $\alpha \equiv (v\tau_c)^{-1}$
Double streaming ¹	$\alpha > \frac{\lambda_v}{L_{ac}} > 1$	$(\frac{\lambda_v}{L_{ac}})^{-1}$	$\langle b^2 \rangle v_t^2 \tau_c$	any ³
collision-less	$\frac{\lambda_v}{L_{ac}} > \alpha > 1$	1	$D_m v_t$	any ³
Rechester-Rosenbluth	$1 > \frac{\lambda_v}{L_K} > \alpha^{1/2}$	$\frac{\lambda_v}{L_K}$	$D_m D_{\parallel} / L_K$	τ_K
Kadomtsev-Pogutse ²	$\alpha^{1/2} > \frac{\lambda_v}{L_K} > (\frac{L_{ac}}{L_K}) \alpha^{1/2}$	$\alpha^{1/2}$	$D_m (D_{\parallel} D_{\perp})^{1/2} / \delta_{\perp}$	$\tau_{\perp}$
Fluid	$\alpha^{1/2} > \frac{\lambda_v}{L_{ac}} > (\frac{\lambda_v}{L_{ac}}) \alpha^{1/2}$	$\frac{\lambda_v}{L_{ac}}$	$D^{(f)} \equiv \langle b^2 \rangle D_{\parallel}$	$\tau_{\perp}$
Double streaming ¹	$\alpha > 1 > \frac{\lambda_v}{L_{ac}}$	$(\frac{\lambda_v}{L_{ac}})^{-1}$	$D^{(f)} (v\tau_c)$	any ³

Table 1: Regimes for the radial particle diffusion coefficient in a specified static magnetic field. For $L_K < L_{ac}$, replace L_K by L_{ac} . Notes: ¹regime absent for $\tau_c = \tau_K$ and $L_{ac} < L_K$; ²regime absent for $L_K < L_{ac}$; ³either $\tau_{\perp}$, τ_K , or a temporal decorrelation time may be used.

Figure Captions

Fig. 1: Sketch of a possible form for the dimensionless particle dispersion function $d(t)$. The figure illustrates that $d(t)$ reaches its long-time limit in a perpendicular decorrelation time τ_c .

Fig. 2: The coarse-grained distance axis for spatial diffusion of magnetic lines. The diffusion law $\langle \delta x^2 \rangle \sim D_m \ell$ holds for $\ell > \Delta \ell > L_{ac}$.

Fig. 3: Coarse-graining for the case $\lambda_v < L_{ac}$. Parallel diffusion ensues for $\ell \gtrsim \delta \ell > \lambda_v$. Radial diffusion ensues for $\ell \gtrsim \Delta \ell > L_{ac}$ (see Fig. 2).

Fig. 4: Coarse-graining for the case $L_{ac} < \lambda_v$. Radial diffusion ensues for $\ell \gtrsim \Delta \ell > L_{ac}$. Parallel diffusion ensues for $\ell \gtrsim \delta \ell > \lambda_v$.

Fig. 5: The functions $d(t)$ and $d_0(t)$ are coincident for $t < \tau_c$. Thus, $d(\infty) \approx d_0(\tau_c)$.

Fig. 6: (a) Second order approximation to the response function. (b) Fourth-order approximation. See App. B for further discussion.

Fig. 7: Higher order terms in the expansion of the response function.

Fig. 8: (a) The Dyson equation. (b) Representation of the collision operator Σ in terms of the scattering matrix K . (c) Relation of the renormalized vertex function Γ to the scattering matrix. (d) Representation of Σ in terms of Γ . (e) The direct-interaction approximation to Σ .

Fig. 9: (a) The Bethe-Salpeter equation. (b) First vertex renormalization. (c) Approximation to the first vertex renormalization.

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Fig. 1

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Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7