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INDEPENDENT PARTICLE SCHRÖDINGER FLUID MOMENTS OF INERTIA[†]

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ABSTRACT

This philosophy of the Single Particle Schrödinger Fluid, especially as regards the velocity fields which find such a natural role therein, is applied to the study of the moments of inertia of independent Fermion system. It is shown that three simplified systems exhibit the rigid-body rotational velocity field in the limit of large A , and that the leading deviations, both

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on the average and fluctuating, from this large A limit can be described analytically, and verified numerically. For a single particle in a Hill-Wheeler box the moments are studied numerically, and their large fluctuations identified with the specific energy level degeneracies of its parallelepiped shape.

The full assemblage of these new and old results is addressed to the question of the necessary and sufficient condition that the moment have the rigid value. Counterexamples are utilized to reject some conditions, and the conjecture is argued that *Unconstrained Shape Equilibrium* might be the necessary and sufficient condition. The spheroidal square well problem is identified as a promising test case.

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1. Introduction

This paper is the second of a series which investigates the theory of the Independent Particle Schrödinger Fluid. In an earlier paper¹⁾, we have formulated the problem of the Schrödinger Fluid and investigated its general properties[†], with emphasis on those aspects related to nuclear collective motions²⁻⁶⁾. In the present paper we apply this formulation to a particular kind of nuclear collective motion, the rotational motion of deformed nuclei.

Nuclear collective rotation is a topic some twenty-five years old^{7,8)} which has grown steadily both in the sophistication of its theory and in the range of data to which it relates⁹⁻¹⁹⁾. In the present study, we focus on the most central parameter of collective rotation, the moment of inertia, and upon the velocity fields for the rotational motion which arise naturally in the Schrödinger fluid description. Since the Schrödinger Fluid theory is at present an independent particle model, our discussions are confined mainly to the "cranking model" approximation⁹⁾ for the velocity fields and the inertia. Still, the novel viewpoint of the Single Particle Schrödinger Fluid allows one a fresh look at some old, and still not fully answered, questions.

In particular, one central question is: What are the necessary and sufficient conditions for the cranking model moment to be equal to the rigid body moment?^{11,20,21)} For two specific idealized cases, the independent particle

[†]This paper was the first (I) in a series of which the present is II. Paper III, dealing with shape vibration and dilatation in the Schrödinger fluid is in preparation. Further, sometimes more detailed, discussion of some of these topics is found in refs. 2) through 6).

cranking moment of inertia is known to have a value equal to the rigid moment of inertia: (a) Any number of independent particles in the anisotropic simple harmonic oscillator potential at the equilibrium deformation¹¹⁾, and (b) an infinite number of particles in an Amado-Brueckner box (cubical square well with periodic boundary conditions)^{22,23)}. In addition, Migdal²⁴⁾ extracts the rigid body value from the cranking model by means of several assumptions.

In the context of the Single Particle Schrödinger Fluid, one is led naturally to a subquestion concerning the velocity fields for these rigid body cases: Are they identical with that for rigid body rotation, or does the rigid moment result from a more subtle conspiracy in which some velocity field other than that of rigid flow somehow provides for a kinetic energy such that the inertia which *just happens to have the rigid value*?

In sect. 2 we give a brief summary of the theory of the Schrödinger Fluid, and especially of the results relevant to inertial parameters. In sect. 3 the velocity fields and inertial properties of single particle states in a rotating harmonic oscillator potential and a rotating Hill-Wheeler box⁸⁾ are studied. We find that the irrotational velocity fields, χ_{θ} , of the various single particle states possess line vortices, and bear no evident relationship to the velocity field for rigid body rotation.

Nevertheless, the moments of inertia always have the rigid value for an oscillator particle at its equilibrium shape. In contrast, the moments of particles in the Hill-Wheeler box vary widely, from values much smaller than the rigid body value to values much larger.

Section 4 addresses the limit $A \rightarrow \infty$ of large nuclei. To the well-known result that the moment of inertia for $A \rightarrow \infty$ particles in the lowest levels of an Amado-Brueckner box approaches the rigid value one adds the demonstration

that the corresponding moment for the Hill-Wheeler box also approaches the rigid body value, but slowly, and with substantial fluctuation. In addition, the velocity fields for both the oscillator and the Hill-Wheeler box are seen to approach the rigid body rotational field, $\Omega \times \mathbf{r}$, in the limit of $A \rightarrow \infty$.

Section 5 describes a study of the finite A corrections to the large A , "rigid body" limit for $\mathcal{F}$ in the case of the Amado-Brueckner box. The result is a smooth average value to which a term fluctuating with A is added. It is compared (favorably) with an exact numerical calculation of the cranking model for the Amado-Brueckner box. Similar exact results for the Hill-Wheeler box are reported.

Section 6 assesses the implications of the present results for several conjectures about the *sufficient* conditions for an independent particle system to have the rigid body moment of inertia. Two conjectures are rejected, because the present results include counter examples. The qualitative elements of a new *Unconstrained Shape Equilibrium* conjecture are proposed, and a specific test is suggested. In addition, the assumptions which Migdal made in extracting the rigid moment are discussed, and shown to be more than sufficient.

Section 7 summarizes the results and conclusions.

Appendix A demonstrates the validity of the vortex expansion of sect. 3.

Appendix B discusses the instability of equilibria near the "displaced spherical degeneracies" of the Hill-Wheeler box. Appendix C and D analyze the total velocity fields for the Hill-Wheeler box and the Simple Harmonic Oscillator in the limit $A \rightarrow \infty$. Appendix E details the evaluation of certain sums in the finite- A corrections of the Amado-Brueckner box. Appendix F demonstrates the equivalence of

the equilibrium condition and the equi-distribution of kinetic energy for a whole class of (separable scaling) potentials. Appendix G provides a proof and some discussion of the parallel axis theorem for rotations in the cranking model approximation.[†]

[†] M. G. Mustafa and K. Kumar²⁵⁾ have utilized the parallel axis theorem on the basis of our proof to obtain some results for the overall moment of inertia in a two-center model from cranking model calculations in the two separate wells. The authors appreciate the opportunity to have discussed this theorem and its application to the two-center model with Dr. Mustafa before the publication of their paper. (But, see also the cautionary note in Appendix G, based on our later appreciation of this theorem as applied to non-rigid systems.)

2. The Single Particle Schrödinger Fluid

The philosophy behind the Single Particle Schrödinger Fluid is to extend into the dynamical realm the approach which has yielded such success in the study of nuclear structure: Describe first the motion of the individual nucleons and build upon that the description of the collective properties of the nucleus as a whole. It begins therefore with a single particle in a time-dependent potential well (Cf., the time-independent potential well of the Nuclear Shell Model); and its first contact with the N-body system comprises the sum of N such non-interacting single particles' motions. Paper I has advanced the discussion to this stage¹⁾. For convenient reference to the basic equations, we summarize the formulation of the Schrödinger Fluid which have been discussed in detail there.

2.1. THE FLUID DYNAMICAL EQUATIONS

We write the time-dependent k-th single particle wave function in the polar form

$$\psi_k(\mathbf{r}, \alpha(t), t) = \phi_k(\mathbf{r}, \alpha(t)) \exp\left(-i \frac{M}{\hbar} S_k(\mathbf{r}, \alpha(t)) - \frac{i}{\hbar} \int^t \epsilon_k(\alpha(t')) dt'\right) \quad (2.1)$$

where α represents some time-dependent collective parameters, ϕ and S are real functions and ϕ is assumed (without loss of generality) to be positive.

In the case of rotation, the parameter α becomes the angle of rotation, $\alpha = \Omega t$, where Ω is the angular velocity. Using eq. (2.1), we separate the real and imaginary parts of the time-dependent Schrödinger equation,[†]

$$H(\mathbf{r}, \mathbf{p}; \alpha(t)) \psi(\mathbf{r}, \alpha(t)) = i \hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, \alpha(t)), \quad (2.2)$$

in which Hamiltonian, H , is α -dependent through its potential, and represents

[†]We suppress the single particle label, k , whenever no ambiguity results, since all the quantities here refer to one single particle state.

a nuclear system undergoing a certain collective motion. There results

two equations: first, the continuity equation,

$$\rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = - \frac{\partial \rho}{\partial t}, \quad (2.3)$$

involving a density equal to the probability density of the single particle distribution,

$$\rho = \phi^2, \quad (2.4)$$

an irrotational velocity field

$$\mathbf{v} = \mathbf{v}_\phi \equiv -\nabla S; \quad (2.5)$$

and second, a dynamically modified Schrödinger equation,

$$(H + V_{\text{dyn}}) \phi = \epsilon_1 \phi, \quad (2.6)$$

where the "dynamical modification potential" is related to the velocity field by

$$V_{\text{dyn}} = -\hbar \left(\frac{\partial S}{\partial t} - \frac{1}{2} \mathbf{v}_\phi^2 \right). \quad (2.7)$$

On the basis of eqs. (2.3) to (2.7), we follow Madelung²⁶⁾ and conceptualize the quantum state of the single particle as a dynamical fluid[†].

2.2. COMPRESSIBLE IRROTATIONAL FLOW WITH LINE VORTICES

We note that the fluid dynamical equations (2.3)-(2.7) imply no condition of incompressible flow ($\nabla \cdot \mathbf{v} = 0$). In fact, $\mathbf{v}_\phi$ is generally compressible^{††}.

We have shown in Paper I that an additional incompressible condition upon $\mathbf{v}_\phi$ will seriously limit the applicability of the Schrödinger Fluid^{†††}. Hence, an

[†]Since this early paper, this fluid mechanical interpretation of the Schrödinger theory has recurred frequently. Many references are cited in Paper I.

^{††}For shape vibrations, compressibility is especially evident near single particle level crossings, and provides the dominant contribution to the vibrational inertia. See refs. 4), 27) and 28).

^{†††}Precisely the assumption of such an incompressibility condition constitutes the essential distinction between Hill-Wheeler dynamics⁸⁾, and the more general Single Particle Schrödinger Fluid. See Appendix A of Paper I.

approach of adequate generality must deal with compressible flows.

On the other hand, as implied by eq. (2.5), the velocity field v_ω is irrotational wherever it is differentiable. At any point where v_ω is not differentiable it is singular. Such singularity can occur only where the density vanishes, as is evident from the expression for the irrotational velocity field,

$$v_\omega = -\nabla S = \frac{i\hbar}{2M} \frac{1}{\rho} (\psi \nabla \psi^* - \psi^* \nabla \psi), \quad (2.8)$$

obtained by writing S in terms of ψ , as follows:

$$S = \frac{i\hbar}{2M} \ln(\psi/\psi^*). \quad (2.9)$$

In Paper I, the singularities of v_ω are shown to comprise line vortices (singularities in the circulation distribution) in the fluid. Also, the circulation around any such line vortex is quantized.

2.3. THE VELOCITY FIELD OF RIGID BODY ROTATION

In addition to v_ω , which is a unique object determined from the fluid dynamical equations (2.3)-(2.7), other velocity fields which satisfy only the continuity equation (2.3) of the Schrödinger equations, plus additional a priori conditions, prove to be convenient. Among those of special interests discussed in Paper I, are the *incompressible* velocity field v_ω (satisfying the incompressible condition $\nabla \cdot v_\omega = 0$), the *Regular* velocity field v_R (regular everywhere in finite space), and the *Geometric* velocity field v_G (simultaneously a special case of v_ω and v_R , which is, in addition, the same for every singular particle state). Although these velocity fields do not solve the other Schrödinger fluid dynamical equation (2.6), they can be utilized to express, in a convenient way, some physical quantities, such as the collective kinetic energy (as in eq. (2.26) below).

For rotations, the rigid body velocity field,

$$v_{\text{rig}} = \Omega \times \mathbf{r}, \quad (2.10)$$

is of particular importance. The field v_{rig} is incompressible, trivially, since

$$\nabla \cdot (\Omega \times \mathbf{r}) \equiv 0. \quad (2.11)$$

It is also a regular velocity field, and, indeed, also a geometric velocity field, as we shall see.

We now demonstrate that for a pure collective rotation, v_{rig} satisfies the continuity equation. We consider a pure collective rotation to be a motion in which the density rotates at a uniform angular velocity, Ω , in time, without changing its shape. Then the density distributions $\rho^s(\mathbf{r}, t)$ and $\rho^b(\mathbf{r}')$ (constant in time) observed in the space-fixed coordinate system ($\mathbf{r}$) and the body-fixed coordinate system ($\mathbf{r}'$), respectively, are related by the equation,

$$\rho(\mathbf{r}') = \rho^s(\mathbf{r}, t) \quad (2.12)$$

for every i , where

$$\mathbf{r}' = \mathcal{R}(t)\mathbf{r}. \quad (2.13)$$

Here $\mathcal{R}$ is the rotation transformation matrix, which, for a rotation about the x -axis of (x, y, z) into (x', y', z') , is

$$\mathcal{R}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Omega t & \sin \Omega t \\ 0 & -\sin \Omega t & \cos \Omega t \end{pmatrix}. \quad (2.14)$$

Utilizing eqs. (2.12)-(2.14), we can calculate the time derivative

$$\frac{\partial}{\partial t} \rho^s(\mathbf{r}, t) = \frac{d\mathbf{r}'}{dt} \cdot \nabla' \rho^s(\mathbf{r}') = v_{\text{rig}} \cdot \nabla \rho^s(\mathbf{r}, t). \quad (2.15)$$

Since by (2.11) v_{rig} is incompressible, eq. (2.15) implies that v_{rig} satisfies the continuity equation (2.3).

Besides being incompressible, v_{rig} is also regular in the finite space as is evident in eq. (2.10). Therefore v_{rig} is simultaneously a special case of v_G and of v_R . Because of the fact that ρ^S depends on t only through ξ^R [eq. (2.12)], v_{rig} is also a geometric velocity field v_G . (See Paper I for further discussion of geometric velocity fields.) Hence we have for a pure collective rotation,

$$v_G = v_R = v_G = v_{rig} = \Omega \times r. \quad (2.16)$$

2.4. THE ADIABATIC APPROXIMATION

In the adiabatic limit ($\alpha \rightarrow 0$), the k -th single particle wave function is approximated by

$$\psi_k = u_k + i\mu_k \quad (2.17)$$

where $u_k(r, \alpha)$ is the quasi-static wave function satisfying the quasi-static Schrödinger equation

$$Hu_k = \epsilon_k u_k = \epsilon_k |k\rangle \quad (2.18)$$

and μ_k is the first order time-dependent perturbation correction, equal to

$$\mu_k = \Omega \sum_{j(\neq k)} \frac{\langle j | J_x | k \rangle}{\epsilon_j - \epsilon_k} u_j \quad (2.19)$$

for rotation about the x -axis.

From eq. (2.17), we can obtain the (approximate) fluid dynamical quantities

$$\rho_k = u_k^2 + \mu_k^2 \quad (2.20)$$

$$S_k = \frac{\hbar}{m} \arctan (\mu_k / u_k) \quad (2.21)$$

and

$$v_{Gk} = -\frac{\hbar}{m} \frac{1}{\rho_k} (\mu_k \nabla u_k - u_k \nabla \mu_k). \quad (2.22)$$

As we have shown in Paper I, these quantities satisfy eqs. (2.6) and (2.7) to order α and eq. (2.3) to order α^2 .

2.5. THE ADIABATIC COLLECTIVE KINETIC ENERGY

The collective kinetic energy of a single particle in the nucleus is defined as the α -dependent part of the expectation value $\langle \psi_1 | H | \psi_1 \rangle$. In the adiabatic limit, this collective kinetic energy proportional to $(\dot{\alpha})^2$ and has the value

$$T_1 = \frac{1}{2} \Omega^2 \mathcal{I} \quad (2.23)$$

where $\mathcal{I}$, "the cranking model moment of inertia" is given by Inglis' expression⁹⁾,

$$\mathcal{I} = 2M\hbar^2 \sum_j |\langle j | J_x | 1 \rangle|^2 / (\epsilon_j - \epsilon_1). \quad (2.24)$$

In the theory of the Schrödinger Fluid, we can express the adiabatic collective kinetic energy in terms of the fluid dynamical quantities, (ρ, v_G) , as

$$T_1 = \frac{M}{2} \int \rho_1 v_{G1}^2 d\tau - \frac{\hbar}{2} \int (\phi_1 / u_1) \nabla \cdot (\mu_1 \phi_1 v_{G1}) d\tau. \quad (2.25)$$

The first term of (2.25) is the classical kinetic energy of a compressible irrotational fluid. The second term is a contribution from the singularities of v_{G1} . However, in spite of the fact that v_{G1} possesses singularities, the first term on the right side of (2.25) is always finite.[†] Since the left side of (2.25) is also finite, the second term on the right is also finite. Hence, the expression of (2.25) in terms of v_G does not introduce any divergence.

This collective kinetic energy is greatly simplified when the regular velocity field v_R is introduced by means of the continuity equation. Then (2.25) takes the form

[†] Cf. ref. 1), eq. (4.20).

$$T_1 = \frac{M}{2} \int \rho_1 \mathbf{v}_{\odot 1} \cdot \mathbf{v}_{R1} d\tau. \quad (2.26)$$

Since for rotations $\mathbf{v}_{R1}$ is given by eq. (2.16), we have

$$T_1 = \frac{M}{2} \int \rho_1 \mathbf{v}_{\odot 1} \cdot (\mathbf{R} \times \mathbf{r}) d\tau. \quad (2.27)$$

The collective kinetic energy T for the entire nucleus is a sum of the single particle contributions which reduces to the form,

$$T = \frac{M}{2} \int \rho_T \mathbf{v}_T \cdot (\mathbf{R} \times \mathbf{r}) d\tau, \quad (2.28)$$

where ρ_T is the total density distribution of the nucleus

$$\rho_T = \sum_{i=occ} \rho_i \quad (2.29)$$

and $\mathbf{v}_T$ the total velocity field

$$\mathbf{v}_T = \frac{1}{\rho_T} \sum_{i=occ} \rho_i \mathbf{v}_{\odot i}. \quad (2.30)$$

We emphasize that we consider eq. (2.26) and its extension (2.28) to be remarkable in its simplicity and its symmetry. As an integral of a density-weighted quadratic form in velocities, it conforms to the classical structure of a continuum kinetic energy, but involves two distinct velocity fields, rather than simply the square of a single velocity field. The occurrence of these two distinct velocity fields, we believe, reflects the two essential aspects of the cranking motion: (a) the rotation of the potential well, which can (just as can the density's variation in time for pure rotations) be described by the regular velocity field, $\mathbf{R} \times \mathbf{r}$, and (b) the response of the individual particle to that motion of the potential, described by $\mathbf{v}_{\odot i}$. This structure seems eminently natural, and, perhaps, sufficiently general to encompass a wide variety of kinetic energy forms. Therefore, we suggest that eq. (2.26)-(2.28) might be a fruitful starting point for generalization to cases where the density and the potential obey a more intricate relationship than that which prevails in pure rotations.

Most of the subsequent investigation in this paper is based on expressions (2.27) and (2.28) for the collective rotational kinetic energy.

3. Single Particle Schrödinger Velocity Fields and One Particle Rotational Inertiae

In this section we study the collective rotation of a single particle in a simple harmonic oscillator (SHO) potential and in the infinite square well, rectangular parallelepiped potential (which we refer to as the "Hill-Wheeler box"). We show that the irrotational velocity fields, $\mathbf{v}_{\odot i}$, contain vortices and do not at all resemble the velocity field for rigid body rotation, eq. (2.10), even in the SHO case where the collective kinetic energy for such a particle (at its equilibrium deformation) is equal to the rigid rotation value¹¹⁾,

$$T_{Rot} = \frac{1}{2} \Omega^2 J_{rig} = \langle y^2 + z^2 \rangle \frac{AM\Omega^2}{2}. \quad (3.1)$$

Nor does the total velocity field, $\mathbf{v}_T$, eq. (2.30), equal the rigid body field for more than one particle (or any finite number of particles) in an oscillator potential. For one particle in a Hill-Wheeler box, *neither* the rotational inertia nor the velocity field resemble the rigid body values.

3.1. SINGLE PARTICLE IN THE HARMONIC OSCILLATOR POTENTIAL

3.1.1. Single particle wave functions

The single particle oscillator wave functions are taken to be products of three one-dimensional oscillator functions of the form,

$$u_{lmm}(x, y, z) = u_l(\xi) u_m(\eta) u_n(\zeta) \quad (3.2)$$

where

$$u_n(\zeta) = (2^n n!)^{-1/2} (m\omega_z/\pi\hbar)^{1/4} H_n(\zeta) \exp(-\zeta^2/2). \quad (3.3)$$

Similar equations hold for $u_\xi(\xi)$ and $u_\eta(\eta)$. In (3.3) H_n is the Hermite polynomial, and the dimensionless variables ξ , η and ζ are defined as

$$(\xi, \eta, \zeta) = \left(\sqrt{\frac{m\omega_\xi}{\hbar}} x, \sqrt{\frac{m\omega_\eta}{\hbar}} y, \sqrt{\frac{m\omega_z}{\hbar}} z \right). \quad (3.4)$$

Here, we restrict the discussion to the axially symmetric geometry for simplicity. The intrinsic energy of the single particle state is

$$\epsilon_{lmm} = \hbar\omega_y(l+m+1) + \hbar\omega_z(n+1/2). \quad (3.5)$$

Using eq. (2.19), we can calculate the cranking correction to the wave function explicitly, obtaining

$$\begin{aligned} \mu_{lmm}(x, y, z) &= u_l(\xi) \mu_{mn}(\eta, \zeta) \\ &= -\frac{\zeta}{2\sqrt{\omega_y\omega_z}} u_l \left\{ \gamma\sqrt{mn} u_{m-1} u_{n-1} + \frac{1}{\gamma} \sqrt{m(n+1)} u_{m-1} u_{n+1} \right. \\ &\quad \left. + \frac{1}{\gamma} \sqrt{(m+1)n} u_{m+1} u_{n-1} + \gamma\sqrt{(m+1)(n+1)} u_{m+1} u_{n+1} \right\}. \end{aligned} \quad (3.6)$$

Here, the functions with subscripts l , m and n are of arguments ξ , η , and ζ , respectively, and

$$\gamma = \frac{\omega_y - \omega_z}{\omega_y + \omega_z} \quad (3.7)$$

is a measure of the deformation of the potential.

3.1.2. The irrotational velocity field for SHO states

We now calculate the irrotational velocity fields, $v_{\otimes}(lmm)$, from eq. (2.22), and separate it into two parts, as follows,

$$v_{\otimes}(lmm) = v_I + v'_{\otimes}(lmm). \quad (3.8)$$

Here v_I , the same for all particles, describes the irrotational incompressible flow of a liquid drop with a fixed ellipsoidal shape, which rotates about the

x-axis with angular velocity Ω :

$$v_I = -\Omega \gamma (z e_y^x + y e_z^x). \quad (3.9)$$

This velocity field is shown in fig. 1.

The remainder is

$$\begin{aligned} v'_{\otimes}(lmm) &= \Omega \frac{1-\gamma}{\gamma} (z^{n+m} m! n!)^{-1} (u_{l00}^2 / \rho_{lmm}) \\ &\quad \cdot \left[z \left(m H_n^2 H_n^2 - m H_{m-1} H_{m+1} H_n^2 - n(1+2m) \frac{1}{\zeta} H_m^2 H_{n-1} \right. \right. \\ &\quad \left. \left. + 2m \frac{1}{\zeta} H_{m-1} H_{m+1} H_{n-1} H_n \right) e_y^x + y [m \leftrightarrow n, n \leftrightarrow \zeta] e_z^x \right], \end{aligned} \quad (3.10)$$

$$\text{where } \rho_{lmm} = u_{l00}^2 + \rho_{lmm}'.$$

The velocity field, $v'_{\otimes}$, varies (via m and n) from one single particle state (l, m, n) to another. In particular, $v'_{\otimes}$ vanishes identically for single particle states with $m=n=0$. In such cases, therefore, the irrotational incompressible[†] velocity field provides the complete description:

$$v_{\otimes}(l00) = v_I. \quad (3.11)$$

This is the SHO special case of the more general result that irrotational (and incompressible) flow always results for nodeless wave functions, first demonstrated by Wick²⁹⁾, and sometimes in the early days of the collective model expected⁷⁻¹⁰⁾, erroneously, to apply to all wave functions.

For single particle states whose quantum numbers m and n are nonzero, $v'_{\otimes}$ is finite. We show in figs. 2 and 3 the velocity field $v_{\otimes}$ for $|l01\rangle$ and $|l12\rangle$, respectively. These figures show interesting vortex structures for the single particle velocity fields.

[†]Frequently in the literature of the nuclear collective model, the term "irrotational" has been used in such a way as to include implicitly also the "incompressible" assumption. See the remarks of sect. 2.2 and refs. 7), 8) and 11).

Since v_1 is regular, the vortices in the velocity field must come from $v'_{\theta}(z_{mn})$. In fact one can always separate v'_{θ} into a sum of vortex terms, as follows:

$$v'_{\theta} = \sum_k^N v_{\text{vort},k} = \frac{1-\gamma}{\gamma} (u_{\theta 0}^2 / \rho_{\theta mn}) \sum_k^N f_k(n, \zeta) [-(z - \bar{z}_k) \hat{e}_y + (y - \bar{y}_k) \hat{e}_z], \quad (3.12)$$

where f_k are regular functions and N is the total number of vortices in v_{θ} . In this equation, each $v_{\text{vort},k}$ contains one compressible line vortex which is parallel to the x -axis (the axis of rotation) and is located at the point $(y, z) = (\bar{y}_k, \bar{z}_k)$ where the density $\rho_{\theta mn}$ goes to zero. The streamlines of $v_{\text{vort},k}$ are circular with their centers at the line vortex.

For the one- and two-vortex cases, the separation (3.12) can be done explicitly:

$$v'_{\theta}(z_{01}) = \frac{1-\gamma}{\gamma} (u_{\theta 0}^2 / \rho_{\theta 01}) (-z \hat{e}_y + y \hat{e}_z) \quad (3.13)$$

$$v'_{\theta}(z_{02}) = \frac{1-\gamma}{\gamma} (u_{\theta 0}^2 / \rho_{\theta 02}) \left\{ \left(\zeta + \frac{1}{\sqrt{2}} \right)^2 [-(z - \bar{z}) \hat{e}_y + y \hat{e}_z] + \left(\zeta - \frac{1}{\sqrt{2}} \right)^2 [-(z + \bar{z}) \hat{e}_y + y \hat{e}_z] \right\} \quad (3.14)$$

with $\bar{z} = (\hbar/2m\omega_z)^{1/2}$. For cases with three or more vortices, eq. (3.12) is proven in Appendix A.

3.1.3. Rigid (equilibrium-) oscillator moments of inertia from fluid dynamical viewpoint

We now examine the cranking moment of inertia in terms of the velocity fields. Bohr and Mottelson¹¹⁾ show that for the simple Harmonic Oscillator case, at the equilibrium deformation, where

$$\frac{d}{dy} \sum_{i=1}^A \epsilon_{l_1 m_1 n_1} = 0, \quad (3.15)$$

the cranking moment of inertia is identically equal to the rigid moment of inertia:

$$\mathcal{J} = \mathcal{J}_{\text{rig}} \equiv \sum_{i=1}^A M \langle y_1^2 + z_1^2 \rangle. \quad (3.16)$$

In terms of expression (2.28) involving the velocity fields, this result asserts the equality of the collective kinetic energy of the Schrödinger Fluid and that of a rigidly rotating classical fluid:

$$\frac{M}{2} \int \rho_T v_T \cdot (\hat{\Omega} \times \hat{r}) d\tau = \frac{1}{2} \mathcal{J}_{\text{rig}} \Omega^2 \equiv \frac{M}{2} \int \rho_T (\hat{\Omega} \times \hat{r})^2 d\tau, \quad (3.17)$$

at the equilibrium deformation. We emphasize that eqs. (3.16) and (3.17) hold for any number of nucleons occupying any set of single particle SHO states at the deformation defined by the equilibrium condition (3.15). In particular, it holds for a one particle state. For this case, eq. (3.17) becomes

$$\frac{M}{2} \int \rho_1 v_{\theta 1} \cdot (\hat{\Omega} \times \hat{r}) d\tau = \frac{M}{2} \int \rho_1 (\hat{\Omega} \times \hat{r})^2 d\tau \quad (3.18)$$

at the equilibrium deformation of the single particle state $|1\rangle \equiv |z_{mn}\rangle$.

Equation (3.18) is a remarkable identity. The scalar product of $v_{\theta 1}$ and $\hat{\Omega} \times \hat{r}$ which occurs on the left side is replaced on the right side, by the absolute square of $\hat{\Omega} \times \hat{r}$. It forces one to inquire whether the irrotational field $v_{\theta 1}$ is equal to $\hat{\Omega} \times \hat{r}$. The answer, of course, is "No." For, as we have discussed above [Cf., eqs. (3.8) and (3.10) and figs. 1 to 3], $v_{\theta 1}$ possesses compressible line vortices. It can, therefore, never be equal to the velocity field for rigid rotation $v_{\text{rig}} = \hat{\Omega} \times \hat{r}$, which has no singularity and is everywhere incompressible, and rotational. Despite this qualitative difference between $v_{\theta 1}$ and v_{rig} , eq. (3.18) shows that, as regards their effects under the integral upon the overall kinetic energy (or the inertial parameter), these two velocity fields are equivalent at the equilibrium deformation.

3.1.4. Cranked moments, rigid moments and spherical moments

We note that the cranking moment of inertia $\mathcal{J}$ and the rigid moment of inertia $\mathcal{J}_{\text{rig}}$ are equal only when the SHO well is at the equilibrium deformation. At other deformations, they can, and do, deviate substantially from one another, as we now discuss.

Figure 4 exhibits the functional dependence of $\mathcal{J}$ (as calculated from the cranking model) and of $\mathcal{J}_{\text{rig}}$. Figures 4a and 4b show, as a function of q , the following expressions for $\mathcal{J}$ and $\mathcal{J}_{\text{rig}}$:

$$\mathcal{J} = \frac{E}{\omega_0} \frac{1}{6+2\gamma} \left(\frac{1+\gamma}{1-\gamma} \right)^{1/3} \left[\gamma^2 (1+q) + \frac{1}{\gamma} (1-q) \right] \quad (3.19)$$

and

$$\mathcal{J}_{\text{rig}} = \frac{E}{\omega_0} \frac{1}{6+2\gamma} \left(\frac{1+\gamma}{1-\gamma} \right)^{1/3} [(1+q) + \gamma(1-q)], \quad (3.20)$$

where E is the total single particle energy,

$$E = \sum_{\text{occ}} [2\hbar\omega_y (m+1/2) + \hbar\omega_z (n+1/2)], \quad (3.21)$$

and q , the ratio of the summed single particle quanta in the y and z directions,

$$q \equiv \frac{\sum_{\text{occ}} (m+1/2)}{\sum_{\text{occ}} (n+1/2)}. \quad (3.22)$$

For $q \neq 1$, the last term of $\mathcal{J}$, eq. (3.19), tends to infinity when the SHO well tends to the spherical shape ($\gamma \rightarrow 0$), no matter how small the deviation, $q - 1$. In contrast, $\mathcal{J}_{\text{rig}}$ is finite at $\gamma = 0$.

The infinite value of $\mathcal{J}$ at $\gamma = 0$ arises in the cranking formula (2.24) as the effect from the degeneracies of the states $|l, m, n\rangle$, $|l, m-1, n+1\rangle$ and $|l, m+1, n-1\rangle$ at the spherical shape.[†] They introduce infinite contributions to $\mathcal{J}$ as the

[†]Analogous degeneracies lead to large contributions to the moment for the Hill-Wheeler box, for which no spherical shape occurs. They lead us to believe that the sharp edges of the Hill-Wheeler (and the Amado-Brueckner) box are the source of the substantial fluctuation in the moment from one A value to another. See sects. 5, 6 and Appendix E.

corresponding energy denominators diminish towards zero. In terms of the single particle Schrödinger Fluid, we can see from eq. (3.12) that the strength of each vortex field $y_{\text{vort},k}$ tends to infinity as $\gamma \rightarrow 0$, implying infinite collective kinetic energy, for fixed rotational frequency, Ω , and an infinite moment of inertia.

Any approach of the system to a spherical shape can be portrayed as a curve in the (q, γ) space approaching the point $(1, 0)$. The "equilibrium" choice of this curve, at which $q = (1-\gamma)/(1+\gamma)$ by (3.15), yields for the moment represented by the dashed curve labelled $\mathcal{J}_e$, in figs. 4a, 4b. This particular curve is the trajectory of the point of intersection between the two families of curves $\mathcal{J}$ (in fig. 4a) and $\mathcal{J}_{\text{rig}}$ [in fig. 4b]. It shows that so long as the single particle levels are not isotropically occupied, ($q \neq 1$), no matter how small the anisotropy, the corresponding curves $\mathcal{J}$ and $\mathcal{J}_{\text{rig}}$ always intersect at a finite value of $\mathcal{J}$. That is, $\mathcal{J}$ is equal to the rigid value at the equilibrium deformation for any anisotropic configuration of the particles in the HO well. This is shown in fig. 4 by the fact that $\mathcal{J}_e$ tends to the finite rigid body value (marked by an open circle) as $\gamma \rightarrow 0$.

When the particle configuration is isotropic[†] ($q \equiv 1$), the value of $\mathcal{J}$ in (3.19) is indeterminate at $\gamma = 0$, but approaches the value zero^{††} as deformation diminishes from a finite value to its equilibrium value $\gamma = 0$. We note that for a spherical configuration, $q = 1$, the rotation operator, $\exp(i\mathcal{L}_x)$, can effect only the identity transformation upon the wave function and the probability density. As discussed in Paper I such an identical transformation can give rise to no

[†]Note that this condition eliminates the infinite terms in the cranking formula since all of the degenerate states are occupied.

^{††}This was the first result obtained from Inglis' Cranking Formula⁹⁾. It (falsely) reinforced the then prevailing expectation^{7), 8), 10)} that the irrotational moment was the correct inertia for nuclear rotation.

physical current. Therefore, only the value zero for the moment of inertia can result from the cranking of such a spherical configuration. This physics is reflected in the fact that $\mathcal{J}$ approaches the value zero as $\gamma \rightarrow 0$ along the $q = 1$ curve.

Clearly, any limiting value may be realized mathematically for the moment $\mathcal{J}$ at $\gamma = 0$ by appropriate selection of the approach to the limit (and therefore at the limiting value of the term, $(1-q)/\gamma$, in (3.19)), which is indeterminate at $q = 1$, $\gamma = 0$. The two limits of special physical interest and denoted in fig. 4a for $\gamma = 0$ by the open and closed points, correspond respectively to the rigid spherical value and zero value already discussed. They alone correspond to combinations of configuration and shape which honor the equilibrium condition. Here the exceptionality of configurations which are precisely spherically symmetric, and their non-continuity with equilibria of non-spherical configurations is especially to be noted.

3.2. MOMENTS OF INERTIA FOR ONE PARTICLE IN THE HILL-WHEELER BOX

3.2.1. Moments at equilibrium deformation

We consider the rotation of a Hill-Wheeler box⁸⁾, an infinite square well potential with the shape of a rectangular parallelepiped, with edges of length, L_x , L_y and L_z . Volume conservation implies

$$L_x L_y L_z = V, \quad (3.23)$$

and we consider only shapes with square cross section, $L_x = L_y$, for simplicity. Then the shape is conveniently characterized by a single deformation parameter,

$$\beta = L_z/L_y. \quad (3.24)$$

The single particle wave functions u_{lm} are sinusoidal functions with $l = 1$, $m = 1$ and $n = 1$ nodes in the x -, y - and z -directions, respectively.

The cranking moment of inertia (2.24) can be calculated directly with the single particle wave functions of the Hill-Wheeler box. The required matrix elements are

$$\langle l'm'n' | \mathcal{L}_x | lmn \rangle = \begin{cases} i\hbar\beta \frac{32}{\pi^2} \frac{m'm'n'n}{(m'^2 - m^2)(n'^2 - n^2)} \left[\frac{1}{\beta^2(m'^2 - m^2)} - \frac{1}{n'^2 - n^2} \right] \delta_{l'l} & \text{for } m' + m \text{ and } n' + n \text{ odd} \\ 0, & \text{otherwise.} \end{cases} \quad (3.25)$$

Then eq. (2.24) gives, for a single particle state (lmn) ,

$$\mathcal{J}_{lmn} = ML_z^2 \frac{4}{\pi^2} \frac{6}{m^2 n^2} \beta^{-2} \sum_{\substack{m'n' \\ (m'+m=\text{odd}) \\ (n'+n=\text{odd})}} \frac{m'^2 n'^2 [\beta^2(m'^2 - m^2) - (n'^2 - n^2)]^2}{(m'^2 - m^2)^4 (n'^2 - n^2)^4 [\beta^2(m'^2 - m^2) + (n'^2 - n^2)]}. \quad (3.26)$$

This contribution to the moment of inertia is a function of β , which can be fixed by the equilibrium condition, that for the sum of the particle energies have its minimum value. For a nucleus containing just one particle in state $|lmn\rangle$, the equilibrium condition

$$\frac{d}{d\beta} \epsilon_{lmn} = 0 \quad (3.27)$$

implies the equilibrium deformation,

$$\beta_0 = n/m. \quad (3.28)$$

We have evaluated (3.26), and the rigid body moment defined in (3.16)

$$\mathcal{J}_{\text{rig}}^{\text{HW}}(m, m, n) = ML_z^2 [n^2(m^2 + n^2) - 12]/(12\pi^2 n^2), \quad (3.29)$$

using the density distribution of the state (m, m, n) at its equilibrium shape. The values of the moment $\mathcal{J}_{lmn}$, its ratio to rigid value, and of the least energy denominator which enters are listed in Table 1. Because

under the interchange of m and n one finds

$$J_{mn} = (m^2/n^2) \cdot J_{nm} \quad (3.30)$$

for both the Hill-Wheeler and the rigid moments, only the values for $n \geq m$ are given in Table 1. The results of Table 1 demonstrate that a single particle in the Hill-Wheeler box, in marked contrast to the anisotropic simple harmonic oscillator (for which the first equality in eq. (3.16) always holds at the equilibrium shape) can have, even at its equilibrium deformation, cranking moment of inertia quite different in magnitude from the rigid body moment. Moreover, the ratio J/J_{rig} may be either much larger or much smaller than unity, depending upon the state in question. Indeed, as many entries in Table 1 show, it is often negative. We turn now to a detailed consideration of these far-from rigid moments, both large and small.

3.2.2. Nodeless wave function has small moment

In qualitative accord with the discussion in sect. 3.2, where it was seen that the velocity field for a nodeless oscillator wave function has no vortices and therefore exhibits an incompressible, irrotational flow, the (111) wave function in the Hill-Wheeler box exhibits a moment much smaller than the rigid body value. We consider this case to be exceptional in Table 1. Other $l = m = n$ moments are generally comparable to the rigid value. (Cf., also, ref. 29.)

3.2.3. Strongest variation moments follow smallest energy denominator

The strongest variations of the moments in Table 1 follow quite directly the behavior of the smallest of the energy denominators in eqs. (2.19) and (3.26),

$$d_b^a = [\epsilon_{m,m+a,n+b} - \epsilon_{nmn}]_{\beta=(n/m)} \quad (3.31)$$

In particular d_{-1}^{+1} is very small ($\approx +1/m^2$), when $n = m+1$, and becomes negative at $n = m+2$. The result is very large moments for states with $n = m+1$, and (except for $m=1$) negative moments for states $n > m+1$. For larger n , a transition back to positive values occurs as the denominator d_{-1}^{+3} becomes comparable (within a factor of 2: items ** in Table 1) with d_{-1}^{+1} , and then for still larger n , much smaller (items * in Table 1).

3.2.4. Quantitative treatment may shift minimum obviating excessive moment

Although the presence of small energy denominator provides a good basis for understanding large contributions to the moment, a physically correct treatment must take care to treat these large terms accurately. Such accuracy requires that the cranking formula be generalized to handle the degenerate perturbation theory which is required near a level crossing. When the levels come too close, consistency requires one first to diagonalize the perturbation operator Ω_x^Q in the degenerate subspace of the two nearly degenerate states. Such a treatment is formulated in Appendix B. We recount only the results here for the special case depicted in fig. 5.

When Ω_x^Q is so diagonalized the eigenvalues arising from (223) and (214) develop a splitting proportional to the strength, Ω , of the perturbation. Since the minimum of the unperturbed (223) level lies near to the crossing point, it may be substantially shifted (or may disappear altogether) in this diagonalization process. Then the single particle state in question will no longer be in equilibrium at the deformation corresponding to the minimum of the unperturbed level, but will shift to a new equilibrium deformation (in this case at a much elongated shape defined by the minimum of the level (214)).

In this way a critical angular velocity, Ω_{crit} , enters the description, corresponding to the least angular velocity at which the relevant energy

minimum disappears. This critical Ω_{crit} quantifies the stability of the deformation against rotation: When the angular velocity is less than Ω_{crit} the deformation is stable, otherwise it is unstable.

But because the angular momentum is quantized, arbitrarily small finite angular velocities are not allowed in actual physical states. The lowest possible nonzero angular velocity is of order

$$\Omega_{\text{min}} = \hbar/3. \quad (3.32)$$

In Appendix B, we show that when Ω_{min} is estimated from (3.32) with the rigid moment, we obtain $\Omega_{\text{min}}/\Omega_{\text{crit}} \approx 10^3$ for the state $|223\rangle$. Hence, we conclude that the deformation determined by applying the equilibrium condition to the unperturbed (223) level is unstable against any quantized physical rotation. The large cranking moment of inertia for this state is therefore unphysical, in the sense that it can never be expected to be manifest in any actual experimental situation.

Thus, we conclude that the very large moments for a single particle with quantum numbers $(m, m, m \pm 1)$ in the Hill-Wheeler box are not of practical concern for their (gedanken) experimental rotational levels. On the other hand, as counter examples for conjectured conditions for rigid moments they demand concern for their place in the logical structure of the theory. We address this question in sect. 6.

4. Rotation of an Infinite Nucleus

We turn now to the study of the collective rotation in the Limit $A \rightarrow \infty$.

We consider the matter flow in the nucleus for three distinct independent particle situations: (a) the uniform potential, with periodic boundary conditions applied on the surfaces of a rectangular parallelepiped box (the Amado-Brueckner box), (b) the Hill-Wheeler box (infinite square well parallelepiped box with $\psi = 0$ boundary condition), and (c) the harmonic oscillator potential.

Amado and Brueckner²²⁾, and Rockmore²³⁾ have shown that the moment of inertia is equal to the rigid value for one special case of (a), the cube[†]. For case (c), we have already mentioned in sect. 3 that the moment of inertia is equal to the rigid value when the nucleus is at equilibrium deformation regardless whether A is finite or infinite. On the other hand, we also see in sect. 3 that for a single particle in the harmonic oscillator potential, the rigid moment does not imply that the matter flow inside the nucleus follows the velocity field of rigid body rotation (2.10). The inquiry into the matter flow in the infinite system is therefore not a trivial one, even though the moment of inertia for cases of (a) and (c) is already known to be equal to the rigid value (and will be shown below to be rigid also for case (b)).

One expects, as compared with the one particle results, the total velocity field for the many particle system to be a much smoother function of position, and to become more and more smooth as increasing A introduces more and more vortices of the single particle velocity fields into the average, eq. (2.30).

[†] A theorem to be presented in sect. 5.1 implies that this result obtained from the cube guarantees also that the moment is rigid for any other (equilibrium) shape of the parallelepiped.

which defines the total velocity field. One thus has a basis to expect, that in spite of the singular vortex structure of the single particle velocity fields, the total velocity v_T in the limit $A \rightarrow \infty$, could become a smooth function. This is indeed the case for all three independent particle potentials (a), (b) and (c). Indeed, we shall show that in each case the total velocity field tends to the velocity field (2.10) for the rigid body rotation, in the limit $A \rightarrow \infty$, at the equilibrium deformation.

For simplicity, we here evaluate the total velocity field only for the Amado-Brueckner box. The evaluations for the Hill-Wheeler box and the harmonic oscillator potential[†] are given in Appendices B and C, respectively.

4.1. AMADO-BRUECKNER BOX

4.1.1. Single particle states, the Fermi spheroid and the equilibrium condition

We use the plane wave functions normalized in the volume $V = L_x L_y L_z$,

$$u_{lmn} = V^{-1/2} \exp \left[2\pi i \left(\frac{lx}{L_x} + \frac{my}{L_y} + \frac{nz}{L_z} \right) \right], \quad (4.1)$$

as convenient eigenfunctions of the Amado-Brueckner box. Here the quantum numbers l , m and n take both positive and negative integral values. The wave functions in (4.1) satisfy the periodic boundary condition, rather than the condition that they vanish at the surface. As did Amado and Brueckner, we expect an intuitive physical basis that this deviation from a more realistic (with the wave function vanishing on the boundary) boundary condition will not grossly

[†]M. Radomski³⁰⁾ first obtained this result by a somewhat different method.

affect the values computed for extensive quantities like the moment of inertia[†].

The single particle energies are

$$e_{lmn} = \frac{2\pi^2 \hbar^2}{M} \left(\frac{l^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2} \right). \quad (4.2)$$

As before, we assume $L_x = L_y$, and use $\beta = L_z/L_y$ to describe the deformation of the box.

The occupied single particle states are taken to be those whose quantum numbers lie inside the Fermi spheroid

$$\frac{l^2 + m^2}{\lambda^2/b^2} + \frac{n^2}{\lambda^2} = 1. \quad (4.3)$$

The quantity b gives the shape of the Fermi spheroid as the ratio between its two semi-axes. It is also equal to the ratio of the root mean square values of the quantum numbers l and m within the Fermi spheroid; i.e.,

$$b^2 = \sum_{lmn}^{\text{occ}} \frac{l^2}{\sum_{lmn}^{\text{occ}} l^2} = \sum_{lmn}^{\text{occ}} \frac{m^2}{\sum_{lmn}^{\text{occ}} m^2}. \quad (4.4)$$

The quantity λ measures the size of the Fermi spheroid and is related to the total number of particles A of the system by

$$A = \frac{4}{3} \pi \lambda^3 b^{-2}. \quad (4.5)$$

In order to study the system at arbitrary deformations, we keep the shape of the Fermi spheroid, b , independent of the deformation, β , of the spatial box. Thus, in general, our Fermi spheroid does not enclose the lowest single particle states at a given deformation. On the other hand, the imposition of the equilibrium condition,

[†]This expectation is supported by the general similarity between the exact numerical results for the Hill-Wheeler box and for the Amado-Brueckner box, shown in figs. 6 and 7.

$$\frac{d}{d\beta} \sum_{lmn}^{\text{occ}} \epsilon_{lmn} = 0, \quad (4.6)$$

requires the box to assume the shape at which the occupied Fermi spheroid specifies the lowest energy configuration. According to (4.2) and (4.6), this occurs when the equilibrium deformation, β_0 , has just the value

$$\beta_0 = \left[\sum_{lmn}^{\text{occ}} n^2 / \sum_{lmn}^{\text{occ}} m^2 \right]^{1/2} = b \quad (4.7)$$

by (4.4).

4.1.2. The single particle currents

The matrix elements of the angular momentum $\mathcal{L}_x = i\hbar(z\partial/\partial y - y\partial/\partial z)$, are calculated directly using the wave functions (4.1), with the result

$$\langle l'm'n' | \mathcal{L}_x | lmn \rangle = i\hbar \delta_{ll'} \left[\frac{1}{3} \frac{n}{m'-m} (-1)^{m-m'} \delta_{nn'} (1 - \delta_{mm'}) - \frac{m}{n'-n} (-1)^{n-n'} \delta_{mm'} (1 - \delta_{nn'}) \right]. \quad (4.8)$$

With these values, we obtain μ_{lmn} according to (2.19) as

$$\mu_{lmn} = \frac{ML^2}{2\pi\hbar} \left[\beta^{-3} \sum_{m'} (-1)^{m-m'} \frac{n}{(m'-m)(m'^2-m^2)} u_{lm'n} - \beta \sum_{n'} (-1)^{n-n'} \frac{m}{(n'-n)(n'^2-n^2)} u_{lmn'} \right]. \quad (4.9)$$

We next calculate the collective current [$i \equiv (lmn)$]

$$J_i = \rho_1 \nabla_{\theta i} = -\frac{\hbar}{M} (\mu_i \nabla u_i - u_i \nabla \mu_i) \quad (4.10)$$

from eq. (2.22). We note that this current vanishes for $\Omega = 0$ since each term in (4.10) is proportional to μ_i and, therefore, proportional to Ω , by eq. (4.9).

Hence, we refer to J_i as the "collective" current, in contrast to the full current, defined by

$$J = (i\hbar/2M) (\psi_1 \nabla \psi_1^* - \psi_1^* \nabla \psi_1). \quad (4.11)$$

(Since we are here using the propagating waves (4.1) with periodic boundary conditions as a basis the full current need not vanish[†], even in the absence of rotation.)

From eqs. (4.1) and (4.9), we obtain at any shape, β ,

$$J_{lmn} = -\frac{\Omega}{\pi V} \{ L_z [\beta^{-2} n A_m(\eta) - \beta^2 m A_n(\zeta)] e_{\eta x} + L_z [\beta^{-2} B_m(\eta) - \beta^2 m^2 A_n(\zeta)] e_{\eta y} - L_y [\beta^2 B_n(\zeta) - \beta^{-2} n^2 A_m(\eta)] e_{\eta z} \}. \quad (4.12)$$

Here, we have introduced the notation

$$(\xi, \eta, \zeta) = \left(\frac{\pi}{L_x} x, \frac{\pi}{L_y} y, \frac{\pi}{L_z} z \right), \quad (4.13)$$

and the functions A_k and B_k are given by

$$A_k(x) \equiv \sum_{k'} \frac{2(-1)^{k'-k}}{(k'-k)^2 (k'+k)} \sin 2(k'-k)x \quad (4.14)$$

and

$$B_k(x) \equiv \sum_{k'} \frac{k(-1)^{k'-k}}{(k'-k)^2} \sin 2(k'-k)x. \quad (4.15)$$

4.1.3. The total current

We compute the total current of the nucleus by summation of J_{lmn} over all occupied states inside the Fermi spheroid (4.3). Note first that the x-component of J_{lmn} is odd in l , therefore vanishes under the (symmetric) summation over l .

[†]But in fact, J will vanish in an even-even ground state in which positive and negative integers are equally occupied, and ought to be made to vanish by appropriate choice of zero order wave function, even for the odd-A case.

There are two terms in (4.12), which involve B_k , $k = m$ or n . Since each summand in B_k is odd in k and $k' - k$ [see eq. (4.15)], these two terms give no contribution to the total current when both k and k' are summed over symmetrically.

Then the total current J_T becomes

$$J_T \equiv \sum_{l,m,n} J_{l,m,n} = \frac{\Omega}{\pi V} \left\{ L_z \beta^2 \sum_{l,m,n} m^2 A_m(\zeta) e_{y,l} - L_y \beta^{-2} \sum_{l,m,n} n^2 A_n(\zeta) e_{z,l} \right\}. \quad (4.16)$$

Consider the sum,

$$F(\zeta) \equiv \sum_{l,m,n} m^2 A_n(\zeta) = \sum_{l,m,n} 2m^2 \sum_{n'=-\bar{n}}^{\bar{n}} \int_{n'}^{\bar{n}} \frac{(-1)^{n'-n}}{(n'-n)^2 (n'+n)} \sin 2(n'-n)\zeta, \quad (4.17)$$

where $\bar{n}$ is the maximum value of n , and $\bar{n}$ is determined by the Fermi spheroid

(4.3) as

$$\bar{n} = [\lambda^2 - b^2(\beta^2 + m^2)]^{1/2}. \quad (4.18)$$

In eq. (4.17), since the summand is odd in $n' - n$, the terms for which both n and n' take on the values between $-\bar{n}$ and $\bar{n}$ cancel one another. On the other hand, because of the denominator $(n' - n)^2$, the major contribution to $F(\zeta)$ comes from the terms where n' is close to n . Such terms can occur when $|n|$ and $|n'|$ are close to $\bar{n}$. Therefore, we can approximate $n' + n = 2\bar{n}$. The sum (4.17) can be put into a more convenient form by introducing the new summation variable, $p = n' - n$, and making use of the symmetry of contributions from positive and negative n . We find

$$F(\zeta) = 4 \sum_{l,m} m^2 \left\{ \sum_{p=1}^{\infty} \sum_{n=\bar{n}-p}^{\bar{n}} - \sum_{p=2\bar{n}+1}^{\infty} \sum_{n=\bar{n}-p}^{\bar{n}} \right\} \frac{(-1)^p}{p^2 2\bar{n}} \sin 2p\zeta. \quad (4.19)$$

4.1.4 Total current for infinite nucleus

As the number of nucleons A tends to infinity, $\bar{n}$ becomes large. As a result, the second term in (4.19) becomes vanishingly small and can be dropped for the case of an infinite nucleus. (This second term contributes for a large, but finite, nucleus. The contribution of this term to the moment of inertia is discussed in sect. 5.) We can then evaluate $F(\zeta)$ as

$$\begin{aligned} F(\zeta) &\xrightarrow{\bar{n} \rightarrow \infty} 2 \sum_{l,m} m^2 \sum_{p=1}^{\infty} \sum_{n=\bar{n}-p}^{\bar{n}} \frac{(-1)^p}{p^2 \bar{n}} \sin 2p\zeta = 2 \sum_{l,m} \frac{m^2}{\bar{n}} \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \sin 2p\zeta \\ &= -\frac{2\pi}{L_z} z \sum_{l,m} \frac{m^2}{\bar{n}}. \end{aligned} \quad (4.20)$$

In obtaining the last line in this equation, a Fourier representation for the function $f(z) = z$ in the interval $(-1/2, 1/2)$ was utilized. The remaining sum in (4.20) has to be carried out for the single particle states inside the Fermi spheroid (4.3). It can be executed by replacing the sum by an integral. We find

$$\sum_{l,m} \frac{m^2}{\bar{n}} = \frac{2}{3} \pi \lambda^3 \frac{1}{b^4} = \frac{A}{2b^2}. \quad (4.21)$$

Therefore, we obtain the y -component of the current J_T for the infinite nucleus as

$$\frac{\Omega}{\pi V} L_z \beta^2 F_y(\zeta) = -\Omega \rho_T \left(\frac{b}{b}\right)^2 z, \quad (4.22)$$

where $\rho_T = A/V$ is the total matter density of the nucleus.

The z -component of J_T can be calculated in exactly the same way, except that instead of the sum in eq. (4.21), we have $\sum_{l,m} n^2/\bar{m}$ where $\bar{m}$ is the value of m at the Fermi spheroid (4.3), and is equal to

$$\bar{m} = [b^2(\lambda^2 - n^2) - \ell^2]^{1/2}. \quad (4.23)$$

The sum can be carried out in a similar way as in (4.21), yielding

$$\sum_{l=0}^{\infty} \frac{n^2}{m} = \frac{2}{3} \pi \lambda^3 = \frac{1}{2} b^2 A. \quad (4.24)$$

Summing up these results, we obtain the total current for the infinite nucleus as:

$$J_T \xrightarrow{A \rightarrow \infty} \Omega \rho_T \left[-\left(\frac{b}{b}\right)^2 z e_y + \left(\frac{b}{b}\right)^2 y e_z \right]. \quad (4.25)$$

It implies a total velocity field as

$$v_T \xrightarrow{A \rightarrow \infty} \Omega \rho_T \left[-\left(\frac{b}{b}\right)^2 z e_y + \left(\frac{b}{b}\right)^2 y e_z \right]. \quad (4.26)$$

At equilibrium (4.7) when $\beta = \beta_0 = b$,

$$v_T \xrightarrow{A \rightarrow \infty} \Omega (-z e_y + y e_z) = v_{rig} = \Omega \times r. \quad (4.27)$$

Thus the total velocity field for the $A \rightarrow \infty$ Amado-Brueckner box is identical to the velocity field for rigid-body rotation (2.10) at the equilibrium shape (4.7).

Exactly the same result is established in Appendices C and D for the Hill-Wheeler box and the Simple Harmonic Oscillator potential.

5. Moments of Inertia for Large but Finite Nuclei

We have shown that for the Amado-Brueckner box, the Hill-Wheeler box and the SHO potential, the total velocity field for the nucleus at the equilibrium deformation tends to the velocity field (2.10) for rigid body rotation as the number of particles A tends to infinity. Then from the kinetic energy expression eq. (2.28) of the theory of Independent Particle Schrödinger Fluid, it follows directly that the moment of inertia for such a nucleus tends to the rigid body value $\mathcal{I}_{rig}$ as $A \rightarrow \infty$.

On the other hand, we have seen in sect. 2 that the moment of inertia for each single particle state can be very different from the rigid value. The question naturally arises how the two cases, of small A and $\lim(A \rightarrow \infty)$, are connected. We study in this section the moment of inertia for case of large, but finite A . We first evaluate the moment of inertia *exactly* for the Amado-Brueckner box and the Hill-Wheeler box. Then, an approximate analytic formula for large, but finite, A is derived for the Amado-Brueckner case. It consists of an average value, which depends smoothly on A , and a sawtooth fluctuation term which arises from the discreteness of the terms in the sum which describes the moment.

We consider only cubical shapes in this section, for simplicity. But for large A the result for the smooth average moment of a cube defines also the smooth average moment for other ground state deformations of the Amado-Brueckner and the Hill-Wheeler box, as we now demonstrate.

Consider the moment of inertia of an A -particle box of any shape, β . It is given by (cf., eq. (3.26)) by a sum of the form,

$$\bar{J} = L_z^2 \sum_{\substack{\ell, m, n \\ ((\ell^2 + m^2) + n^2 \leq \lambda^2)}} f(\beta \ell, \beta m, n), \quad (5.1)$$

where β is the deformation parameter of the box at equilibrium.

If we consider the cubical configuration of A-particles whose equilibrium deformation is $\beta_0 = 1$, then its (smooth average) moment of inertia $\bar{J}_0$ is related to the smooth average of $\bar{J}$, $\bar{J}$, by the following equality,

$$\bar{J}/(L_z^2 A) = \bar{J}_0/(L_0^2 A_0) \quad (5.2)$$

where

$$A = \int_{\beta^2(\ell^2 + m^2) + n^2 \leq \lambda^2} d\ell \, dm \, dn, \quad (5.3)$$

and A_0 is also given by this expression for $\beta = 1$.

We prove eq. (5.2) as follows. Consider eq. (5.1) divided by AL_z^2

$$\bar{J}/(AL_z^2) = \frac{\sum_{\ell, m, n} f(\beta \ell, \beta m, n)}{[\beta^2(\ell^2 + m^2) + n^2 \leq \lambda^2]} \bigg/ \frac{\sum_{\ell, m, n} 1}{[\beta^2(\ell^2 + m^2) + n^2 \leq \lambda^2]} \quad (5.4)$$

Then the smooth average value is given by approximating the sums by integrals to obtain

$$\bar{J}/(AL_z^2) = \frac{\int_{\beta^2(\ell^2 + m^2) + n^2 \leq \lambda^2} d\ell \, dm \, dn \, f(\beta \ell, \beta m, n)}{\int_{\beta^2(\ell^2 + m^2) + n^2 \leq \lambda^2} d\ell \, dm \, dn} \quad (5.5)$$

The substitution $\ell' = \beta \ell$, $m' = \beta m$ and $n' = n$ transform this into

$$\frac{1}{\beta^2} \int_{\ell'^2 + m'^2 + n'^2 \leq \lambda^2} d\ell' \, dm' \, dn' \, f(\ell', m', n') \bigg/ \frac{1}{\beta^2} \int_{\ell'^2 + m'^2 + n'^2 \leq \lambda^2} d\ell' \, dm' \, dn', \quad (5.6)$$

which is identical with $\bar{J}_0/(A_0 L_0^2)$. This establishes the identity (5.2),

and allows us to restrict our considerations to cubical shapes with no loss in generality for the smooth average moments of ground state configurations.

5.1. MOMENT OF INERTIA AS A FUNCTION OF A

We execute the sums over eq. (3.26) numerically to obtain the moment of inertia for A-particles in the Hill-Wheeler box, and similarly evaluate eq. (5.1) for the Amado-Brueckner box. $\bar{J}_{\text{rig}}$ is evaluated in each case from eq. (3.16). The resulting values of $\bar{J}/\bar{J}_{\text{rig}}$ are plotted against A (dots) in figs. 6 and 7 for these two cases, respectively.

One sees that the values of $\bar{J}/\bar{J}_{\text{rig}}$ fluctuate within a certain band about an A-dependent average value.

We construct below a theoretical formula for the smooth average of the Amado-Brueckner case, plotted in fig. 6 as a thin solid curve. For this case, also, a theoretical correction in the form of a sawtooth function of A, is constructed and plotted (dashed curve, fig. 6). Finally the smoothed average of the exact numerical results is plotted in fig. 6 (as a thick curve) for comparison with the theoretical smooth average (thin curve).

For the Hill-Wheeler box (fig. 7), theoretical expressions for the leading finite A corrections corresponding to those in fig. 6 are not available. The exact results are, nevertheless, presented to illustrate their general similarity with those in fig. 6. In particular, one sees in figs. 6 and 7 no reason to suspect that periodic boundary conditions of Amado-Brueckner alter the basic physical content of the Hill-Wheeler box. Also fig. 7 shows clearly that the Hill-Wheeler box need not give the rigid moment at its equilibrium shape.

The fluctuations in the moment of inertia can be understood qualitatively as the effect of the large contribution from certain single particle states near the surface of the Fermi sphere. As can be seen from eq. (5.7) below, the single particle contribution in the Amado-Brueckner box is proportional to m^2 . Therefore, the single particle states with the largest m-values provide the largest contribution. Since m is an integer, this largest contribution increases stepwise while the radius of the Fermi sphere increases

continuously. This results in a fluctuation of the exact moment of inertia as compared with the more smoothly A -dependent rigid moment. (The quantitative treatment is given below.)

For the case of Hill-Wheeler box, Table 1 leads one to expect large contributions from the near crossings of single particle levels near the surface of the Fermi sphere. As can be seen from eq. (3.26), the factor $[(m'^2 - n^2) + (n'^2 - n^2)]$ in the denominator becomes the smallest for a state with $m = n$ just inside the Fermi surface, because its energy is nearly equal to that of the two unoccupied states ($m' = m \pm 1$, $n' = n \mp 1$). Therefore, the state with $m = n$ on the Fermi surface gives a large contribution to the moment of inertia. As in the Amado-Brueckner case this leads to fluctuations in the moment of inertia because these large contributions enter the calculated moment discretely, even when the Fermi sphere is growing smoothly.

5.2. FINITE- A CORRECTION FOR THE MOMENT OF INERTIA

We now calculate the moment of inertia $\mathcal{I}$ for the Amado-Brueckner (cubical) box of large but finite A . Since we are interested in corrections to the large A limit, we keep all terms in $\mathcal{I}$ larger than $A^{-1/3}$ times the leading term. The correction that emerges is negative and of the form $A^{-1/3}(\ln A)$, so that convergence to the $A \rightarrow \infty$ limit is very slow.[†]

[†]The emergence of a correction of order $(\ln x) x^{-1}$ is reminiscent of similar terms discovered in statistical mechanics some years ago^{29, 30, 31}. Their occurrence here could encourage the suspicion that such terms are ubiquitous in many-body problems

The moment of inertia can be obtained by taking the scalar product of the current $\mathbf{j}_T$ in eq. (4.16) with $\nabla \times \mathbf{r}$ and integrating over space. The result (for a cube) is

$$\mathcal{I} = \frac{4M_0^2}{\pi^2} \sum_{\ell m}^{\text{occ}} m^2 I, \quad (5.7)$$

where

$$I = \sum_{n=-\bar{n}}^{\bar{n}} \sum_{n'=n+1}^{\infty} \frac{1}{(n' + n)(n' - n)^3}, \quad (5.8)$$

($n \neq 0$)

at the equilibrium deformation. We should note that this expression is essentially the same as that obtained by Amado and Brueckner²² before any large A approximation is introduced. In order to carry out the sum in (5.8) for finite $\bar{n}$, we break it up into three sums:

$$I \equiv I_1 + I_2 - I_3 \quad (5.9)$$

$$I_1 = \sum_{p=1}^{2\bar{n}} \frac{1}{p^3} \sum_{n=\bar{n}+1-p}^{\bar{n}} \frac{1}{2n+p} \quad (5.10)$$

$$I_2 = \sum_{p=2\bar{n}+1}^{\infty} \frac{1}{p^3} \sum_{n=-\bar{n}}^{\bar{n}} \frac{1}{2n+p} \quad (5.11)$$

$$I_3 = \sum_{n=\bar{n}+1}^{\infty} \frac{1}{n^4}. \quad (5.12)$$

Here the new variable $p = n' - n$ has been introduced, and $\bar{n} = (\lambda^2 - m^2 - \ell^2)^{1/2}$, with λ as the radius of the Fermi sphere. The evaluation of I_1 and I_2 is described in Appendix E. In the second non-vanishing order $(2\bar{n}+1)^{-2}$, they are given by

$$I_1 = \frac{\pi^2}{6(2\bar{n}+1)} - \frac{1}{2(2\bar{n}+1)^2} + O[(2\bar{n}+1)^{-3}] \quad (5.13)$$

$$I_2 = \frac{1}{2(2\bar{n}+1)^2} + O[(2\bar{n}+1)^{-5}]. \quad (5.14)$$

From (5.12), one can see that I_3 is of the order $(\bar{n})^{-3}$ and can be neglected in comparison with I_1 and I_2 . Moreover, I_2 cancels exactly the second term of I_1 . Hence, we obtain

$$I = \frac{\pi^2}{6(2\bar{n}+1)} = \frac{\pi^2}{12\bar{n}} - \frac{\pi^2}{24\bar{n}(\bar{n}+\frac{1}{2})}, \quad (5.15)$$

where we have separated the leading order term in $(1/\bar{n})$, and an exact remainder.

By eqs. (5.7) and (5.8), the moment of inertia is

$$J = MI_0^2 \left\{ \frac{1}{3} \sum_{\ell m}^{\text{occ}} \frac{m^2}{\bar{n}} - \frac{1}{6} \sum_{\ell m}^{\text{occ}} \frac{m^2}{\bar{n}(\bar{n}+\frac{1}{2})} \right\}. \quad (5.16)$$

As Amado and Brueckner showed, the first term in this equation gives the rigid value if the sums over ℓ and m are replaced by integrals over the entire Fermi sphere. However, such a replacement is only accurate for the terms of leading order in $A^{-1/3}$. To obtain correctly the second non-vanishing order, the sums must be evaluated more accurately.

5.2.1. INTEGRATION OVER THE FERMI SPHERE

There are certain points to be observed in approximating the sum over ℓ and m by an integral. We must choose the domain of integration carefully so that terms smaller by $A^{-1/3}$ are not omitted from the integral. In addition, because $\bar{n}$ occurs in the denominator, the domain of integration must avoid values of $\bar{n}$ smaller than the smallest possible value of n , ($m, n \geq 1$); otherwise, unreasonably large (and false) contribution will be introduced into the calculated moment of inertia.

To fix the domain of integration, we consider the Fermi sphere, $\ell^2 + m^2 + n^2 \leq \lambda^2$. (See fig. 8.) In approximating a sum over the lattice points, $\ell, m, n = \text{integer}$, inside the Fermi sphere by an integral, we can choose a domain of integration,

satisfying the requirement $n \geq 1$, as the volume bounded by the Fermi sphere and the planes $n = 1$ and $m = 1/2$ (represented by the shaded volume in fig. 8). As one can easily check, the volume of this domain corresponds to the correct number of particles in the system, to order λ^2 ,

$$A = \frac{4}{3} \pi \lambda^3 - 3 \pi \lambda^2. \quad (5.17)$$

Therefore, this domain is appropriate for our purpose.

Using this domain of integration, we can evaluate the second term of (5.16) as

$$\begin{aligned} \sum_{\ell m}^{\text{occ}} \frac{m^2}{\bar{n}(\bar{n}+\frac{1}{2})} &= 4 \int_{1/2}^{\lambda} m^2 dm \int_0^{\sqrt{\lambda^2 - m^2 - 1}} \frac{dn}{\bar{n}(\bar{n}+\frac{1}{2})} \\ &= \pi \lambda^2 \ln \lambda - \pi \left(\frac{1}{2} + \ln \frac{3}{2} \right) \lambda^2. \end{aligned} \quad (5.18)$$

to order λ^2 .

For the first term in eq. (5.16), we first carry out the ℓ sum by integration

$$\sum_{\ell} \frac{1}{\bar{n}} = 2 \int_0^{\sqrt{\lambda^2 - m^2 - 1}} \frac{d\ell}{\bar{n}} = \pi - \frac{2}{\sqrt{\lambda^2 - m^2}}. \quad (5.19)$$

Then, the summation over m for the first term of (5.16) can be carried out exactly

$$\sum_m \pi m^2 = 2\pi \sum_{m=1}^{[\lambda]} m^2 = \frac{\pi}{3} [\lambda] ([\lambda] + 1) (2[\lambda] + 1), \quad (5.20)$$

where $[x]$ denotes the integral part of the real number x . The sum over m in the second term is replaced by an integral:

$$-2 \int_{1/2}^{\lambda} \frac{m^2}{\sqrt{\lambda^2 - m^2}} dm = -2 \int_{1/2}^{\lambda} \frac{m^2 dm}{\sqrt{\lambda^2 - m^2}} = -\pi \lambda^2 \quad (5.21)$$

to order λ^2 .

The quantity in (5.20) which is a step function of λ , can be conveniently separated into a smooth component and a sawtooth oscillating component. When this is done and all the results in eqs. (5.18), (5.20) and (5.21) are col-

lected, we obtain $\bar{J}$ as

$$\bar{J} = \bar{J} + J_{osc} \quad (5.22)$$

with the smooth component $\bar{J}$ given by

$$\bar{J} = \frac{2\pi}{9} ML_0^2 \lambda^3 \left\{ 1 - \frac{3}{4} \frac{\ln \lambda}{\lambda} + \left(-\frac{9}{8} + \frac{3}{4} \ln \frac{3}{2} \right) \frac{1}{\lambda} \right\} \quad (5.23)$$

and the oscillating component J_{osc} ,

$$J_{osc} = \frac{2\pi}{9} ML_0^2 \left\{ [\lambda]([\lambda] + 1) \left([\lambda] + \frac{1}{2} \right) - \lambda^3 \right\}. \quad (5.24)$$

Dividing the result for $\bar{J}$ by the rigid moment $J_{rig} = \frac{1}{6} A ML_z^2$ where A is given by eq. (5.17), we obtain

$$\begin{aligned} \bar{J}/J_{rig} &= 1 - \frac{3}{4} \frac{\ln \lambda}{\lambda} + \left(\frac{9}{8} + \frac{3}{4} \ln \frac{3}{2} \right) \frac{1}{\lambda} \\ &= 1 - 1.209 A^{-1/3} \left(\frac{1}{3} \ln A - 2.382 \right). \end{aligned} \quad (5.25)$$

One sees that the smooth part of the correction to $\bar{J}/J_{rig}$ is of the order $A^{-1/3} \ln A$. It diminishes rather slowly as A increases. The quantity $\bar{J}/J_{rig}$ is plotted in fig. 6 (thin curve). It approximates the curve obtained by averaging the exact calculation (the solid curve) very well in large A values ($A \gtrsim 25,000$, or, equivalently $\lambda \gtrsim 18$).

5.2.2. THE OSCILLATING CORRECTION, J_{osc}

For the sawtooth oscillating correction J_{osc} , we calculate the frequency and the amplitude of the oscillation. Since $[\lambda]$ has a discontinuity whenever λ crosses over an integer value, we calculate J_{osc} at the value of λ just slightly smaller than an integer ($\lambda = [\lambda] + 1 - \epsilon$, $\epsilon \rightarrow 0$) and when λ is equal to an integer ($\lambda = [\lambda]$):

$$J_{osc}/J_{rig} = \mp \frac{3}{2\lambda}, \quad \text{for } \lambda = \begin{cases} [\lambda] + 1 - \epsilon \\ [\lambda] \end{cases} \quad (5.26)$$

to order λ^{-1} . This implies that J/J_{rig} oscillates about its smooth average with the amplitude,

$$\text{Amplitude} = \frac{3}{2\lambda} = (9\pi)^{1/3} A^{-1/3}. \quad (5.27)$$

The period of oscillation is equal to the step width of the step function $[\lambda]$.

From this width, $\Delta\lambda = 1$, the period of oscillation as a function of A can be found,

$$\text{Period} = (36\pi)^{1/3} A^{2/3}. \quad (5.28)$$

This sawtooth oscillating correction is added to on the exact average of J/J_{rig} and the sum represented by the dashed curve in fig. 6. One sees that this curve represents the gross variations of the moment of inertia as a function of A rather satisfactorily.

6. Necessary and Sufficient Conditions for Rigid Moments of Inertia in Quantum Many-Independent-Particle Systems

We turn now to the implications of these results for our understanding of the moment of inertia problem, by considering two conjectures about the conditions for obtaining the rigid moment, and one analysis from which the rigid moment emerges as the leading term.

6.1. CONJECTURED CONDITIONS

A. deShalit and H. Feshbach³⁴⁾ have pointed out that for the harmonic oscillator potential, the average kinetic energy of the particles in the body-fixed frame of reference is distributed equally in the x-, y- and z-degrees of freedom, i.e.,

$$\sum_{\text{occ}} \langle \frac{1}{2M} p_x^2 \rangle = \sum_{\text{occ}} \langle \frac{1}{2M} p_y^2 \rangle = \sum_{\text{occ}} \langle \frac{1}{2M} p_z^2 \rangle. \quad (6.1)$$

These authors argue from (6.1) that the velocity field in the body-fixed frame is isotropic, and, therefore, that there must be no net current at any given point in this frame. Therefore, they say, in the space-fixed frame the velocity field must be that of a rigid rotator. Since this argument is independent of any special form of the potential, they suggest that the condition of "Equidistribution of Kinetic Energy," as we shall call their proposed condition (6.1), leads in general to rigid body values for the moment of inertia.

A. Bohr and B. Mottelson, in their later review of the simple harmonic oscillator case³⁵⁾, utilize a self-consistency requirement between the density and the potential in place of the condition (3.15) to define the equilibrium shape. They demand that the anisotropy of the density to be equal to the anisotropy of the equi-potential surfaces, obtaining for the oscillator the condition:

$$\left[\sum_{k=1}^A \langle x^2 \rangle_k \right] / L_x^2 = \left[\sum_{k=1}^A \langle y^2 \rangle_k \right] / L_y^2 = \left[\sum_{k=1}^A \langle z^2 \rangle_k \right] / L_z^2 \quad (6.2)$$

where L_x , L_y and L_z here denote the principal axes of the (ellipsoidal) equi-potential surfaces. For the oscillator, (6.2) leads to identically the same results as the energy minimization condition, eq. (3.15).

The question naturally arises whether this version of the oscillator equilibrium condition, which we refer to as the "Potential-Density Self-Consistent Anisotropy" condition, might offer some difference from and possible advantage over (3.15) in application to systems other than the oscillator.

Migdal²⁴⁾ analyzes the rotational moments in superfluidic nuclei. In the limit of weak pairing, when the pairing gap, Δ , becomes small, he exhibits the leading term of the moment in the form,

$$J_0 = (M/M_{\text{eff}}) \sum_{jk} |\langle k | J_x | j \rangle|^2 \delta(\epsilon_j). \quad (6.3)$$

Equation (6.3) already represents an averaged version of the pure independent particle model cranking model formula (2.24) since it incorporates both the specific structure of the pairing model and the replacement of the discrete sum over (quasi-) particle levels by a smoothed approximation which explicitly demands that "many" levels be found within a gap-width, Δ , near the Fermi energy. But this latter assumption is violated when the pairing gap, Δ , becomes smaller than the single particle level spacing. It follows that the limit $\Delta \rightarrow 0$ which yields Migdal's equation (6.3) is a conditioned limit, under which the single particle level spacing is assumed implicitly to approach zero faster than Δ . This condition is violated in the independent particle models we consider here, since Δ is zero, but the level spacing (for finite A) remains finite.)

6.2. MIGDAL'S SMOOTH ISOTROPY ASSUMPTIONS OVER-SUFFICE

Consider first Migdal's condition that the velocity field in the body-fixed frame is everywhere isotropic[†]. This is indeed a strong assumption involving a condition on the field at every point in space. It is also an adequate basis in itself for the immediate inference of the rigid-body rotational velocity field as the actual velocity field in the space-fixed frame and therefore, of rigid-body value of the moment of inertia.

On the other hand, it might for the same reasons be suspected of being so strong an assumption as to rule out interesting physical possibilities, including perhaps the *actual* behavior of rotating independent-particle systems. For, as we have seen in sect. 3.1 and figs. 1, 2, 3, the velocity field, $\mathbf{v}_0$, associated with the time-dependent wave function is certainly quite different from that of a rotating rigid body.

[†] A. Bohr and B. Mottelson¹¹⁾, arguing from the earlier work of N. Bohr²⁰⁾, show that for a slowly rotating classical gas in thermodynamical equilibrium is sufficient to guarantee (to order n^2) the isotropy of the velocity distribution, everywhere in the rest frame of the gas and the immediate inference that in the laboratory frame the velocity is given by (2.10), $\mathbf{v} = \mathbf{R} \times \mathbf{r}$. This result cannot, however, be immediately extrapolated to the quantum mechanical case, as inspection of figs. 2 and 3 will show. There one sees that because of the vortex structure of the quantum velocity field no transformation to a frame rotating at constant velocity will bring the particle current to zero at every point. (Recall that the current is the product of the (scalar) density and this irrotational velocity field. Therefore, the streamlines shown in figs. 1 to 3 apply equally to the current and to the velocity field.) Thus for Schrödinger fluids, the conditions under which the current will be zero in the rotating frame remain to be established. We have shown here that in several cases, this condition prevails in the limit $A \rightarrow \infty$, on the average.

To these pre-conditions, Migdal adds the explicit assumption that in the body-fixed system all directions of the momentum, $\mathbf{p}$, are equally probable at every point in space—enough alone to guarantee the isotropy of the velocity field everywhere in the body-fixed system, and therefore the rigid-body value of the inertia. Indeed, the rigid-body value is subsequently exhibited as the values of (6.3).

We here examine these proposals critically, and guided by the new results we have obtained, suggest a simple generalization of the equilibrium condition which is a plausibly sufficient condition for obtaining rigid-body moments in independent particle system. Moreover, it conforms with all the results now available, it is specific enough to suggest (at least one) further immediate test, and simple enough to offer, if it were verified, a promising basis for a deeper understanding of the rotational matter-flow problem.

We remark nevertheless that the association of smooth averaged systems in Migdal's analysis with rigid-body moments of independent-particle systems seems to be a sound intuition for the general problem. For in each case where many particles have been involved in a microscopic analysis (including the Anisotropic S. E. Oscillator, and the $A \rightarrow \infty$ limits of both the Amado-Brueckner and the Hill-Wheeler box, analyzed here) the rigid-body moment has emerged as the correct result *on the average*. Thus, Migdal's inference of the rigid moment from the combination of isotropy and a smooth average approximation conforms to all the microscopic results available at present.

We therefore identify the "smoothness" assumption as a plausible candidate for a condition to be intimately connected with rigid moments. The isotropy condition on the velocity at each point in space, on the other hand, is perhaps excessive. It is surely sufficient alone, and it is certainly not a *necessary* condition: the one-particle S.E.O. velocities violate it but still yield rigid moments. (Cf., sect. 3.)

In summary, we suggest that the rigid-body value might be expected to be the correct moment of inertia for properly *smoothed* systems, a possibility which conforms to all of the new results presented here both for one-particle and many-particle systems. We shall below conjecture unrestricted minimization of the energy, i.e., *Unconstrained Shape Equilibrium*, as the condition at which the system is "properly smoothed" in this sense.

6.3. EQUI-DISTRIBUTION OF KINETIC ENERGY ASSUMPTION IS INSUFFICIENT

In contrast, the condition of Equi-distribution of Kinetic Energy, eq. (6.1) must be judged *insufficient* to give rigid moment on the basis of the present results, since both conditions are met by single particles in the Hill-Wheeler box at its equilibrium shape and by a large (but not infinite) number of particles in the Amado-Brueckner box, but the rigid moment is *not* obtained.

Specifically, the state $(l(-m), n, n)$ in the Hill-Wheeler box whose moment has *not* the rigid value meets the equilibrium condition (3.15) [see also (3.27)] when $L_x (=L_y)$ and L_z in

$$\epsilon_{l,mn} = \frac{\pi^2 n^2}{2M} \left[\frac{m^2}{L_y^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2} \right] \quad (6.4)$$

satisfy the condition

$$\frac{L_y}{L_z} = \frac{m}{n}. \quad (6.5)$$

Then the kinetic energy

$$\epsilon_{l,mn} = \frac{\pi^2 n^2}{2ML_z^2} \left[\frac{n^2}{2} \cdot \frac{m^2}{n^2} + \frac{n^2}{2} \cdot \frac{m^2}{n^2} + n^2 \right] \quad (6.6)$$

is equally distributed in accordance with condition (6.1). Table 1 documents this insufficiency of this condition for the rigid-body moment in greater detail.

Regarding the many-particle Amado-Brueckner box the numerical results of fig. 6, and the analytical result (5.22)-(5.24) confirm this counterexample. In addition, the numerical results for the Hill-Wheeler box in fig. 7 show that for large finite- A , it, too, provides a counterexample to the Equi-distribution of Kinetic Energy Conjecture.

Regarding the original arguments for the equi-distribution condition, we note that what Migdal takes as an *assumption* (the isotropy of the velocity in the body-fixed frame) becomes here an *inference*, drawn from the equi-distribution condition (6.1). But it is, from the viewpoint of commensurability[†], a dubious inference, drawing as it does a statement of isotropy of *every* point in a three-dimensional space from the simple equality (6.1) of three real numbers. [†]We appeal here to the Principle of Commensurability³⁶⁾ which states the obvious fact that a theory can validly provide only results which are "commensurate" with its structure and its input. This principle recommends, in practice, the inspection of theories, and especially of theoretical conjectures, from the input-output viewpoint as a useful method of evaluating plausibility.

A more explicit rejection of this inference can be based on the velocity fields plotted in figs. 2 and 3. Although the average kinetic energy for the oscillator obeys eq. (6.1), these velocity fields are by no means equal to $\hat{p} \times \hat{x}$.

Figures 2 and 3 are therefore counterexamples to the conjecture that Equi-distribution of Kinetic Energy is a sufficient condition for the rigid rotational velocity field.

6.4. POTENTIAL-DENSITY SELF-CONSISTENT ANISOTROPY CONDITION IS INSUFFICIENT (AND AMBIGUOUS)

We consider the condition (6.2) also for the single particle in a Hill-Wheeler box, and conclude that it, too, is (at best) insufficient to guarantee the rigid moment. For here the equi-potential surfaces are parallelepipeds with edge-lengths, $(L_x = L_y, L_y, L_z)$. Then the condition of consistent potential-density anisotropies for the state $(l = m, n)$ is given (provided one selects the second moment as the measure of extension) by

$$\langle y^2 \rangle / L_y^2 = \text{Constant} = \langle z^2 \rangle / L_z^2 \quad (6.7)$$

or

$$L_y^2 [1/12 - 1/2\pi^2 m^2] / L_y^2 = L_z^2 [1/12 - 1/2\pi^2 n^2] / L_z^2 = \text{Constant}. \quad (6.8)$$

This condition is entirely independent of L_y, L_z , and simply requires $m = n$. But for this case, Table 1 shows that the moment is *not* equal to the rigid-body value. Thus for the (few) states in the Hill-Wheeler box which can meet this condition at all, it fails to provide the rigid moment. For the other states (nearly all) the fact that it can not be satisfied renders it at best useless.

We note, however, that the generalization of eq. (6.2) from the oscillator to the square well box is by no means unambiguous. An alternative, generalization is suggested by the sequence of Separable Scaling Potentials defined by

$$V_n = \sum_{i=1}^3 V_i(x_i) = U_0 \sum_{i=1}^3 (\alpha_i)^{-2} (x_i/\alpha_i)^n \quad (6.9)$$

which includes both the anisotropic simple harmonic oscillator ($n=2$) and the Hill-Wheeler box ($n=\infty$) as special cases.

This family of potentials is discussed further in Appendix F, for each of which it is shown that the Minimal Energy condition (3.15) implies *both* the Equi-distribution of Kinetic Energy (6.1) and the condition

$$\sum_{k=1}^A \langle x_k^n \rangle / (\alpha_k)^{n+2} = \text{Constant} \quad (1 = 1, 2, 3) \quad (6.10)$$

which is one possible natural generalization of the Potential-Density Self-Consistent Anisotropy Condition, for this family. Note that for $n=2$, the identification, $\alpha_i^{-4} = \omega_i^2$, recovers the oscillator results[†] (6.2).

It follows immediately that the use of generalization (6.10) of the Potential-Density Self-Consistent Anisotropy Condition for this family of separable potentials reduces that condition to the Equi-distribution of Kinetic Energy Condition (6.1) and the condition of Minimal Energy (3.15). Then the Hill-Wheeler box moments summarized in Table 1 provides simultaneous counterexamples to all three of these conditions as prospective sufficient conditions for the

[†]We note also that for the simple harmonic oscillator case this equivalence of the Potential-Density Self-Consistent Anisotropy Condition, the Equi-distribution of Kinetic Energy Condition, and the Minimal Energy Condition, persists beyond the adiabatic small Ω limit considered in this paper to arbitrarily large Ω . One can easily check that the Minimal Energy Condition (3.15) (at constant volume) for the exact solution of the cranked oscillator discussed in ref. 37), gives exactly the same generalization to finite Ω , as is demonstrated in that work to follow from the other two conditions.

rigid value of the moment of inertia.

We note in passing that the equivalence of a variety of reasonable conditions for the rigid moment, such as prevails in the S.H.O. case (at all rotational frequencies) would be an entirely satisfactory characteristic for the correct necessary and sufficient condition to exhibit.

6.5. UNCONSTRAINED SHAPE EQUILIBRIUM AS CONDITION FOR RIGID MOMENTS

What, then, is the necessary and sufficient condition that a system of independent particles in a potential exhibit the rigid-body moment of inertia? On the basis of the present new results on the systematics of the moments in some special cases, we suggest that the *Unconstrained Shape Equilibrium*,

$$\delta E = 0, \quad (6.11)$$

as a plausible necessary and sufficient condition that an independent particle system have the rigid-body value for the moment. By *Unconstrained Shape Equilibrium* we mean specifically a generalization of (3.15) in which the energy is stationary with respect to *arbitrary* (volume conserving) variations of the shape of the equi-potential potential surfaces, rather than just the simplified limited parametric variations which are envisaged in eq. (3.15).

In particular, let us separate the consideration of variations of the shape of the equi-potentials from that of the spacings between the equi-potentials. For the infinite square well spacing of the equi-potentials (with all equi-potentials, except those of value $-V_0$, lying in the same enveloping surface), this generalization to *Unconstrained Shape Equilibrium* extends the consideration of the parallelepiped Hill-Wheeler box to encompass boxes of all possible shapes, including, for example, the ellipsoids. In this extended realm, one must expect the equilibrium condition (3.15) in general to lead to some shape other than a parallelepiped as the minimal energy shape for a pre-

chosen configuration[†]. We conjecture that at this *Unconstrained Shape Equilibrium* the cranking moment will have the rigid value.

Unfortunately, the evaluation of the moment of inertia for one, or many, particles at shape equilibrium in, say, a spheroidal square well box is a substantial numerical task. However, the suspicion that some such a shape is the *Unconstrained Shape Equilibrium* for an axially symmetric independent particle system, and that, therefore, one expects to calculate the rigid-body value of the moment may encourage the work, or perhaps, lead one to find some alternative test of the conjecture.

5.6. ARGUMENTS IN FAVOR OF UNCONSTRAINED SHAPE EQUILIBRIUM CONJECTURE

What recommends the hope that such a simple generalization of the "parametric equilibrium conditions" to the *Unconstrained Shape Equilibrium* condition might suffice to guarantee the rigid result? We offer three cogent (we think), but not conclusive, arguments, each based on a fact:

- (a) All $A \rightarrow \infty$ system studied have rigid-body moments.

We interpret this result as suggesting that when the effects of shape become negligible, *all* systems exhibit rigid moments. Conversely, we infer, *any* restriction of the shape may prevent the realization of the rigid value. [A special trivial case is a restriction which prevents even the attainment

[†]More specifically, we envisage a wave function characterized by a given nodal topology and we seek by infinitesimal alterations of the shape of the equi-potential surfaces to minimize the energy of that particular nodal topology. The absolute minimum is then the *Unconstrained Shape Equilibrium* for that "configuration." See also footnote in sect. 6.6. Further discussion of the nodal structure as a useful characterization of wave function "configurations" is found in refs. 2), 3), 4) and 38), and in Paper III of this series (in preparation).

of parametric equilibrium, as fig. 4 demonstrates.] Specifically, the restriction to a particular parametric family of shapes (e.g., Hill-Wheeler parallelepipeds) seems a probable cause for non-rigidity.

Note that this result indicates also an insensitivity of the rigid moment condition to the specific radial dependence of the potential in the nuclear interior. This suggests that the angular shape of the equi-potential surfaces might be the property to which the moment ratio to rigid is most sensitive. Nor is this observation inconsistent with the idea that extensive properties of nuclei, as "leptodermous" systems³⁹⁾, ought not to depend heavily on the details of the nuclear surface.

- (b) The fluctuation around the rigid-body values summarized in Table 1 and fig. 7, are associated systematically with specific eigenvalue degeneracies of the parallelepiped.

This result suggests that any deviation in shape which tends to lift these degeneracies[†] would reduce the fluctuations of the moments. Moreover, in a one-parameter family of shapes which includes the sphere the near-vanishing energy denominators which lead here to such large moments would occur at the spherical shape, and so would not arise in calculating the moments for non-spherical systems. Thus one can expect that an alteration of the parallelepiped shape towards a smoother shape (e.g., a spheroid) ought to lead both to a lower

[†]As, e.g., the rotation itself is shown to do in sect. 3.33 and Appendix B, for the worst degeneracies, demonstrating that a physically rotating system in equilibrium ought never to exhibit the most extreme fluctuations of moments calculated in these simple models.

energy[†] (conforming therefore to the *Unconstrained Shape Equilibrium* condition) and to a smoother and expectedly, a more nearly rigid, moment of inertia.

- (c) The result that one or more particles in an *oscillator* well at its (parametric) equilibrium shape *always* give exactly the rigid-body moment.

This result suggests that when the ratio $\mathcal{I}/\mathcal{I}_{\text{rigid}}$ is perfectly smooth in A, it also has the value, one. Combined with the preceding argument towards smoothness at the *Unconstrained Shape Equilibrium*, this suggests also the rigid value at *Unconstrained Shape Equilibrium*.

In addition, this result suggests that, if any single one-parameter family of shapes is to comprise *all* of the *Unconstrained Shape Equilibrium* shapes than that family must be the ellipsoids. [But note the possibility, discussed below, that the equilibrium shape depends upon the spacing of equi-potentials.]

[†]As a simple example for the effect of lowering the energy by letting the nucleus to assume a shape smoother than the Hill-Wheeler box, one can consider a volume conserving alternation of this box in which the eight corners are cut off by planes at 45° to the faces, and the dimension of the remaining figure is scaled up to maintain the original volume. The eigenenergies of this cut box can be easily calculated by boundary perturbation⁴⁰⁾. Taking the cubical geometry ($L_x = L_y = L_z = L$), we find that in the lowest order perturbation the energy of each single particle increases by $(16/105)\pi^2 k^2 m^2 n^2 \epsilon^7 / L^9$ by cutting off the⁹ corners, but decreases by $(8/9)\pi^2 (k^2 + m^2 + n^2) \epsilon^3 / L^5$ from the increase in the dimension of the box to conserve volume, where $\epsilon^3/6$ is the volume of one cut corner. Therefore, the cut box has a lower energy than the original Hill-Wheeler box.

6.7. QUESTION OF "RADIAL DEPENDENCE" IS OPEN

We thus arrive at a strongly focussed interest in the moments of inertia for spheroidal boxes, based, however, on a discussion which has omitted the possible role of the equi-potential spacing or "radial" dependence of the potential. But if the "radial" dependence is essential, the spheroidal box is again the most promising study to demonstrate that fact.

What if the attainment of the rigid-body value requires also a specific "radial" dependence of the potential, as well as a particular shape of its equi-potential? Then, of course, the problem is much more complicated, since the constant-volume condition, which is the only point at which the binding properties of the nucleus has so far entered, is too simple to specify also the radial dependence of the potential. Nevertheless one would expect that for some class (perhaps the leptodermous³⁹) radial dependences, the rigid-body condition would be, at least in first approximation, independent of the details in the nuclear surface. Then, for a corresponding class or "radial" dependences, the *Unconstrained Shape Equilibrium* condition plus the condition on the acceptable class of "radial" dependences, might suffice for rigidity.

The possibility exists also that the Potential-Density Self-Consistent Anisotropy Condition and/or the Equi-Distribution of Kinetic Energy condition might follow from the *Unconstrained Shape Equilibrium* condition, thus constituting necessary, though not sufficient, conditions for the rigid moment.

7. Summary and Conclusions

In this paper, we have approached the problem of moments of inertia of independent particle systems from the viewpoint of the Single Particle Schrödinger Fluid, and the velocity fields which it naturally involves. We have shown that the total velocity fields for the oscillator, the Amado-Brueckner box and the Hill-Wheeler box approach the rigid-body value, $\bar{Q} \times r$, as $A \rightarrow \infty$. Properties of the moments for the oscillator away from equilibrium are studied and used as a basis for later inference about the general problem. Exact numerical calculations of these moments are reported for large, but finite A . For the Amado-Brueckner box, an improved analytical approximation is shown to give good overall agreement with exact numerical results both as regards the average deviation from the rigid-body value (of order $A^{-1/3} \ln A$) and the fluctuations about the average.

Several conjectures from the literature are examined in the light of these results and rejected as the necessary and sufficient conditions for the rigid moment. A generalized *Unconstrained Shape Equilibrium* conjecture is put forward, together with various arguments in its favor. The moments of the infinite square well box of spheroidal shape are identified as having special interest for this conjecture.

Several appendices detail some miscellaneous results, including a study of the instability against rotation of a nuclear equilibrium near a level crossing, and a proof of the parallel axis theorem for quantum systems which obey the continuity equation.

Appendix

A. Validity of the vortex expansion of the velocity field

To prove eq. (3.12) for the case of more than two vortices, we note that all the singularities in v'_θ come from the zeroes of ρ_{lmn} . It is therefore sufficient to show that for a given set of $N > 2$ points r_k in the yz -plane, it is always possible to write any nonsingular two-dimensional flow $v(r)$ in the form

$$v(r) \equiv A(r)e_y + B(r)e_z = \sum_{k=1}^N f_k(r)e_x \times (r - r_k). \quad (A.1)$$

For our purpose, r_k are the zeroes of ρ_{lmn}/u_l^2 . One observes here that eq. (A.1) can be proved if we can find two sets of constants $\{a_k\}$ and $\{b_k\}$, such that

$$e_y = \sum_{k=1}^N a_k e_x \times (r - r_k) \quad (A.2)$$

$$e_z = \sum_{k=1}^N b_k e_x \times (r - r_k), \quad (A.3)$$

and that it implies

$$f_k(r) = a_k A(r) + b_k B(r). \quad (A.4)$$

Equation (A.2) is equivalent to three simultaneous linear equations:

$$\begin{cases} \sum_{k=1}^N a_k = 0 \\ \sum_{k=1}^N a_k y_k = 0 \\ \sum_{k=1}^N a_k z_k = 1. \end{cases} \quad (A.5)$$

These equations have nontrivial solutions for a_k if $N > 2$ and r_k are (a) not on a straight line in the yz -plane, or (b) all lying on the y -axis, or (c) all lying on the z -axis. Equation (A.3) gives exactly the same conclusion for

b_k . Therefore, for the case where r_k are not on a straight line, eqs. (A.2) and (A.3) can always be solved simultaneously. If r_k lie on a straight line, we can choose this straight line to be the y -axis (or equally well, the z -axis), then eqs. (A.2) and (A.3) can again be solved. Hence, eq. (A.1) always holds if $N > 2$. Combining this result with the explicit separation for $N = 1$ and 2, given in eqs. (3.13) and (3.14) we conclude that the separation of v'_θ into vortex terms in the form of eq. (3.13) is always possible[†].

[†]This separation is evidently *not* unique for $N > 3$ (nor for $N = 3$ if r_k lie on a straight line), as one can see from eq. (A.4) where the number of unknowns (a_k) then exceeds the number of equations.

B. Instability of equilibrium near a level crossing

Consider a level crossing between single particle levels $|1\rangle$ and $|2\rangle$ at a deformation, α_c , near the minimum of the energy ϵ_1 of level 1 at deformation, α_1 . In the calculation of moment of inertia for state $|1\rangle$ at the equilibrium deformation, the usual cranking formula (based on the non-degenerate perturbation theory) breaks down. To circumvent this difficulty, we diagonalize the cranking Hamiltonian $H - \Omega \mathcal{L}_x$ in the subspace spanned by $|1\rangle$ and $|2\rangle$ before the cranking perturbation formula is applied. [Figure 9 portrays this situation and the terminology we use for the energies and deformations of this discussion.]

B.1. Diagonalization of degeneracy for the cranking perturbation

Since $|1\rangle$ and $|2\rangle$ are time reversal invariant intrinsic states, and since the angular momentum, $\mathcal{L}_x$, is odd under time reversal, its diagonal elements in $|1\rangle$ and $|2\rangle$ vanish. Therefore in the subspace ($|1\rangle, |2\rangle$) the cranking Hamiltonian is of the form

$$H - \Omega \mathcal{L}_x = \begin{pmatrix} \epsilon_1 & -\Omega\delta \\ -\Omega\delta^* & \epsilon_2 \end{pmatrix}, \quad (B.1)$$

where $\delta \equiv \langle 1 | \mathcal{L}_x | 2 \rangle$, and ϵ_1 and ϵ_2 are the single particle energies of $|1\rangle$ and $|2\rangle$, respectively. When the Hamiltonian is diagonalized in this subspace, we obtain new states

$$\begin{cases} |a\rangle = A|1\rangle + B|2\rangle \\ |b\rangle = -B|1\rangle + A|2\rangle, \end{cases} \quad (B.2)$$

with eigenvalues

$$\begin{Bmatrix} \epsilon_a \\ \epsilon_b \end{Bmatrix} = \frac{\epsilon_1 + \epsilon_2}{2} \mp \left[\left(\frac{\epsilon_1 - \epsilon_2}{2} \right)^2 + \Omega^2 |\delta|^2 \right]^{1/2}. \quad (B.3)$$

The coefficients in eq. (B.2) are given by

$$\begin{cases} A = \Omega\delta / [(\epsilon_1 - \epsilon_a)^2 + \Omega^2 |\delta|^2]^{1/2} \\ B = (\epsilon_1 - \epsilon_a) A / \Omega\delta. \end{cases} \quad (B.4)$$

Next we apply the first order perturbation in the space spanned by all single particle states, $|a\rangle, |b\rangle, |3\rangle, |4\rangle, \dots$, and obtain the approximate eigenstates to the cranking Hamiltonian as

$$|\psi_a\rangle = |a\rangle + \sum_{j \geq 3} \frac{\langle j | \Omega \mathcal{L}_x | a \rangle}{\epsilon_j - \epsilon_a} |j\rangle - \frac{1}{2} \sum_{j \geq 3} \frac{|\langle j | \Omega \mathcal{L}_x | a \rangle|^2}{(\epsilon_j - \epsilon_a)^2} |a\rangle. \quad (B.5)$$

This state has the energy

$$\epsilon_a \equiv \langle \psi_a | H | \psi_a \rangle = \langle a | H | a \rangle + \sum_{j \geq 3} \frac{|\langle j | \Omega \mathcal{L}_x | a \rangle|^2 (\epsilon_j - \langle a | H | a \rangle)}{(\epsilon_j - \epsilon_a)^2}. \quad (B.6)$$

We can express $\langle a | H | a \rangle \equiv \epsilon_a^0$ in various ways in terms of $\epsilon_1, \epsilon_2, \epsilon_a$ and ϵ_b by means of eqs. (B.2) - (B.4):

$$\langle a | H | a \rangle = \epsilon_a^0 = \epsilon_1 + \frac{\Omega^2 |\delta|^2 (\epsilon_2 - \epsilon_1)}{(\epsilon_b - \epsilon_a)(\epsilon_2 - \epsilon_a)} \quad (B.7a)$$

$$= \epsilon_2 + \frac{\Omega^2 |\delta|^2 (\epsilon_1 - \epsilon_2)}{(\epsilon_b - \epsilon_a)(\epsilon_1 - \epsilon_a)} \quad (B.7b)$$

$$= \epsilon_a + \frac{2\Omega^2 |\delta|^2}{\epsilon_b - \epsilon_a}. \quad (B.7c)$$

Although these three expressions are completely identical, eq. (B.7a) is more convenient to apply to the left of the crossing point; eq. (B.7b), to the right. Equation (B.7c) is a convenient symmetric form.

B.2. Equilibrium deformation and its instability

Since ϵ_a is the energy of state $|\psi_a\rangle$ we expect that this state is in equilibrium at the deformation where ϵ_a is minimum. However, if we assume that

the energies, $\epsilon_3, \epsilon_4, \dots$, differ from ϵ_a and ϵ_b by amounts greater than the differences, $\epsilon_a - \epsilon_b$, then their large energy denominators guarantee that the terms in the summation, eq. (B.6), are small compared to the Ω^2 term of $\epsilon_a^0 = \langle a | H | a \rangle$. Hence the energy ϵ_a can be approximated by ϵ_a^0 . In this approximation the equilibrium condition becomes

$$\frac{d\epsilon_a^0}{d\alpha} = 0, \quad (B.8)$$

and defines the equilibrium deformation α_a .

If Ω is very small, a minimum in ϵ_1 near the crossing point, may still result in a nearby minimum in ϵ_a^0 . But as Ω increases, this minimum in ϵ_a^0 may disappear, resulting in a shift of the stable point to a drastically different value. This shift occurs at a critical angular velocity, Ω_{crit} , determined by the condition

$$\frac{d^2 \epsilon_a^0}{d\alpha^2} = 0. \quad (B.9)$$

To obtain a simple explicit expression for Ω_{crit} , we assume that ϵ_1 varies quadratically and ϵ_2 varies linearly with α near the crossing; i.e., we write

$$\epsilon_1 = S(\alpha - \alpha_c)^2 - 2S(\alpha_1 - \alpha_c)(\alpha - \alpha_c) + \epsilon_c \quad (B.10)$$

and

$$\epsilon_2 = -t(\alpha - \alpha_c) + \epsilon_c, \quad (B.11)$$

where S and t are constant. Thus ϵ_1 has a minimum at α_1 (and would therefore be the equilibrium deformation if the degeneracy did not require the more careful treatment) and ϵ_c is the value of ϵ_1 (and ϵ_2) at the crossing point $\alpha = \alpha_c$.

We can expand $\epsilon_a^0(\alpha)$ near α_a in Taylor series,

$$\epsilon_a^0(\alpha) = \epsilon_a^0(\alpha_a) + \frac{d\epsilon_a^0}{d\alpha} \bigg|_{\alpha_a} (\alpha - \alpha_a) + \frac{1}{2} \frac{d^2 \epsilon_a^0}{d\alpha^2} \bigg|_{\alpha_a} (\alpha - \alpha_a)^2 + \dots \quad (B.12)$$

and utilize eqs. (A.8) and (A.9) to obtain

$$\epsilon_a^0(\alpha) = \epsilon_a^0(\alpha_a) + O((\alpha - \alpha_a)^3). \quad (B.13)$$

Since α_1 and α_c are nearly equal, we expect $\alpha_1 - \alpha_a$ and $\alpha_c - \alpha_a$ to be small, so that when $\alpha \approx \alpha_1$ or $\alpha \approx \alpha_c$ the cubic corrections in eq. (B.13) can be neglected.

Therefore, we obtain the approximate condition,

$$\epsilon_a^0(\alpha_1) = \epsilon_a^0(\alpha_c), \quad (B.14)$$

which involves the various parameters of the problem and the critical frequency, Ω_{crit} , solving eq. (B.14) (using eqs. (B.7a), (B.10) and (B.11) and assuming the δ is independent of α), one finds

$$\Omega_{crit} = \frac{\sqrt{st}(\alpha_c - \alpha_1)^{3/2} [t + s(\alpha_c - \alpha_1)]}{|\delta| [t - s(\alpha_c - \alpha_1)]}. \quad (B.15)$$

Hence, when the rotational angular velocity, Ω , is smaller than this value, the equilibrium deformation is stable against the rotation; if $\Omega > \Omega_{crit}$ the equilibrium deformation is unstable.

B.3. A specific level crossing: Ω_{crit} for Hill-Wheeler box levels (223) and (214)

In fig. 5 we show the Hill-Wheeler box levels (223) and (214). As a function of the deformation coordinate, $\beta = L_z/L_y$, corresponding to the deformation coordinate α used here, these two levels cross each other near the minimum of level (223). In that region, the energies $\epsilon_1 \equiv \epsilon_{(223)}$ and $\epsilon_2 \equiv \epsilon_{(214)}$ can be well described by the approximations (B.10) and (B.11), respectively, with the constants

$$\left\{ \begin{array}{l} \alpha_c - \alpha_1 = .0275 \\ \tau = 8.95 \epsilon \\ S = .1436 \epsilon \\ \epsilon_c = 27.0 \epsilon, \end{array} \right. \quad (\text{B.16})$$

where $\epsilon \equiv \pi^2 \hbar^2 / 2ML_{z1}^2$ and L_{z1} is the L_z value at α_1 . The matrix element

$\delta = \langle 214 | \mathcal{H}_x | 223 \rangle$ at α_1 can be calculated readily from eq. (3.25). We obtain

$$\delta = 1.61 \hbar \quad (\text{B.17})$$

and this value changes only slightly as α varies from α_1 to α_c . Substitute the values in (B.14) and (3.15), we find the critical angular velocity Ω_{crit} to be

$$\Omega_{\text{crit}} = .01 \hbar / ML_{z1}^2. \quad (\text{B.18})$$

B.4. Minimum is unstable for least quantized angular momentum

Since the angular momentum must be quantized at integral multiples of $\hbar$ we are not free to assume arbitrarily small angular velocities. Indeed the lowest possible non-zero angular velocity Ω_{min} will be of the rough magnitude

$$\Omega_{\text{min}} \sim \hbar / \mathcal{I}_{\text{rig}}, \quad (\text{B.19})$$

where $\mathcal{I}_{\text{rig}}$ is the rigid moment of inertia. For state $|223\rangle$ we have

$$\mathcal{I}_{\text{rig}} = \langle 223 | y^2 + z^2 | 223 \rangle = .109 ML_{z1}^2. \quad (\text{B.20})$$

Then (B.18) implies that the lowest possible angular velocity is

$$\Omega_{\text{min}} \sim 9.2 \hbar / ML_{z1}^2. \quad (\text{B.21})$$

Thus from eqs. (B.17) and (B.20) we see that

$$\Omega_{\text{min}} \sim 10^3 \times \Omega_{\text{crit}}. \quad (\text{B.22})$$

It follows that the deformation determined by the minimum of level (223) is unstable against any physical rotation.

C. Total velocity field for the infinite Hill-Wheeler box

In this appendix we show that the total velocity field for the rotating many-particle Hill-Wheeler box tends to the velocity field for rigid-body rotation as $A \rightarrow \infty$.

C.1. The total current J_T

Our task is to calculate the total current. With the help of the matrix elements given in eq. (3.25), one can readily calculate the single particle currents. Summing up these single particle currents for the occupied states, we obtain the total current as

$$J_T = \sum_{\ell mn}^{\text{occ}} \sum_{m' n'}^{\text{unocc}} C_{\ell mn, m' n'} X_{\ell}(\xi) [A_{mm'}(\eta) B_{nn'}(\zeta) \xi_y + \beta^{-1} A_{nn'}(\zeta) B_{mm'}(\eta) \xi_z] \quad (\text{C.1})$$

($m' + m = \text{odd}$)
($n' + n = \text{odd}$)

with

$$C_{\ell mn, m' n'} = \frac{2^9}{\pi^3} \Omega_{\text{min}}^2 \frac{mm'nn'[(m'^2 - m^2)\beta^2 - (n'^2 - n^2)]}{(m'^2 - m^2)^2(n'^2 - n^2)^2[(m'^2 - m^2)\beta^2 + (n'^2 - n^2)]} \quad (\text{C.2})$$

$$X_{\ell}(\xi) = \frac{1}{2} - \frac{1}{2} \cos 2\ell(\xi - \frac{1}{2}) \quad (\text{C.3})$$

$$A_{kk'}(x) = \kappa \sin k'(x - \frac{1}{2}) \cos k(x - \frac{1}{2}) - k' \cos k'(x - \frac{1}{2}) \sin k(x - \frac{1}{2}) \quad (\text{C.4})$$

$$B_{kk'}(x) = \sin k'(x - \frac{1}{2}) \sin k(x - \frac{1}{2}). \quad (\text{C.5})$$

In (C.1), the symbols ξ , η and ζ are as defined in (4.13).

C.2. The Pauli principle suggests an approximation scheme

From eqs. (C.1) and (C.2), one can see that large contributions to J_T arise from terms with m and m' over n and n' near the Fermi surface. This fact simplifies the calculation, because then by introducing the new summation variables,

$$\begin{cases} \hat{m} = \frac{1}{2}(m+n'); & p = m' - m \\ \hat{n} = \frac{1}{2}(n+n'); & q = n' - n, \end{cases} \quad (C.6)$$

we can obtain an approximate expression for (C.2) by retaining the terms in the lowest order of p and q :

$$C_{l_{mr}, m'n'} = \frac{2^5}{\pi^3 V} \Omega_L z \frac{\beta^2 \hat{m} p - \hat{n} q}{p^2 q^2 (\beta^2 \hat{m} p + \hat{n} q)}. \quad (C.7)$$

Now we let the system occupy the Fermi spheroid (4.3) which has semi-axes equal to λ/b and λ in the m and n directions, respectively. For large A , we replace the sums in (C.1) by integrals:

$$\sum_{mm'n'} \rightarrow \int d\hat{m} \int d\hat{n} \sum_{pq} = \frac{1}{b} \int_0^{\pi/2} d\theta \sum_{pq} \int_{v_{\min}}^{v_{\max}} v dv. \quad (C.8)$$

Here we have introduced the polar coordinates

$$m\hat{b} = v \cos \theta, \quad \hat{n} = v \sin \theta. \quad (C.9)$$

The limits of the integral, $v_{\max}$ and $v_{\min}$, are the maximum and minimum possible values of v for fixed p and q , respectively, and are determined by the limits of summation in the original m , n , m' and n' sums. They are found to be

$$\begin{cases} v_{\max} \\ v_{\min} \end{cases} = \lambda \pm \frac{1}{2} \sqrt{2b^2 + q^2} \cos[\theta - \tan^{-1}(q/pb)] \\ = \lambda \pm \frac{1}{2}(p \cos \theta + q \sin \theta), \quad (C.10)$$

where

$$\lambda = \sqrt{\lambda^2 - b^2 \ell^2}. \quad (C.11)$$

Since m and n correspond to the occupied state while m' and n' correspond to the unoccupied states, the sum over p and q have to run through the lattice points outside the ellipse shown in fig. 10. For large A , this can be approximated by summing over the lattice points outside the tangent to the ellipse for a fixed θ (i.e., the lattice points in the shaded area in fig. 10). Therefore, the p and q sums depend on θ , and cannot be moved outside the integral.

C.3. Evaluation of the y -component of J_T

Let us consider the y -component of J_T . We first write $A_{mm'}(\eta)$ and $B_{nn'}(\zeta)$ in terms of $\hat{m}$, $\hat{n}$, p and q , as

$$A_{mm'}(\eta) = \hat{m} \sin p(\eta - \frac{1}{2}) - \frac{p}{2} \sin 2\hat{n}(\eta - \frac{1}{2}) \quad (C.12)$$

$$B_{nn'}(\zeta) = \frac{1}{2} \cos q(\zeta - \frac{1}{2}) - \frac{1}{2} \cos 2\hat{m}(\zeta - \frac{1}{2}). \quad (C.13)$$

The y -component J_y can be calculated by multiplying (C.3), (C.7), (C.12) and (C.13) together and performing the sums and integrals of (C.8).

Each of the three functions $X_\ell(\xi)$, $A_{mm'}(\eta)$ and $B_{nn'}(\zeta)$, as given in eqs. (C.3), (C.12) and (C.13) contains two terms. However, one can easily see that only their first terms contribute to the leading order of A , as follows. The function $X_\ell(\xi)$ is proportional to the density distribution along the x -axis. The first term of (C.3) is the volume term of this density, while the second term contributes only to the density ripples which is vanishingly small in the interior of the system as $A \rightarrow \infty$. For the second terms in (C.12) and (C.13) we use again the fact that only the terms with $\hat{m}$ and $\hat{n}$ near the Fermi surface are important. However, for large A , the values of $\hat{m}$ and $\hat{n}$ near the Fermi surface are large. This implies that the second terms in (C.12) and (C.13) are oscillating with high frequencies near the Fermi surface and are integrated to zero when the integral over $\hat{m}$ and $\hat{n}$ are carried out. Therefore, we only need to consider

the first terms in (C.3), (C.12) and (C.13) in calculating J_y .

Now we have

$$J_y = \frac{8\Omega L}{\pi^2 V b^2} \sum_{\ell} \int_0^{\pi/2} d\theta \sum_{pq} \int_{v_{\min}}^{v_{\max}} v dv \frac{\beta^2 \cos \theta - bq \sin \theta}{2^2 (\ell^2 p \cos \theta + bq \sin \theta)} v \cos \theta \sin p(\eta - \frac{1}{2}) \cos q(\zeta - \frac{1}{2}) \quad (C.14)$$

We can perform the v integration obtaining

$$\int_{v_{\min}}^{v_{\max}} v^2 dv = \Lambda^2 (pb \cos \theta + q \sin \theta), \quad (C.15)$$

to lowest order of p and q .

Inserting eq. (C.15) into (C.14), one observes that the resulted summand is invariant under the transformation, $(p, q) \rightarrow (-p, -q)$, which takes a point in the shaded area to the unshaded area in fig. 10. Therefore, by introducing a factor $1/2$ we can make the p, q sum unrestricted and move it outside of the integral.

The sum over ℓ involves only Λ^2 and can be carried out easily. To leading order of λ , we have

$$\pi \sum_{\ell} \Lambda^2 = \frac{2\pi\lambda^3}{3b} = 4b\Lambda. \quad (C.16)$$

The last equality follows because the total number of particles in the Hill-Wheeler box is equal to $\frac{1}{6} \pi \lambda^3 b^{-2}$ (cf., eq. (4.5)).

To proceed further, we let the system assume its equilibrium deformation, i.e., we let $b = \beta$. The integration over θ simplifies and it leads, together with the result in (C.16), to

$$J_y = \frac{8\Omega L}{\pi^2 V \beta} \sum_{p, q = -\infty}^{\infty} \frac{1}{2^2} \left(\frac{\pi}{2}\right) \beta p - q \sin p(\eta - \frac{1}{2}) \cos q(\zeta - \frac{1}{2}). \quad (C.17)$$

By utilizing the symmetry properties of the summands in terms of p and q , eq. (C.17) can be rewritten in the form

$$J_y = \frac{15\Omega L}{\pi V} \sum_{p=1}^{\infty} \frac{1}{p} \sin p(\eta - \frac{1}{2}) \sum_{q=1}^{\infty} \frac{1}{q} \cos q(\zeta - \frac{1}{2}), \quad (C.18)$$

which shows two sums which are, respectively, a Fourier representation of the function $F(x) = -\frac{\pi}{4}$ and $F(x) = \frac{\pi}{4}x$ in the interval $(-\frac{1}{2}, \frac{1}{2})$. Therefore, we obtain

$$J_y = -\eta \frac{\Lambda}{V} z = -\Omega p z. \quad (C.19)$$

C.4. The total velocity

The z -component of J_T can be calculated in a similar fashion as for the y -component when y and m is exchanged with z and n , respectively. However, instead of the factor $v \cos \theta$ in eq. (C.14), one will have $v \sin \theta$, which leads to a positive sign in the final result. Therefore, the total current is equal to

$$J_T = \Omega p (-z e_{xy} + y e_{xz}). \quad (C.20)$$

Upon dividing by ρ , we arrive at the velocity field of rigid rotation for the total velocity field for the many-particle Hill-Wheeler box in uniform rotation at the limit of $A \rightarrow \infty$.

$$\sum_{k=0}^K \frac{H_k^2}{2^k k!} = \frac{1}{2^{K+1} K!} (H_{K+1}^2 - H_K H_{K+2}). \quad (D.7)$$

One can obtain from (D.5)

$$J_y = -\Omega z \tilde{\rho}, \quad (D.8)$$

where

$$\tilde{\rho} = \sum_{\ell=0}^{\lambda} \sum_{n=0}^{\bar{n}} \sum_{m=0}^{\bar{m}-1} u_{\ell mn}^2 \quad (D.9)$$

is the density for a slightly smaller nucleus which is obtained from the original nucleus by deleting the top most layer of states $|\ell mn\rangle$ from the Fermi sea. Since the number of particles occupying the states in this layer is proportional to $A^{2/3}$ and is negligible in comparison with A as $A \rightarrow \infty$, the difference between $\tilde{\rho}$ and ρ is also negligible. Therefore, we have $\tilde{\rho} \rightarrow \rho$ and $J_y \rightarrow -\Omega z \rho$, as $A \rightarrow \infty$.

D.3. Evaluation of the z-component

In analogy with the evaluation of the y-component, we throw away a term in the z-component whose slowly varying part is of the order n^{-1} . Then, we have

$$J_z = \frac{\Omega}{2} \sqrt{\frac{\pi}{M\omega_y}} \left(\frac{M\omega}{\pi\hbar} \right)^{3/2} \frac{1}{\beta} \sum_{\ell=0}^{\lambda} \sum_{n=0}^{\bar{n}} \frac{\exp(-\sigma^2) H_{\ell}^2}{2^{i+\bar{m}+n} \ell! \bar{m}! n!} H_{\bar{m}+1} H_{\bar{m}} (H'_{n+1} H_n - H_{n+1} H'_n - H_n^2). \quad (D.10)$$

Utilizing the asymptotic form (D.6), we can approximate

$$\begin{aligned} \frac{e^{-\eta^2}}{2^{\bar{m}} \bar{m}!} H_{\bar{m}+1} H_{\bar{m}} &= \frac{1}{\sqrt{\pi}} \left[\sin \frac{\eta}{\sqrt{2\bar{m}+2}} + \sin(2\sqrt{2\bar{m}+2} \eta) \right] \\ &= \frac{\eta}{\sqrt{2\pi\bar{m}}} + \text{rapid osc. term} \end{aligned} \quad (D.11)$$

and

$$\begin{aligned} \frac{\exp(-\zeta^2)}{2^n n!} (H'_{n+1} H_n - H_{n+1} H'_n - H_n^2) &= 2 \sqrt{\frac{2n+2}{\pi}} \left[\cos \frac{\zeta}{\sqrt{2n+2}} + \cos(2\sqrt{2n+2} \zeta) \right] \\ &= \sqrt{\frac{3n}{\pi}} + \text{rapid osc. term.} \end{aligned} \quad (D.12)$$

Using these approximations, we have

$$J_z = \Omega y \left(\frac{M\omega}{\pi\hbar} \right)^{3/2} \frac{1}{\beta\pi} \sum_{\ell=0}^{\lambda} \frac{\exp(-\xi^2)}{2^{\ell} \ell!} H_{\ell}^2 \sum_{n=0}^{\bar{n}} (n/\bar{m})^{1/2}. \quad (D.13)$$

On the other hand, by using the asymptotic form (D.6) and the Stirling formula for the factorials, the density $\tilde{\rho}$ in (D.9) can be approximated by

$$\tilde{\rho} = \left(\frac{M\omega}{\pi\hbar} \right)^{3/2} \frac{1}{\pi} \sum_{\ell=0}^{\lambda} \frac{\exp(-\xi^2)}{2^{\ell} \ell!} H_{\ell}^2 \sum_{n=0}^{\bar{n}} (\bar{m}/n)^{1/2}. \quad (D.14)$$

By replacing sums by integrals, the sum over n in (D.14) can be transformed to the corresponding sum in (D.13):

$$\sum_{n=0}^{\bar{n}} (\bar{m}/n)^{1/2} = \int_0^{\bar{n}} d\bar{m}/n^{1/2} = \frac{1}{\beta} \int_0^{\bar{n}} d\bar{m} (n/\bar{m})^{1/2} = \frac{1}{\beta} \sum_{n=0}^{\bar{n}} (n/\bar{m})^{1/2}. \quad (D.15)$$

From eqs. (D.13) - (D.15), we obtain $J_z \rightarrow \Omega y \rho$, and hence, together with the result on J_y , $J \rightarrow \rho \Omega \times \vec{r}$, as $A \rightarrow \infty$.

E. Evaluation of I_1 and I_2

$$(1) I_1 = \sum_{p=1}^{2\bar{n}} \frac{1}{p^3} \sum_{n=\bar{n}+1-p}^{\bar{n}} \frac{1}{2n+p}$$

Expand $(2n+p)^{-1}$ about $N = 2\bar{n}+1$. The sum over n can be written as

$$\sum_{n=\bar{n}+1-p}^{\bar{n}} \frac{1}{2n+p} = \frac{p}{N} + 2(1-\delta_{1p}) \begin{cases} \sum_{q=1}^{\infty} \sum_{k=1}^{p/2} (2k-1)^{2q} N^{-(2q+1)} & \text{for } p \text{ even} \\ \sum_{q=1}^{\infty} \sum_{k=1}^{(p-1)/2} (2k)^{2q} N^{-(2q+1)} & \text{for } p \text{ odd} \end{cases} \quad (E.1)$$

Then, replace the sum over k by integral

$$\sum_{k=1}^K \rightarrow \int_{1/2}^{K+1/2} dk. \quad (E.2)$$

After performing this integral, we can write I_1 as

$$I_1 = \frac{1}{p} \sum_{p=1}^{2\bar{n}} \frac{1}{p^2} + \sum_{p=2}^{2\bar{n}} \sum_{q=1}^{\infty} \frac{p^{2q-2}}{(2q+1)N^{2q+1}} - \sum_{p=1}^{\bar{n}} \frac{1}{(2p-1)^3} \sum_{q=1}^{\infty} \frac{1}{(2q+1)N^{2q+1}}. \quad (E.3)$$

For the first term in this equation, we use the approximation

$$\sum_{p=1}^{2\bar{n}} \rightarrow \sum_{p=1}^{\infty} - \int_{2\bar{n}+1}^{\infty} dp. \quad (E.4)$$

For the second and the third terms in (E.3), we use

$$\sum_{p=p_0}^P \rightarrow \int_{p_0}^{P+1} dp. \quad (E.5)$$

After the integration, the second term in (E.3) is proportional to N^{-2} , and the remaining sum over q can be carried out and is equal to $\frac{1}{2}$. The sum over q in the third term of (E.3) cannot be carried out analytically, but the terms in the sum are of orders $(1/N)^3$ and higher. Collecting these results, we obtain

$$I_1 = \frac{1}{N} \sum_{p=1}^{\infty} \frac{1}{p^2} - \frac{1}{2N^2} - \frac{5}{6} \frac{1}{N^3} + \dots \quad (E.6)$$

$$(11) I_2 = \sum_{p=2\bar{n}+1}^{\infty} \frac{1}{p^3} \sum_{n=-\bar{n}}^{\bar{n}} \frac{1}{2n+p}$$

The evaluation of I_2 is very similar to that of I_1 . Following the expansion of $(2n+p)$ and the replacement of sum by integral given in (E.2), we obtain

$$\sum_{n=-\bar{n}}^{\bar{n}} \frac{1}{2n+p} = \sum_{q=1}^{\infty} \frac{1}{2q-1} \left(\frac{N}{p}\right)^{2q-1} - \frac{1}{p} \sum_{q=1}^{\infty} \frac{1}{(2q+1)p^{2q}}. \quad (E.7)$$

In obtaining I_2 , the sum over p is again replaced by the integral (E.5). It gives

$$I_2 = \frac{1}{2N^2} - \frac{1}{15N^5} + \dots \quad (E.8)$$

F. Equivalence of the equilibrium condition and the condition of equilibrium distribution of kinetic energy for the case of separable scaling potential

Define the separable scaling potential as

$$V(r) = V_1(x_1) + V_2(x_2) + V_3(x_3) \quad (F.1a)$$

$$V_i(x_i) = \alpha_i^{-2} U(\xi_i) = \alpha_i^{-2} U_0 \xi_i^n, \quad (F.1b)$$

where coordinate x_i , $i = 1, 2, 3$ is scaled by the deformation parameters α_i , i.e., $\xi_i = x_i/\alpha_i$. The deformation parameters satisfy the condition of volume conservation, $\alpha_1 \alpha_2 \alpha_3 = \text{const.}$ This potential includes two special cases which are often studied in nuclear physics, namely, the anisotropic harmonic oscillator potential ($n = 2$) and the Hill-Wheeler box ($n \rightarrow \infty$).

We now show that for this potential the equilibrium deformation implies the equilibrium distribution of the kinetic energy in each x_i -direction, and vice versa. We note first that the Schrödinger equation is separable with this potential. In fact the separated equations,

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{d\xi_i^2} - U(\xi_i) \right] X_i(\xi_i) = \lambda_i X_i(\xi_i), \quad (F.2)$$

have no explicit dependence on α_i because of the scaling property defined in (F.1b). (Note that the index $i = 1, 2, 3$ signifies different equations. They are not to be confused with the index usually used to signify the different eigenvalues for a fixed equation. The latter kind of index is suppressed here.) By the relationship between the energy and the eigenvalues λ_i which are independent of α_i , we can obtain the energy of a single particle state as a function of deformation, as

$$\epsilon = \frac{\lambda_1}{\alpha_1^2} + \frac{\lambda_2}{\alpha_2^2} + \frac{\lambda_3}{\alpha_3^2}. \quad (F.3)$$

For simplicity, we consider the "axially symmetric" case, $\alpha_1 = \alpha_2$, and define the new deformation parameter $\beta = \alpha_3/\alpha_1$. In terms of β , the equilibrium condition is

$$\frac{dE}{d\beta} \equiv \frac{d}{d\beta} \sum_{\text{occ}} \epsilon = 0. \quad (F.4)$$

From eqs. (F.3) and (F.4), we obtain the equilibrium deformation β_0 as

$$\beta_0^2 = \sum_{\text{occ}} \lambda_3 / \sum_{\text{occ}} \lambda_1. \quad (F.5)$$

By eq. (F.5), one obtains

$$\sum_{\text{occ}} \frac{\lambda_1}{\alpha_1^2} = \sum_{\text{occ}} \frac{\lambda_2}{\alpha_2^2} = \sum_{\text{occ}} \frac{\lambda_3}{\alpha_3^2}, \quad (F.6)$$

at the equilibrium deformation.

In order to prove the equilibrium distribution of the kinetic energy, we use the Virial theorem. When applied to the separated equations (F.2), the Virial theorem gives

$$\lambda_i = \frac{n+2}{n} \langle T_{\xi_i} \rangle = \frac{n+2}{2} \langle U(\xi_i) \rangle, \quad (F.7)$$

where T_{ξ_i} is the kinetic energy operator for the coordinate ξ_i and the average is taken on the state $X_i(\xi_i)$. Dividing this equation by α_i^2 , we have the following relationship between the kinetic energy and the potential energy in the ordinary space:

$$\frac{\lambda_i}{\alpha_i^2} = \frac{n+2}{n} \langle T_i \rangle = \frac{n+2}{2} \langle V_i \rangle. \quad (F.8)$$

Since the full wave function for a single particle is simply the product $X_1 X_2 X_3$, we can also regard the average in (F.8) as taken on the full wave function.

Since eq. (F.8) holds for any single particle state, we can substitute it into

each term in eq. (F.6) and obtain

$$\sum_{\text{occ}} \langle T_1 \rangle = \sum_{\text{occ}} \langle T_2 \rangle = \sum_{\text{occ}} \langle T_3 \rangle \quad (\text{F.9})$$

and

$$\sum_{\text{occ}} \langle V_1 \rangle = \sum_{\text{occ}} \langle V_2 \rangle = \sum_{\text{occ}} \langle V_3 \rangle. \quad (\text{F.10})$$

Equation (F.9) shows that the condition for the equi-distribution of kinetic energy is satisfied. We note that from eq. (F.4) to (F.10) every step follows from the previous ones necessarily and sufficiently. Therefore, we conclude that the equilibrium condition and the condition of equi-distribution of kinetic energy are equivalent for the whole family of separable scaling potentials defined by eq. (F.1).

In addition to eq. (F.9), we also obtained (F.10) which shows that at the equilibrium deformation the potential energy of the system is equally distributed in the three directions. For the special case where $n = 2$, this separable scaling potential is the anisotropic harmonic oscillator with angular frequencies $\omega_i = \alpha_i^{-2}$. For this case, eq. (F.10) takes the form

$$\omega_1^2 \sum_{\text{occ}} \langle x_1^2 \rangle = \omega_2^2 \sum_{\text{occ}} \langle x_2^2 \rangle = \omega_3^2 \sum_{\text{occ}} \langle x_3^2 \rangle. \quad (\text{F.11})$$

This is the condition of Potential-Density Self-Consistent Anisotropy for the harmonic oscillator potential used by Bohr and Mottelson³⁵.

For the other potentials in the family, eq. (F.10) implies

$$\alpha_1^{-n-2} \sum_{k=1}^A \langle x_1^n \rangle_k = \alpha_2^{-n-2} \sum_{k=1}^A \langle x_2^n \rangle_k = \alpha_3^{-n-2} \sum_{k=1}^A \langle x_3^n \rangle_k, \quad (\text{F.12})$$

which is one possible generalization of the Potential-Density Self-Consistent Anisotropy Condition in which the y-extension of the matter density is measured by the average of y^n , and the y-extension of the equi-potential surface, $V = V_0$, by $(V_0/U_0)(\alpha_2)^{(n+2)/n}$. Then (F.12) becomes identically the Potential-Density Self-Consistent Anisotropy Condition.

G. The parallel axis theorem for the cranking moment of inertia

The parallel axis theorem for moments of inertia is a fundamental property of classical motion. However, to our knowledge, no parallel axis theorem has been established for rotating quantum systems. We obviate this deficiency, at least for systems which can be described by the cranking model, in the following discussion. We also discuss some of its uses, and some of its limitations.

The parallel axis theorem of classical mechanics states that if $\mathcal{J}$ and $\mathcal{J}'$ are the moments of inertia of the system about an axis (z-axis) through its center of mass and another axis (z'-axis) parallel to the z-axis, respectively, then

$$\mathcal{J}' = \mathcal{J} + AMD^2 \quad (\text{G.1})$$

where AM is the mass of the body and D , the distance between z- and z'-axes.

This theorem, as we show, applies in classical mechanics to any system whose matter flow obeys the continuity equations, (2.3). Hence we can also extend this to quantum moments of inertia calculated by the cranking formula, provided continuity is satisfied.

This transformation (G.1) of the moment of inertia can be obtained by considering the transformation of the velocity fields involved in the rotation. Let O and O' be the origins of the two coordinate systems involved in the transformation. The position $\mathbf{r}'$ in the O' system is related to the position $\mathbf{r}$ in the O system by

$$\mathbf{r}' = \mathbf{r} + \mathbf{R} \quad (\text{G.2})$$

where $\mathbf{R}$ is the position of O in the O' system.

We consider the continuity equation for a classical system whose boundary and density distribution ρ in the body-fixed frame are time independent.

Since $\mathcal{O}$ is a space-fixed frame, we have in $\mathcal{O}$

$$\frac{\partial \rho(\mathbf{r})}{\partial t} = -\mathbf{r} \times \boldsymbol{\omega} \cdot \nabla \rho(\mathbf{r}). \quad (G.3)$$

Then the continuity equation in the $\mathcal{O}$ system becomes

$$\rho(\mathbf{r}) \nabla \cdot \mathbf{v} + \nabla \rho(\mathbf{r}) \cdot \mathbf{v} = -\mathbf{r} \times \boldsymbol{\omega} \cdot \nabla \rho(\mathbf{r}). \quad (G.4)$$

Denoting the quantities in the $\mathcal{O}'$ -system by a prime, we can similarly write down the continuity equation for the rotation about the z' -axis, obtaining

$$\begin{aligned} \rho'(\mathbf{r}') \nabla' \cdot \mathbf{v}' + \nabla' \rho'(\mathbf{r}') \cdot \mathbf{v}' &= -\mathbf{r}' \times \boldsymbol{\omega}' \cdot \nabla' \rho'(\mathbf{r}') \\ &= -(\mathbf{r} \times \boldsymbol{\omega} + \mathbf{R} \times \dot{\mathbf{R}}) \cdot \nabla' \rho'(\mathbf{r}'). \end{aligned} \quad (G.5)$$

Here, we have made use of the transformation (G.2) in obtaining the last line.

However, because of (G.2), we have $\nabla' = \nabla$, and since the same density distribution is considered, we have $\rho'(\mathbf{r}') = \rho(\mathbf{r})$. Hence, eq. (G.5) can be written as

$$\rho(\mathbf{r}) \nabla \cdot \mathbf{v}' + \nabla \rho(\mathbf{r}) \cdot \mathbf{v}' = -(\mathbf{r} \times \boldsymbol{\omega} + \mathbf{R} \times \dot{\mathbf{R}}) \cdot \nabla \rho(\mathbf{r}). \quad (G.6)$$

Comparing this equation with (G.4), we obtain the transformation of the velocity field as

$$\mathbf{v}'(\mathbf{r}') = \mathbf{v}(\mathbf{r}) + \mathbf{R} \times \dot{\mathbf{R}}. \quad (G.7)$$

One can also see that $\mathbf{v}'$ satisfies the boundary condition in the $\mathcal{O}'$ -system since the boundary of the system is assumed to be time independent in the body-fixed frame.

We note that in deriving (G.7), we do not make any specific assumption about the circulation ($\nabla \times \mathbf{v}$). In fact, eq. (G.7) shows that the transformation preserves the circulation since $\mathbf{v}'(\mathbf{r}')$ and $\mathbf{v}(\mathbf{r})$ differ by only a constant velocity field $\mathbf{R} \times \dot{\mathbf{R}}$, i.e., we have

$$\nabla' \times \mathbf{v}'(\mathbf{r}') = \nabla \times \mathbf{v}(\mathbf{r}). \quad (G.8)$$

Since the continuity equation specifies the divergence distribution and since the divergence and the circulation distributions together uniquely determine the velocity field, it follows that $\mathbf{v}'(\mathbf{r}')$ in (G.7) is the unique velocity field describing the system in rotation about the z' -axis.

With the known field $\mathbf{v}'$, we can express the rotational kinetic energy of the system about $\mathcal{O}'$ in terms of the velocity field for rotation about $\mathcal{O}$. We obtain

$$\begin{aligned} T' &= \frac{AM}{2} \int \rho'(\mathbf{r}') \mathbf{v}'^2 d\tau' \\ &= \frac{AM}{2} \int \rho(\mathbf{r}) \mathbf{v}^2 d\tau + \frac{AM}{2} \Omega^2 D^2 \int \rho(\mathbf{r}) d\tau \\ &\quad + AM \mathbf{R} \times \dot{\mathbf{R}} \cdot \int \rho(\mathbf{r}) \mathbf{v} d\tau. \end{aligned} \quad (G.9)$$

Since by definition, $\mathcal{O}$ is the center of mass of the system, the total momentum with respect to $\mathcal{O}$, given by the integral in the third term of (G.9), vanishes identically. Then the transformation of the collective kinetic energy (G.9) reduces to

$$T' = T + \frac{AM}{2} \Omega^2 D^2, \quad (G.10)$$

which is just the parallel axis theorem (G.1) (multiplied by the constant $\Omega^2/2$) for the cranking moment of inertia.

For the quantum cranking model formula, the result (G.1) then follows at once: by eq. (2.27) the cranking model kinetic energy is an integral over a scalar product of the two velocities $\mathbf{v}_{\text{ex}}$ and $\mathbf{R} \times \dot{\mathbf{R}}$, to both of which eq. (G.7) and the continuity condition (G.4) applies. Hence the result (G.10) also follows.

We should remark here that although the parallel axis transformation (G.1) holds for all kinds of flows, one has to be careful in applying it to the case where the flow is not equal to the velocity field for rigid-body rotation. For a rigid body, the parallel axis transformation provides a convenient method in calculating the moment of inertia: one can divide the body into

several parts and then apply the parallel axis transformation to the moments of inertia of the separate parts. This procedure is justified because the boundary of each divided part also rotates rigidly, or in other words, because this boundary is time independent in the body-fixed frame of this divided part.

If, however, the flow is not equal to that of rigid rotation, the boundary of a divided part of the body may deform as time evolves and the parallel axis transformation is not applicable to such a part of the body. Therefore, one can apply the parallel axis transformation to a non-rigid body as a whole, but not necessarily to separate parts of such a body.

In calculating the moment of inertia of a deformed nucleus by the two center model[†], the parallel axis transformation has sometimes been applied to two separate parts of the nucleus. According to the above remark, the moment of inertia calculated in this way may be in error because the cranking model velocity field need not be equal to the velocity field of rigid rotation.

[†]See ref. 25), and footnote to sect. 1.

References

- 1) K.-K. Kan and J. J. Griffin, Phys. Rev. C15 (1977) 1126.
- 2) K.-K. Kan, U. of Maryland Ph.D. thesis, College Park, Maryland (1975).
- 3) J. J. Griffin and K.-K. Kan, Rev. Mod. Phys. 48 (1976) 467.
- 4) J. J. Griffin and K.-K. Kan, Proc. Third Int. Conf. on the physics and chemistry of fission, Rochester, 1973 (I.A.E.A., Vienna, 1974) vol. I, p. 533.
- 5) K.-K. Kan and J. J. Griffin, Bull. Am. Phys. Soc. 18 (1973) 647; 19 (1974) 526; 19 (1974) 996; 20 (1975) 581; and 22 (1977) 529.
- 6) J. J. Griffin and K.-K. Kan, Proc. Third Int. Workshop on gross properties of nuclei, Hirschegg, 1975, ed. W. J. Myers (Inst. f. Kernphysik, Tech. Hoch. Darmstadt, report AED-Conf.-75-009-000 (unpublished) p. 56).
- 7) A. Bohr, Dan. Mat.-Fys. Medd. K. Dan. Vid. Selsk. 26 (1952) no. 14; Rotational states of atomic nuclei (E. Munksgaards Forlag, Copenhagen (1954)); and A. Bohr and B. R. Mottelson, Dan. Mat.-Fys. Medd. K. Dan. Vid. Selsk. 27 (1953) no. 16.
- 8) J. D. Hill and J. A. Wheeler, Phys. Rev. 89 (1953) 1102, especially p. 1109 and figs. 10-12, pp. 1124-1125.
- 9) D. R. Inglis, Phys. Rev. 96 (1955) 1059; Phys. Rev. 103 (1956) 1786.
- 10) K. W. Ford, Phys. Rev. 90 (1953) 29, and Phys. Rev. 95 (1954) 1250.
- 11) A. Bohr and B. R. Mottelson, Dan. Mat.-Fys. Medd. K. Dan. Vid. Selsk. 30 (1955) no. 1.
- 12) S. T. Belyaev, Dan. Mat.-Fys. Medd. K. Dan. Vid. Selsk. 31 (1959) no. 11.
- 13) J. J. Griffin and M. Rich, Phys. Rev. Lett. 3 (1959) 342.
- 14) J. J. Griffin and M. Rich, Phys. Rev. 118 (1960) 850.
- 15) L. S. Kisslinger and R. A. Sorensen, Dan. Mat.-Fys. Medd. K. Dan. Vid. Selsk. 32 (1960) no. 9.

- 16) S. G. Nilsson and O. Prior, Dan. Mat.-Fys. Medd. K. Dan. Vid. Selsk. 32 (1961) no. 16.
- 17) M. Mariscotti, G. Scharf-Goldhaber and B. Buck, Phys. Rev. 178 (1969) 864.
- 18) R. A. Sorensen, Rev. Mod. Phys. 45 (1973) 353.
- 19) Heavy ions high-spin states and nuclear structure, vols. 1 and 2, lectures at Trieste, 1973 (I.A.E.A., Vienna, 1975).
- 20) M. Bohr, Studier over metallernes elektrontheori (Thanning og Appel, Copenhagen, 1911) as reproduced in Niels Bohr Collected Works, vol. 1, edited by J. R. Nielsen (North-Holland, Amsterdam, 1972) p. 167.
- 21) L. Wilets, Theories of nuclear fission (Clarendon Press, Oxford, 1964) p. 73.
- 22) R. D. Amado and K. A. Brueckner, Phys. Rev. 115 (1959) 778.
- 23) R. M. Rockmore, Phys. Rev. 116 (1959) 469.
- 24) A. B. Migdal, Nucl. Phys. 13 (1959) 655.
- 25) M. G. Mustafa and K. Kumar, Phys. Lett. 49B (1974) 405.
- 26) E. Madelung, Z. Phys. 40 (1926) 322.
- 27) J. J. Griffin, Proc. Second Int. Conf. on the physics and chemistry of fission (I.A.E.A., Vienna, 1969) pp. 12-14.
- 28) J. J. Griffin, Nucl. Phys. A170 (1971) 395.
- 29) G. C. Wick, Phys. Rev. 73 (1948) 51.
- 30) M. Radomski, Phys. Rev. C14 (1976) 14, and Stanford U. Ph.D. thesis (1974), unpublished.
- 31) K. Kawasaki and I. Oppenheim, Phys. Rev. 139 (1965) 1763.
- 32) J. R. Dorfman and E. G. D. Cohen, J. of Math. Phys. 8 (1967) 282.
- 33) Y. Kan and J. R. Dorfman, Phys. Rev. A (in press).
- 34) A. deShalit and H. Feshbach, Theoretical nuclear physics I: nuclear structure (John Wiley and Sons, Inc., New York, 1974) p. 425.
- 35) A. Bohr and B. R. Mottelson, Nuclear structure, vol. 2 (W. A. Benjamin, Inc., New York, 1975) chapt. 4.

- 36) J. J. Griffin, Proc. Fall Creek Falls Meeting on heavy ion collisions, Pikeville, 1977 (Oak Ridge National Laboratory report CONF-770602, Oak Ridge, Tennessee); and Proc. Int. Workshop on gross properties of nuclei, IV, Eirschegg, 1977 (Inst. f. Kernphysik Tech. Hoch. Darmstadt, report AED-Conf.-75-015-000).
- 37) G. Ripka, J. P. Blaizot and H. Kassis, in Heavy ion, high-spin states and nuclear structure, lectures at Trieste, 1973 (I.A.E.A., Vienna, 1975) vol. 1, p. 455.
- 38) J. Riess, Ann. of Phys. (N.Y.) 57 (1970) 301 and 67 (1971) 347; Phys. Rev. 2D (1970) 647 and 31 (1976) 3862.
- 39) W. D. Myers and W. J. Swiatecki, Ann. of Phys. (N.Y.) 55 (1969) 395 and 84 (1973) 186.
- 40) P. M. Morse and H. Feshbach, Method of theoretical physics, ch. 9.2 (McGraw-Hill, New York, 1953) p. 1060.

Table Caption

The table lists the cranking model moment of inertia, $\mathcal{J}_{(m,n)}^{HW}$ its ratio to rigid [in square parentheses] and the least energy denominator, d_{MIN} , involved in the moment calculation for single particles in a Hill-Wheeler box. It illustrates (a) that the moments deviate substantially from the rigid values, (b) that the nearly vanishing energy denominators which occur systematically for states with $n = m \pm 1$ lead to very large moments (column 2), (c) that the sign and magnitude of the moments tracks qualitatively with the inverse of the smallest energy denominator, d_{MIN} , except when the two smallest denominators have different signs and similar (within 2x) magnitudes (entries **). For the prolate shapes, the smallest denominator is either d_{-1}^{+1} or d_{-1}^{+3} (indicated by *). The moments are in units (ML_z^2) ; the energies, $(\hbar^2 \pi^2 / 2ML_z^2)$. All results are at the equilibrium shape. Moments for $n < m$ can be obtained by using the $n \leftrightarrow m$ symmetry mentioned in the text.

Table 1. Moments of Inertia for One Particle in a Hill-Wheeler Box (a,m,n)-State

[illegible]

Figure Captions

Figure 1: Irrotational incompressible velocity field v_I for the rotation of the harmonic oscillator potential at a prolate shape. This velocity field comprises the complete irrotational velocity field, v_0 , for the single particle state $|2mn\rangle = |200\rangle$, but constitutes only the regular (vortex-free) part of v_0 for any other single particle states. [See eqs. (3.8) - (3.11).] In this and the following two figures, the boundary curve shows the shape of the nuclear equi-potential surfaces, and the distances between the streamlines are chosen arbitrarily.

Figure 2: The irrotational velocity field v_0 for rotation of the harmonic oscillator state $|2mn\rangle = |001\rangle$, at its equilibrium deformation. This state exhibits one vortex at the origin. See also the caption for fig. 1.

Figure 3: Irrotational velocity field v_0 for rotation of the harmonic oscillator state $|2mn\rangle = |112\rangle$, at its equilibrium deformation. The wealth of possible structure is evident here, with seven vortices in evidence. See also the caption for fig. 1.

Figure 4: Moments of inertia for the case of harmonic oscillator potential shown as functions of the deformation of the potential, $\gamma = (\omega_y - \omega_z)/(\omega_y + \omega_z)$, and the anisotropy of the configuration, $q = \sum_{occ} (m+1/2) / \sum_{occ} (n+1/2)$. Figure (a) is for the cranking moment of inertia $\mathcal{J}$, and fig. (b), for the rigid moment $\mathcal{J}_{rig}$. In both (a) and (b), the trajectory of the point of intersection between the two families of curves $\mathcal{J}$ and $\mathcal{J}_{rig}$ is also shown, and is denoted by $\mathcal{J}_e$. The value $\mathcal{J}_e$ is also the moment of inertia of the nucleus

at the shape equilibrium deformation of a fixed q . The open circle at the left end of the curve $\mathcal{J}_e$ and the black dot at the origin in (a) signify the fact that the equilibrium moment of inertia for the spherical configuration ($q=1$) is zero, when the spherical limit is approached from finite deformation, γ . If, on the other hand, the spherical limit is approached by diminishing the anisotropy q of the configuration systematically towards the isotropic value, $q=1$, and calculating the moment at the appropriate small equilibrium deformation, then the "spherical" limit is given by the rigid spherical value, indicated by the open circle. The discontinuity between the black dot and the open circle at $\gamma=0$, therefore reflects the difference between a system whose equilibrium deformation is identically zero, and one whose equilibrium deformation is approximately zero. In one case a quantum symmetry applies and the moment is identically zero. In the other, no exact spherical symmetry enters and the moment has the rigid value.

Figure 5: Single particle levels of the Hill-Wheeler box as functions of $\beta = L_z/L_y$, in the vicinity of the minimum of the level, $(2mn) = (223)$. Only the levels which have non-zero matrix elements of the x-component of the angular momentum, $\mathcal{L}_x$, with the level (223) are shown. The dotted curves represent the levels obtained when the operator, $\mathcal{L}_x$, is diagonalized between levels, (223) and (214). We note that for a sequence of spheroidal shapes (rather than parallelepipeds) the (214) - (223) degeneracy would occur at the spherical shape, causing, therefore, no strong variation of any deformed moment of inertia.

Figure 6: Ratio of the cranking moment of inertia with the rigid value as a function of A for the Amado-Brueckner box with cubical geometry; the dots represent the exact numerical values. The heavy solid curve is a smooth average obtained from the exact values. The middle light solid curve shows the analytic smooth average for large A [eq. (5.25)]. The upper and lower solid curves are the analytic estimate for the limits of fluctuation about the smooth average. The dashed sawtooth curve shows the oscillation described by eq. (5.26) added to the average value (heavy solid curve).

Figure 7: Ratio of the cranking moment of inertia to the rigid value as a function of A for the Hill-Wheeler box with cubical geometry. (Note that eight times as many particles are accommodated up to a given Fermi energy in an Amado-Brueckner box $(\pm l, \pm m, \pm n)$ as by a Hill-Wheeler box $(+l, +m, +n)$. Therefore, the number of particles in this figure is eight times less than the comparable value of A in fig. 6.) Figures 6 and 7 suggest that the choice between periodic and Dirichlet boundary conditions does not alter the qualitative behavior of the moment. In both cases, substantial fluctuation about the average and a slow approach towards the rigid value as $A \rightarrow \infty$ characterize the exact results. Figures 6 and 7 clearly contradict the conjecture that at equilibrium the Hill-Wheeler box moments are equal to their rigid values. (See sect. 6 of text.)

Figure 8: Fermi sphere for the Amado-Brueckner box. For simplicity, only an octal of the Fermi sphere is shown. Two slabs, one across the lm -plane and the other across the ln -planes, are cut out from the

sphere as to guarantee that (a) $n \geq 1$, and (b) the remaining volume to be equal to the correct number of particles in the box, to order λ^2 .

Figure 9: A crossing of two single particle levels near the minimum of one of the two. The energies ϵ_1 and ϵ_2 of states $|1\rangle$ and $|2\rangle$ are shown to cross each other at the deformation α_c near the minimum of ϵ_1 at the deformation α_1 . At α_c , $\epsilon_1 = \epsilon_2 = \epsilon_c$. The curves e_a and e_b denote the eigenvalues of the states $|a\rangle$ and $|b\rangle$ which diagonalize the cranking Hamiltonian, $\tilde{H} = \Omega \mathcal{L}_x$, in the subspace of $|1\rangle$ and $|2\rangle$. The curve ϵ_a^0 denotes the energy $\langle a|H|a\rangle$. The position of the minimum of ϵ_a^0 , before it disappears as Ω increases to a sufficient value, is denoted by α_a .

Figure 10: Fermi spheroid in the space of $\hat{m}$ and $\hat{n}$ for a fixed l . Also shown are the quantities p, q and v, θ defined in eqs. (C.6) and (C.9), respectively. The shaded area represents the domain of the pq -summation in eq. (C.8).

Figure 11: Triangular shaped Fermi sea for the harmonic oscillator potential with a fixed l value. The dots represent the occupied states of the system. The larger dots represent those occupied states with the maximum m value for fixed l and n . The arrows represent the terms to be summed in calculating the total current for the system in rotation.

IRROTATIONAL INCOMPRESSIBLE VELOCITY FIELD, S.H.O.

Fig. 1

S.H.O. IRROTATIONAL VELOCITY FIELD

$$|l_{mn}\rangle = |001\rangle$$

Fig. 2

S.H.O. IRROTATIONAL VELOCITY FIELD

$$|\ell mn\rangle = |112\rangle$$

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11