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Dynamo Theory: Can Amplification of Magnetic- Field Profiles Arise from a Cross-Field Alpha Effect?

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DYNAMO THEORY: CAN AMPLIFICATION OF MAGNETIC FIELD
PROFILES ARISE FROM A CROSS-FIELD ALPHA EFFECT?

by

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ABSTRACT

The answer to the title question is here provided in cylindrical geometry for an essentially arbitrary radial dependence of this alpha effect, except that it is subject to a simple, physically required constraint. The interest in this type of alpha effect derives from its connection with compressible turbulence, which is usually not considered in kinematic dynamo theory.

I. INTRODUCTION: THE ALPHA EFFECT

It can be shown that the reversed -field pinch cannot exist in a cylindrically symmetric, resistive steady state. Consequently, there must be superposed on such steady, symmetric field profiles either a time-dependent sea of fluctuations, or some amount of asymmetry. (An example of the latter is the "Helical Ohmic State.") In either case, there will appear in Ohm's law for the cylindrically symmetric portions of the fields, an extra term arising from the products of fluctuations,

$$\vec{E} + \langle \delta \vec{V} \times \delta \vec{B} \rangle = \eta \vec{J} \quad , \quad (1)$$

in which the brackets signify an appropriate average over the space or time structure of the fluctuations. This extra term gives rise to what is known as the alpha effect.

Now, if one considers the Ohm's law for the fluctuating quantities, and takes the curl, one finds

$$\frac{\partial}{\partial t} \vec{\delta B} - D \nabla^2 \vec{\delta B} = \vec{B} \cdot \nabla \vec{\delta V} - \vec{B} \nabla \cdot \vec{\delta V} \quad , \quad (2)$$

in which products of the small fluctuating quantities have been neglected, and $D = \eta/\mu_0$ is the resistive diffusion coefficient. The mean field, $\vec{B}$, is here assumed to vary on a spatial scale much larger than the characteristic lengths of the fluctuations.

For fluctuations of short wavelength compared to the mean field profiles, one can write the formal solution to Eq. (2) as

$$\vec{\delta B}(\underline{r}, t) = \frac{1}{(2\pi)^3} \int_0^t dt' \int d^3K \exp[i\underline{K} \cdot \underline{r} - (t-t')K^2 D] \{ (\vec{B} \cdot \underline{K}) \vec{\delta U}(\underline{K}, t') - \vec{B} [\underline{K} \cdot \vec{\delta U}(\underline{K}, t')] \} \quad , \quad (3)$$

where a transient term has been ignored (requiring $D \neq 0$) and $\vec{\delta U}(\underline{K}, t)$ represents the Fourier transform of $\vec{\delta V}(\underline{r}, t)$.

From the general form of Eq. (3), we can see that $\langle \vec{\delta V} \times \vec{\delta B} \rangle$ must have the general form $\vec{\alpha} \cdot \vec{B}$, where $\vec{\alpha}$ is a pseudo-tensor that depends on the turbulent velocity field (which in turn depends upon the mean magnetic field profiles in a self-consistent sense). Some simple examples of this alpha effect will now be discussed.

II. THE PARALLEL ALPHA EFFECT

For a statistically spherically symmetric and incompressible field of velocity fluctuations, it can be shown that $\vec{\alpha}$ is a simple (pseudo) scalar object. Then, Eq. (1) becomes

$$\vec{E} + \alpha \vec{B} = 0 \quad , \quad (4)$$

provided we ignore resistivity. This equation then governs the evolution of the mean-field profiles. To exhibit a simple example, we shall suppose that α is a constant.

Take the curl of Eq. (4). Then

$$-\frac{\partial \vec{B}}{\partial t} + \alpha \nabla \times \vec{B} = 0 \quad . \quad (5)$$

Take the time derivative of Eq. (5) and use Eq. (5). Then

$$-\frac{\partial^2 \vec{B}}{\partial t^2} + \alpha^2 \nabla \times \nabla \times \vec{B} = 0 \quad . \quad (6)$$

If we look at a Fourier component of the mean field, say $\vec{B}(\vec{k})$, we find, from Eq. (6),

$$\frac{\partial^2 \vec{B}}{\partial t^2} - \alpha^2 k^2 \vec{B} = 0 \quad , \quad (7)$$

with the time dependence $e^{\gamma t}$ where $\gamma = \pm |\alpha| |k|$.

Thus, for general initial conditions, the parallel alpha effect yields amplification of some constituents, regardless of the sign of α . Including resistivity, one likewise finds amplification (exponentially growing field profiles) for those Fourier components with $|\alpha| > |k|D$. The following observation (made by Leaf Turner) points up an interesting connection of this amplification phenomenon with force-free fields and their helicities.

Substitution of the time dependence $\exp(\gamma t)$ back into Eq. (5) yields, for the field having that particular $\gamma(\vec{k})$,

$$\nabla \times \vec{B}_\gamma = \frac{\gamma}{\alpha} \vec{B}_\gamma \quad . \quad (8)$$

Thus, in the parallel alpha effect, each growing portion of the mean field

profiles will be force free, with a current helicity ($\vec{B}_Y \cdot \nabla \times \vec{B}_Y$) whose sign is the same as the sign of α . (The damped solution has a helicity of the opposite sign.) Thus, the parallel alpha effect displays the possibility of sustainment of global field profiles against their resistive evolution.

An example of the amplifying solution of Eq. (8) is (for a coordinate system $\hat{x}, \hat{y}, \hat{k} = \hat{z}$),

$$B_x = B_0 \cos \left(\frac{Y}{\alpha} z \right) \quad \text{and}$$

$$B_y = - B_0 \sin \left(\frac{Y}{\alpha} z \right) ,$$

which constitutes a stationary circularly polarized magnetic structure. This structure resembles a left-handed screw along z , for positive α , and a right-handed screw for negative α .

III. THE CROSS-FIELD ALPHA EFFECT

From Eq. (3), we note the possibility that $\langle \delta \vec{V} \times \delta \vec{B} \rangle$ can have a contribution of the form $(\vec{\alpha} \times \vec{B})$. This is especially obvious from the form of the term that explicitly involves compressible fluctuations $(\vec{K} \cdot \delta \vec{U}) \vec{B}$.

In an infinite, homogeneous magnetoplasma, the vector $\vec{\alpha}$ in $\vec{\alpha} \times \vec{B}$ does not "know" which way to point (since $\vec{B}$ is the only preferred direction). Thus, it must vanish. However, in a finite plasma cylinder, wherein the turbulence is "quasi-local", with some "knowledge" of the global geometry, the vector $\vec{\alpha}$ does "know" which way to point, because $\vec{B}$ is not the only preferred direction. In fact, in a cylindrically symmetric pinch in which the mean field profiles depend only upon the cylindrical radius r , we must have

$$\vec{\alpha} = \alpha(r) \hat{r} , \tag{9}$$

so that $\vec{\alpha} \times \vec{B}$ is coupled to an inductive electric field lying within the cylindrical flux surfaces of the mean magnetic field. The other possibility, that the vector $\vec{\alpha}$ lies in the flux surface, produces a radial electric field of vanishing curl that precludes amplification of magnetic profiles.

Furthermore, in order that the radial vector $\vec{\alpha}$ be well defined at the origin, we must have $\alpha(r) \rightarrow 0$ as $r \rightarrow 0$.

As the simplest possibility, we shall take

$$\alpha(r) = \sigma_0 r \quad (r \rightarrow 0) , \quad (10)$$

where σ_0 is a constant.

Then, the governing equation is

$$\vec{E} + \vec{\alpha} \times \vec{B} = 0 , \quad (11)$$

where we ignore resistivity for simplicity. In cylindrical coordinates, the curl of this equation then reads

$$-\frac{\partial \vec{B}}{\partial t} - \frac{\vec{B}}{r} \frac{\partial}{\partial r} (r\alpha) + \alpha \frac{B_\theta}{r} \hat{\theta} - \alpha \frac{\partial \vec{B}}{\partial r} = 0 , \quad (12)$$

where $\hat{\theta}$ is a unit vector in the azimuthal direction, and $\alpha(r)$ has an arbitrary radial dependence except for the constraint at the origin [Eq. (10)].

The vector equation, Eq. (12), constitutes two uncoupled equations for the components B_θ and B_z . We seek solutions subject to the physical restrictions imposed by cylindrical symmetry of the mean fields, namely,

$$B_\theta \rightarrow r, B_z \rightarrow \text{const.}, \text{ as } r \rightarrow 0 . \quad (13)$$

The B_θ equation can be reduced to

$$\frac{\partial(\alpha B_\theta)}{\partial t} + \alpha \frac{\partial(\alpha B_\theta)}{\partial r} = 0 , \quad (14)$$

with the general wave-type or hyperbolic solution

$$B_{\theta}(r,t) = \frac{\text{const.}}{\alpha(r)} f_{\theta} \left(\int_{r_0}^r \frac{dr}{\alpha(r)} - t \right) , \quad (15)$$

where $f_{\theta}(y)$ is, thus far, an arbitrary function.

Because $\alpha \rightarrow \sigma_0 r$ as $r \rightarrow 0$, it follows that

$$\int_{r_0}^r \frac{dr}{\alpha} \rightarrow \frac{1}{\sigma_0} \ln r \text{ as } r \rightarrow 0 .$$

Therefore, in order that $B_{\theta} \rightarrow r$ as $r \rightarrow 0$, the function $f_{\theta}(y)$ must have the form

$$f_{\theta}(y) = \exp (2 \sigma_0 y) . \quad (16)$$

Therefore, Eq. (15) becomes

$$B_{\theta}(r,t) = \frac{\text{const.}}{\alpha(r)} \exp [2 \sigma_0 \left(\int_{r_0}^r \frac{dr}{\alpha(r)} - t \right)] . \quad (17)$$

Clearly, the length r_0 is not significant, because a change in r_0 is equivalent to a change in the multiplicative constant factor in front.

The B_z component of Eq. (12) can be reduced to

$$\frac{\partial(\alpha r B_z)}{\partial t} + \alpha \frac{\partial(\alpha r B_z)}{\partial r} = 0 , \quad (18)$$

which is identical to Eq. (14) with B_{θ} replaced by $(r B_z)$.

Since $(rB_z) \rightarrow r$ as $r \rightarrow 0$, the solution for (rB_z) must be given again by Eq. (17), to within a multiplicative constant.

IV. DISCUSSION

The solutions for B_θ and B_z clearly grow together exponentially in time provided $\sigma_0 < 0$. This means that the vector $\vec{\alpha}$ points radially inwards near the origin.

However, these fields satisfy a very special initial condition, namely, $B_\theta(r) = \text{const.} \times [r B_z(r)]$. From this, it can be shown that they are not, in general, force-free.

Also, these growing solutions do not admit field reversal. To see this, note that $\alpha(r)$ would have to change sign from negative to positive, say linearly at $r = R$. (Take $\alpha = (\Omega)(r-R)$ near $r = R$.) If we choose $r_0 = R - \epsilon$, then, for $r < R - \epsilon$, and for $r \rightarrow R - \epsilon$,

$$\int_{r_0}^r \frac{dr'}{\alpha(r')} = \int_{R-\epsilon}^r \frac{dr'}{(\Omega)(r'-R)} \rightarrow \frac{1}{\Omega} \ln \left[\frac{(R-r)}{\epsilon} \right] ,$$

so that

$$B_z \sim \frac{(R-r)^{\left(\frac{2\sigma_0}{\Omega}\right)}}{\Omega(R-r)} \sim (R-r) \left[\frac{2\sigma_0}{\Omega} - 1 \right] .$$

Since $\alpha(r)$ is negative but increasing towards zero, we must have $\Omega > 0$. For growing (in time) fields, we have $\sigma_0 < 0$. Therefore, the dependence $(R-r)[2\sigma_0/\Omega - 1]$ is singular and inadmissible at $r \rightarrow R$. This behavior of $\alpha(r)$ can only be associated with fields that vanish everywhere!

In conclusion, we have shown that the cross-field alpha effect, in isolation, can produce amplification of special field profiles, just as can the parallel alpha effect. The significance for the more general problem including both effects together with resistivity is under investigation.