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*THERM: A Three-Dimensional Transient
Heat Conduction Computer Program*

Los Alamos

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*THERM: A Three-Dimensional Transient
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NOMENCLATURE

c	Specific heat or heat capacity [J/(kg K)].
D^α	Element matrix defined by Eq. (21).
D	Global matrix of D^α .
e_r	Unit vector in r direction of the cylindrical coordinate system.
e_ϕ	Unit vector in ϕ direction of the cylindrical coordinate system.
F^α	Element matrix defined by Eq. (20).
F	Global matrix of F^α .
$G^{\alpha\beta}$	Element matrix defined by Eq. (19).
G	Global matrix of $G^{\alpha\beta}$.
h	Convective heat-transfer coefficient.
i	Unit vector in x direction of the Cartesian coordinate system.
j	Unit vector in y direction of the Cartesian coordinate system.
k	Unit vector in z direction of the Cartesian coordinate system.
k_x	Conductivity for x direction [W/(m K)]. Similar conductivities are defined for the y , z , r , and ϕ directions.
k_{ji}	Conductivity tensors.
kg	Kilograms.
K	Degrees Kelvin.
J	Joules.
m	Meters.
n	Unit normal vector.
n_i	Component of the normal tensor.
p^α	Finite-element shape functions.
q_n	Normal heat flux (W/m ²).
q	Heat flux vector.
q^i	Tensor component of the heat flux.
Q	Internal heat source density (W/m ³).
r	Radial coordinate of cylindrical coordinates.
$R^{\alpha\beta}$	Element matrix defined by Eq. (22).
R	Global matrix of $R^{\alpha\beta}$.
s	Seconds.
$S^{\alpha\beta}$	Element matrix defined by Eq. (18).
S	Global matrix of $S^{\alpha\beta}$.
S	Surface of the body (m ²).

S_q	Surface of the body with heat flux boundary condition (m^2).
S_θ	Surface of the body with temperature boundary condition (m^2).
t	Time (s).
V	Volume of the body (m^3).
W	Watts.
x^i	Curvilinear coordinates. For this report they are Cartesian coordinates x, y, z or cylindrical coordinates r, ϕ, z (m).
x	Cartesian coordinate (m).
y	Cartesian coordinate (m).
z	Cartesian coordinate (m).
α	Time integration constant that varies from 0 (explicit) to 1 (greater than 0, implicit).
δ	Variational operator.
Δ	Designates incremental quantities.
θ	Temperature of the body (K).
θ_s	Temperature of the surface of the body (K).
θ_f	Temperature of the flow for convective boundary (K).
ρ	Mass density (kg/m^3).
ρc	Thermal capacity [$J/(m^3 K)$].
ϕ	Circumferential coordinate of cylindrical coordinates.
∇	Differential operator, defined by Eq. (1a).
∇_k	Differential operator with conductivity, defined by Eq. (1b).
$;$	Designates covariant differentiation.

Repeated Latin indices indicate summation 1 through the number of coordinates.

Repeated Greek indices indicate summation 1 through the number of element nodal points.

Subscript i signifies inner.

Subscript o signifies outer.

THERM: A THREE-DIMENSIONAL TRANSIENT HEAT CONDUCTION COMPUTER PROGRAM

by

W. A. Cook

ABSTRACT

THERM is a three-dimensional finite-element computer program for solving transient heat conduction problems. This report presents the techniques used to develop THERM. The theory described consists of a governing equation, boundary conditions, and an equivalent variational principle. The matrix equations used in THERM are derived using both vector and tensor analysis. These equations use finite-element approximations for the geometry and a finite-difference approximation for the time. THERM has finite-element formulations using both Cartesian or cylindrical coordinates. Several example problems are included to demonstrate that the THERM formulations are correct and that THERM can be used to solve meaningful problems.

I. INTRODUCTION

This report presents the techniques used to develop THERM, a three-dimensional computer program which solves transient heat conduction problems. This computer program uses the finite-element method to approximate geometry and the finite difference method to approximate time. THERM has continuum and boundary finite elements using both Cartesian and cylindrical coordinates.

Both vector analysis and tensor analysis are used to present the governing equation, boundary conditions, variational principle, and to derive the finite-element equations. Each derivation is general and can be used for any coordinate system. Appendix A presents the significant equations in Cartesian coordinates and Appendix B presents them in cylindrical coordinates. Appendix C presents the Christoffel symbols for Cartesian and cylindrical coordinates required for covariant differentiation in the tensor analysis approach.

The program's ability to solve thermal problems was verified with simple rectangular and cylindrical example problems, and a reactor example problem was used to check its performance. THERM is listed in Appendix D with input instructions included as comments in the listing.

Temperature is represented as θ , heat flux as q , time as t , heat source as Q , volume as V , surface as S , conductivity as k , and thermal capacity as ρc . A complete listing of symbols is contained in the nomenclature.

II. THEORY

This section presents the governing equation and boundary conditions for heat conduction and the corresponding variational principle. Both vector analysis and tensor analysis are used in developing the finite-element approximations. This section concludes with the finite difference approximation of the time and an estimate of accuracy.

A. Vector Analysis Approach

This subsection uses vector analysis to derive general finite-element equations for thermal heat conduction. Appendices A and B present the significant equations of this subsection in Cartesian and cylindrical coordinates, respectively.

1. Heat Conduction Equation and Boundary Conditions. The governing equation for heat conduction is

$$\nabla \cdot (\nabla_k \theta) + Q = \rho c \frac{\partial \theta}{\partial t} \quad \text{in } V, \quad (1)$$

where, for Cartesian coordinates,

$$\nabla \equiv \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} , \quad (1a)$$

or, for cylindrical coordinates,

$$\nabla \equiv \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\varphi \frac{1}{r} \frac{\partial}{\partial \varphi} + \mathbf{k} \frac{\partial}{\partial z} , \quad (1a)$$

and, for Cartesian coordinates,

$$\nabla_k \equiv \mathbf{i} k_x \frac{\partial}{\partial x} + \mathbf{j} k_y \frac{\partial}{\partial y} + \mathbf{k} k_z \frac{\partial}{\partial z} , \quad (1b)$$

or, for cylindrical coordinates,

$$\nabla_k \equiv \mathbf{e}_r k_r \frac{\partial}{\partial r} + \mathbf{e}_\varphi \frac{k_\varphi}{r} \frac{\partial}{\partial \varphi} + \mathbf{k} k_z \frac{\partial}{\partial z} . \quad (1b)$$

The boundary conditions are

$$\nabla_k \theta = \mathbf{q} \text{ on } S_q \quad (2)$$

and

$$\theta = \theta_s \text{ on } S_\theta . \quad (3)$$

2. Variational Principle. In this subsection the variational principle is shown to be equivalent to the governing equation and the boundary conditions.

Consider the following variational principle:

$$\int_v [(\nabla_k \theta) \cdot \delta(\nabla \theta)] dV - \int_v \left(Q - \rho c \frac{\partial \theta}{\partial t} \right) \delta \theta dV - \int_{S_q} (\mathbf{n} \cdot \mathbf{q}) \delta \theta dS = 0 . \quad (4)$$

The Gauss divergence theorem is

$$\int_v (\nabla \cdot \mathbf{A}) dV = \int_s (\mathbf{n} \cdot \mathbf{A}) dS . \quad (5)$$

Let

$$\mathbf{A} = (\nabla_k \theta) \delta \theta , \quad (6)$$

then

$$\int_v \left\{ \nabla \cdot [(\nabla_k \theta) \delta \theta] \right\} dV = \int_s \left\{ \mathbf{n} \cdot [(\nabla_k \theta) \delta \theta] \right\} dS . \quad (7)$$

The first integral of Eq. (7) can be written as two integrals such that Eq. (7) becomes

$$\int_v [(\nabla_k \theta) \cdot \delta(\nabla \theta)] dV = \int_s [\mathbf{n} \cdot (\nabla_k \theta)] \delta \theta dS - \int_v [\nabla \cdot (\nabla_k \theta)] \delta \theta dV . \quad (8)$$

Substituting Eq. (8) into Eq. (4) gives

$$-\int_v \left\{ [\nabla \cdot (\nabla_k \theta)] + Q - \rho c \frac{\partial \theta}{\partial t} \right\} \delta \theta dV + \int_s \left\{ \mathbf{n} \cdot [(\nabla_k \theta) - \mathbf{q}] \right\} \delta \theta dS = 0 . \quad (9)$$

Eq. (9) is zero in V if

$$\nabla \cdot (\nabla_k \theta) + Q - \rho c \frac{\partial \theta}{\partial t} = 0 . \quad (10)$$

When $\theta = \theta_s$, $\delta \theta = 0$ on S_θ ; and on S_q ,

$$\nabla_k \theta = \mathbf{q} . \quad (11)$$

3. Finite-Element Formulation. Equation (4) is the variational principle used to develop the finite-element model. The finite-element assumption is that, over each element m , the temperature varies as follows:

$$\theta = p^\alpha(\xi^i) \theta_\alpha , \quad (12)$$

where repeated Greek letters are summed over all element nodal points. Thus,

$$\nabla \theta = \left\{ \nabla [p^\alpha(\xi^i)] \right\} \theta_\alpha . \quad (13)$$

Let $\mathbf{q}$ be defined as

$$\mathbf{n} \cdot \mathbf{q} = q_n - h (\theta - \theta_f) . \quad (14)$$

Let these quantities be constant on the boundary of element m :

$$h = h_m , \quad \theta_f = (\theta_f)_m , \quad \text{and} \quad q_n = (q_n)_m . \quad (15)$$

Let the heat source, Q , be a constant material quantity and thus, for element m ,

$$Q = Q_m . \quad (16)$$

Let thermal capacity, ρc , and conductivities, k_x or k_r , k_y or k_θ , and k_z , be constant with respect to the geometry of element m but vary with temperature.

Substituting Eqs. (12) through (16) into Eq. (4), we obtain

$$\sum_m \left\{ \left[(S^{\alpha\beta})_m + (G^{\alpha\beta})_m \right] \theta_\beta - (F^\alpha)_m - (D^\alpha)_m + \left[(R^{\alpha\beta})_m \right] \left(\frac{\partial \theta_\beta}{\partial t} \right) \right\} \delta \theta_\alpha = 0 , \quad (17)$$

where

$$(S^{\alpha\beta})_m = \int_{(V)_m} \left[(\nabla_k p^\beta) \cdot (\nabla p^\alpha) \right] dV , \quad (18)$$

$$(G^{\alpha\beta})_m = h_m \int_{(S)_m} (p^\alpha p^\beta) dS , \quad (19)$$

$$(F^\alpha)_m = [h_m(\theta)_{f_m} + (q_n)_{m}] \int_{(S_t)_m} (p^\alpha) dS , \quad (20)$$

$$(D^\alpha)_m = Q_m \int_{(V)_m} (p^\alpha) dV , \quad (21)$$

and

$$(R^{\alpha\beta})_m = \rho c \int_{(V)_m} (p^\alpha p^\beta) dV . \quad (22)$$

Define Θ as the global nodal point temperature matrix. Similarly define S , G , F , D , and R as the global matrices for the element matrices defined by Eqs. (18), (19), (20), (21), and (22). With these definitions, Eq. (17) can be written as

$$(S + G) \Theta + R \frac{\partial \Theta}{\partial t} = F + D . \quad (23)$$

B. Tensor Analysis Approach

This subsection uses tensor analysis to derive general finite-element equations for thermal heat conduction. Appendix C presents the particulars for covariant differentiation in Cartesian and cylindrical coordinates. Appendices A and B present the significant equations of this subsection in Cartesian and cylindrical coordinates, respectively.

1. Heat Conduction Equation and Boundary Conditions. The governing tensor equation for heat conduction is

$$(k^{ji} \theta_{;i})_{,j} + Q = \rho c \frac{\partial \theta}{\partial t} \text{ in } V , \quad (1)$$

where ";" designates covariant differentiation with respect to x^i (see Appendix C).

The boundary conditions are

$$k^{ji}\theta_{;i} = q^j \text{ on } S_q \quad (2)$$

and

$$\theta = \theta_s \text{ on } S_\theta . \quad (3)$$

2. Variational Principle. In this subsection the variational principle is shown to be equivalent to the governing equation and the boundary conditions.

Consider the following variational principle:

$$\int_v \left[(k^{ji}\theta_{;i}) \delta(\theta_{;j}) \right] dV - \int_v \left[\left(Q - \rho c \frac{\partial \theta}{\partial t} \right) \delta \theta \right] dV - \int_{S_q} \left[(q^j n_j) \delta \theta \right] dS = 0 . \quad (4)$$

The Gauss divergence theorem is

$$\int_v (f^j_{;j}) dV = \int_s (f^j n_j) dS . \quad (5)$$

Let

$$f^j = (k^{ji}\theta_{;i}) \delta \theta , \quad (6)$$

then

$$\int_v \left\{ \left[(k^{ji}\theta_{;i}) \delta \theta \right]_{;j} \right\} dV = \int_s \left\{ \left[(k^{ji}\theta_{;i}) \delta \theta \right] n_j \right\} dS . \quad (7)$$

The first integral of Eq. (7) can be written as two integrals such that Eq. (7) becomes

$$\int_v \left[(k^{ji}\theta_{;i}) \delta(\theta_{;j}) \right] dV = - \int_v \left\{ \left[(k^{ji}\theta_{;i})_{;j} \right] \delta \theta \right\} dV + \int_s \left\{ \left[(k^{ji}\theta_{;i}) \delta \theta \right] n_j \right\} dS . \quad (8)$$

Substituting Eq. (8) into Eq. (4) gives

$$-\int_V \left\{ \left[\left(k^{ji} \theta_{;i} \right)_{;j} + Q - \rho c \frac{\partial \theta}{\partial t} \right] \delta \theta \right\} dV + \int_S \left\{ \left[\left(k^{ji} \theta_{;i} - q^j \right) n_j \right] \delta \theta \right\} dS = 0 . \quad (9)$$

Eq. (9) is zero in V if

$$\left(k^{ji} \theta_{;i} \right)_{;j} + Q = \rho c \frac{\partial \theta}{\partial t} . \quad (10)$$

When $\theta = \theta_s$, $\delta \theta = 0$ on S_θ ; and on S_q ,

$$k^{ji} \theta_{;i} = q^j . \quad (11)$$

3. Finite-Element Formulation. Equation (4) is the variational principle used to develop the finite-element model. The finite-element assumption is that, over each element m , the temperature varies as follows:

$$\theta = p^\alpha \left(\xi^i \right) \theta_\alpha , \quad (12)$$

where repeated Greek letters are summed over all element nodal points. Thus,

$$\theta_{;j} = \left\{ \left[p^\alpha \left(\xi^i \right) \right]_{;j} \right\} \theta_\alpha . \quad (13)$$

Let $\mathbf{q}$ be defined as

$$q^j n_j = q_n - h \left(\theta - \theta_f \right) . \quad (14)$$

Let these quantities be constant on the boundary of element m :

$$h = h_m , \quad \theta_f = \left(\theta_f \right)_m , \quad \text{and} \quad q_n = \left(q_n \right)_m . \quad (15)$$

Let the heat source, Q , be a constant material quantity and thus, for element m ,

$$Q = Q_m . \quad (16)$$

Let thermal capacity, ρc , and conductivities, k_x or k_r , k_y or k_θ , and k_z , be constant with respect to the geometry of element m but vary with temperature.

Substituting Eqs. (12) through (16) into Eq. (4), we obtain

$$\sum_m \left\{ \left[(S^{\alpha\beta})_m + (G^{\alpha\beta})_m \right] \theta_\beta - (F^\alpha)_m - (D^\alpha)_m + \left[(R^{\alpha\beta})_m \right] \left(\frac{\partial \theta_\beta}{\partial t} \right) \right\} \delta \theta_\alpha = 0 , \quad (17)$$

where

$$(S^{\alpha\beta})_m = \int_{(V)_m} \left[k^{ji} (p^\alpha_{;i}) (p^\beta_{;j}) \right] dV , \quad (18)$$

$$(G^{\alpha\beta})_m = h_m \int_{(S)_m} (p^\alpha p^\beta) dS , \quad (19)$$

$$(F^\alpha)_m = \left[h_m (\theta_f)_m + (q_n)_m \right] \int_{(S)_m} (p^\alpha) dS , \quad (20)$$

$$(D^\alpha)_m = Q_m \int_{(V)_m} (p^\alpha) dV , \quad (21)$$

and

$$(R^{\alpha\beta})_m = \rho c \int_{(V)_m} (p^\alpha p^\beta) dV . \quad (22)$$

Define Θ as the global nodal point temperature matrix. Similarly define $\mathbf{S}$, $\mathbf{G}$, $\mathbf{F}$, $\mathbf{D}$, and $\mathbf{R}$ as the global matrices for the element matrices defined by Eqs. (18) through (22). With these definitions Eq. (17) can be written as

$$(\mathbf{S} + \mathbf{G}) \Theta + \mathbf{R} \frac{\partial \Theta}{\partial t} = \mathbf{F} + \mathbf{D} . \quad (23)$$

C. Formulation of Time Integration

This section presents the finite difference operator used in this report (see Ref. 1). Define the time integration constant α as follows:

$$\frac{\partial \Theta[x^i, t + \alpha(\Delta t)]}{\partial t} = \Delta \Theta^j / \Delta t , \quad (24)$$

and

$$\Theta[x^i, t + \alpha(\Delta t)] = \Theta^j + \alpha(\Delta \Theta^j) , \quad (25)$$

where the superscript j refers to time increment. Thus,

$$\Theta(x^i, t + \Delta t) = \Theta^{j+1} = \Theta^j + \Delta \Theta^j . \quad (26)$$

Substituting Eqs. (24) through (26) into Eq. (23) gives

$$[\alpha(\mathbf{S} + \mathbf{G}) + \mathbf{R} / \Delta t] \Delta \Theta^j = \mathbf{F} + \mathbf{D} - (\mathbf{S} + \mathbf{G}) \Theta^j . \quad (27)$$

When α is 0, the time integration is explicit (the Euler forward method); when α is 0.5, the time integration is implicit (the trapezoidal rule); and when α is 1, the time integration is again implicit (the Euler backward method).

Thus the time integration is a finite difference incremental approximation. The explicit time step must be small enough to maintain stability and the implicit time step small enough to preserve the accuracy of the solution (see Ref. 2). If the thermal capacity, conductivity, heat source, or convective heat-transfer coefficients vary with temperature, they could be modified with the iteration technique in THERM. The iteration technique is only meaningful with implicit time integration and has not been completed.

D. Accuracy Estimate

The incremental temperature nodal matrix ($\Delta\theta$) times the right-hand side of the master matrix equation (Eq. 27) gives an estimate of accuracy. This estimate is checked with an example problem in Sec. III. The errors/node calculation and accuracy estimate are shown in Table III. The accuracy estimate, as a comparison number, agrees favorably with the errors/node calculation for the example problem given in Table III. In stress analysis, this type of accuracy estimate is proportional to the potential energy.

III. EXAMPLE PROBLEMS

This section presents several example problems that were used to test various portions of THERM. The last example is a reactor problem having special significance in reactor safety.

A. Rectangular Problem

A one-dimensional steady-state problem with rectangular geometry is described by the following governing equation:

$$k \frac{d^2\theta}{dx^2} + Q = 0 . \quad (28)$$

The boundary conditions are

$$\left[k \frac{d\theta}{dx} \right]_{x=x_0} = q_0 \quad (29)$$

and

$$[\theta]_{x=x_1} = \theta_i . \quad (30)$$

Equation (28) can be solved with the boundary conditions in Eqs. (29) and (30) to give

$$\theta = \theta_i + \frac{q_0}{k} (x - x_i) + \frac{Q}{k} \left(\frac{x^2}{2} - x_i x_0 + x x_0 - \frac{x^2}{2} \right) . \quad (31)$$

THERM was able to solve the problem exactly, using Cartesian continuum and boundary elements and the quadratic shape functions to model the solution exactly. When α equals 1 and Δt is infinite, a steady-state solution is obtained with THERM (see Table I).

B. Cylindrical Problems

1. Radial Variation. Reference 3, p. 165, presents a one-dimensional heat equation in cylindrical coordinates that allows only radial temperature variation. The boundary and general initial temperature distribution presented are for a solid cylinder. This is a one-dimensional geometry transient problem. Problem 2 in Ref. 3, p. 168, presents the temperature solution for a particular initial temperature distribution. THERM solved this problem accurately.

The governing equation is

$$\frac{\partial \theta}{\partial t} = k \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) . \quad (32)$$

The boundary condition is

$$\left[\theta(t) \right]_{r=r_0} = B , \quad (33)$$

TABLE I
SOLUTION TO THE RECTANGULAR PROBLEM

Constants		x (m)	θ (K)
q_0	11,250. (W/m ²)	1.0	200.
θ_i	200. (K)	2.0	2,000.
k	225. [W/(m K)]	3.0	3,300.
Q	112,500(W/m ³)	4.0	4,100.
		5.0	4,400.

where B is a constant. The initial conditions are

$$\begin{aligned} [\theta]_{t=t_0} &= A, & 0 \leq r < \frac{r_0}{2} \\ [\theta]_{t=t_0} &= A + B, & r = \frac{r_0}{2} \\ [\theta]_{t=t_0} &= B, & \frac{r_0}{2} < r \leq r_0 \end{aligned} \quad (34)$$

where A and B are constants. The solution to this problem is

$$\theta(r,t) = B + \left(\frac{A-B}{2} \right) \left(\sum_{j=1}^{\infty} \frac{J_1(\lambda_j)}{\lambda_j [J_1(2\lambda_j)]^2} J_0(\lambda_j r) e^{-k(\lambda_j)^2 t} \right), \quad (35)$$

where $\lambda_1, \lambda_2 \dots$ etc., are the positive roots of $J_0(2\lambda)$.

A computer program was written to solve for 60 roots, and then using these roots, another computer program was written to calculate the series solution in Eq. (35). Due to the discontinuous temperature distribution, this was a very demanding problem but THERM solved it quite accurately with four equally spaced quadratic elements in the radial direction.

A one-dimensional steady-state problem with cylindrical geometry is described by the following governing equation:

$$\frac{k}{r} \frac{d}{dr} \left(r \frac{d\theta}{dr} \right) + Q = 0. \quad (36)$$

The boundary conditions are

$$\left[k \frac{d\theta}{dr} \right]_{r=r_0} = q_0 \quad (37)$$

and

$$[\theta]_{r=r_i} = \theta_i. \quad (38)$$

Equation (36) can be solved with the boundary conditions in Eqs. (37) and (38) to give

$$\theta = \theta_i - \frac{Q}{4k}(r^2 - r_i^2) + \frac{r_o}{k}\left(\frac{r_o}{2}Q + q_o\right)\left[\log_e\left(\frac{r}{r_i}\right)\right] . \quad (39)$$

THERM's solution to Eq. (39) is given in Table II.

THERM solved this problem approximately using the linear and quadratic shape functions. The incremental temperature nodal matrix times the right-hand side of the master matrix equation, $\Delta\Theta \bullet (\text{RHS})$, gives an estimate of accuracy. This estimate is shown in Table III to compare solutions. This error estimate is a reasonable one because it is a scalar that increases as the errors/node decreases, as shown in Table III.

The $\Delta\Theta \bullet (\text{RHS})$ results are plotted in Fig. 1. The results for the 27-nodal-point elements are higher and represent more accurate calculations than the 8-nodal-point elements, which are lower and represent less accurate calculations.

An inspection of Fig. 1 reveals that the accuracy of both the 27-nodal-point elements and the 8-nodal-point elements are nearly linear with respect to the number of radial nodal points. Even though the accuracy for a given number of nodal points is slightly less for the 8-nodal-point elements than for the 27-nodal-point elements, the finite-element calculations are considerably more expensive for 27-nodal-point elements. These types of problems lend themselves to having as few nodal points as possible to model the geometry and approximation functions as simple as possible. If the temperature variations had larger gradients, the higher-order elements would be more efficient.

TABLE II
SOLUTION TO THE CYLINDRICAL PROBLEM WITH RADIAL VARIATION

Constants		r (m)	θ (K)
q_o	11,250. (W/m ²)	0.04	2.0
θ_i	2.0 (K)	0.055	4.210278
k	225. [W/(m K)]	0.07	5.784618
Q	112,500(W/m ³)	0.085	6.950164
		0.10	7.822180

TABLE III
ACCURACY ESTIMATE FOR CYLINDRICAL PROBLEM
WITH RADIAL VARIATION

Element	Radial nodes	$\Delta\Theta \cdot (\text{RHS})$	Errors/node
8-node	3	4967.	0.063575
27-node	3	5157.	0.01547
8-node	5	5469.	0.01328
27-node	5	5636.	0.01324
8-node	7	5876.	0.00507
27-node	7	6087.	0.00026
8-node	25	9302.	0.00017
27-node	25	10060.	0.0000006

Fig. 1.
Accuracy results for cylindrical problem with radial variation.

2. Circumferential Variation. The circumferential variation was checked by analyzing the rectangular problem in Sec. III.A with quadratic finite elements using cylindrical coordinates. The circumferential coordinate approximates the x variation of the problem. The approximation functions of the quadratic finite elements using cylindrical coordinates do not solve this problem exactly but approximate very closely the correct solution.

C. Reactor Example Problem

A typical reactor problem is shown in Fig. 2. The details for this problem were supplied by Ref. 4. The input parameters used for this problem are shown in Tables IV and V. Note that c and ρ were not needed for the steady-state solution.

Fig. 2.
Typical reactor problem.

TABLE IV
MATERIAL AND INTERNAL HEAT GENERATION PROPERTIES
FOR A TYPICAL REACTOR PROBLEM

Material	k [W/(m K)]	c [J/(kg K)]	ρ (kg/m ³)	Q (W/m ³)
Cladding	240.0	900.0	2699.0	0.0
Target	70.96	3308.0	530.0	0.0
Fuel	178.27	845.73	2979.4	1,909,000,000.

TABLE V
THERMAL BOUNDARY CONDITIONS FOR A
TYPICAL REACTOR PROBLEM

Channels	T_f (K)	h [W/(m ² K)]
A-E, G-I	300.0	30,000.0
F	300.0	0.0
Exterior	300.0	100.0

The maximum temperature calculated for the core is plotted in Fig. 3 as a function of the circumferential angle. This calculation used 1410 nodal points and 98 27-nodal-point continuum elements.

Figure 4 is a composite of two plots of a coarse mesh consisting of 906 nodal points. One plot shows the calculation using 62 27-nodal-point continuum elements and the other plot shows the calculation using 496 8-nodal-point continuum elements.

The 8-nodal-point calculation in Fig. 4 required only 3/5 the computer time required for the 27-nodal-point calculation. This type of problem should have as few nodal points as possible to model the geometry and approximation functions as simply as possible. If the temperature variation had larger gradients, the more complicated elements would be more efficient, as shown in Ref. 5 for stress analysis problems.

The $\Delta\Theta \cdot (\text{RHS})$ values for the calculations plotted in Figs. 3 and 4 are shown in Table VI.

Fig. 3.
Calculation of maximum core temperature for a typical reactor problem.

Fig. 4.
Comparison of 27- and 8-nodal-point calculations for a typical reactor problem.

TABLE VI
ACCURACY ESTIMATE FOR A TYPICAL
REACTOR PROBLEM

Nodal Points	Elements	$\Delta\Theta \cdot (\text{RHS})$
906	8-node (cyl.)	6572980.
906	27-node (cyl.)	6729590.
1401	27-node (cart.)	6776610.
1401	27-node (cyl.)	6777010.

IV. CONCLUSIONS

THERM is a three-dimensional finite-element transient heat conduction computer program. THERM solves very complicated geometry problems with extreme temperature variations. The very nature of the finite-element method and the method of integrating the finite elements requires significant computer time. For reactor problems, as shown in Sec. III.C, the computer time requirements are large compared to other methods. The computer time requirements for THERM could be reduced with single-point integration (see Ref. 6), improved matrix-equation solution algorithms, and restrictions to explicit time integrations. With these improvements it is doubtful that THERM would be any better than the HERA computer program that solves these reactor problems very accurately and efficiently (Ref. 7). The types of problems solved so far with THERM lend themselves to having as few nodal points as possible to model the geometry and approximation functions that are as simple as possible. If the temperature variations had larger gradients, the higher-order elements would be more efficient.

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APPENDIX A CARTESIAN COORDINATES

The governing equation for heat conduction, Eq. (1) of Sec. II, is

$$\frac{\partial}{\partial x} \left(k_x \frac{\partial \theta}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial \theta}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial \theta}{\partial z} \right) + Q = \rho c \frac{\partial \theta}{\partial t} \quad \text{in } V. \quad (\text{A-1})$$

The boundary conditions, Eqs. (2) and (3) of Sec. II, are

$$k_x \frac{\partial \theta}{\partial x} \mathbf{i} + k_y \frac{\partial \theta}{\partial y} \mathbf{j} + k_z \frac{\partial \theta}{\partial z} \mathbf{k} = \mathbf{q} \quad \text{on } S_q \quad (\text{A-2})$$

and

$$\theta = \theta_s \quad \text{on } S_\theta. \quad (\text{A-3})$$

The variational principle, Eq. (4) of Sec. II, is

$$\begin{aligned} & \int_V \left[k_x \frac{\partial \theta}{\partial x} \delta \left(\frac{\partial \theta}{\partial x} \right) + k_y \frac{\partial \theta}{\partial y} \delta \left(\frac{\partial \theta}{\partial y} \right) + k_z \frac{\partial \theta}{\partial z} \delta \left(\frac{\partial \theta}{\partial z} \right) \right] dV \\ & - \int_V \left(Q - \rho c \frac{\partial \theta}{\partial t} \right) \delta \theta dV - \int_{S_q} (\mathbf{n} \cdot \mathbf{q}) \delta \theta dS = 0. \end{aligned} \quad (\text{A-4})$$

Eqs. (13) of Sec. II are

$$\frac{\partial \theta}{\partial x} = \left\{ \frac{\partial [p^\alpha(\xi^i)]}{\partial x} \right\} \theta_\alpha, \quad \frac{\partial \theta}{\partial y} = \left\{ \frac{\partial [p^\alpha(\xi^i)]}{\partial y} \right\} \theta_\alpha, \quad \text{and} \quad \frac{\partial \theta}{\partial z} = \left\{ \frac{\partial [p^\alpha(\xi^i)]}{\partial z} \right\} \theta_\alpha. \quad (\text{A-5})$$

Eq. (18) of Sec. II is

$$(S^{\alpha\beta})_m = \int_{(v)_m} \left(k_x \frac{\partial p^\alpha}{\partial x} \frac{\partial p^\beta}{\partial x} + k_y \frac{\partial p^\alpha}{\partial y} \frac{\partial p^\beta}{\partial y} + k_z \frac{\partial p^\alpha}{\partial z} \frac{\partial p^\beta}{\partial z} \right) dV . \quad (A-6)$$

APPENDIX B CYLINDRICAL COORDINATES

The governing equation for heat conduction, Eq. (1) of Sec. II, is

$$\frac{\partial}{\partial r} \left(k_r \frac{\partial \theta}{\partial r} \right) + \frac{k_r}{r} \frac{\partial \theta}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left(k_\varphi \frac{\partial \theta}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial \theta}{\partial z} \right) + Q = \rho c \frac{\partial \theta}{\partial t} \quad \text{in } V \quad (\text{B-1a})$$

or

$$\frac{1}{r} \frac{\partial}{\partial r} \left(k_r r \frac{\partial \theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left(k_\varphi \frac{\partial \theta}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial \theta}{\partial z} \right) + Q = \rho c \frac{\partial \theta}{\partial t} \quad \text{in } V . \quad (\text{B-1b})$$

The boundary conditions, Eqs. (2) and (3) of Sec. II, are

$$k_r \frac{\partial \theta}{\partial r} \mathbf{e}_r + \frac{k_\varphi}{r} \frac{\partial \theta}{\partial \varphi} \mathbf{e}_\varphi + k_z \frac{\partial \theta}{\partial z} \mathbf{k} = \mathbf{q} \quad \text{on } S_q \quad (\text{B-2})$$

and

$$\theta = \theta_s \quad \text{on } S_\theta . \quad (\text{B-3})$$

The variational principle, Eq. (4) of Sec. II, is

$$\begin{aligned} & \int_V \left[k_r \frac{\partial \theta}{\partial r} \delta \left(\frac{\partial \theta}{\partial r} \right) + \frac{k_\varphi}{r^2} \frac{\partial \theta}{\partial \varphi} \delta \left(\frac{\partial \theta}{\partial \varphi} \right) + k_z \frac{\partial \theta}{\partial z} \delta \left(\frac{\partial \theta}{\partial z} \right) \right] dV \\ & - \int_V \left(Q - \rho c \frac{\partial \theta}{\partial t} \right) \delta \theta dV - \int_{S_q} (\mathbf{n} \cdot \mathbf{q}) \delta \theta dS = 0 . \end{aligned} \quad (\text{B-4})$$

Eqs. (13) of Sec. II are

$$\frac{\partial \theta}{\partial r} = \left\{ \frac{\partial}{\partial r} [p^\alpha(\xi^i)] \right\} \theta_\alpha , \quad \frac{\partial \theta}{\partial \varphi} = \left\{ \frac{\partial}{\partial \varphi} [p^\alpha(\xi^i)] \right\} \theta_\alpha , \quad \text{and} \quad \frac{\partial \theta}{\partial z} = \left\{ \frac{\partial}{\partial z} [p^\alpha(\xi^i)] \right\} \theta_\alpha . \quad (\text{B-5})$$

Eq. (18) of Sec. II is

$$\left(S^{\alpha\beta}\right)_m = \int_{(V)_m} \left(k_r \frac{\partial p^\alpha}{\partial r} \frac{\partial p^\beta}{\partial r} + \frac{k_\phi}{r^2} \frac{\partial p^\alpha}{\partial \phi} \frac{\partial p^\beta}{\partial \phi} + k_z \frac{\partial p^\alpha}{\partial z} \frac{\partial p^\beta}{\partial z} \right) dV . \quad (\text{B-6})$$

APPENDIX C

COVARIANT DIFFERENTIATION

Covariant differentiation of a scalar like the temperature, θ , is

$$\theta_{;i} = \frac{\partial \theta}{\partial x^i} . \quad (C-1)$$

Covariant differentiation of a contravariant first order tensor like $(k^{ji}\theta_{;i})_{;k}$ is

$$(k^{ji}\theta_{;i})_{;k} = \frac{\partial (k^{ji}\theta_{;i})}{\partial x^k} + \left\{ \begin{matrix} j \\ ks \end{matrix} \right\} (k^{si}\theta_{;i}) . \quad (C-2)$$

$\left\{ \begin{matrix} j \\ ks \end{matrix} \right\}$ are Christoffel symbols of the second kind.

For Cartesian coordinates all Christoffel symbols of the second kind are zero.

For cylindrical coordinates the only nonzero Christoffel symbols of the second kind are

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = -r , \quad \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} = \frac{1}{r} , \quad \text{and} \quad \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \frac{1}{r} . \quad (C-3)$$

The conductivity tensors for Cartesian coordinates, in terms of physical conductivities, are

$$k^{ji} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} . \quad (C-4)$$

The conductivity tensors for cylindrical coordinates, in terms of physical conductivities, are

$$k^{ji} = \begin{bmatrix} k_r & 0 & 0 \\ 0 & \frac{k_\phi}{r^2} & 0 \\ 0 & 0 & k_z \end{bmatrix} . \quad (C-5)$$

The heat flux tensors and normal tensors for Cartesian coordinates, in terms of physical heat fluxes and physical normal vector components, are

$$q^i = \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} \quad \text{and} \quad n_i = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} . \quad (C-6)$$

The heat flux and normal tensors for cylindrical coordinates, in terms of physical heat fluxes and physical normal vector components, are

$$q^i = \begin{pmatrix} q_r \\ \frac{q_\phi}{r} \\ q_z \end{pmatrix} \quad \text{and} \quad n_i = \begin{pmatrix} n_r \\ r n_\phi \\ n_z \end{pmatrix} . \quad (C-7)$$

APPENDIX D

THERM PROGRAM LISTING

```

      program Therm (inp3d,out3d,tape5=inp3d,tape6=out3d)
C*****
C
C      An incremental and iteration nonlinear continuum thermal
C      analysis code using Lagrangian interpolation formulas.
C
C              **** August 1990 ****
C
C*****
C
      common /control/ title(20), ncn, ncns, np, ncord, mband, alpha,
1 time, Delt
      common /matp/ emat(10,6)
      common /nodes/ cord(1410,3), cord1(1410,3), T(1410), dT(1410)
      common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
      common /bdy/ nby, nbc(100), Ts(100)
      common /gid/ xg(9), wg(9), ngaus
      common /stiff/ fo(1410), fl(1410), ss(1410,595)
      common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
C
C*****
C
C              input instructions
C
C      card 0 (one card), i5
C          nprob = number of problems
C
C      card 1 (one card), 20a4
C          problem title, and control data
C
C      card 2 (one card), 11i5
C          nmat = number of materials
C          np = number of nodal points,
C          nce = number of continuum elements ** nce+ncel .le. 98 **
C          ncel = number of continuum elements (r,theta,z)
C          nbe = number of boundary elements
C          nbel = number of boundary elements
C          nlag = order of Lagrangian interpolation
C          nby = number of designated temperature nodal points
C          ngaus = number of gauss points in one direction
C          ninc = number of increments
C          nite = number of iterations
C
C      card 3 (one card), 2f10.3
C          alpha = time intergration constant
C          alpha = 0.0 [Eulerian forward, (explicit method)],
C              = 0.5 [trapezoidal rule, (implicit method)], and
C              = 1.0 [Eulerian backward, (implicit method)].
C          time = maximum time of the problem
C
C

```

```

c      card 4 (nmat cards), i5,5x,6f10.3
c          n = material number
c          emat(n,1) = specific heat
c          emat(n,2) = density
c          emat(n,3) = heat source density
c          emat(n,4) = conductivity x or r
c          emat(n,5) = conductivity y or theta
c          emat(n,6) = conductivity z
c
c      card 5 (np cards), 2i5,4f10.3
c          n = node number
c          ics is "0" for (x,y,z) and "1" for (r,theta,z)
c          cord(n,1) = x or r component of the n-th node
c          cord(n,2) = y or theta component of the n-th node
c          cord(n,3) = z component of the n-th node
c          T = initial temperature data
c
c      card 6 (nce cards), 16i5 (8 and 27 nodal point elements)
c          n = element number
c          mat(n) = material number.
c          nc(1) = 1st corner node number
c          nc(2) = next node
c          .
c          .
c          .          *** These elements use Cartesian coordinates ***
c          .
c
c      card 7 (ncel cards), 16i5 (8 and 27 nodal point elements)
c          n = element number
c          mat(n) = material number.
c          nc(1) = 1st corner node number
c          nc(2) = next node
c          .
c          .
c          .          *** These elements use cylindrical coordinates ***
c          .
c
c      card 8 (nbe cards), i5,5x,9i5 (4 and 9 nodal point elements)
c          n = element number
c          nbq(1) = 1st corner node number
c          nbq(2) = next node
c          .
c          .
c          .          *** These elements use Cartesian elements ***
c          .
c
c      card 9 (nbcl cards), i5,5x,9i5 (4 and 9 nodal point elements)
c          n = element number
c          nbq(1) = 1st corner node number
c          nbq(2) = next node
c          .
c          .
c          .          *** These elements use cylindrical elements ***
c          .
c
c      card 10 ( nbe+nbcl or nbe+nbcl cards), i5,5x,3f10.3
c          n = boundary element
c          qn = Normal heat flux

```

```

c          hf = Convective heat transfer coefficient
c          Tf = Temperature of the flow for convective boundary
c
c          card 11 (nby cards), i5,5x,f10.3
c          nbc(i) = number of the i-th boundary condition node
c          Ts(i) is designated temperature at nodal point i
c
c*****
c
c          read (5,90) nprob
c
c          problem loop -- nprob = total number of problems
c
c          do 80 iprob=1,nprob
c
c          read title and control data
c
c          read (5,110) title
c          write (6,140) title
c          read (5,90) nmat,np,nce,ncel,nbe,nbel,nlag,nby,ngaus,ninc,nite
c          read (5,100) alpha,time
c          write (6,150) nmat,np,nce,ncel,nbe,nbel,nlag,nby,ngaus,ninc,nite
1  ,alpha,time
c          if (nce+ncel.gt.98) stop
c          if (nbe+nbel.gt.60) stop
c          ncord=3
c          inc=ninc
c          Delt=time/inc
c          time=0.0
c
c          read material properties - specific heat density Q conductivity
c
c          write (6,170)
c          do 10 l=1,nmat
c          read (5,120) n, (emat(n,i),i=1,6)
c          write (6,130) n, (emat(n,i),i=1,6)
10 continue
c
c          mcn=ngaus*ngaus*ngaus
c          ncn=(nlag+1)*(nlag+1)*(nlag+1)
c          ncns=(nlag+1)*(nlag+1)
c          write (6,160) ncn,mcn,np
c
c          call gdata
c
c          increment loop
c
c          do 70 inc=1,ninc
c
c          increment the time
c
c          time=time+Delt
c
c          iteration loop
c
c          do 60 ite=1,nite
c
c          zero ss and fl

```

```

c      do 30 n=1,np
c      fl(n)=0.0
c      do 20 j=1,mband
c      ss(n,j)=0.0
20 continue
30 continue

c      if (nbe.ne.0) call Belemt
c      if (nbel.ne.0) call Belemt1
c      if (nce.ne.0) call Celemt
c      if (ncel.ne.0) call Celemt1

c      save RHS in fo

c      do 40 n=1,np
c      fo(n)=f1(n)
40 continue

c      call solve

c      write (6,180) inc,ite,time,Delt
c      write (6,200)
c      do 50 n=1,np
c      T(n)=T(n)+dT(n)
50 write (6,190) n,T(n),dT(n)

c      call check

c      60 continue
c      ** ends iteration loop **
c      70 continue
c      ** ends increment loop **
c      80 continue
c      ** ends problem loop **

c      stop

c      90 format (11i5)
100 format (2f10.3)
110 format (20a4)
120 format (i5,5x,6f10.3)
130 format (i5,5x,1p6e12.5)
140 format (1h1,5x,7hTherm -,2x,20a4)
150 format (1h0,30x,18hcontrol parameters/
1 10x,30hnumber of materials-----,i3//
2 10x,30hnumber of nodal points-----,i5//
3 10x,30hnumber of continuum elements--,i3*   (x,y,z)*//
4 10x,30hnumber of continuum elements--,i3*   (r,theta,z)*//
5 10x,30hnumber of boundary elements---,i3*   (x,y,z)*//
6 10x,30hnumber of boundary elements---,i3*   (r,theta,z)*//
7 10x,30horder of Lag. interpolation---,i3//
8 10x,30hnumber of displacement b.c.---,i3//
9 10x,30hnumber of gauss points-----,i3//
* 10x,30hnumber of increments-----,i3//
1 10x,30hnumber of iterations-----,i3//
2 10x,30halpha (time int. constant)----,f5.2/
3 10x,*alpha = 0.0 (Eulerian forward)*18x*0.5 (trapezoidal rule)*

```

```

4 /18x*1.0 (Eulerian backward)*//
5 10x,30htime (maximum) of problem-----,1pe13.5)
160 format (1h0,14hnodes/element=,i2,2x,17h gaussian points=,i3,2x,15h
1total unknowns=,i4)
170 format (1h0,20h material properties/3x*mat1*6x*sp. heat*5x*density
1 *5x*Q*7x*conductivity kx or kr, ky or ktheta, kz*)
180 format (1h0,1x*increment,iterate*2i3,5x*time and dt*1p2e13.5)
190 format (i10,3x,1p2e13.5)
200 format (1h0,7x*node*3x*temperature*)
end

```

```

      subroutine gdata
      common /control/ title(20), ncn, ncns, np, ncord, mband, alpha,
1 time, Delt
      common /nodes/ cord(1410,3), cordl(1410,3), T(1410), dT(1410)
      common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
      common /bdy/ nby, nbc(100), Ts(100)
      common /gid/ xg(9), wg(9), ngaus
      dimension x(3)

c
c      this subroutine reads geometry (nodes and elements),
c                      reads problem parameters,
c                      reads boundary and initial conditions, and
c                      generates integration data
c
c      read nodal point coordinate and initial temperature data
c
      if (nce.ne.0) write (6,290)
      if (ncel.ne.0) write (6,300)
10 continue
      read (5,230) n,ics,(x(i),i=1,ncord),T(n)
      if (ics.ne.0) go to 30

c
c      Cartesian coordinates
c
      do 20 i=1,ncord
      cord(n,i)=x(i)
20 continue
      cordl(n,1)=sqrt(x(1)**2+x(2)**2)
      cordl(n,2)=atan3(x(2),x(1))
      cordl(n,3)=cord(n,3)
      go to 50

c
c      cylindrical coordinates
c
30 continue
      do 40 i=1,ncord
      cordl(n,i)=x(i)
40 continue
      cord(n,1)=x(1)*cos(x(2))
      cord(n,2)=x(1)*sin(x(2))
      cord(n,3)=cordl(n,3)
50 continue
      if (nce.ne.0) write (6,240) n,(cord(n,j),j=1,ncord),T(n)
      if (ncel.ne.0) write (6,240) n,(cordl(n,j),j=1,ncord),T(n)
      if (n.lt.np) go to 10

c
c      read continuum element data and calculate mband
c
      mband=0
      if (nce.eq.0) go to 80
      write (6,320)
      do 70 m=1,nce
      read (5,260) n,mat(n),(nc(n,j),j=1,ncn)
      write (6,260) n,mat(n),(nc(n,j),j=1,ncn)
      mn=1000
      mx=0
      do 60 k=1,ncn

```

```

        mn=min0(nc(n,k),mn)
        mx=max0(nc(n,k),mx)
60  continue
    nq=mx-mn+1
    if (nq.gt.mband) mband=nq
70  continue
c
c      read continuum element data and calculate mband
c
80  continue
    if (ncel.eq.0) go to 110
    write (6,320)
    do 100 m=1,ncel
        read (5,260) n,mat(nce+n),(nc(nce+n,j),j=1,ncn)
        write (6,260) n,mat(nce+n),(nc(nce+n,j),j=1,ncn)
        mn=1000
        mx=0
        do 90 k=1,ncn
            mn=min0(nc(n,k),mn)
            mx=max0(nc(n,k),mx)
90  continue
        nq=mx-mn+1
        if (nq.gt.mband) mband=nq
100 continue
110 continue
    write (6,370) mband
c
c      read boundary element data
c
    if (nbe.eq.0) go to 130
    write (6,330)
    do 120 m=1,nbe
        read (5,260) n,(nbq(n,j),j=1,ncns)
        write (6,260) n,(nbq(n,j),j=1,ncns)
120 continue
130 continue
c
c      read boundary element data
c
    if (nbel.eq.0) go to 150
    write (6,330)
    do 140 m=1,nbel
        read (5,260) n,(nbq(nbe+n,j),j=1,ncns)
        write (6,260) n,(nbq(nbe+n,j),j=1,ncns)
140 continue
150 continue
c
c      read nodal point qn, hf, and Tf data
c
    if (nbe+nbel.eq.0) go to 190
    write (6,310)
    k=0
160 continue
    read (5,250) n,qn(n),hf(n),Tf(n)
    k=k+1
170 continue
    if (k.eq.n) go to 180
    qn(k)=qn(k-1)

```

```

        hf(k)=hf(k-1)
        Tf(k)=Tf(k-1)
        write (6,250) k,qn(k),hf(k),Tf(k)
        k=k+1
        go to 170
180    continue
        write (6,250) n,qn(n),hf(n),Tf(n)
        if (n.lt.nbe+nbel) go to 160
190    continue
c
c        read boundary data
c
c        if (nby.le.0) go to 210
        write (6,360)
        do 200 i=1,nby
            read (5,270) nbc(i),Ts(i)
            write (6,340) nbc(i),Ts(i)
            T(nbc(i))=Ts(i)
            Ts(i)=0.0
200    continue
210    continue
c
c        generate gaussian integration data
c
c        write (6,350)
        call gpoints
        do 220 i=1,ngaus
            write (6,280) xg(i),wg(i)
220    continue
c
c        return
c
230    format (2i5,6f10.3)
240    format (i5,5x,1p5e11.3)
250    format (i5,5x,5f10.3)
260    format (16i5)
270    format (i5,5x,f10.3)
280    format (2f15.8)
290    format (2x*node          x          y      *7x*z*10x*To*)
300    format (2x*node          r          theta*7x*z*10x*To*)
310    format (3x*be           qn          hf      Tf*)
320    format (1h1,7h  el      ,6hmat1  ,10x,5hnodes)
330    format (1h1,6h  be      ,*boundary element nodes*)
340    format (8x*nodal point*i5,8x*Designated temperature*f13.5)
350    format (32h0 gaussian integration constants)
360    format (1h0* Designated nodal point Temperatures*)
370    format (5x*mband*i5)
        end

```



```
function atan3 (y,x)
  r=x*x+y*y
  if (r.eq.0.) go to 10
  atan3=atan2(y,x)
  return
10 atan3=0.
  return
end
```

```

subroutine Belemt
common /control/ title(20), ncn, ncns, np, ncard, mband, alpha,
1 time, Delt
common /matp/ emat(10,6)
common /nodes/ cord(1410,3), cord1(1410,3), T(1410), dT(1410)
common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
common /gid/ xg(9), wg(9), ngaus
common /stiff/ fo(1410), fl(1410), ss(1410,595)
common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)

c
c   this subroutine calculates boundary elements S and F
c   area=0.0
c
c   boundary - formation of stiffness and force(effective)
c
c   do 100 mm=1,nbe
c
c   specify element coordinates xe
c
c   do 10 k=1,ncns
c     Te(k)=T(nbq(mm,k))
c     do 10 i=1,ncard
c       xe(i,k)=cord(nbq(mm,k),i)
10  continue
c
c   zero F and G
c
c   do 20 ia=1,ncns
c     F(ia)=0.0
c     do 20 ib=1,ncns
c       G(ia,ib)=0.0
20  continue
c
c   integration loops
c
c   do 50 igauss=1,ngaus
c     a=xg(igauss)
c     do 50 jgauss=1,ngaus
c       c=xg(jgauss)
c
c       call sfuncts (a,c)
c       call auxm (mm)
c
c       ds=det*wg(igauss)*wg(jgauss)
c       area=area+ds
c
c   integrate for G and F
c
c   do 40 ia=1,ncns
c     do 30 ib=1,ncns
c       G(ia,ib)=G(ia,ib)+p(ia)*p(ib)*ds
30  continue
c     F(ia)=F(ia)+p(ia)*ds
40  continue
50  continue

```

```

c
c      store F and -G*Ti into RHS
c
      do 70 ia=1,ncns
      sum=0.0
      do 60 ib=1,ncns
      sum=sum+G(ia,ib)*hf(mm)*T(nbq(mm,ib))
60  continue
      fl(nbq(mm,ia))=fl(nbq(mm,ia))+F(ia)*(qn(mm)+hf(mm)*Tf(mm))-sum
70  continue
c
c      store G into global "stiffness" matrix
c
      do 90 ia=1,ncns
      do 80 ib=1,ncns
      kk=nbq(mm,ib)-nbq(mm,ia)+1
      if (kk.le.0) go to 80
      ss(nbq(mm,ia),kk)=ss(nbq(mm,ia),kk)+alpha*G(ia,ib)*hf(mm)
80  continue
90  continue
c
100 continue
      write (6,110) area
c
      return
c
110 format (1h0,5x,5harea=,e13.7)
      end

```

```

      subroutine Belemt1
      common /control/ title(20), ncn, ncns, np, ncord, mband, alpha,
1 time, Delt
      common /matp/ emat(10,6)
      common /nodes/ cord(1410,3), cordl(1410,3), T(1410), dT(1410)
      common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
      common /gid/ xg(9), wg(9), ngaus
      common /stiff/ fo(1410), fl(1410), ss(1410,595)
      common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)

C
C      this subroutine calculates boundary elements S and F
C      area=0.0
C
C      boundary - formation of stiffness and force(effective)
C
C      do 100 mm=1,nbel
C
C      specify element coordinates xe
C
C      do 10 k=1,ncns
C      Te(k)=T(nbq(nbe+mm,k))
C      do 10 i=1,ncord
C      xe(i,k)=cordl(nbq(nbe+mm,k),i)
10 continue
C
C      zero F and G
C
C      do 20 ia=1,ncns
C      F(ia)=0.0
C      do 20 ib=1,ncns
C      G(ia,ib)=0.0
20 continue
C
C      integration loops
C
C      do 50 igauss=1,ngaus
C      a=xg(igauss)
C      do 50 jgauss=1,ngaus
C      c=xg(jgauss)
C
C      call sfuncts (a,c)
C      call auxml (nbe+mm)
C
C      ds=det*wg(igauss)*wg(jgauss)
C      area=area+ds
C
C      integrate for F and G
C
C      do 40 ia=1,ncns
C      do 30 ib=1,ncns
C      G(ia,ib)=G(ia,ib)+p(ia)*p(ib)*ds
30 continue
C      F(ia)=F(ia)+p(ia)*ds
40 continue
50 continue

```

```

c
c      store F and -G*Ti into RHS
c
      do 70 ia=1,ncns
      sum=0.0
      do 60 ib=1,ncns
      sum=sum+G(ia,ib)*hf(nbe+mm)*T(nbq(nbe+mm,ib))
60 continue
      fl(nbq(nbe+mm,ia))=fl(nbq(nbe+mm,ia))+F(ia)*(qn(nbe+mm)+hf(nbe+mm)
1 *Tf(nbe+mm))-sum
70 continue
c
c      store G into global "stiffness" matrix
c
      do 90 ia=1,ncns
      do 80 ib=1,ncns
      kk=nbq(mm,ib)-nbq(mm,ia)+1
      if (kk.le.0) go to 80
      ss(nbq(nbe+mm,ia),kk)=ss(nbq(nbe+mm,ia),kk)+alpha*G(ia,ib)*hf(nbe
1 +mm)
80 continue
90 continue
c
100 continue
      write (6,110) area
c
      return
c
110 format (1h0,5x,5harea=,e13.7)
      end

```

```

      subroutine sfuncts (a,c)
      common /control/ title(20), ncn, ncns, np, ncord, mband, alpha,
1 time, Delt
      common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
      common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
      dimension h(2,3), dh(2,3)
c
c      a and c denote xi1 and xi2 values at the point considered
c
      go to (10,20), nlag
c
c      linear one-dimensional shape functions and derivatives
c
10 continue
      h(1,1)=.5*(1.-a)
      h(1,2)=.5*(1.+a)
      h(2,1)=.5*(1.-c)
      h(2,2)=.5*(1.+c)
      dh(1,1)=-.5
      dh(1,2)=.5
      dh(2,1)=-.5
      dh(2,2)=.5
      go to 30
c
c      quadratic one-dimensional shape functions and derivatives
c
20 continue
      h(1,1)=-.5*(1.-a)*a
      h(1,2)=1.-a*a
      h(1,3)=.5*(1.+a)*a
      h(2,1)=-.5*(1.-c)*c
      h(2,2)=1.-c*c
      h(2,3)=.5*(1.+c)*c
      dh(1,1)=-.5+a
      dh(1,2)=-2.*a
      dh(1,3)=.5+a
      dh(2,1)=-.5+c
      dh(2,2)=-2.*c
      dh(2,3)=.5+c
c
c      Lagrangian two-dimensional shape functions and derivatives
c
30 continue
      do 40 m=1,nlag+1
      do 40 l=1,nlag+1
      ic=l+(m-1)*(nlag+1)
      p(ic)=h(1,1)*h(2,m)
      dp(1,ic)=dh(1,1)*h(2,m)
      dp(2,ic)=h(1,1)*dh(2,m)
40 continue
c
c      calculate dx/dxi
c
      do 60 i=1,ncord
      do 60 j=1,ncord-1

```

```
    dxdxi(i,j)=0.  
    do 50 k=1,ncns  
      dxdxi(i,j)=dxdxi(i,j)+xe(i,k)*dp(j,k)  
50  continue  
60  continue  
    return  
  end
```

```

subroutine auxm (mm)
common /control/ title(20), ncn, ncns, np, ncord, mband, alpha,
1 time, Delt
common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
c
c   this subroutine calculates det
c   and the xp coordinates (integration points)
c
c   calculate coordinates of the integration points , xp
c
do 20 i=1,ncord
xp(i)=0.
do 10 k=1,ncns
xp(i)=xp(i)+xe(i,k)*p(k)
10 continue
20 continue
c
c   calculate det for a surface
c
det=sqrt((dxdxi(2,1)*dxdxi(3,2)-dxdxi(3,1)*dxdxi(2,2))**2+(dxdxi(3
1 ,1)*dxdxi(1,2)-dxdxi(1,1)*dxdxi(3,2))**2+(dxdxi(1,1)*dxdxi(2,2)
2 -dxdxi(2,1)*dxdxi(1,2))**2)
return
end

```



```

subroutine auxml (mm)
common /control/ title(20), ncn, ncns, np, ncord, mband, alpha,
1 time, Delt
common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
c
c   this subroutine calculates det
c   and the xp coordinates (integration points)
c
c   calculate coordinates of the integration points , xp
c
  do 20 i=1,ncord
    xp(i)=0.
    do 10 k=1,ncns
      xp(i)=xp(i)+xe(i,k)*p(k)
10  continue
20  continue
c
c   r*dxdxi(2,n)
c
  do 30 n=1,ncns
    dxdxi(2,n)=xp(1)*dxdxi(2,n)
30  continue
c
c   calculate det for a surface
c
  det=sqrt((dxdxi(2,1)*dxdxi(3,2)-dxdxi(3,1)*dxdxi(2,2))**2+(dxdxi(3
1 ,1)*dxdxi(1,2)-dxdxi(1,1)*dxdxi(3,2))**2+(dxdxi(1,1)*dxdxi(2,2)
2 -dxdxi(2,1)*dxdxi(1,2))**2)
  return
end

```

```

      subroutine Celemt
      common /control/ title(20), ncn, ncns, np, ncord, mband, alpha,
1 time, Delt
      common /matp/ emat(10,6)
      common /nodes/ cord(1410,3), cord1(1410,3), T(1410), dT(1410)
      common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
      common /gid/ xg(9), wg(9), ngaus
      common /stiff/ fo(1410), fl(1410), ss(1410,595)
      common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)

c
c   this subroutine calculates S, D, and R matrices using (x,y,z)
c
      volume=0.0
c
      do 110 mm=1,nce
c
c   specify element coordinates xe
c
      do 10 k=1,ncn
      Te(k)=T(nc(mm,k))
      do 10 i=1,ncord
      xe(i,k)=cord(nc(mm,k),i)
10 continue
c
c   the heat source density and conductivity for this element
c
      Q=emat(mat(mm),3)
      ek(1)=emat(mat(mm),4)
      ek(2)=emat(mat(mm),5)
      ek(3)=emat(mat(mm),6)
c
c   define beta as (thermal capacity)/(dt)
c
      beta=emat(mat(mm),1)*emat(mat(mm),2)/Delt
c
c   zero S, D, and R
c
      do 20 ia=1,ncn
      D(ia)=0.0
      do 20 ib=1,ncn
      S(ia,ib)=0.0
      R(ia,ib)=0.0
20 continue
c
c   integration loops
c
      do 60 igauss=1,ngaus
      a=xg(igauss)
      do 60 jgauss=1,ngaus
      c=xg(jgauss)
      do 60 kgauss=1,ngaus
      e=xg(kgauss)
c
      call sfunct (a,c,e)
      call aux (mm)

```

```

c      dv=det*wg(igaus)*wg(jgaus)*wg(kgaus)
      volume=volume+dv
c
c      calculate S
c
      do 40 ia=1,ncn
      do 40 ib=1,ncn
      do 30 i=1,ncord
      S(ia,ib)=S(ia,ib)+ek(i)*dpdx(i,ia)*dpdx(i,ib)*dv
30 continue
40 continue
c
c      calculate D and R
c
      do 50 ia=1,ncn
      D(ia)=D(ia)+p(ia)*dv
      do 50 ib=1,ncn
      R(ia,ib)=R(ia,ib)+p(ia)*p(ib)*dv
50 continue
c
60 continue
c
c      store D and -S*Ti into RHS
c
      do 80 ia=1,ncn
      sum=0.0
      do 70 ib=1,ncn
      sum=sum+S(ia,ib)*T(nc(mm,ib))
70 continue
      fl(nc(mm,ia))=fl(nc(mm,ia))+D(ia)*Q-sum
80 continue
c
c      store S and R into global "stiffness" matrix
c
      do 100 ia=1,ncn
      do 90 ib=1,ncn
      kk=nc(mm,ib)-nc(mm,ia)+1
      if (kk.le.0) go to 90
      ss(nc(mm,ia),kk)=ss(nc(mm,ia),kk)+alpha*S(ia,ib)+beta*R(ia,ib)
90 continue
100 continue
c
110 continue
      write (6,120) volume
      return
c
120 format (1h0,5x,7hvolume=,e13.7)
      end

```

```

      subroutine Celemt1
      common /control/ title(20), ncn, ncns, np, ncard, mband, alpha,
1 time, Delt
      common /matp/ emat(10,6)
      common /nodes/ cord(1410,3), cordl(1410,3), T(1410), dT(1410)
      common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
      common /gid/ xg(9), wg(9), ngaus
      common /stiff/ fo(1410), fl(1410), ss(1410,595)
      common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
c
c      this subroutine calculates S, D, and R matrices using (r,theta,z)
c
c      volume=0.0
c
c      do 120 mm=1,ncel
c
c      specify element coordinates xe
c
c      do 10 k=1,ncn
c      Te(k)=T(nc(nce+mm,k))
c      do 10 i=1,ncard
c      xe(i,k)=cordl(nc(nce+mm,k),i)
10 continue
c
c      the heat source density and conductivity for this element
c
c      Q=emat(mat(nce+mm),3)
c      ek(1)=emat(mat(nce+mm),4)
c      ek(2)=emat(mat(nce+mm),5)
c      ek(3)=emat(mat(nce+mm),6)
c
c      define beta as (thermal capacity)/(dt)
c
c      beta=emat(mat(nce+mm),1)*emat(mat(nce+mm),2)/Delt
c
c      zero S, D, and R
c
c      do 20 ia=1,ncn
c      D(ia)=0.0
c      do 20 ib=1,ncn
c      S(ia,ib)=0.0
c      R(ia,ib)=0.0
20 continue
c
c      integration loops
c
c      do 70 igauss=1,ngaus
c      a=xg(igauss)
c      do 70 jgauss=1,ngaus
c      c=xg(jgauss)
c      do 70 kgauss=1,ngaus
c      e=xg(kgauss)
c
c      call sfunct (a,c,e)
c      call aux (nce+mm)

```

```

C      dv=xp(1)*det*wg(igaus)*wg(jgaus)*wg(kgaus)
      volume=volume+dv
C
C      (1/r)*dp/dx
C
      do 30 n=1,ncn
      dpdx(2,n)=(1./xp(1))*dpdx(2,n)
30 continue
C
C      calculate S
C
      do 50 ia=1,ncn
      do 50 ib=1,ncn
      do 40 i=1,ncord
      S(ia,ib)=S(ia,ib)+ek(i)*dpdx(i,ia)*dpdx(i,ib)*dv
40 continue
50 continue
C
C      calculate D and R
C
      do 60 ia=1,ncn
      D(ia)=D(ia)+p(ia)*dv
      do 60 ib=1,ncn
      R(ia,ib)=R(ia,ib)+p(ia)*p(ib)*dv
60 continue
C
70 continue
C
C      store D and -S*Ti into RHS
C
      do 90 ia=1,ncn
      sum=0.0
      do 80 ib=1,ncn
      sum=sum+S(ia,ib)*T(nc(nce+mm,ib))
80 continue
      fl(nc(nce+mm,ia))=fl(nc(nce+mm,ia))+D(ia)*Q-sum
90 continue
C
C      store S and R into global "stiffness" matrix
C
      do 110 ia=1,ncn
      do 100 ib=1,ncn
      kk=nc(mm,ib)-nc(mm,ia)+1
      if (kk.le.0) go to 100
      ss(nc(nce+mm,ia),kk)=ss(nc(nce+mm,ia),kk)+alpha*S(ia,ib)+beta*R(ia
1,ib)
100 continue
110 continue
C
120 continue
      write (6,130) volume
      return
C
130 format (1h0,5x,7hvolume=,e13.7)
      end

```

```

      subroutine sfunc (a,c,e)
c
      common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
      common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
      dimension h(3,3), dh(3,3)
c
c      a,c, and e denote xi1,xi2, and xi3 values at the point considered
c
      go to (10,20), nlag
c
c      linear one-dimensional shape functions and derivatives
c
10 continue
      h(1,1)=.5*(1.-a)
      h(1,2)=.5*(1.+a)
      h(2,1)=.5*(1.-c)
      h(2,2)=.5*(1.+c)
      h(3,1)=.5*(1.-e)
      h(3,2)=.5*(1.+e)
      dh(1,1)=-.5
      dh(1,2)=.5
      dh(2,1)=-.5
      dh(2,2)=.5
      dh(3,1)=-.5
      dh(3,2)=.5
      go to 30
c
c      quadratic one-dimensional shape functions and derivatives
c
20 continue
      h(1,1)=-.5*(1.-a)*a
      h(1,2)=1.-a*a
      h(1,3)=.5*(1.+a)*a
      h(2,1)=-.5*(1.-c)*c
      h(2,2)=1.-c*c
      h(2,3)=.5*(1.+c)*c
      h(3,1)=-.5*(1.-e)*e
      h(3,2)=1.-e*e
      h(3,3)=.5*(1.+e)*e
      dh(1,1)=-.5+a
      dh(1,2)=-2.*a
      dh(1,3)=.5+a
      dh(2,1)=-.5+c
      dh(2,2)=-2.*c
      dh(2,3)=.5+c
      dh(3,1)=-.5+e
      dh(3,2)=-2.*e
      dh(3,3)=.5+e
c
c      Lagrangian three-dimensional shape functions and derivatives
c
30 continue
      do 40 n=1,nlag+1
      do 40 m=1,nlag+1
      do 40 l=1,nlag+1

```

```

        ic=1+(m-1)*(nlag+1)+(n-1)*(nlag+1)*(nlag+1)
        p(ic)=h(1,1)*h(2,m)*h(3,n)
        dp(1,ic)=dh(1,1)*h(2,m)*h(3,n)
        dp(2,ic)=h(1,1)*dh(2,m)*h(3,n)
        dp(3,ic)=h(1,1)*h(2,m)*dh(3,n)
40    continue
        return
    end

```

```

      subroutine aux (mm)
c
      common /control/ title(20), ncn, ncns, np, ncord, mband, alpha,
1 time, Delt
      common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxidx(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
c
c      calculate dx/dxi
c
      do 20 i=1,ncord
      do 20 j=1,ncord
      dxidx(i,j)=0.
      do 10 k=1,ncn
      dxidx(i,j)=dxidx(i,j)+xe(i,k)*dp(j,k)
10 continue
20 continue
      do 40 i=1,ncord
      xp(i)=0.
      do 30 k=1,ncn
      xp(i)=xp(i)+xe(i,k)*p(k)
30 continue
40 continue
c
c      calculate determinant, dxi/dλ, and dp/dx
c
      det=dxidx(1,1)*dxidx(2,2)*dxidx(3,3)+dxidx(2,1)*dxidx(3,2)*dxidx(1
1 ,3)+dxidx(1,2)*dxidx(2,3)*dxidx(3,1)-dxidx(1,3)*dxidx(2,2)*dxidx(
2 3,1)-dxidx(1,2)*dxidx(2,1)*dxidx(3,3)-dxidx(2,3)*dxidx(3,2)*dxidx
3 (1,1)
      if (det) 50,60,70
50 continue
      write (6,100) mm
      stop
60 det=.000001
      write (6,110) mm
70 continue
      detp=dxidx(2,2)*dxidx(3,3)-dxidx(2,3)*dxidx(3,2)
      dxidx(1,1)=detp/det
      detp=dxidx(1,2)*dxidx(3,3)-dxidx(1,3)*dxidx(3,2)
      dxidx(1,2)=-detp/det
      detp=dxidx(1,2)*dxidx(2,3)-dxidx(1,3)*dxidx(2,2)
      dxidx(1,3)=detp/det
      detp=dxidx(2,1)*dxidx(3,3)-dxidx(2,3)*dxidx(3,1)
      dxidx(2,1)=-detp/det
      detp=dxidx(1,1)*dxidx(3,3)-dxidx(1,3)*dxidx(3,1)
      dxidx(2,2)=detp/det
      detp=dxidx(1,1)*dxidx(2,3)-dxidx(1,3)*dxidx(2,1)
      dxidx(2,3)=-detp/det
      detp=dxidx(2,1)*dxidx(3,2)-dxidx(2,2)*dxidx(3,1)
      dxidx(3,1)=detp/det
      detp=dxidx(1,1)*dxidx(3,2)-dxidx(1,2)*dxidx(3,1)
      dxidx(3,2)=-detp/det
      detp=dxidx(1,1)*dxidx(2,2)-dxidx(1,2)*dxidx(2,1)
      dxidx(3,3)=detp/det
      do 90 i=1,ncord
      do 90 n=1,ncn
      dpdx(i,n)=0.

```



```

        do 80 j=1,ncord
          dpdx(i,n)=dpdx(i,n)+dp(j,n)*dxidx(j,i)
80      continue
90      continue
        return
c
100 format (1h0,38h negative det stopped at, element no =,i3)
110 format (1h0,23hzero determinant el no ,i3)
      end

```

```

      subroutine gpoints
      common /gid/ xg(9), wg(9), ngaus
c
c      this subroutine calculates Gauss points for integration
c
      go to (10,20,30,40,50,60,70,80,90), ngaus
10  xg(1)=0.
    wg(1)=2.
    return
20  xg(1)=-0.577350269189626
    wg(1)=1.
    go to 100
30  xg(1)=-0.774596669241483
    xg(2)=0.
    wg(1)=0.555555555555556
    wg(2)=0.888888888888889
    go to 100
40  xg(1)=-0.861136311594053
    xg(2)=-0.339981043584856
    wg(1)=0.347854845137454
    wg(2)=0.652145154862546
    go to 100
50  xg(1)=-0.906179845938664
    xg(2)=-0.538469310105683
    xg(3)=0.
    wg(1)=0.236926885056189
    wg(2)=0.478628670499366
    wg(3)=0.568888888888889
    go to 100
60  xg(1)=-0.932469354203152
    xg(2)=-0.661209386466265
    xg(3)=-0.238619186083197
    wg(1)=0.171324492379170
    wg(2)=0.360761573048139
    wg(3)=0.467913934572691
    go to 100
70  xg(1)=-0.949107912342759
    xg(2)=-0.741531185599394
    xg(3)=-0.405845151377397
    xg(4)=0.
    wg(1)=0.129484966168870
    wg(2)=0.279705391489277
    wg(3)=0.381830050505119
    wg(4)=0.417959183673469
    go to 100
80  xg(1)=-0.960289856497536
    xg(2)=-0.796666477413627
    xg(3)=-0.525532409916329
    xg(4)=-0.183434642495650
    wg(1)=0.101228536290376
    wg(2)=0.222381034453374
    wg(3)=0.313706645877887
    wg(4)=0.362683783378362
    go to 100
90  xg(1)=-0.968160239507626
    xg(2)=-0.836031107326636
    xg(3)=-0.613371432700590
    xg(4)=-0.324253423403809

```

```

      xg(5)=0.
      wg(1)=0.081274388361574
      wg(2)=0.180648160694857
      wg(3)=0.260610696402935
      wg(4)=0.312347077040003
      wg(5)=0.330239355001260
100  n=ngaus/2
      do 110 j=1,n
          i=ngaus+1-j
          xg(i)=-xg(j)
110  wg(i)=wg(j)
      return
      end

```

```

      subroutine solve
      common /control/ title(20), ncn, ncns, np, ncard, mband, alpha,
1 time, Delt
      common /nodes/ cord(1410,3), cord1(1410,3), T(1410), dT(1410)
      common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
      common /bdy/ nby, nbc(100), Ts(100)
      common /stiff/ fo(1410), fl(1410), ss(1410,595)
      common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
c
c      this subroutine solves stiffness matrix equations
c
c      zero incremental temperature boundary conditions
c
      if (nby.eq.0) go to 50
      do 40 n=1,np
      do 30 i=1,nby
      if (nbc(i).eq.n) go to 10
      if (nbc(i).lt.n) go to 30
      if (nbc(i).ge.n+mband) go to 30
      ss(n,nbc(i)-n+1)=0.0
      go to 30
10 do 20 j=2,mband
      ss(n,j)=0.0
20 continue
      fl(n)=0.0
30 continue
40 continue
50 continue
c
c      remove zero on the diagonal
c
      do 60 i=1,np
      if (ss(i,1).eq.0.0) ss(i,1)=1.0
60 continue
c
c      gauss elimination equation solver
c
c      decompose banded [ss] matrix
c
      do 90 k=1,np
      do 90 l=2,mband
      if (ss(k,l).eq.0.) go to 90
      i=k+l-1
      pp=ss(k,l)/ss(k,1)
      if (i.gt.np) go to 80
      j=0
      do 70 n=1,mband
      j=j+1
      ss(i,j)=ss(i,j)-pp*ss(k,n)
70 continue
80 continue
      ss(k,l)=pp
90 continue
c
c      decompose fl vector (applied loads)

```

```

c      do 120 k=1,np
      do 100 l=2,mband
      if (ss(k,l).eq.0.) go to 100
      i=k+l-1
      if (i.gt.np) go to 110
      f1(i)=f1(i)-ss(k,l)*f1(k)
100  continue
110  continue
      f1(k)=f1(k)/ss(k,1)
120  continue

c
c      back substitution
c
      do 130 k=1,np
      dT(k)=f1(k)
130  continue
      do 150 m=2,np
      k=np+1-m
      do 140 l=2,mband
      if (ss(k,l).eq.0.) go to 140
      i=k+l-1
      dT(k)=dT(k)-ss(k,l)*dT(i)
140  continue
150  continue
      return
      end

```

```

subroutine check
common /control/ title(20), ncn, ncns, np, ncard, mband, alpha,
1 time, Delt
common /matp/ emat(10,6)
common /nodes/ cord(1410,3), cordl(1410,3), T(1410), dT(1410)
common /element/ nlag, nce, ncel, nc(98,27), mat(98), nbe, nbel,
1 nbq(60,9), qn(60), hf(60), Tf(60)
common /gid/ xg(9), wg(9), ngaus
common /stiff/ fo(1410), fl(1410), ss(1410,595)
common /numel/ p(27), dp(3,27), dpdx(3,27), xe(3,27), xp(3), det,
1 dxdxi(3,3), dxidx(3,3), S(27,27), D(27), R(27,27), G(9,9), F(9),
2 Te(27), ek(3)
C
C   this subroutine checks the problem parameters
C
C   (dT)*(RHS) calculation
C
C   dtrhs=0.0
C   do 10 n=1,np
C     dtrhs=dtrhs+fo(n)*dT(n)
10 continue
C
C   write (6,20) dtrhs
C   return
C
20 format (5x' (dT)*(RHS)'lpe12.5)
end

```

END

**DATE
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