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CHARACTERIZING SOURCE REGIONS WITH SIGNAL
SUBSPACE METHODS: THEORY AND COMPUTATIONAL
METHODS

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Abstract

A mathematical approach is developed for empirically characterizing a given source region using waveforms from a collection of calibration events. A region is considered to be adequately characterized if the waveforms from any event in the source region can be represented as a linear combination of calibration event waveforms. The purpose of such characterizations is to build waveform "recognizers" for specific regions for precision location applications, and to provide a means of separating superimposed waveforms from multiple events in different source regions. The particular form of characterization used is insensitive to variations in the source time function and to anything but changes from the normal range of source mechanisms encountered in the source region. The standard waveform correlation coefficient used to estimate event clustering is generalized to estimate separation between single events and event clusters, and between two clusters of events. The generalized correlation coefficient is insensitive to variations in source time function and, to some extent, mechanism. The statistics of waveform correlation coefficients are developed, and show that conventional estimates made from single station data are often ambiguous due to the transient nature of seismic signals. A correlation coefficient developed for network or array data removes the ambiguity.

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Introduction

The operational problems posed by the use of internal stations to monitor a Comprehensive Test Ban Treaty (CTBT) or a very-low-yield threshold test ban are daunting. The low detection thresholds of internal networks will assure the detection of many tens of thousands of events each year. A conservative monitoring doctrine would require an interpretation for each each detection, to see if it could be a signal from a clandestine nuclear test. Very often the majority of detected regional events are mining explosions. Under this condition, the discrimination problem is driven by the need to screen out mining explosions in the search for clandestine tests.

The monitoring problem is simplified by the fact that the vast majority of explosions will be confined to established mines and quarries. In the vicinity of any one station in the network, the number of such source regions is likely to be modest. In this situation, the event location problem can be recast as a multiple hypothesis test on location at a number of discrete sources (Harris, 1987). The discrimination problem might be solved by a process of attributing events to known legitimate sources.

This report presents a waveform correlation solution to the location and discrimination problems. As a corollary, a solution is also presented to the problem of detecting a weak signal, from a small or distant event, masked by a stronger signal from a local mining explosion.

The operational assumption is that an event may be screened out if it can be located positively at a known mine, and if its source mechanism is consistent with previous explosions at that mine. The waveform correlation approach requires a collection of master events to calibrate each source region. It assumes further that an event of unknown origin may be attributed to a particular mine if the seismograms it generates can be approximated as a linear combination of master event waveforms. The best linear approximation is that which maximizes the correlation with the unlocated event's waveforms.

Mathematically, the signals generated by an event are treated as vectors in a signal vector space. The collection of signals corresponding to a group of master events calibrating a source region span a *signal subspace*. The subspace itself characterizes the source region.

An unlocated event belongs to the region if its signal vector has a substantial projection into the subspace. The projection operation maximizes the correlation coefficient in the manner described above.

If the seismogram from an event can be approximated from a collection of master event recordings, then it also can be removed from the data by subtracting its approximation. This observation provides a means for separating superimposed waveforms from, for example, a close mining explosion and a more distant or much smaller event. It may be useful for sorting out events in aftershock swarms. Specifically, master events waveforms may be used as reference signals in a "noise canceller" (Widrow, et al., 1975) to suppress the seismograms of masking events.

Waveform correlation measurements have been used extensively to bound the separation between pairs of events, and to study event clustering, i.e. the tendency of events to occur in tight spatial groups. For example, Frankel (1982) has observed the clustering of small earthquake pairs that may be precursors to larger events. Pechman and Kanamori (1982) have examined clusters that may represent fault asperities. Such approaches are similar to the multiple hypothesis test on location proposed in this report. They depend upon the fact that similar waveforms frequently result from closely located sources with nearly identical source functions and mechanisms. Frequently, clustering studies rely upon an hypothesis advanced by Geller and Mueller (1980), which suggests that events producing similar waveforms must be located within a quarter of the shortest wavelength to which the similarity extends.

The notion of using calibration event waveforms as an empirical basis for representing seismograms from a particular source region is similar to empirical Green's function applications in strong motion studies (e.g. Hartzell, 1978; Joyner and Boore, 1986; Hutchings and Wu, 1989) for seismic hazard assessment. These studies make the assumption that small earthquakes are impulsive point sources, producing a flat spectrum over the frequency band of interest. The resulting seismograms approximate earth impulse responses in this case and can be convolved with an hypothesized source time function to synthesize seismograms for a large event. Such methods require a number of small events that adequately sample the source volume. It is essential that the events be carefully controlled to have uniform focal mechanisms (Hutchings and Wu, 1989).

The synthesis process may be reversed to use small calibration events to estimate the rupture process of large earthquakes (Fukuyama and Irikura, 1986). In this case, waveforms from a collection of small well-located earthquakes provide a basis for decomposing the seismogram of a much larger event. The weights and, particularly, the delays in the decomposition may present a picture of the spread of the rupture front in the larger event. This approach is an empirical equivalent of inversion methods that use synthetic seismograms generated for elements of a grided fault plane (e.g. Takeo, 1987).

All of these applications have one thing in common: the use of calibration event waveforms as an empirical basis for representing the waveforms from events in a particular source region. The correlation coefficient is a measure of the accuracy of this representation. Consequently, it is a measure of how close one event is to another or, with suitable modification, to a group of events. So modified, it is also a measure of the degree of certainty that one event belongs to a cluster.

Characteristics of An Empirical Basis

A waveform basis should be complete, in the sense that the waveforms from any event in the source region be representable from the basis. The requirement implies that the calibrating event set should adequately spatially sample the source volume that is being characterized. In addition, it must contain events that span the full range of source mechanisms and frequency bands developed by events in the region.

One contribution of this report will be to extend the waveform representation and the calculation of the correlation coefficient to collections of events. A complete characterization is difficult or impossible to achieve with a single event. Existing correlation methods compare events pairwise, and correspond to single-event source region characterizations. A collection of events will form a much better representation, with defects of excitation in one event likely to be compensated by different excitation in another.

The source region characterization must be insensitive as far as possible to variations in the source time function, to waveform registration errors, and to anything but changes in the source mechanism from what is normal for the region in question. In contrast to many scientific investigations, the location and verification applications require no estimates of source time function or source mechanism. Instead, they require empirical waveform

“recognizers” for location and changes in mechanism.

Another contribution of this report is to ensure that the waveform basis and corresponding correlation coefficient are suitably insensitive to the source time function. Existing waveform correlation methods are sensitive to source function and normal mechanism variations. These sensitivities can result in low correlation values for closely located events. The proposed methods draw upon adaptive signal processing concepts (cf. Alexander, 1986) to ensure some degree of independence from the source function.

The empirical basis for a source region should be unique to that region. Uniqueness eliminates problems with ambiguity or uncertainty in the location and attribution process. It also ensures that the desired signal will not be cancelled in the event separation problem. Waveform representations and correlation estimators that use data recorded by a single sensor (one component at a single recording site) have problems with uniqueness. Single-channel waveforms do not provide enough data or sufficiently diverse signals to reliably discriminate different source regions.

A third contribution of this report is to extend waveform representation and the correlation coefficient to use data from arrays or networks of stations, i.e. a spatial aperture. Recording over an aperture improves the uniqueness of the characterization by increasing the diversity of the characterizing signals. The term “diversity” is intended to suggest that the collection of signals has more degrees of freedom to change when the source location or mechanism is changed. In contrast to a single station, an array exhibits greater diversity by introducing patterns of propagation delays across the aperture that coarsely distinguish different source regions on the basis of direction. In general, a collection of source-receiver paths, with potential for interactions with a greater number of scatterers, improves uniqueness over a single path characterization.

Finally, the waveform representation and correlation coefficient must be statistically reliable. If short time windows are used, for example, the estimated correlation coefficient may be statistically unreliable (Thorbjarnardottir and Pechman, 1987). A correlation “floor” (bias phenomenon) develops. Any correlation value below the level of the floor is ambiguous. It could be an indication of event proximity, or it could be an artifact of the small quantity of data. In this report, the statistics of the correlation coefficient are shown to depend on the time-bandwidth product (TB), where T is the duration of the

data analysis window and B is the signal bandwidth. A time-bandwidth product of about 100 is required for reliable separation of source regions.

Another contribution of this report is to show that the use of array or network data improves the statistical reliability of correlation estimates. By computing the correlation coefficient over a spatial aperture, the time-bandwidth product is effectively multiplied by the number of independent sensors. This approach brings about a marked reduction in bias and variance in the correlation estimate.

Source Region Characterization

We first examine the assumptions behind a signal subspace characterization for a source region. In this report, no attempt will be made to prove that a complete characterization is possible. It will be assumed that a basis for all waveforms generated by events in a particular region can be constructed from a realizable (finite) collection of calibrating events. We will speculate on the necessary conditions for such a basis: adequate spatial sampling of the source region, full illumination of the range of mechanisms and frequency bands exhibited by events in the region. It is not shown that these conditions are sufficient. Nonetheless, it is appropriate to discuss reasonable requirements for a calibrating event set. It is also desirable that a basis be kept specific to particular source region, so that it can be used to resolve events in different regions. A choice of array or network data can be made to improve resolution.

A source region Ω is assumed to be characterized by the waveforms generated by a collection of N_M events at locations $\{\underline{y}_j\}$ in Ω . A sensor at location $\underline{x}$ receives the signal $r_k(t, \underline{x}, \underline{y}_j)$. The subscript index $k \in \{v, n, e\}$ indicates the sensor orientation. If the source region characterization is complete, then the signal $r_k(t, \underline{x}, \underline{y})$ produced by an arbitrary event at location $\underline{y} \in \Omega$ can be written as a combination of the calibrating waveforms:

$$r_k(t, \underline{x}, \underline{y}) \approx \hat{r}_k(t, \underline{x}, \underline{y}) = \sum_{j=1}^{N_M} \int_{-\infty}^{\infty} dt' a_j(t') r_k(t - t', \underline{x}, \underline{y}_j) \quad (1)$$

In practice, the degree to which this equation can be solved for the weights a_j , is the measure of confidence that an unlocated event originates in the calibrated source region. This representation equation can be viewed as a test of location.

Equation (1) does not represent a simple linear combination of calibration waveforms. The weights, a_j , are filters in a convolution operation. This complication is necessary to compensate for differences in source function and for small errors in registration between the waveforms. Waveforms from the master events and the event to be represented must be aligned for an efficient representation. The equation above assumes that alignment is accurate to the duration of the filters.

What really characterizes the source volume from an empirical standpoint are the Green's functions, $g_{ik}(t, \underline{x}, \underline{y})$, which relate source functions to the received signal. Any given source is representable as a superposition of elemental forces, f_i (sums of real and equivalent body forces), resulting in the received signal:

$$r_k(t, \underline{x}) = \sum_i \int_{-\infty}^{\infty} dt' \int_V d\underline{\xi} f_i(t', \underline{\xi}) g_{ik}(t - t', \underline{x}, \underline{\xi}) \quad (2)$$

Here $V \subset \Omega$ is the source volume over which the f_i are non-zero [Aki and Richards, 1980; Stump and Johnson, 1977]. Throughout this report, all events will be assumed to act as point sources. Under this assumption, the volume integral disappears from the signal representation:

$$r_k(t, \underline{x}, \underline{y}) = \sum_i \int_{-\infty}^{\infty} dt' f_i(t') g_{ik}(t - t', \underline{x}, \underline{y}) \quad (3)$$

and $\underline{y} \in \Omega$ is the point source location. Likewise, the Fourier Transform of the received signal may be represented by:

$$R_k(\omega, \underline{x}, \underline{y}) = \sum_i F_i(\omega) G_{ik}(\omega, \underline{x}, \underline{y}). \quad (4)$$

A complete characterization of a source region may be possible provided calibration events are available which adequately sample the source volume and fully "illuminate" the range of relevant Green's functions. If a basis of modest dimension is to be constructed, the Green's functions must be spatially continuous and sufficiently slowly varying that they can be interpolated from samples at a finite number of sample points:

$$G_{ik}(\omega, \underline{x}, \underline{y}) \approx \sum_{j=1}^{N_M} \alpha_j(\omega, \underline{x}, \underline{y}) G_{ik}(\omega, \underline{x}, \underline{y}_j) \quad (5).$$

In an area, such as a fault zone, where strong spatial discontinuities occur, it may be sufficient that the area be susceptible to partition into a small number of continuous regions. In addition, it may be possible to choose the source region so that a limited number of source mechanisms are represented by the events in it. For example, the source region may be restricted to a specific mine which fires a limited number of charge patterns.

This kind of situation reduces the dimension of the basis still further, since only a small number of specific linear combinations of Green's functions may have to be characterized.

If the range of source mechanisms and Green's functions is adequately sampled, then the remaining requirement is that the basis exhibit adequate spectral coverage. In the frequency domain, the signal is represented by:

$$\begin{aligned}
 R_k(\omega, \underline{x}, \underline{y}) &\approx \sum_{j=1}^{N_M} A_j(\omega) R_k^j(\omega, \underline{x}, \underline{y}_j) \\
 \sum_i F_i(\omega) G_{ik}(\omega, \underline{x}, \underline{y}) &\approx \sum_{j=1}^{N_M} A_j(\omega) \left[\sum_i F_i^j(\omega) G_{ik}(\omega, \underline{x}, \underline{y}_j) \right] \quad (6)
 \end{aligned}$$

If the calibration event source functions $F_i^j(\omega)$ have complete spectral coverage as a group, then the weights $A_j(\omega)$ can be chosen to satisfy the representation.

Although it is desirable that a collection of master event waveforms be sufficiently flexible to represent waveforms from any event in the calibrated source region, it must not be so flexible as to form a basis for waveforms from events in other regions. There is a potential problem with equation (6), in that the source functions $F_i^j(\omega)$ and the Green's functions $G_{ik}(\omega)$ occur ambiguously as products. The weights $A_j(\omega)$ are intended to compensate for differences in source functions between the master events and the event being represented by them. But, they may just as well compensate for differences in the Green's functions, if the "new" event is in some completely different region. This ambiguity is undesirable, since it is the uniqueness of the Green's functions which gives waveform correlation methods their ability to estimate location. This problem may be solved in two different ways: the signals may be collected over an aperture (i.e. and array or network), and the filters, represented by the weights $A_j(\omega)$, may be constrained to have short durations.

If waveforms are collected over an aperture, the result is a vector signal of the form:

$$\begin{aligned} \underline{r}(t) &= \begin{bmatrix} r_v(t, \underline{x}_1) \\ r_n(t, \underline{x}_1) \\ r_e(t, \underline{x}_1) \\ \vdots \\ r_v(t, \underline{x}_{N_A}) \\ r_n(t, \underline{x}_{N_A}) \\ r_e(t, \underline{x}_{N_A}) \end{bmatrix} \\ &= \sum_i \int_{-\infty}^{\infty} dt' f_i(t') \underline{g}_i(t - t', \underline{y}) \end{aligned}$$

where $\underline{g}_i(t, \underline{y})$ is the corresponding vector of Green's functions

$$\underline{g}_i(t, \underline{y}) = \begin{bmatrix} g_{iv}(t, \underline{x}_1, \underline{y}) \\ g_{in}(t, \underline{x}_1, \underline{y}) \\ g_{ie}(t, \underline{x}_1, \underline{y}) \\ \vdots \\ g_{iv}(t, \underline{x}_{N_A}, \underline{y}) \\ g_{in}(t, \underline{x}_{N_A}, \underline{y}) \\ g_{ie}(t, \underline{x}_{N_A}, \underline{y}) \end{bmatrix}$$

and N_A is the number of stations sites (which may or may not be three-component stations). Taking appropriate Fourier Transforms to arrive at the analog to equation 6:

$$\begin{aligned} \underline{R}(\omega) &\approx \sum_{j=1}^{N_M} A_j(\omega) \underline{R}^j(\omega) \\ &= \sum_{j=1}^{N_M} A_j(\omega) \left[\sum_i F_i^j(\omega) \underline{G}_i(\omega, \underline{y}_j) \right] \end{aligned} \quad (7)$$

The source functions are still scalars, and can be compensated with scalar filters. The Green's functions are now vectors. Provided the aperture is sufficiently large, the extra complexity this change gives to the Green's functions should make it all but impossible for the weights to compensate for variations in the Green's functions. Thus, the use of network or array data improves the uniqueness of the source region representation.

In the time domain representation for the vector waveform is:

$$\underline{r}(t) = \sum_{i=1}^{N_M} \int_{-\infty}^{\infty} dt' a_i(t') \underline{r}_i(t-t') \quad (8)$$

The durations of the filters $\{a_i\}$ may be chosen to favor compensation of the source time-functions rather than the Green's functions in many cases. For example, the duration of quarry explosions is often much less than a second, but the scattered signal persists (with coda) for many seconds or tens of seconds. If the filter duration approximates the source duration, then it cannot compensate for the later-time Green's function contributions to the recorded signal. This observation suggests that the filters used in waveform representation should be designed in the time domain.

Correlation Coefficient

Waveform correlation coefficients are used to measure the similarity between single-channel waveforms from pairs of events, and to infer bounds on event separation. How is this measurement to be extended to multi-channel waveforms and to compare waveforms from one event to those of a cluster of events?

Most investigators (e.g. Pechman and Kanamori, 1982) use some discrete-time variant of the correlation coefficient:

$$\rho = \max_{\tau \in [-T_c/2, T_c/2]} \frac{\langle r_1, (r_2)_\tau \rangle}{\langle r_1, r_1 \rangle^{\frac{1}{2}} \langle r_2, r_2 \rangle^{\frac{1}{2}}} \quad (9)$$

for the signal pair (r_1, r_2) . The correlation coefficient is a normalized inner-product of the two signals

$$\langle r_1, (r_2)_\tau \rangle = \int_0^T r_1(t) r_2(t + \tau) dt$$

where the signals are assumed to be zero mean:

$$\int_0^T r(t) dt = 0$$

and defined on the interval $[0, T]$. The correlation coefficient is maximized over a range of relative delays between the two signals, to allow for possible errors of alignment. The correlation "window" is chosen to have a length $T_c = T/2$ in order to minimize edge effects.

The correlation coefficient proposed here between a single multichannel signal $\underline{r}(t)$ and a collection of multichannels signals $\{\underline{r}_i(t)\}$ is

$$\hat{\rho} = \max_{\{a_i\}} \frac{\langle \underline{r}, (\hat{\underline{r}}) \rangle}{\langle \underline{r}, \underline{r} \rangle^{\frac{1}{2}} \langle \hat{\underline{r}}, \hat{\underline{r}} \rangle^{\frac{1}{2}}} \quad (10)$$

where

$$\hat{\underline{r}}(t) = \sum_{i=1}^{N_M} \int_{-T_a/2}^{T_a/2} a_i(t') \underline{r}_i(t - t') dt'$$

where T_a is the filter duration, and the inner product is modified to average over the spatial aperture of an array or a network:

$$\begin{aligned}\langle \underline{r}_1, \underline{r}_2 \rangle &= \int_0^T [\underline{r}_1(t)]^T \underline{r}_2(t) dt \\ &= \sum_{i=1}^{N_A} \int_0^T r_1(t, \underline{x}_i) r_2(t, \underline{x}_i) dt\end{aligned}$$

The superscript T denotes the vector transpose operation. The dependence of the received signals on source position $\underline{y}$ has been suppressed for clarity.

Note that this correlation coefficient optimally combines a filtered combination of reference event waveforms. The optimal choice for the collection of filters is that which maximizes the correlation. The solution to the mathematical problem that this optimization poses is the topic of the next section. The added complexity of this form of correlation provides relative immunity to source function variations, an opportunity to characterize a range of source mechanisms, and a mechanism for adjusting small registration errors.

The correction for registration errors is easiest to see. If $\underline{r}(t)$ is a delayed copy of one of the other signals, for example $A\underline{r}_p(t - t_o)$, then the filter $a_p(t) = \delta(t + t_o)$ will align them to produce a correlation coefficient of one. The registration error must be smaller than the duration of the filter ($t_o < T_a/2$).

The use of multiple master events with separate weightings enables the optimization process to select among a range of events with different mechanisms to match the mechanism of the event in question. The filters permit some compensation for source function variations. This point is worth reiterating in the context of the correlation coefficient. For simplicity, consider the case of just two signals ($N_M = 1$) with only one Green's function excited, and recorded with only a single station. The correlation coefficient can be shown to be:

$$\hat{\rho} = \frac{\int d\omega F_1^*(\omega) F_2(\omega) A(\omega) G^*(\omega, \underline{x}, \underline{y}_1) G(\omega, \underline{x}, \underline{y}_2)}{\left(\int d\omega |F_1(\omega)|^2 |G(\omega, \underline{x}, \underline{y}_1)|^2 \right)^{1/2} \left(\int d\omega |A(\omega)|^2 |F_2(\omega)|^2 |G(\omega, \underline{x}, \underline{y}_2)|^2 \right)^{1/2}} \quad (11).$$

The conventional correlation coefficient selects $A(\omega) = 1$, and, consequently, measures

variations in source function as much as variations in the Green's functions. However, the correlation is maximized (for two close sources) by choosing $A(\omega) = F_1(\omega)/F_2(\omega)$, which is possible with the new definition. With this choice, the correlation coefficient compares Green's functions, as it should.

Signal Subspace Characterization

A convenient vector space machinery can be devised for representing the waveforms generated by events in a specific source region, and for visualizing the representation and generalized correlation coefficient geometrically. The signals corresponding to a source region are a basis for a subspace or hyperplane in a vector space. The correlation coefficient can be viewed in terms of the angle between hyperplanes representing different source regions.

Signal Vector Space and Subspace Definitions

The vector of waveforms produced by a source at position $\underline{y}$ and recorded at locations $\{\underline{x}_j, j = 1, \dots, N_A\}$ has been defined as $\underline{r}(t)$. The sampled-data equivalent of this signal is

$$\underline{r}[n] = \underline{r}(n\Delta t) = \begin{bmatrix} r(n\Delta t, \underline{x}_1, \underline{y}) \\ r(n\Delta t, \underline{x}_2, \underline{y}) \\ \vdots \\ r(n\Delta t, \underline{x}_{N_A}, \underline{y}) \end{bmatrix} \quad (12)$$

where Δt is the data sampling interval. If the waveform data are defined on a time interval $[0, T]$ T seconds long, then a finite number of samples $\{\underline{r}[0], \underline{r}[1], \dots, \underline{r}[N]\}$ are available with $T = N\Delta t$. It is essential to think of such finite sequences as vectors, using the notation

$$\underline{v} = \begin{bmatrix} \underline{r}[0] \\ \underline{r}[1] \\ \vdots \\ \underline{r}[N] \end{bmatrix} \quad (13)$$

The signal vectors are defined in a vector space, S , formally of dimension $d = N_A(N + 1)$ for collections of vertical stations, and three times that for three-component stations.

Each signal defines a subspace of dimension one in this vector space:

$$\mathcal{V} = \{\underline{u} : \underline{u} = \alpha \underline{v}, \alpha \in \mathbb{R}\} \quad (14)$$

where $\mathbb{R}$ denotes the set of real numbers. A collection of p signals $\{\underline{v}_i\}$ defines a subspace

of dimension $\leq p$:

$$\overline{\mathcal{V}} = \{ \underline{u} : \underline{u} = \sum_{i=1}^p \alpha_i \underline{v}_i, \alpha_i \in \mathbb{R} \} \quad (15)$$

The dimension is possibly less than p because of the potential for redundancy in the signals. A matrix notation is convenient for operations involving collections of signals:

$$\underline{V} = [\underline{v}_1 \mid \underline{v}_2 \mid \dots \mid \underline{v}_p] \quad (16)$$

The associated subspace is the set of those vectors in the column space of the data matrix $\underline{V}$, i.e. those vectors $\underline{u}$ expressible as

$$\underline{u} = \underline{V} \underline{\alpha} \quad (17)$$

where $\underline{\alpha}$ is the vector of the p weights $\{\alpha_i\}$.

The source region characterization must be capable of representing signals as a combination of filtered master event waveforms. In discrete time notation:

$$\underline{r}[n] \approx \sum_{i=1}^{N_M} \sum_{k=-M}^M a_i[k] \underline{r}_i[n-k] \quad (18)$$

The filter duration in seconds is $T_a = 2M\Delta t$. Consequently, it is necessary to introduce a vector space machinery for describing convolutions. The convolution operation is simply a linear combination of shifted copies of a signal. This observation suggests that the signal subspace be modified to include vectors representing shifted copies of an event's waveforms:

$$(\underline{v})_m = \begin{bmatrix} \underline{r}[m] \\ \underline{r}[1+m] \\ \vdots \\ \underline{r}[N+m] \end{bmatrix} \quad (19)$$

Values of the signal outside of the interval $[0, \dots, N]$ may be taken to be zero. Then the convolution

$$\hat{\underline{r}}[n] \approx \sum_{k=-M}^M a[k] \underline{r}[n-k] \quad ; \quad n = 0, \dots, N \quad (20)$$

is expressible as (Alexander, 1986):

$$\hat{\underline{v}} \approx \sum_{k=-M}^M a[k] (\underline{v})_k \quad (21)$$

Thus, the data available to represent a given source region may be summarized in the signal data matrix:

$$\underline{V} = [(\underline{v}_1)_{-M} \mid \dots (\underline{v}_1)_0 \dots (\underline{v}_1)_M \dots (\underline{v}_p)_{-M} \dots (\underline{v}_p)_0 \dots \mid (\underline{v}_p)_M] \quad (22)$$

The corresponding subspace $\overline{V}$ may be said to completely characterize the source region, if any event in that region produces a signal that lies in $\overline{V}$.

Correlations as Inner Products in a Vector Space

The standard correlation measure between two signals can be viewed in terms of inner products in a vector space and in terms of the angle between the signal vectors. The correlation coefficient can be generalized to measure the similarity between one signal and a collection of signals, and between two separate collections of signals.

Recall that the correlation coefficient for two scalar signals $r_1[n]$ and $r_2[n]$ is:

$$\rho = \frac{\sum_{n=0}^N r_1[n] r_2[n]}{\left(\sum_{n=0}^N r_1^2[n] \right)^{\frac{1}{2}} \left(\sum_{n=0}^N r_2^2[n] \right)^{\frac{1}{2}}} \quad (23)$$

It can be thought of as a ratio of inner products defined on the corresponding signal vectors:

$$\begin{aligned} \rho &= \frac{\langle \underline{v}_1, \underline{v}_2 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle^{1/2} \langle \underline{v}_2, \underline{v}_2 \rangle^{1/2}} \\ &= \frac{\langle \underline{v}_1, \underline{v}_2 \rangle}{\|\underline{v}_1\| \|\underline{v}_2\|} \end{aligned} \quad (24)$$

where, for multichannel signals, the inner product is defined by

$$\langle \underline{v}_1, \underline{v}_2 \rangle = \underline{v}_1^T \underline{v}_2 = \sum_{n=0}^N r_1^T[n] r_2[n]$$

and the Euclidean norm $\|\cdot\|$ (i.e. vector "length") by

$$\|\underline{v}\| = \langle \underline{v}, \underline{v} \rangle^{1/2}.$$

Consulting any introductory text on linear algebra (e.g. Strang, 1976), the correlation coefficient may be interpreted as the cosine of the angle θ between vectors $\underline{v}_1$ and $\underline{v}_2$.

By extension, it is also the cosine of the minimum angle between the subspaces spanned by the two signal vectors. The notion of defining the correlation coefficient in terms of the minimum angle between signal subspaces is an important geometric interpretation simplifying comparisons involving many signals.

The correlation coefficient between a single signal $\underline{v}$ and the subspace $\overline{V}$ defined by the collection of calibration signals $\{\underline{v}_i\}$ is the cosine of the minimum angle between them. The cosine is maximized (and the angle minimized) by maximizing the correlation between $\underline{v}$ and all vectors $\hat{\underline{v}}$ in the subspace:

$$\hat{\rho}^2 = \max_{\hat{\underline{v}} \in \overline{V}} \frac{\langle \underline{v}, \hat{\underline{v}} \rangle^2}{\langle \underline{v}, \underline{v} \rangle \langle \hat{\underline{v}}, \hat{\underline{v}} \rangle} \quad (25)$$

Any vector in the subspace, $\hat{\underline{v}}$, is a linear combination of the calibration signals, and is expressible in terms of the data matrix $\underline{V}$ by:

$$\hat{\underline{v}} = \underline{V} \underline{\alpha} \quad (26)$$

The coefficients $\underline{\alpha}$ are then chosen to maximize the correlation between $\hat{\underline{v}}$ and $\underline{v}$:

$$\hat{\rho}^2 = \max_{\underline{\alpha}} \frac{1}{\underline{v}^T \underline{v}} \cdot \frac{(\underline{v}^T \underline{V} \underline{\alpha})^2}{\underline{\alpha}^T (\underline{V}^T \underline{V}) \underline{\alpha}} \quad (27)$$

Provided the columns of the data matrix are linearly independent, the maximizing $\underline{\alpha}$ is

$$(\underline{V}^T \underline{V})^{-1} \underline{V}^T \underline{v} \quad (28)$$

and the maximizing subspace signal is

$$\hat{\underline{v}}_o = \underline{V} (\underline{V}^T \underline{V})^{-1} \underline{V}^T \underline{v}. \quad (29)$$

The correlation coefficient is:

$$\hat{\rho}^2 = \frac{\underline{v}^T \underline{V} (\underline{V}^T \underline{V})^{-1} \underline{V}^T \underline{v}}{\underline{v}^T \underline{v}} \quad (30)$$

The signal $\hat{v}_o$ which maximizes the correlation is also the least squares approximation to $\underline{v}$ in the subspace $\overline{\mathcal{V}}$, i.e. it satisfies:

$$\min_{\hat{v} \in \overline{\mathcal{V}}} \|\hat{v} - \underline{v}\|. \quad (31)$$

Projection Operators

Projection operators (Golub and Van Loan, 1989, Alexander, 1986) provide a more compact mechanism for describing the generalized correlation operation defined above. The projection operator

$$\underline{P}_{\overline{\mathcal{V}}} = \underline{V} (\underline{V}^T \underline{V})^{-1} \underline{V}^T \quad (32)$$

maps (“projects”) an arbitrary vector $\underline{v}$ into its least-square residual approximation in the subspace $\overline{\mathcal{V}}$:

$$\hat{v}_o = \underline{P}_{\overline{\mathcal{V}}} \underline{v} \quad (33)$$

All projection matrices are idempotent, which gives them the following properties: $\underline{P}^T = \underline{P}$ and $\underline{P} \underline{P} = \underline{P}$.

The projection operator provides an attractive interpretation for the correlation coefficient. Substituting the definition of $\underline{P}_{\overline{\mathcal{V}}}$ into equation 30, and using the properties of an idempotent matrix:

$$\begin{aligned} \hat{\rho}^2 &= \frac{\underline{v}^T \underline{P}_{\overline{\mathcal{V}}} \underline{v}}{\underline{v}^T \underline{v}} \\ &= \frac{(\underline{P}_{\overline{\mathcal{V}}} \underline{v})^T (\underline{P}_{\overline{\mathcal{V}}} \underline{v})}{\underline{v}^T \underline{v}} \end{aligned} \quad (34)$$

Thus:

$$|\hat{\rho}| = \frac{\|\underline{P}_{\overline{\mathcal{V}}} \underline{v}\|}{\|\underline{v}\|} \quad (35)$$

The correlation coefficient is the ratio between the length of the vector’s projection into the subspace and the length of the vector itself. Heuristically, it represents the “fraction” of the vector that lies in the subspace.

The projection operator and its orthogonal complement operator $\underline{P}_{\overline{\mathcal{V}}}^\perp = \underline{I} - \underline{P}_{\overline{\mathcal{V}}}$ divide the entire vector space $\mathcal{S}$ into two orthogonal subspaces $\overline{\mathcal{V}}$ and $\overline{\mathcal{V}}^\perp$. Any vector $\underline{v}$ in $\mathcal{S}$ can

be written uniquely in terms of its projections into these two subspaces:

$$\begin{aligned}\underline{v} &= \underline{P}_{\overline{V}} \underline{v} + \underline{P}_{\overline{V}}^{\perp} \underline{v} \\ &= \hat{\underline{v}}_o + \hat{\underline{v}}_o^{\perp}\end{aligned}\tag{36}$$

These two terms represent the part of the signal $\underline{v}$ that is predictable from the calibration event signals and which “fits in” to the source region, and the part which contains new information. This perspective will be useful later for separating superimposed signals from different source regions.

When linear dependencies exist among the source signals, the singular value decomposition can be used to find an orthogonal basis for the subspace and a corresponding projection operator. Let the SVD be defined by:

$$\underline{V} = \underline{U} \underline{\Sigma} \underline{Y}^T \tag{37}$$

where $\underline{\Sigma}$ is the diagonal matrix of singular values. If the columns of $\underline{V}$ are linearly dependent, then the dimension of an orthogonal basis of the column space may be determined as the number of significantly non-zero singular values, $d_V \leq p$ (Golub and Van Loan, 1989). An orthonormal basis is obtainable by partitioning the matrix of left singular vectors consistent with the principal singular values:

$$\underline{U} = \left[\underline{U}^s \mid \underline{U}^{\perp} \right] \tag{38}$$

where the number of columns in $\underline{U}^s$ is d_V and in $\underline{U}^{\perp}$ is $p - d_V$. The columns of $\underline{U}^s$ form the basis for $\overline{V}$ and the projection operator is:

$$\underline{P}_{\overline{V}} = \underline{U}^s (\underline{U}^s)^T. \tag{39}$$

As a practical matter, $\underline{U}^s$ may be stored as a characterization for the source region.

Vector Space Interpretation of Correlation Coefficients

At this point it is possible to state succinctly the difference between the standard method of computing the correlation coefficient (with compensation for registration errors) and the method offered in this report. Considering the simple two-signal case, the standard methods for estimating correlation maximize the sample correlation over a range of relative delays between the signals:

$$\rho = \max_{m \in [-M, \dots, M]} \frac{\langle \underline{v}_1, (\underline{v}_2)_m \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle^{1/2} \langle (\underline{v}_2)_m, (\underline{v}_2)_m \rangle^{1/2}} \tag{40}$$

Let the projection operators for the subspaces corresponding to individual shifted copies of the signal $(\underline{v}_2)_m$ be denoted by $\underline{P}_{\underline{v}_{2m}}$. The conventional correlation estimate is:

$$\rho = \max_m \frac{\|\underline{P}_{\underline{v}_{2m}} \underline{v}_1\|}{\|\underline{v}_1\|} \quad (41)$$

It requires maximization over many individual projections. The proposed method involves a single projection:

$$\hat{\rho} = \frac{\|\underline{P}_{\underline{v}_2} \underline{v}_1\|}{\|\underline{v}_1\|} \quad (42)$$

where $\underline{P}_{\underline{v}_2}$ is the projection operator for the single subspace spanned by the collection of delayed copies of the signal $\underline{v}_2$.

Source Region Comparison with Canonical Correlations

Two source regions represented, respectively, by collections of signals $\underline{v}_{1i}$ and $\underline{v}_{2i}$ can be compared provided a suitable measure can be devised for the distance between their subspaces. The minimum distance is a measure of how close they come to intersecting. This distance is determined by the maximum correlation between two signals, $\hat{\underline{v}}_1$ and $\hat{\underline{v}}_2$, which lie, respectively, in the first and second subspaces. These vectors are chosen to be as nearly equal as possible, excluding the trivial zero solution:

$$\begin{aligned} \hat{\underline{v}}_1 &= \sum_{i=1}^{p_1} \alpha_{1i} \underline{v}_{1i} \approx \sum_{i=1}^{p_2} \alpha_{2i} \underline{v}_{2i} = \hat{\underline{v}}_2 \\ \hat{\underline{v}}_1 &= \underline{V}_1 \underline{\alpha}_1 \approx \underline{V}_2 \underline{\alpha}_2 = \hat{\underline{v}}_2 \end{aligned} \quad (43)$$

These vectors are chosen to maximize the correlation coefficient:

$$\begin{aligned} \hat{\rho}^2 &= \max_{(\underline{\alpha}_1, \underline{\alpha}_2)} \frac{\langle \hat{\underline{v}}_1, \hat{\underline{v}}_2 \rangle^2}{\langle \hat{\underline{v}}_1, \hat{\underline{v}}_1 \rangle \langle \hat{\underline{v}}_2, \hat{\underline{v}}_2 \rangle} \\ &= \max_{(\underline{\alpha}_1, \underline{\alpha}_2)} \frac{\left(\underline{\alpha}_1^T (\underline{V}_1^T \underline{V}_2) \underline{\alpha}_2 \right)^2}{\underline{\alpha}_1^T (\underline{V}_1^T \underline{V}_1) \underline{\alpha}_1 \quad \underline{\alpha}_2^T (\underline{V}_2^T \underline{V}_2) \underline{\alpha}_2} \end{aligned} \quad (44)$$

The solution of this problem is obtained as the the solution to the generalized eigenvalue problem (Björck and Golub, 1973):

$$\begin{bmatrix} \underline{0} & \underline{V}_1^T \underline{V}_2 \\ \underline{V}_2^T \underline{V}_1 & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \end{bmatrix} = \lambda \begin{bmatrix} \underline{V}_1^T \underline{V}_1 & \underline{0} \\ \underline{0} & \underline{V}_2^T \underline{V}_2 \end{bmatrix} \begin{bmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \end{bmatrix} \quad (45)$$

The correlation coefficient, which is known as the maximum canonical correlation, can be obtained from the maximum eigenvalue:

$$\hat{\rho} = \sqrt{\lambda_{max}} \quad (46)$$

If it is one, then the subspaces intersect, i.e. a vector can be found which lies in both $\overline{V}_1$ and $\overline{V}_2$. If the correlation is zero, then the subspaces are completely orthogonal.

Subspaces are more complicated than this single measurement implies. In fact, two subspaces may intersect, but may have components which are not near to intersection. The full range of eigenvalues λ_i is a more complete characterization of the range of angles between the two subspaces:

$$\cos(\theta_i) = \sqrt{\lambda_i}. \quad (47)$$

For example, if two source regions overlap, the principal canonical correlation is expected to be close to one, but the minimum canonical correlation may still be small. This situation could arise if the source regions have disjoint subregions, and may be a useful indicator of that condition. It could also arise if the two regions entirely overlapped, but a disparity existed in source mechanisms, i.e. if one event collection contained an event with an unusual mechanism.

A much better method for measuring canonical correlations utilizes the concept of measuring the cosine of the angle between the two signal subspaces. In this method, due to Björck and Golub, the two data matrices are factored using the SVD:

$$\begin{aligned} \underline{V}_1 &= \underline{U}_1 \underline{\Sigma}_1 \underline{Y}_1^T \\ \underline{V}_2 &= \underline{U}_2 \underline{\Sigma}_2 \underline{Y}_2^T \end{aligned} \quad (48)$$

The two matrices of left singular vectors are partitioned to correspond to their respective significantly non-zero singular values:

$$\begin{aligned} \underline{U}_1 &= \left[\underline{U}_1^s \mid \underline{U}_1^\perp \right] \\ \underline{U}_2 &= \left[\underline{U}_2^s \mid \underline{U}_2^\perp \right] \end{aligned} \quad (49)$$

The matrix

$$\underline{C} = (\underline{U}_1^s)^T \underline{U}_2^s \quad (50)$$

is formed, which is the matrix of all inner products between basis vectors of the two subspaces. The canonical correlation coefficient is the maximum singular value of $\underline{C}$:

$$\hat{\rho} = \sigma_{max}(\underline{C}). \quad (51)$$

The collection of angles between the two subspaces is given by the complete set of singular values:

$$\cos(\theta_i) = \sigma_i. \quad (52)$$

$\underline{C}$ is usually a matrix of fairly small dimension, so that the extra SVD calculation to get the singular values is computationally inexpensive.

Fundamental Problems with Transient Signals

If one could design the signals used to characterize a given source region, certainly one objective would be to make them very specific to that region to eliminate ambiguities in location applications. Ideally, the subspaces representing different source regions would be orthogonal. The transient nature of seismic signals makes this situation difficult to attain. For example, the correlation coefficients computed with short signals exhibit a “floor” below which non-zero sample correlation values may be a processing artifact (Thorbjarnardottir and Pechman, 1987). This floor sets an effective limit on the minimum correlation that can be measured, and, consequently, the maximum separation between events that can be discerned statistically.

This issue can be examined through an analysis of the correlation coefficient. The vector space approach provides a useful perspective for understanding the problem. Short signals have few degrees of freedom and occupy a vector space of small dimension. Intuitively, vectors (representing the signals) in a low-dimension vector space are likely to be “near” each other. “Nearness” or similarity is measured by the amount one vector projects onto the other, i.e. upon the correlation coefficient. Heuristically, the chance that two vectors will correlate substantially, even when chosen at random, is large in a low-dimensional space. The vectors simply do not have much “room” to get away from each other.

Consider the extreme example of signals $T = 0$ seconds long, represented by a single sample $N = 1$. The dimension of this space is one. Vectors in this space always project perfectly upon each other (they are colinear), and always have correlation coefficients equal to one, i.e. the correlation floor is one. Consequently, there is no way to distinguish spatially correlated events from those that are uncorrelated in a data space of dimension one. The situation does not improve rapidly as the number of dimensions is increased. The floor is lowered by increasing the dimension of the data space, but several hundred dimensions are required for a really satisfactory floor level ($\approx .2$). This assertion is substantiated later.

How is the dimension of a data space measured? From the above discussion, it is clear that the dimension depends upon signal duration. But it also depends upon the bandwidth of the data. Consider the class of sinusoidal signals at a single frequency, another extreme class occupying a vector space of low dimension. These signals have zero bandwidth and

two degrees of freedom: amplitude and initial phase. Independent of the length of the signals, the correlation coefficient for two signals in this class is just the cosine of the difference between the initial phase angles. If that difference is chosen at random, a large proportion of resulting correlation coefficients will be near one. In order to increase the dimension of the data space, the signals must have some non-vanishing bandwidth. As shown in the appendix, the approximate dimension of the signal vector space is $2TB$, where T is the signal duration, and B is the signal bandwidth (positive frequency bandwidth). This is a standard result (cf. Franks, 1981), but is reproduced in the appendix for tutorial purposes.

Maximizing the correlation coefficient over a range of lags compounds the problem. The difficulty is severe enough when a signal is represented by a single vector in a low-dimension vector space. When it is represented by several vectors (i.e. the collection of delayed signals $\{ (\underline{y})_m : m \in [-M, \dots, M] \}$) for the conventional correlation coefficient, the problem becomes worse. The correlation bias increases as the number of subspaces representing a signal increases. As the subspaces begin to “fill” the complete vector space $\mathbb{R}^{N+1}$, the chance of spurious correlation between two signals increases. One signal is more likely to have a substantial projection into one of the subspaces of the other. Another way to view the problem is that by shifting one signal past the other, the number of opportunities for two signals to appear similar (even if uncorrelated) is increased.

The situation is nearly the same for the generalized correlation coefficient proposed in this report, with the exception that the collection of $2M + 1$ subspaces of dimension one is replaced by a single subspace of dimension $2M + 1$. The problem becomes critical when the subspace dimension becomes a considerable fraction of the dimension of the signal vector space. To solve this problem, the best approach is to increase the vector space dimension, i.e. to increase the amount of data. Since the bandwidth and the duration of the signals are fixed, the problem can be solved only by increasing the number of stations used in data collection. Array or network data is necessary.

Asymptotic Statistics of ρ

The correlation floor phenomenon places a fundamental limitation on the usefulness of correlation measurements. It can be examined quantitatively by developing the probability

distribution for the correlation coefficient computed for completely uncorrelated signals. This distribution represents a lower bound for the case of correlated signals, and, thus, defines the floor.

For signals with large time-bandwidth product, the probability density function (pdf) for the correlation coefficient can be developed analytically. It is worthwhile to do so, since the asymptotic form of the pdf does show what data parameters the correlation floor depends upon. Except for very large TB it tends to underpredict the floor.

The detailed argument for the asymptotic form of the pdf is relegated to the appendix. Assuming Gaussian statistics for the signal data, the pdf approaches:

$$p(\rho) = (2T_c B) f(\rho) (F(\rho))^{2T_c B - 1}$$

where $f(\cdot)$ is the Gaussian pdf with zero mean and standard deviation

$$\sigma = \frac{1}{\sqrt{2TB - 3}}$$

and $F(\cdot)$ is the normal error function:

$$F(\rho) = \int_{-\infty}^{\rho} dx f(x).$$

Thus, the pdf of the sample correlation coefficient depends on just three parameters in the limit: the duration of the signal T , the signal bandwidth B , and the duration T_c of the delay window over which the correlation is maximized.

As TB is increased, the variance of f declines, which causes the pdf to be concentrated near zero. The correlation floor declines. As the correlation window duration T_c is increased (and the subspaces fill the vector space), the shape of F is such that the pdf moves away from zero. The correlation floor increases. When multichannel data is used T is effectively replaced by $N_A T$, where N_A is the number of spatially independent data channels. A large N_A will substantially reduce the correlation floor.

Small-Sample Statistics

Bias effects (i.e. the floor) are especially pronounced for data with low TB , where the asymptotic form for the pdf is inaccurate. Unfortunately, this is the typical region of

operation with seismic data. To examine the statistics of the sample correlation coefficient in this region, the only option is Monte-Carlo simulation. Simulations in this section examine three issues; the actual distribution of ρ under the standard estimation scheme ($T_c = T/2$) for TB varying over the range expected in normal seismic problems (10-160), the effect of variations in center frequency, and the effect of reducing the correlation lag window, T_c . The results are summarized graphically in Figures (1- 3).

These figures display frequency distributions (histograms) of sample correlation coefficients for 3000-9000 realizations of statistically independent random processes using various window lengths and bandwidths. The data were produced by a Gaussian random number generator and filtered into various bands with 4-pole Butterworth recursive filters. The standard correlation coefficient (9) was used. The histograms were normalized to approximate the pdfs of sample correlation coefficients by scaling them to integrate numerically to one.

Figure 1 shows the variation of pdfs for sample correlations with a fixed T of 10 seconds, a fixed correlation lag window of 5 seconds ($T_c = T/2$), and varying bandwidth. It demonstrates the distribution of ρ over the likely range of time-bandwidth products. For small time-bandwidth products, i.e. 10-40, large bias and variance result: the sample pdf's are strongly displaced from their desired position around $\rho = 0$. One may expect the range of values .2-.6 for sample correlations computed with data from a single sensor. For example, with data having a TB product of 10, correlations would rarely drop below .3, and only values above .6 would mark two signals as being truly correlated. This effect strongly restricts the usefulness of correlations calculated from signals of low dimensionality. Even larger time-bandwidth products (80-160) will result in correlations in the range .15-.25 for uncorrelated signals. This is the likely range for correlations computed from array or network data. The increase of the time-bandwidth product into this range markedly increases the reliable operating region of the waveform correlation method.

Figure 2 demonstrates that the correlation statistics, and, in particular, the "correlation floor" are independent of center frequency. The figure shows two sets of histograms: one for two cases with $TB = 10$ and one for two cases with $TB = 40$. Within each set, the bandwidths, data window lengths, and correlation window lengths were held constant; only the center frequency was varied. For example the two $TB = 10$ histograms were cal-

culated for the bands 4-5 Hz and 10-11 Hz. The histograms overlay quite well, indicating invariance to center frequency.

Often it is not possible to do much about the signal duration or bandwidth, but it is possible to vary the lag window over which the correlation coefficient is maximized. The larger the lag window, the more opportunities there are for the estimated correlation coefficient to become large: bias therefore increases. To reduce bias the lag window can be shortened. This result is indicated by Figure 3, which shows sample correlation histograms for $T = 10$ seconds, $B = 1$ Hz and lag window durations ranging from $T/2$ (the normal choice) to $T/16$. The choice of the lag window length involves a tradeoff between bias and the possibility of not achieving the correct waveform alignment. In contrast to bias, the variance of correlation values appears to depend weakly on correlation lag window duration, T_c , and more strongly on TB .

Thorbjarnardottir and Pechman (1987) recognized the bias problem and suggested that bias depends on center frequency and the fact that correlations are maximized over the lag window. They calculated correlation coefficients in the following bands: 1-2 Hz, 2-4 Hz, 4-8 Hz, and 8-16 Hz for a suite of events at a quarry, and noticed that the collection of estimated correlations exhibited the floor which decreased with increasing center frequency. They attributed this effect to the supposition that "higher frequency signals should be less likely to yield high cross-correlations by chance." In fact, bandwidth, not center frequency, was the critical parameter in their study. The impression that center frequency controlled the bias in their results was incidental to the fact that they increased the bandwidth as they increased the center frequency of their filters.

Figure 1. The time-bandwidth product (TB) of the data strongly controls the distribution of the sample correlation coefficient. The histograms shown are computed with segments of uncorrelated Gaussian noise that have been filtered to give the specified time-bandwidth product. The distributions are displaced away from the origin ($\rho = 0$). The bias (offset) is the correlation “floor” phenomenon that limits the effectiveness of correlation methods when short data records are used. The bias gets worse with decreasing time-bandwidth product.

Figure 2. The distribution of the sample correlation coefficient is unaffected by the center frequency of the data (all other things being equal). Here, histograms for data at two center frequencies (4.5 and 10.5 Hz) are compared for each of two time-bandwidth products. The distributions vary with bandwidth, but not center frequency.

Figure 3. The correlation window size has a strong effect on bias, but less on variance. These correlation histograms were constructed for data having a time-bandwidth product of 10 ($T = 10$ second, $B = 1$ Hz). The correlation window, i.e. the delay window over which the sample correlation is maximized was varied from ± 2.5 seconds to ± 0.3125 seconds. A larger correlation window provides more opportunities for an erroneous high correlation, and causes the distribution to shift to higher values of ρ .

Discussion

The principal treaty verification application of source region characterization is the process of attributing events to known legitimate sources, as mentioned in the introduction. Under the hypothesis that many in-country monitoring stations will be located in regions with a modest number (a few tens) of mines, and that the bulk of detected events will be mining explosions, the principal verification task will be to screen out mining explosions. Assuming a data base of past, well-located explosion waveforms is available, it should be possible to construct projection operators $\underline{P}_i$ for the signal subspaces representing each of the mines. The waveforms from some unlocated event, represented by $\underline{v}$ can then be used in a multiple hypothesis test on source location. The likely source is that for which

$$\frac{||\underline{P}_i \underline{v}||}{||\underline{v}||}$$

is maximum.

The multiple hypothesis test on location would probably be used as the second phase of a two-phased location approach. The first would use network triangulation or array locations that do not depend on the availability of master events to provide a "ballpark" estimate of location. Then a data base search for nearby mines would identify the collection of mines to be used in the hypothesis test.

Because the master events span the range of historical shooting practices, this multiple hypothesis test should test for mechanism as well as location. One of the concerns is that a decoupled nuclear explosion might be placed close to a large chemical mining explosion, and that the chemical explosion would effectively mask the nuclear shot. The mechanisms of the two might be different (and their locations not coincident), so that a suitably specific waveform basis may discriminate the two.

The number of events required to form an adequate characterization is an important data base issue. It may be solved initially by computing the SVD of the data matrix $\underline{V}$ for a collection of events, and noting the number d_V of significantly non-zero singular values $\{\sigma_i\}$. Many tests for this dimension exist, such as the F test used by Vasco (1989), and statistical model order tests (e.g. Akaike, 1974). Once the dimension of the subspace is

identified, then the minimal basis is constructed from the partitioned matrix of left singular vectors, $\underline{U}^s$ as previously discussed.

In an operational situation, continual acquisition of new events is expected. With each newly acquired event for which the location can be independently confirmed, the question of whether to add the event's waveforms to the basis will arise. If the correlation coefficient is large, the event is completely "explained" in light of past explosions, and it would contribute nothing to a revised basis. If the correlation coefficient is relatively small, then it should be used to revise the basis. The computational method would involve a reorthogonalization using the augmented matrix:

$$\underline{A} = [\underline{U}^s \mid \underline{v}_{-M} \cdots \underline{v}_M]$$

using the SVD:

$$\underline{A} = \underline{U} \underline{\Sigma} \underline{Y}^T$$

where, again, the vectors $\underline{v}_i$ represent shifted versions of the new event signal. The new $\underline{U}$ matrix would be partitioned to reflect the increased dimension of the revised basis.

A low correlation coefficient is also a flag that the event should be examined in more detail. If the event is located at a particular mine, the portion of the signal not representable by the basis may contain waveforms from a second event masked by the first. Consequently, the waveform in the orthogonal complement of the basis

$$\underline{v}_o^\perp = \underline{P}_i^\perp \underline{v}$$

should be interpreted for a possible second event. The effect of the orthogonal projection operator will be to strip away the signals corresponding to a masking event, revealing any waveforms from events in another location.

Appendix

Degrees of Freedom

For convenience, the available data are treated as a continuous-time sample of a Gaussian random process, $v(t)$, defined on a finite interval $[0, T]$, which is T seconds long. A finite duration segment of a random process can be represented by a Fourier sine-cosine series:

$$v(t) = \sum_{i=-\infty}^{\infty} v_{si} \sin\left(\frac{2\pi i}{T} t\right) + \sum_{i=-\infty}^{\infty} v_{ci} \cos\left(\frac{2\pi i}{T} t\right) \quad (53)$$

For sufficiently large T , the coefficients of the series are approximately statistically independent (Davenport and Root, 1987). Furthermore, the variance of the coefficients in the series weighted by the window length asymptotically approaches samples of the spectrum of the random process:

$$\lim_{T \rightarrow \infty} T E \{ v_{si}^2 \} = \lim_{T \rightarrow \infty} T E \{ v_{ci}^2 \} = S_v\left(\frac{2\pi i}{T}\right) \quad (54)$$

The coefficients are also zero mean:

$$E \{ v_{si} \} = E \{ v_{ci} \} = 0.$$

For a bandlimited random process, only a limited number of the coefficients are appreciable (non-zero). Suppose the data have been filtered to have a bandwidth of B Hertz in the interval Ω_B . Then, only the coefficients satisfying

$$\frac{2\pi i}{T} \in \Omega_B$$

have appreciably non-zero variances. The number of these coefficients is the number of degrees of freedom of the data sample: $2TB$. If we further assume that the spectrum of the random process is flat over Ω_B , then the Fourier coefficients are approximately identically distributed statistically independent Gaussian random variables.

Asymptotic Form of Correlation Statistics

The correlation coefficient, ρ , is a ratio of inner products of the form:

$$\langle v, w \rangle = \int_0^T dt v(t) w(t) = \sum_i v_{ci} w_{ci} + \sum_i v_{si} w_{si}. \quad (55)$$

The correlation coefficient is

$$\rho = \frac{\sum_i v_{ci} w_{ci} + \sum_i v_{si} w_{si}}{\left(\sum_i v_{ci}^2 + \sum_i v_{si}^2 \right)^{\frac{1}{2}} \left(\sum_i w_{ci}^2 + \sum_i w_{si}^2 \right)^{\frac{1}{2}}} \quad (56)$$

The sums are taken over the approximately $2TB$ appreciable Fourier coefficients.

The probability density function for ρ is extremely complicated (Kendall and Stuart), but a simple approximation is available for the transformed variable:

$$z = \frac{1}{2} \ln \left(\frac{1 + \rho}{1 - \rho} \right) \quad (57)$$

when the number of coefficients is moderately large (≥ 10). This transformation is known as the Fisher z -transformation. For uncorrelated sample data, the pdf of z is approximately Gaussian with zero mean and standard deviation

$$\sigma_z = \frac{1}{\sqrt{2TB - 3}} \quad (58)$$

The correlation coefficient is maximized over a delay window T_c seconds long:

$$\rho = \max_{\tau \in [-T_c/2, T_c/2]} \rho(\tau) = \max_{\tau \in [-T_c/2, T_c/2]} \frac{\langle v, (w)_\tau \rangle}{\langle v, v \rangle^{\frac{1}{2}} \langle w, w \rangle^{\frac{1}{2}}} \quad (59)$$

Because the z -transformation is monotonic, maximizing ρ is the same as maximizing z . Although it is difficult to approximate the distribution of ρ , one can approximate the pdf of the corresponding z -transformation variable:

$$z = \max_{\tau \in [-T_c/2, T_c/2]} z(\tau) = \max_{\tau \in [-T_c/2, T_c/2]} \frac{1}{2} \ln \left(\frac{1 + \rho(\tau)}{1 - \rho(\tau)} \right) \quad (60)$$

The distribution for a single value of z is normally distributed, $N(0, \sigma_z)$. However, $z(\tau)$ varies over the continuous interval $[-T_c/2, T_c/2]$. One approach is to split the range of delays τ into bins for which the samples of z (and ρ) should be mutually uncorrelated. The continuous maximization problem can be approximated as a problem of maximization over a finite number of identically distributed Gaussian random variables, each representing z within a single bin. Observe that the numerator of ρ is the cross-correlation function:

$$\int dt v(t) w(t + \tau) \quad (61)$$

From standard spectral analysis theory, it is known that samples of the estimated cross-correlation function are approximately uncorrelated provided they are taken at separations greater than the auto-correlation length τ_l of the two random processes (Jenkins and Watts, 1968). Consequently,

$$E \{ \langle u, w_{\tau_1} \rangle \langle u, w_{\tau_2} \rangle \} \approx 0 \quad (62)$$

for

$$|\tau_1 - \tau_2| > \tau_l.$$

For a bandpass random process, the correlation length is approximately the reciprocal of the process bandwidth, i.e. $\tau_l \approx B^{-1}$. This means that the bins are spaced $1/B$ seconds apart, and that their number is approximately $T_c/(B^{-1}) = T_c B$. Since the correlation is a completely general function, it can be represented as a Fourier series of the form of eq. (53), i.e. as a Fourier sine-cosine series. The sine and cosine series are independent, which means they are independent within a single bin. The correlation value within a particular bin has two degrees of freedom, and, consequently, two opportunities to contribute a maximum correlation value.

Therefore, the transformed correlation coefficient function may be thought of as a collection of $2T_c B$ independent Gaussian random variables z_i . For small T_c , these are uniformly distributed with zero mean and variance $(2TB - 3)^{-1}$. Let $p(\cdot)$ be the normal pdf:

$$f(\zeta) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-\zeta^2}{2\sigma^2}} \quad (63)$$

with $\sigma^2 = 2TB - 3$, which is the pdf of each of the z_i . From Papoulis (1965),

$$z = \max_i z_i \quad (64)$$

has the pdf

$$p_z(\zeta) = (2T_c B) f(\zeta) (F(\zeta))^{2T_c B - 1} \quad (65)$$

This form is the desired asymptotic result (all the arguments require TB large). To get some idea of how large TB must be to achieve this asymptotic form in practice, Monte-Carlo simulations were used to estimate frequency distributions for comparison with (65) above. To avoid the effects of end conditions in calculating the correlation coefficients, the data were not assumed to be zero outside the estimation window.

Figure (2) shows six pdfs for values of TB ranging from 10 to 320. Note that for very small time-bandwidth product, the asymptotic form is a poor approximation to the true statistics of the sample correlation coefficient. It underestimates the bias and overestimates the variance. However, for values greater than about $TB = 100$, the asymptotic form is an excellent approximation. The asymptotic pdf of z is a good approximation for the actual pdf of ρ when the time-bandwidth product is in this range. As TB increases, the pdf concentrates around zero. In the limit the pdfs for ρ and z are identical, since to good approximation

$$z \approx \rho ; \quad |\rho| < .2 \quad (66)$$

Figure 4. The theoretical asymptotic distribution for the sample correlation coefficient agrees with Monte-Carlo simulations for $TB > 100$. The theoretical probability density function for the Fisher transformation variable z (maximized over a correlation window of duration $T/2$) is compared against Monte-Carlo histograms for TB ranging from 10 to 320. Above $TB = 100$ the theoretical distribution is useful for predicting performance of correlation estimators.

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