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A DEFORMED POTENTIAL MANY-PARTICLE THEORY

by

Jerry Paul Draayer

Ph.D. Thesis, August 1968

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A DEFORMED POTENTIAL MANY-PARTICLE THEORY

by

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A DEFORMED POTENTIAL MANY-PARTICLE THEORY*

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ABSTRACT

The relationship between the $U(3) \supset R(3)$ state labeling scheme introduced by Elliott (1) for a description of many particles moving in an isotropic harmonic oscillator potential and the Gel'fand $U(3) \supset U(2) \times U(1)$ state labeling scheme introduced by Nagel and Moshinsky (2) is established. A rule complementary to the one given by Elliott for the quantum number K required in the $U(3) \supset R(3)$ reduction is given and the use of K in state labeling is clarified. An explicit expression for the transformation brackets between the two state labeling schemes is obtained and the results used to derive an explicit expression for both the $U(3) \supset R(3)$ coupling coefficients and projected tensor matrix elements.

Application of the invariance of the commutators of the infinitesimal generators of $U(3)$ to a canonical transformation is used to generalize these results to a deformed potential many-particle theory. An expansion of the deformed potential many-particle states in terms of the nondeformed potential many-particle states is made. A phenomenological description of the motion of nucleons in a deformed potential

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created by a deformed core which itself is allowed to rotate adiabatically is presented. An application to the $A = 18$ system is made, treating O^{18} as having two particles moving in a deformed potential created by the O^{16} core.

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1. J. P. Elliott, Proc. Roy. Soc. 245A (1958) 128; 245A (1958) 562
 2. J. G. Nagel and M. Moshinsky, J. Math. Phys. 6 (1965) 682

1. INTRODUCTION

The ultimate goal of nuclear physics is to obtain a complete understanding of the structure of nuclei. Since nuclei are in general many-nucleon systems for which a complete mathematical description is as yet neither possible nor practical, nuclear structure physicists have contented themselves with developing models which are both mathematically simple and physically significant in the sense that they are capable of describing nuclear observables. The most successful of such models take as their starting point the single-particle shell model in which it is assumed that each nucleon moves in an average potential created by the other nucleons of the nuclear system. To be sure there are theoretical difficulties inherent in the assumption that the strong internucleon forces could average out in such a simple manner. This is precisely the reason that this simple model, proposed as early as 1930 (1) in direct analogy with the single-particle model proven to be useful in atomic physics, was not fully appreciated until 1949 when Mayer (2) and Haxel, Jensen and Suess (3) independently demonstrated the effectiveness of the single-particle shell model in describing, predicting and cataloging nuclear data. Although a complete theoretical justification for such a simple average nucleon interaction is still pending, the usefulness of the technique remains and has led nuclear structure physicists to continue to

develop useful modifications and extensions.

Outside of the hierarchy of models built upon the simple phenomenological models possessing the descriptive titles rigid rotor and spherical vibrator, the most promising developments lie in attempts to take into account within the framework of the shell model the many-particle characteristics of nuclei. This may be accomplished by treating a nucleus as a system of Fermions moving in a common potential. For example, the three-dimensional isotropic harmonic oscillator potential has long been known to serve as a mathematically convenient choice. In fact, as early as 1940 Jauch and Hill (4) pointed out the invariance of the corresponding particle Hamiltonian to unitary transformations in three dimensions, $U(3)$, or equivalently to unitary unimodular transformations in three dimensions, $SU(3)$. It remained however for Elliott (5, 1958) to exploit this invariance in firmly establishing the physical significance of the three-dimensional isotropic harmonic oscillator in many-particle shell model calculations. Some four years later $SU(3)$ was also recognized as being of importance in the classification of elementary particles (6). Spurred on by these discoveries physicists have been led to devote considerable effort to the investigation of the unitary group in n -dimensions, $U(n)$ (7), the fundamental properties of which have been available for many years in the pioneering researches of Weyl (8) and Murnaghan (9).

The most complete comprehension of $U(n)$ has been obtained by establishing the connection between the abstract Weyl (8) basis vectors and the boson calculus familiar from the work of Schwinger (10) and Bargmann (11). This together with the adaptation by Nagel and Moshinsky (12) of the work of Gel'fand and Zetlin (13) has given a canonical setting to the state labeling problem for $U(n)$. This labeling scheme involves the natural chain of subgroups $U(n) \supset U(n-1) \supset \dots \supset U(2) \supset U(1)$. In each step of the chain the reduction $U(m) \supset U(m-1) \times U(1)$ must be considered. In particular, for $U(3)$ one must consider the reduction $U(3) \supset U(2) \times U(1)$ or equivalently in terms of the unitary unimodular groups, the reduction $SU(3) \supset SU(2) \times U(1)$.

The canonical decomposition $SU(3) \supset SU(2) \times U(1)$ is particularly useful in elementary particle theory since the $SU(2)$ and the $U(1)$ refer respectively to the isotopic spin and hypercharge of a given $SU(3)$ multiplet. For use in nuclear physics, however, it is essential to recognize that this $SU(2)$ is not equivalent to the physical angular momentum subgroup $R(3)$. The required reduction is $SU(3) \supset R(3)$ and involves a well-known multiplicity problem since a given irreducible representation, IR, of $R(3)$ may be contained more than once in a given IR of $SU(3)$. One of Elliott's (5) main contributions was a resolution of this problem by a projection technique which introduced a quantum number K

that may be given the physical interpretation of being analogous to the projection of the total physical orbital angular momentum on an intrinsic axis of a nuclear many-body system. More recently, Bargmann and Moshinsky (14) have also resolved this problem by introducing an auxiliary operator Ω related to the Casimir operators of $SU(3)$ whose eigenvalue ω may be given the physical interpretation of the projection of the mass quadrupole moment along the direction of the total orbital angular momentum also in the nuclear many-body system. In both solutions to the multiplicity problem the requisite quantum number introduced has simple selection rules but the states to which they refer are in general nonorthogonal. This is a result of the fact demonstrated by Racah (15) that it is impossible to define an auxiliary quantum number in the reduction $SU(3) \supset R(3)$ which yields both simple selection rules and state orthogonality.

Although the physically different requirements for elementary particle theory and for nuclear theory have led to the development of the properties of $U(3)$ along seemingly different paths, whether one works in the $SU(3) \supset SU(2) \times U(1)$ scheme or in the $SU(3) \supset R(3)$ scheme one is interested in such quantities as coupling coefficients and tensorial matrix elements. To avoid a duplication of effort it is therefore necessary to have an explicit expression for the transformation brackets between the two state labeling schemes. In

preparation for the derivation of such an expression, Section 2 of this thesis is devoted to a review of the angular momentum projection techniques introduced by Elliott (5) and Section 3 is devoted to a review of the canonical state labeling scheme as adapted by Nagel and Moshinsky (12) from the work of Gel'fand and Zetlin (13). In Section 4 the connection between the two state labeling schemes is established and a complete reinterpretation and generalization of Elliott's work is given in terms of the canonical Gel'fand state labeling scheme. These results in turn are used to derive an explicit expression for the transformation brackets. In Section 5 the transformation brackets are used to establish a basis for the development of a complete $SU(3) \supset R(3)$ tensorial algebra. Such an algebra is necessary if the $U(3) \supset R(3)$ states are to become as useful in nuclear physics as, for example, $R(3)$ angular momentum states have proven to be.

Having established in Sections 4 and 5 the connection between the Elliott $SU(3) \supset R(3)$ state labeling scheme and the canonical Gel'fand state labeling scheme, in Section 6 it is shown that the results of Sections 2 through 5 are equally applicable to the consideration of nucleons moving in an average deformed nuclear potential. Such a situation is known to exist physically since many nuclei possess nonzero quadrupole moments, a measure of nuclear deformation. In fact, perhaps the most often used theoretical energy levels

are those of Nilsson (16) who gave a description of the motion of a single particle moving in an axially symmetric deformed potential created by a core of nucleons which itself is allowed to rotate adiabatically giving rise to rotational bands built upon the single particle energy levels. Newton (17) generalized Nilsson's work to the case of a non-axially symmetric deformed potential and Section 6 of this thesis may well serve as the mathematical basis for the generalization of Newton's work to deformed potential many-particle states. In essence then the theory is a generalization of the work of Nilsson and Newton to deformed potential many-particle states in much the same way as Elliott's work is a generalization of single-particle shell model to many-particle shell model states.

In Section 7 the deformed potential many-particle states are expanded in terms of the nondeformed potential many-particle states. The expansion is carried out in three steps. First, the deformed potential $SU(3) \supset SU(2) \times U(1)$ many-particle states are expanded in terms of the nondeformed $SU(3) \supset SU(2) \times U(1)$ many-particle states. Secondly, eigenstates of the physical orbital angular momentum are obtained from the deformed potential $SU(3) \supset SU(2) \times U(1)$ many-particle states. And thirdly, eigenstates of the physical orbital angular momentum are obtained from the deformed potential $SU(3) \supset R(3)$ many-particle states. Such expansions are basic to a study

of the extent to which the deformed potential many-particle states represent particle-hole structure found to be necessary for an adequate description of nuclear energy levels in nondeformed potential many-particle theory (18).

In Section 8 a theoretical description of the motion of nucleons coupled to that of a deformed core which itself is allowed to rotate adiabatically is presented. This is in preparation for Section 9 in which an actual calculation for the $A = 18$ nuclear system is performed treating O^{18} as having two nucleons moving in an axially symmetric deformed potential created by the $A = 16$ core of nucleons.

Section 10 is reserved for an analysis of the theory developed and the calculations performed in this thesis. A projective resume of additional research possibilities is also included.

2. $U(3) \supset R(3)$ BASIS

In firmly establishing the importance of the three-dimensional isotropic harmonic oscillator in nuclear many-particle shell model calculations by exploiting the invariance of the corresponding many-particle Hamiltonian to unitary transformations in three dimensions, Elliott (5) gave a prescription for performing the reduction of $U(3)$ with respect to the rotation group in three dimensions, $R(3)$, generated by the three components of the physical orbital angular momentum. This prescription serves as a basic tool for many developments in this thesis. To be sure, modifications and generalizations will be presented but the fundamental importance of Elliott's work remains and this section will be devoted to a review of both the basic concepts introduced by Elliott and the usefulness of the techniques as demonstrated in the literature. No attempt will be made at completeness nor will the notation introduced by Elliott be strictly adhered to; rather, the material will be presented in a form appropriate for the reinterpretations and generalizations to be given in Section 4.1.

2.1. Infinitesimal generators

If n denotes the total number of particles moving in an m -dimensional isotropic harmonic oscillator potential

and α and β are cartesian tensor indices then the infinitesimal generators of $U(m)$ may be written as

$$C_{\alpha\beta} = \sum_{s=1}^n A_{\alpha}^{s+} A_{\beta}^s \quad (2.1)$$

where the A_{α}^{s+} and A_{β}^t are oscillator creation and annihilation operators satisfying the commutation relations

$$[A_{\alpha}^s, A_{\beta}^{t+}] = \delta_{\alpha\beta}^{st},$$

$$[A_{\alpha}^{s+}, A_{\beta}^{t+}] = [A_{\alpha}^s, A_{\beta}^t] = 0, \quad 1 \leq \{\alpha\} \leq m, 1 \leq \{t\} \leq n \quad (2.2)$$

with $\delta_{\alpha\beta}^{st}$ equal to one if $\alpha = \beta$ and $s = t$ and zero otherwise. For $m = 3$ the total number of such infinitesimal generators is $3 \times 3 = 9$ and these may be conveniently divided in a manner familiar from the theory of $R(3)$ irreducible tensorial sets (19) into three sets containing one, three and five members respectively. Explicitly these are the oscillator Hamiltonian

$$H = \sum_{\alpha=1}^3 C_{\alpha\alpha}, \quad (2.3)$$

the three components of the physical orbital angular momentum

$$L_{\alpha} = -i \sum_{\beta, \gamma=1}^3 \epsilon_{\alpha\beta\gamma} C_{\beta\gamma} \quad (2.4)$$

and the five components of a second rank spherical tensor

$$Q_0 = 2C_{33} - C_{11} - C_{22}, \quad (2.5a)$$

$$Q_{\pm 1} = \mp \sqrt{3/2} [(C_{13} + C_{31}) \pm i(C_{23} + C_{32})], \quad (2.5b)$$

$$Q_{\pm 2} = \sqrt{3/2} [(C_{11} - C_{22}) \pm i(C_{12} + C_{21})]. \quad (2.5c)$$

the zeroth component of which is related to the mass quadrupole moment of the oscillator. In Equation 2.4 $\epsilon_{\alpha\beta\gamma}$ is the third rank Levi-Civita tensor. The operators L_α of Equation 2.4 satisfy the usual R(3) commutations relations

$$[L_\alpha, L_\beta] = iL_\gamma, \quad (\alpha\beta\gamma \text{ cyclic}), \quad (2.6)$$

explicitly demonstrating that R(3) is a proper subgroup of U(3).

The particular linear combination of infinitesimal generators given by

$$\Lambda_1 = \frac{1}{2}(C_{12} + C_{21}), \quad (2.7a)$$

$$\Lambda_2 = -\frac{i}{2}(C_{12} - C_{21}), \quad (2.7b)$$

$$\Lambda_3 = \frac{1}{2}(C_{11} - C_{22}) \quad (2.7c)$$

also satisfy the commutation relations

$$[\Lambda_\alpha, \Lambda_\beta] = i\Lambda_\gamma, \quad (\alpha\beta\gamma \text{ cyclic}), \quad (2.8)$$

explicitly demonstrating that they too generate a proper subgroup of U(3). This is the SU(2) subgroup in the reduction

$U(3) \supset SU(3) \supset SU(2) \times U(1)$. The IR of $SU(3)$ differ from those of $U(3)$ by nothing more than a multiplicative phase factor. The infinitesimal generators of this $SU(2)$ subgroup may be written in terms of the L 's and the Q 's as

$$\Lambda_1 = \frac{-i}{2\sqrt{6}}(Q_{+2} - Q_{-2}), \quad (2.9a)$$

$$\Lambda_2 = \frac{1}{2}L_3, \quad (2.9b)$$

$$\Lambda_3 = \frac{1}{2\sqrt{6}}(Q_{+2} + Q_{-2}). \quad (2.9c)$$

Equation 2.9b expresses the relationship of the $SU(2)$ infinitesimal generator Λ_2 to the $R(3)$ infinitesimal generator L_3 . This correspondence forms the basis for the proof that the $U(3) \supset R(3)$ basis states defined by Elliott span the $U(3)$ IR space. It will also prove to be essential for the development of Section 4.2.

2.2. Irreducible representations

An IR of $U(3)$ may be labeled by a three-rowed Young pattern (20) or equivalently by three numbers, $H_{\alpha 3}$, specifying the length of each row in the Young pattern. That is, by $[H_{13}, H_{23}, H_{33}]$ or by

$$\begin{array}{|c|} \hline H_{13} \\ \hline H_{23} \\ \hline H_{33} \\ \hline \end{array} \quad (2.10)$$

The corresponding SU(3) IR is obtained by subtracting H_{33} from each row yielding $[H_{13}-H_{33}, H_{23}-H_{33}]$ with a Young pattern

$$\begin{array}{|c|} \hline H_{13}-H_{33} \\ \hline H_{23}-H_{33} \\ \hline \end{array} . \quad (2.11)$$

In the more usual (λ_μ) notation with $\lambda = H_{13}-H_{23}$ and $\mu = H_{23}-H_{33}$ this is simply

$$\begin{array}{|c|} \hline \lambda+\mu \\ \hline \mu \\ \hline \end{array} . \quad (2.12)$$

Three additional labels are necessary to specify the rows within the IR. In the reduction $U(3) \supset U(2) \times U(1)$ or equivalently $SU(3) \supset SU(2) \times U(1)$ these correspond to the IR label ε of $U(1)$, the IR label Λ of $SU(2)$ and the row label Λ_0 within the $SU(2)$ IR. In terms of the infinitesimal generators of $U(3)$ as defined in Section 2.1 these are the eigenvalues of $Q_0, \Lambda^2 = \Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2$ and Λ_0 respectively, where Λ_0 is one of the components Λ_1, Λ_2 or Λ_3 . The rules for determining the allowable values of ε, Λ and Λ_0 given the values of λ and μ are

$$\varepsilon = 2\lambda + \mu - 3(p+q), \quad (2.13a)$$

$$\Lambda = (\mu+p-q)/2, \quad (2.13b)$$

$$\Lambda_0 = -\Lambda + r \quad (2.13c)$$

where p , q and r may take on the values

$$p = 0, 1, 2, \dots, \lambda, \quad (2.14a)$$

$$q = 0, 1, 2, \dots, \mu, \quad (2.14b)$$

$$r = 0, 1, 2, \dots, 2\lambda. \quad (2.14c)$$

In the $SU(3) \supset R(3)$ scheme Elliott (5) has determined that three additional labels K , L and M may be obtained from a Hill-Wheeler (21) projection integral of the form

$$\int d\Omega D_{MK}^{L*}(\Omega) R(\Omega) |(\lambda_\mu) \varepsilon \Lambda \Lambda_0\rangle \quad (2.15)$$

where $|(\lambda_\mu) \varepsilon \Lambda \Lambda_0\rangle$ is an $SU(3) \supset SU(2) \times U(1)$ basis state and $R(\Omega)$ and $D_{MK}^{L*}(\Omega)$ are respectively an $R(3)$ rotation and an $R(3)$ rotation matrix with Ω specifying three Euler angles. In Equation 2.15 the eigenvalue of $L^2 = L_1^2 + L_2^2 + L_3^2$ is $L(L+1)$ so that L labels an IR of $R(3)$, M is the eigenvalue of L_3 labeling the row within that IR and K is a label that distinguishes possible multiple occurrences of that IR in the reduction of the (λ_μ) IR of $SU(3)$. The rule for determining the allowable K values is

$$K = \min(\lambda_\mu), \min(\lambda_\mu)-2, \dots, 1 \text{ or } 0. \quad (2.16)$$

For each K , L may have the values

$$L = K, K+1, \dots, K+\max(\lambda_\mu), K \neq 0, \quad (2.17a)$$

$$L = \max(\lambda_\mu), \max(\lambda_\mu)-2, \dots, 1 \text{ or } 0, K = 0. \quad (2.17b)$$

Elliott's proof that these projected states span the $SU(3)$

IR space hinges upon the fact that the $SU(3) \supset SU(2) \times U(1)$ state involved in the projection is the one having $\varepsilon = \varepsilon_{\max} = 2\lambda + \mu$ if $\lambda \geq \mu$ and $\varepsilon = \varepsilon_{\min} = -\lambda - 2\mu$ if $\lambda < \mu$. There is however an incompleteness in Elliott's proof and this will be rectified in the generalization of Section 4.1 where a simplified rule for determining the allowable K values for projection from $\varepsilon_{\min}$ for both $\lambda \geq \mu$ and $\lambda < \mu$ is given.

2.3. Permutation symmetry

Since a nucleus is a system of Fermions the overall wavefunction must be antisymmetric under simultaneous exchange of both spatial and spin-isospin coordinates of a pair of particles. It is therefore necessary to require that each IR of $SU(3)$ be associated with a definite particle permutation symmetry so that multiplication by the conjugate spin-isospin symmetry leads to overall antisymmetry (22). Such a requirement makes it necessary to consider the reduction of the symmetric group in n-dimensions, $S(n)$, with respect to $SU(3)$ where n is the number of particles. Since an IR of $S(n)$ may also be labeled by a Young pattern (23) the required reduction may most easily be studied by constructing many-particle spatial wavefunctions from those of a single particle by taking direct products. A simple example will serve to demonstrate the technique.

Consider a single particle state for the 2s-1d oscillator

shell. It has an $S(1)$ symmetry label $\square$ and an $SU(3)$ symmetry label $\square\square$ corresponding to two oscillator phonons. Taking the direct product of two such states we find by using Littlewood's rules (24) for multiplying Young patterns that

$$\square \times \square = \square\square + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$$

$$([1] \times [1] = [2] + [1,1]) \quad (2.18)$$

$$(6 \times 6 = 21 + 15)$$

for $S(2)$ and

$$\square\square \times \square\square = \square\square\square\square + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

$$((2,0)sd \times (2,0)sd = (4,0)sdg + (2,1)pdf + (0,2)sd) \quad (2.19)$$

$$(6 \times 6 = 15 + 15 + 6)$$

for $SU(3)$.

In Equations 2.18 and 2.19 we have indicated the dimensionality (20) of the IR in each term of the direct product and the spectroscopic notation for the L values contained in each of the IR of $SU(3)$. Since p , d and f are the only antisymmetric orbital angular momentum states consistent with the configurations s^2 , sd and d^2 , the $(2,1)$ IR of $SU(3)$ must be associated with the antisymmetric IR $[1,1]$ of $S(2)$. The complete reduction $S(2) \supset SU(3)$ for this example must therefore

be $[2] (4,0)sdg (0,2)sd$ and $[1,1] (2,1)pdf$; the dimensionality in each case serves as a check.

The continuation of such a procedure for more than two particles can be used to determine the IR of $SU(3)$ contained in a given IR of $S(n)$. Elliott (5) has done this for the complete 2s-1d oscillator shell and has given a table listing the results. Notice that there may be multiple occurrences of a given IR of $SU(3)$ in the reduction of a particular IR of $S(n)$. In the 2s-1d shell this first occurs for the $[3,2]$ IR of $S(5)$ which contains the $(4,0)$ IR of $SU(3)$ twice. A resolution of this multiplicity problem is not available and although Section 3 of this thesis gives a more complete mathematical definition of the problem, the problem itself remains unsolved.

2.4. Physical significance

Since the Elliott $SU(3) \supset R(3)$ projected wavefunctions form a complete set satisfying the boundary conditions imposed upon bound state nuclear wavefunctions they form a basis in terms of which such wavefunctions may be expanded. Whether the basis is physically significant or not is determined by the number of basis states required in the expansion. A determination of the expansion coefficients requires a knowledge of the matrix elements of operators considered to be of physical significance in describing nucleon

interactions. Since this is a somewhat more difficult task to perform in the $SU(3) \supset R(3)$ projected basis than in either the $j-j$ or $l-s$ coupled basis such calculations are limited. However, by using the mass quadrupole interaction Q_0 of Section 2.1, Elliott (5) and Elliott and Harvey (25) have explicitly demonstrated for $2s-1d$ shell nuclei that the eigenstates of the nuclear Hamiltonian are nearly pure $SU(3) \supset R(3)$ projected states obtained from the most symmetric IR of $SU(3)$. More recently other authors have also performed calculations using the $SU(3) \supset R(3)$ projected states. For positive (26) as well as negative (27) parity levels the results again indicate that these states are indeed physically significant.

The model Hamiltonian,

$$H = \frac{1}{2} \sum_{s=1}^n [(P^s)^2 + (Q^s)^2], \quad (2.20)$$

when written in terms of the center of mass coordinates

$$Q_\alpha = \sum_{s=1}^n Q_\alpha^s / n$$

is

$$H = H \text{ intrinsic} + H \text{ center of mass} \quad (2.21)$$

$$= H \text{ intrinsic} + \frac{1}{2} \left[\frac{P^2}{n} + nQ^2 \right]. \quad (2.22)$$

This result expresses the fact that the center of mass moves

in a harmonic potential. Spurious excitations necessarily result and must be removed since they are unphysical.

Verhaar (28) has developed a group theoretical technique based upon the $SU(3)$ symmetry of the states for the elimination of these spurious excitations. To demonstrate the technique consider a $1-\hbar\omega$ excitation in O^{16} . The $SU(3)$ configurations which result are

$$(0,1) \times (2,0) = (2,1) + (1,0). \quad (2.23)$$

Since a $1-\hbar\omega$ excitation of the center of mass carries a $(1,0)$ IR of $SU(3)$ which when coupled to the $(0,0)$ $SU(3)$ IR of the ground state yields

$$(0,0) \times (1,0) = (1,0), \quad (2.24)$$

the $(1,0)$ IR of $SU(3)$ in Equation 2.23 is completely spurious and must be eliminated. The generalization of the method is equally straightforward and allows one to circumvent the difficulties which arise in the elimination of such spurious excitations in other coupling schemes based upon the shell model (29).

3. $U(3) \supset U(2) \times U(1)$ BASIS

Since a thorough understanding of the unitary group in n -dimensions, $U(n)$, is basic to developments in the remaining sections of this thesis, this section is devoted to a review of the basic properties of $U(n)$ as reported by Moshinsky and his associates (30) in a series of papers studying the unitary groups and their relationship to nuclear structure. No attempt at completeness will be made; rather, the material presented will be limited to that which will prove to be useful in the following sections.

3.1. Infinitesimal generators

If n denotes the total number of particles moving in an m -dimensional isotropic harmonic oscillator potential then the infinitesimal generators of $U(mn)$ may be written as

$$C_{\alpha\beta}^{st} = A_{\alpha}^{s+} A_{\beta}^t, \quad 1 \leq \{\alpha\} \leq m, \quad 1 \leq \{s\} \leq n \quad (3.1)$$

where the A_{α}^{s+} and A_{β}^t are defined in Section 2.1. By using Equation 2.2 it is not difficult to show that these operators do satisfy the appropriate commutation relations, namely,

$$[C_{\alpha\beta}^{st}, C_{\rho\sigma}^{uv}] = \delta_{\beta\rho}^{tu} C_{\alpha\sigma}^{sv} - \delta_{\sigma\alpha}^{vs} C_{\rho\beta}^{ut}. \quad (3.2)$$

Furthermore, if we define

$$C_{\alpha\beta} \equiv \sum_{s=1}^n C_{\alpha\beta}^{ss} \quad (3.3)$$

and

$$C^{st} \equiv \sum_{\alpha=1}^m C_{\alpha\alpha}^{st} \quad (3.4)$$

then by using Equation 3.2 it is not difficult to show that

$$[C_{\alpha\beta}, C_{\rho\sigma}] = \delta_{\beta\rho} C_{\alpha\sigma} - \delta_{\sigma\alpha} C_{\rho\beta} \quad (3.5)$$

and

$$[C^{st}, C^{uv}] = \delta^{tu} C^{sv} - \delta^{vs} C^{ut}. \quad (3.6)$$

In addition, we have that

$$[C_{\alpha\beta}, C^{st}] = 0. \quad (3.7)$$

The commutation relations, Equations 3.5 through 3.7, express the fact that the $C_{\alpha\beta}$ and C^{st} are respectively the generators of the unitary groups $U(m)$ and $U(n)$ in the reduction

$$U(mn) \supset U(m) \times U(n). \quad (3.8)$$

Although other such reductions are possible this particular one is physically significant in the sense that it separates the particle labels from the cartesian tensor labels.

3.2. Irreducible representations

Using the results of Equation 3.5 we find for $U(m)$ that

$$[C_{\alpha\alpha}, C_{\beta\beta}] = 0. \quad (3.9)$$

It therefore follows that the basis states of a given IR of $U(m)$ may be labeled simultaneously by the m eigenvalues, w_α , of the $C_{\alpha\alpha}$. These eigenvalues, called weights, may be considered to form the m components of a weight vector. Although to each basis state there corresponds a unique weight vector, the weight vectors do not uniquely specify all the basis states (15). Additional state labels are necessary. A solution to this state labeling problem, given first by Weyl (8), consists of recognizing that

$$U(m) \supset U(m-1) \supset \dots \supset U(2) \supset U(1) \quad (3.10)$$

forms a complete canonical reduction of $U(m)$ in the sense that in each step where the reduction $U(p) \supset U(p-1) \times U(1)$ must be considered, $1 \leq p \leq m$, a given IR of $U(p)$ contains at most one occurrence of any IR of $U(p-1)$ (31). Such a reduction therefore allows the basis states of a given IR of $U(m)$ to be labeled uniquely in terms of the IR labels of its canonical subgroups $U(p)$.

A particularly convenient form for expressing this result is to label each state vector by a Gel'fand pattern (12, 13, 14)

$$|H_{\alpha\beta}\rangle = \left| \begin{array}{ccccccc} H_{1m} & H_{2m} & \cdot & \cdot & \cdot & \cdot & H_{mm} \\ & H_{1m-1} & H_{2m-1} & \cdot & \cdot & \cdot & H_{m-1} & m-1 \\ & & \cdot & \cdot & \cdot & \cdot & & \\ & & & H_{12} & H_{22} & & & \\ & & & & H_{11} & & & \end{array} \right\rangle, \quad 1 \leq \alpha \leq \beta \leq m. \quad (3.11)$$

The $H_{\alpha\beta}$ are integers satisfying the betweenness conditions

$$H_{\alpha\beta} \geq H_{\alpha\beta-1} \geq H_{\alpha+1\beta} \geq 0 \quad (3.12)$$

made evident by the geometrical construction of the pattern. . . .

In terms of the reduction given by Equation 3.10, the $H_{\alpha\beta}$ specify the length of each row in the Young pattern (20) labeling for the IR of $U(p)$ to which the state belongs.

The weights of these Gel'fand states defined by

$$C_{\alpha\alpha} |H_{\alpha\beta}\rangle = W_{\alpha} |H_{\alpha\beta}\rangle \quad (3.13)$$

are

$$W_{\alpha} = \sum_{\beta=1}^{\alpha} H_{\beta\alpha} - \sum_{\beta=1}^{\alpha-1} H_{\beta\alpha-1}. \quad (3.14)$$

The corresponding $SU(m)$ Gel'fand states are obtained in precisely the same manner as indicated in Section 2.2 for $U(3)$, that is, by subtracting H_{mm} from each of the $H_{\alpha\beta}$. Notice that the allowable values of ε , Λ and Λ_0 of that section are nothing more than the betweenness conditions for $U(3)$ given by Equation 3.12.

Gel'fand states of particular interest are those for which $H_{\alpha\beta} = H_{\gamma}$, $H_{\gamma} = H_{\gamma m}$ (32). These are called extremal states and are $m!$ in number if all of the H_{γ} are distinct. These states have the property of being uniquely specified by their weight vectors. That particular extremal state for which $H_{\alpha\beta} = H_{\alpha}$, $1 \leq \alpha \leq \beta \leq m$, is called the highest weight state since from Equation 3.14 together with Equation 3.12 we have that $W_1 = H_1 \geq W_2 = H_2 \geq \dots \geq W_m = H_m$. Explicitly,

$$|H_{\alpha}\rangle = \left| \begin{array}{ccccccc} H_1 & H_2 & & \dots & & & H_m \\ & H_1 & H_2 & & \dots & & H_{m-1} \\ & & & \dots & \dots & \dots & \\ & & & & H_1 & H_2 & \\ & & & & & H_1 & \end{array} \right\rangle. \quad (3.15)$$

Lowering operators constructed from the infinitesimal generators of $U(m)$ which allow any of the Gel'fand states in the $[H_{\alpha}]$ IR of $U(m)$ to be obtained from this state are particularly simple since by construction

$$C_{\alpha\beta} |H_{\alpha}\rangle = 0, \quad \alpha < \beta \leq m. \quad (3.16)$$

A Hilbert space representation of these Gel'fand basis vectors for $U(m)$ may be given in terms of the creation operators A_{α}^{s+} operating on the vacuum ket $|0\rangle$ defined by

$$A_{\alpha}^s |0\rangle = 0, \quad 1 \leq \alpha \leq m, \quad 1 \leq \beta \leq n. \quad (3.17)$$

Since the $A_\alpha^{S^+}$ form the components of a single vector in the mn -dimensional space, products of the $A_\alpha^{S^+}$ operating on $|0\rangle$ can only form bases for the completely symmetric IR of $U(mn)$ (33). In particular, the IR of $U(mn)$ labeled by $[H]$ is spanned by the set of linearly independent homogeneous polynomials of degree H in the $A_\alpha^{S^+}$ operating on $|0\rangle$, that is, states of H oscillator phonons. This places a restriction on the allowable IR of $U(m)$ and $U(n)$ which occur in the reduction given by Equation 3.8. In particular, if for convenience we assume $m \geq n$ then only those IR $[H_\alpha]$ of $U(m)$ and $[H^S]$ of $U(n)$ can occur for which $H_\alpha = 0$ if $\alpha > n$ and $H_\alpha = H^\alpha$ for $1 \leq \alpha \leq n$ with

$$\sum_{\alpha=1}^n H_\alpha = H.$$

In other words, the IR labels for $U(m)$ and $U(n)$ must be identical and the H_α must form a partition of H . It is also possible to show that if $D_m [H_\alpha]$, $D_n [H^S]$ and $D_{mn} [H]$ denote respectively the dimensionalities of the $[H_\alpha]$ IR of $U(m)$, the $[H^S]$ IR of $U(n)$ and the $[H]$ IR of $U(mn)$ then

$$D_{mn}[H] = \sum_{[H_\alpha]} g[H_\alpha] D_m[H_\alpha] D_n[H^\alpha] \quad (3.18)$$

with $g[H_\alpha] = 1$. The sum is over all partitions $[H_\alpha]$ of H . This demonstrates that each partition $[H_\alpha]$ of H occurs precisely once in the reduction (34). Therefore, using the

reduction schemes given by Equations 3.8 and 3.10 we have that

$$|H_{\alpha\beta}; H^{st}\rangle = \begin{array}{c} \begin{array}{ccccccc} H_1 & H_2 & \cdot & \cdot & \cdot & \cdot & H_m \\ & H_{1m-1} & H_{2m-1} & \cdot & \cdot & \cdot & H_{m-1m-1} \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & H_{12} & & H_{22} & & \\ & & & & H_{11} & & \end{array} \\ \\ \begin{array}{ccccccc} H^1 & H^2 & \cdot & \cdot & \cdot & \cdot & H^n \\ & H^{1n-1} & H^{2n-1} & \cdot & \cdot & \cdot & H^{n-1n-1} \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & H^{12} & & H^{22} & & \\ & & & & H^{11} & & \end{array} \end{array} \quad ; \quad (3.19)$$

with $H_\alpha = 0$ if $\alpha > n$, $H_\alpha = H^\alpha$ for $1 \leq \alpha \leq n$ and

$$\sum_{\alpha=1}^n H_\alpha = H$$

affords a complete labeling scheme. The subsets of these states having fixed $H_{\alpha\beta}$ or H^{st} form, respectively, a representation of the Gel'fand basis vectors for the IR $[H_\alpha]$ of $U(m)$ and $[H^S]$ of $U(n)$.

In addition to forming a Hilbert space representation of the Gel'fand basis vectors for $U(m)$ and $U(n)$, these states are physically significant because they represent a unique distribution of H oscillator phonons among the m

cartesian directions and over the n particles. The distributions are given by the weights in $U(m)$ and $U(n)$ respectively. Explicitly there are W_α phonons in the direction α and W^s phonons on particle s . That particular state which is highest weight with respect to both $U(m)$ and $U(n)$ may be constructed explicitly (35). It is

$$\begin{aligned} |H_\alpha\rangle &= |H_\alpha; H^\alpha\rangle \\ &= N[H_\alpha] (\Delta_1^1)^{H_1-H_2} (\Delta_{12}^{12})^{H_2-H_3} \dots (\Delta_{12\dots r}^{12\dots r})^{H_r} |0\rangle \end{aligned} \quad (3.20)$$

where r equals the minimum of m and n and

$$\Delta_{12\dots\alpha}^{12\dots\alpha} \equiv \det \begin{bmatrix} A_1^{1+} & A_1^{2+} & \dots & A_1^{\alpha+} \\ A_2^{1+} & A_2^{2+} & \dots & A_2^{\alpha+} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_\alpha^{1+} & A_\alpha^{2+} & \dots & A_\alpha^{\alpha+} \end{bmatrix}. \quad (3.21)$$

$N[H_\alpha]$ is the normalization given by

$$\begin{aligned} N[H_\alpha] &= \left[\prod_{\beta=2}^r \prod_{\alpha=1}^{\beta-1} (H_\alpha - H_\beta + \beta - \alpha)! \right]^{1/2} \\ &\times \left[\prod_{\alpha=1}^r (H_\alpha + r - \alpha)! \right]^{1/2}. \end{aligned} \quad (3.22)$$

Since the $C_{\alpha\beta}$ and C^{st} commute, any of the $D_{mn}[H]$ states $|H_{\alpha\beta}; H^{st}\rangle$ can be constructed from this highest weight state by simply applying independently the appropriate lowering operators for $U(m)$ and $U(n)$. Notice that the $|H_{\alpha\beta}; H^{st}\rangle$ are necessarily orthogonal since each is labeled by a distinct set of IR labels for the subgroups $U(p)$ and $U(q)$ of $U(m)$ and $U(n)$ respectively. The normalization follows from the normalization factor for the lowering operators as given by Nagel and Moshinsky (12).

3.3. Permutation symmetry

Although the basis vectors introduced in Section 3.2 as a Hilbert space representation of the Gel'fand basis vectors are uniquely labeled and represent a unique distribution of H oscillator phonons in the m cartesian oscillator directions and over the n particles, they are not appropriate for describing a system of Fermions since they in no way take into account the Pauli principle. That is, these basis vectors possess no definite particle permutation symmetry (30). In order to resolve this fundamental difficulty it is necessary to consider the reduction $U(n) \supset S(n)$ rather than the canonical reduction used in Equation 3.19. This reduction, however, leads to another difficulty; namely, a given IR of $S(n)$ may occur more than once in any given IR of $U(n)$. This is the multiplicity problem for $U(n) \supset S(n)$ and necessitates

finding additional labels with which to distinguish multiple occurrences of the IR of $S(n)$ which occur in the reduction of any particular IR of $U(n)$.

To obtain a more thorough understanding of this $U(n) \supset S(n)$ multiplicity problem, consider the $|H_{\alpha\beta}; H^{st}\rangle$. Since the energy of particle s is $\hbar\omega W^s$ we see that the shell structure of the n -particle system is determined by the weights of the states. In particular, if by a weight structure, $\bar{U} = (U^1, U^2, \dots, U^n)$ with $U^1 \geq U^2 \geq \dots \geq U^n$, we mean the set of all weight vectors which are equivalent to $\bar{U}$ under permutation of weight components then the set of all $|H_{\alpha\beta}; H^{st}\rangle$ having weight vectors $\bar{W} = (W^1, W^2, \dots, W^n)$ belonging to a given weight structure form a basis for the shell structure having one particle of energy $\hbar\omega U^1$, one particle of energy $\hbar\omega U^2$ and so on. In particular, the $|H_{\alpha\beta}; H^{st}\rangle$ having the weight structure $\bar{U} = (U, U, \dots, U)$ form a basis for describing the motion of n particles moving in the U oscillator shell. Therefore, choosing a particular weight structure determines the shell structure and limits the number of basis states, $|H_{\alpha\beta}; H^{st}\rangle$, from which states of definite particle permutation symmetry must be formed. Group theoretically this reduction is equivalent to saying that the weight structure specifies the IR of a subgroup $A(n)$ of $U(n)$. The semidirect product (36) of this group with $S(n)$ yields a subgroup $K(n)$ of $U(n)$ whose IR labels

may be used to label state vectors. That is, the states may be labeled by the chain

$$U(n) \supset K(n) \supset S(n) \quad (3.23)$$

where the IR of $K(n)$ determine the shell structure and the IR of $S(n)$ the particle permutation symmetry (37). For $n = 1$ and 2 these results are sufficient to yield a unique labeling scheme. For $n \geq 3$, however, additional labels may be necessary. Although standard projection techniques (30) are applicable, a truly elegant reduction including $S(n)$ is as yet unavailable.

3.4. Physical applications

As an example, complementary to that of Section 2.3, which will serve to illustrate the concepts introduced in Sections 3.1 through 3.3 as well as lay the ground work for the study of the $A = 18$ system in Sections 8 and 9, consider the case of two particles moving in the $2s-1d$ oscillator shell. The required reduction analogous to Equation 3.8 is

$$U(6) \supset U(3) \times U(2). \quad (3.24)$$

Since two particles moving in the $2s-1d$ oscillator shell corresponds to $H = 4$ oscillator phonons, the IR of $U(6)$ under consideration is the $[4]$. The allowable $U(3)$ and $U(2)$ IR consistent with the constraints that the H_α form a partition of 4 and that $H_1 = H^1$, $H_2 = H^2$ and $H_3 = 0$ are the $[4]$, $[3,1]$ and $[2,2]$. Using the fact that the dimension-

ality of the completely symmetric IR $[H]$ of $U(n)$ is given in terms of a binomial coefficient by (20)

$$D_n[H] = \binom{n + H - 1}{H} \quad (3.25)$$

and the fact that the dimensionality of the $[H_\alpha]$ IR of $U(3)$ and $U(2)$ are given by (34)

$$D_3[H_1 H_2 H_3] = (H_1 - H_2 + 1)(H_2 - H_3 + 1)(H_1 - H_3 + 2)/2 \quad (3.26)$$

and

$$D_2[H_1 H_2] = H_1 - H_2 + 1, \quad (3.27)$$

respectively, we find that

$$\begin{aligned} D_6[4] &= \binom{6 + 4 - 1}{4} \\ &= 126 \end{aligned}$$

and

$$\begin{aligned} &D_3[4]D_2[4] + D_3[31]D_2[31] + D_3[22]D_2[22] \\ &= (15)(5) + (15)(3) + (6)(1) \\ &= 126 \end{aligned}$$

explicitly demonstrating the validity of Equation 3.18 and therefore the completeness of the reduction for the case under consideration.

The $|H_{\alpha\beta}; H^{st}\rangle$ given by

$$|H_{\alpha\beta}; H^{st}\rangle = \left| \begin{array}{ccc|ccc} H_1 & H_2 & 0 & H_1 & H_2 & \\ & H_{12} & H_{22} & & H^{11} & \\ & & H_{11} & & & \end{array} \right\rangle \quad (3.28)$$

are all obtainable from the highest weight state

$$|H_{\alpha}\rangle = \left| \begin{array}{ccc|ccc} H_1 & H_2 & 0 & H_1 & H_2 & \\ & H_1 & H_2 & & H_1 & \\ & & H_1 & & & \end{array} \right\rangle$$

$$= N[H_1 H_2] (\Delta_1^1)^{H_1-H_2} (\Delta_{12}^{12})^{H_2} |0\rangle \quad (3.29)$$

with

$$N[H_1 H_2] = \left[\frac{(H_1-H_2+1)(H_1+2)(H_2+1)}{(H_1+2)!(H_2+1)!} \right]^{1/2} \quad (3.30)$$

by application of the normalized lowering operators for U(3) and U(2). Using the definitions of Equations 3.3, 3.4 and 3.17 together with the commutation relations of Equations 2.2, 3.5, 3.6 and 3.7 it is not difficult to explicitly demonstrate for this example the results given by Equation 3.14, namely, $W_1 = H_{11}$, $W_2 = H_{12} + H_{22} - H_{11}$, $W_3 = H_1 + H_2 + H_3 - H_{12} - H_{22}$, $W^1 = H^{11}$ and $W^2 = H^1 + H^2 - H^{11}$. For the highest weight state $|H_{\alpha}\rangle$ this reduces to $W_1 = W^1 = H_1$, $W_2 = W^2 = H_2$ and $W_3 = 0$. In particular, the highest weight state for the [4] IR of U(3) and U(2) corresponds to one particle

carrying 4 phonons of excitation and vibrating in the first cartesian oscillator direction with the other particle in the oscillator ground state.

Restricting the particles to the $2s-1d$ shell requires $W^1 = H^{11} = 2$. Since only one of the $D_2 [H_\alpha]$ states for the IR $[H_\alpha]$ of $U(2)$ has $H^{11} = 2$, this shell structure requirement reduces the number of states from 126 to $15 + 15 + 6 = 36$ which is equivalent to the result found in Section 2.3. Explicitly these states are

$$|H_{\alpha\beta}; H^{st}\rangle = \left| \begin{array}{ccc} H_1 & H_2 & 0 \\ & H_{12} & H_{22} \\ & & H_{11} \end{array} ; \begin{array}{c} H_1 & H_2 \\ & 2 \end{array} \right\rangle \quad (3.31)$$

obtainable from

$$\begin{aligned} |H_\alpha; H^{st}\rangle &= \left| \begin{array}{ccc} H_1 & H_2 & 0 \\ & H_1 & H_2 \\ & & H_1 \end{array} ; \begin{array}{c} H_1 & H_2 \\ & 2 \end{array} \right\rangle \\ &= \left[\frac{(2-H_2)!}{(H_1-2)!(H_1-H_2)!} \right]^{1/2} (C^{21})^{H_1-2} |H_\alpha\rangle \\ &= \left[\frac{(H_1-H_2+1)(H_1+2)(H_2+1)}{(H_1+2)!(H_2+1)!(H_1-2)!} \right]^{1/2} (\Delta_{12}^{12})^{H_1-2} (\Delta_{12}^{12})^{H_2} |0\rangle \end{aligned} \quad (3.32)$$

by application of the normalized lowering operators for $U(3)$.

From this equation it is a simple matter to recognize that the states with H_2 even, $[4]$ and $[2,2]$, correspond to the symmetric $[2]$ IR of $S(2)$ and the state with H_2 odd, $[3,1]$, corresponds to the antisymmetric $[1,1]$ IR of $S(2)$. That is, the states for 2 particles in the $2s-1d$ shell may be uniquely labeled by $|H_{\alpha\beta}; [H_{\alpha}](2,2)[f]\rangle$ corresponding to the IR labels of the groups in the chain $U(2) \supset K(2) \supset S(2)$. The notation is usually abbreviated to $|H_{\alpha\beta}; [f]\rangle$ since the $[H_{\alpha}]$ is redundant and the shell structure is assumed known in any particular physical problem. Although these states are equivalent to those obtained in Section 2.3, the method used to obtain them is different. It gives a more complete comprehension of the multiple shell structure of the n -particle system and also casts the problem of constructing states of definite particle permutation symmetry into precise group theoretical language.

4. $U(3) \supset R(3)$ BASIS PROJECTED FROM THE $U(3) \supset U(2) \times U(1)$ BASIS

In this section $U(3) \supset R(3)$ states will be obtained by applying the angular momentum projection technique introduced by Elliott (Section 2) to the Hilbert space representation of the Gel'fand $U(3)$ basis introduced in Section 3. A rule complementary to the one given by Elliott (Equations 2.16 and 2.17) for the quantum number K will be given and the use of K in state labeling will be clarified. Using the expression for a finite $U(3)$ transformation and the result expressed by Equation 2.9b an explicit expression for the transformation brackets between the bases for an IR of $U(3)$ labeled in the $U(3) \supset R(3)$ scheme and the $U(3) \supset U(2) \times U(1)$ scheme will be given. This result will be used to derive an explicit expression for the normalization and overlaps of the projected states.

4.1. Projected basis

As demonstrated in Section 3.4 for the case of two particles moving in the 2s-1d oscillator shell, the $U(3) \supset U(2) \times U(1)$ basis states are equivalent to the $|(\lambda_\mu)_\varepsilon \Lambda \Lambda_0\rangle$ states used by Elliott (5) in obtaining the $U(3) \supset R(3)$ projected states for the n -particle system if under the reduction $U(n) \supset K(n) \supset S(n)$ only those states having weights belonging to the weight structure

$\bar{U} = (U, U, \dots, U)$ with $U = H/n$ are included. However, from Equations 2.4 and 3.7 we see that $R(\Omega)$ commutes with the lowering operators of $U(n)$ so that a generalization of Elliott's work is possible by defining

$$|H_{\alpha\beta}, KLM; H^{st}\rangle \equiv \int d\Omega D_{MK}^{L*}(\Omega) R(\Omega) |H_{\alpha\beta}; H^{st}\rangle. \quad (4.1)$$

We have retained the complete symbol $H_{\alpha\beta}$ on the left as a reminder of the $U(3) \supset U(2) \times U(1)$ basis state from which the projected states are obtained; however, only the $U(3)$ IR labels H_{α} remain valid state labels. Furthermore, it is possible to abbreviate this to simply

$$|H_{\alpha\beta}; KLM\rangle \equiv \int d\Omega D_{MK}^{L*}(\Omega) R(\Omega) |H_{\alpha\beta}\rangle \quad (4.2)$$

for $|H_{\alpha\beta}\rangle = |H_{\alpha\beta}; H^{\alpha}\rangle$, the highest weight $U(n)$ state. The appropriate physical states may then be obtained directly from these projected states by the techniques outlined in Section 3.3. These generalized $U(3) \supset R(3)$ states correspond to eigenstates of L^2 and L_3 obtained from the multiple-shell configurations of the n -particle system.

As pointed out in Section 2.2, Elliott's rule for determining the allowable K values and the corresponding values of L resolves the multiplicity problem for the $U(3) \supset R(3)$ reduction if the $|(\lambda\mu)\varepsilon\Lambda\Lambda_0\rangle$ states involved in the projection have $\varepsilon = \varepsilon_{\max} = 2\lambda + \mu$ for $\lambda \geq \mu$ and

$\varepsilon = \varepsilon_{\min} = -\lambda - 2\mu$ for $\lambda < \mu$. From Equations 2.5a and 2.7c we see that for Λ_0 chosen to be the eigenvalue of Λ_3 the explicit correspondence between the two labeling schemes may be expressed in terms of the U(3) Gel'fand patterns as

$$\begin{aligned}
 |H_{\alpha\beta}\rangle &= \left| \begin{array}{ccc} H_{13} & H_{23} & H_{33} \\ & H_{12} & H_{22} \\ & & H_{11} \end{array} \right\rangle \\
 &= \left| \begin{array}{ccc} \lambda+\mu+v & \mu+v & v \\ & p+\mu+v & q+v \\ & & r+q+v \end{array} \right\rangle \quad (4.3)
 \end{aligned}$$

with the $H_{\alpha\beta}$ given by Equation 3.12 and the p , q and r given by Equation 2.14. Neglecting phase differences we find that under the unimodular restriction this reduces to

$$\begin{aligned}
 |H_{\alpha\beta}\rangle &= \left| \begin{array}{ccc} H_{13}-H_{33} & H_{23}-H_{33} & 0 \\ & H_{12}-H_{33} & H_{22}-H_{33} \\ & & H_{11}-H_{33} \end{array} \right\rangle \\
 &= \left| \begin{array}{ccc} \lambda+\mu & \mu & 0 \\ & p+\mu & q \\ & & r+q \end{array} \right\rangle \\
 &= |(\lambda\mu)\varepsilon\Lambda\Lambda_0\rangle \quad (4.4)
 \end{aligned}$$

where ε , Λ and Λ_0 are given by Equations 2.13. Using these results we then see that the $\varepsilon = \varepsilon_{\max}$ and $\varepsilon = \varepsilon_{\min}$ correspond respectively to the lowest weight and highest weight state of the $|H_{\alpha\beta}\rangle$ where $\underline{W}_1$ is defined to be of higher weight than $\underline{W}_2$ if $W_{1\alpha} \geq W_{2\alpha}$, $1 \leq \alpha \leq 3$. Clearly these are only 2 of the $3! = 6$ extremal states for the $[H_\alpha]$ IR of $U(3)$. Since the remaining extremal states may be obtained from either of these by a simple permutation of cartesian axis labels, it is clear that a rule must exist which is analogous to the one given by Elliott (5) for projection from any of the extremal states. In particular, it will prove to be convenient to have such a rule for the canonical $U(3) \supset R(3)$ states defined by the projection

$$|H_\alpha; KLM\rangle \equiv \int d\Omega D_{MK}^{L*}(\Omega) R(\Omega) |H_\alpha\rangle \quad (4.5)$$

on the highest weight $U(3)$ state $|H_\alpha\rangle$ given explicitly by

$$\begin{aligned} |H_\alpha\rangle &= N[H_1 H_2 H_3] (\Delta_1^1)^{H_1-H_2} (\Delta_{12}^{12})^{H_2-H_3} (\Delta_{123}^{123})^{H_3} |0\rangle \\ &= N(\lambda\mu) (\Delta_1^1)^\lambda (\Delta_{12}^{12})^\mu |0\rangle. \end{aligned} \quad (4.6)$$

As a first step in the determination of such a rule, consider the origin of the conditions imposed upon ε by Elliott (5) in proving that the projected states of Equation 2.15 span the $SU(3)$ IR space. For $\varepsilon = \varepsilon_{\max}$ and $\varepsilon = \varepsilon_{\min}$ Λ is fixed by Equations 2.13 at $\mu/2$ and $\lambda/2$

respectively. Interpreting Λ_0 as the eigenvalue of $\Lambda_2 = \frac{1}{2}L_3$ the orthogonality of the functions $e^{im\phi}$ occurring in the integral of Equation 2.15 then forces $K = 2\Lambda_0$.

Therefore we have that

$$K = -\mu, -\mu+2, \dots, \mu \quad (4.7a)$$

if $\varepsilon = \varepsilon_{\max}$ and

$$K = -\lambda, -\lambda+2, \dots, \lambda \quad (4.7b)$$

if $\varepsilon = \varepsilon_{\min}$. This set of K values contains all those required by the rule of Equation 2.16 for arbitrary λ and μ if and only if $\varepsilon_{\max}$ is used for $\lambda \geq \mu$ and $\varepsilon_{\min}$ is used for $\lambda < \mu$. Consider a special case which violates these conditions, projection from $\varepsilon_{\max}$ when $\lambda < \mu$. The available values of K are given by Equation 4.7a. The set of K values defined by Equation 2.16 include the positive values in the set defined by Equation 4.7b. This set of K values will be contained in the set given by Equation 4.7a if and only if λ and μ are both even or both odd. For the rule of Equation 2.16 to be completely general it is therefore necessary to require projection from $\varepsilon_{\max}$ if $\lambda \geq \mu$ and $\varepsilon_{\min}$ if $\lambda < \mu$. This is contrary to Elliott's second paper (5) in which it is stated that these conditions on ε are made only so as to involve the fewest possible number of the $[(\lambda\mu)\varepsilon\Lambda_0]$ states. Indeed, this ambiguity manifests itself in the vanishing of

certain normalization integrals given in the paper by Elliott and Harvey (25) in which all the $R(3)$ basis states are obtained by projection from the $|(\lambda\mu)\varepsilon\Lambda\Lambda_0\rangle$ with $\varepsilon = \varepsilon_{\max}$.

The difficulty is associated with the rule of Equations 2.16 and 2.17 and not with Equations 4.7. If one wishes to project from states with $\varepsilon = \varepsilon_{\min}$ for $\lambda \geq \mu$ and $\varepsilon = \varepsilon_{\max}$ for $\lambda < \mu$ then Equations 2.16 and 2.17 must be replaced by

$$K = \max(\lambda\mu), \max(\lambda\mu)-2, \dots, 1 \text{ or } 0 \quad (4.8)$$

where for each K , L may have the values

$$L = K, K+1, \dots, K+\min(\lambda\mu), K \neq 0, \quad (4.9a)$$

$$L = \min(\lambda\mu), \min(\lambda\mu)-2, \dots, 1 \text{ or } 0, K = 0. \quad (4.9b)$$

It is not difficult to convince oneself that this rule and that of Equations 2.16 and 2.17 give the same values of L for a given $(\lambda\mu)$ except that L will be associated in general with a different value of K . For example consider the IR (5,2). According to Equations 2.16 and 2.17

$$K = 2, L = 2, 3, 4, 5, 6, 7$$

$$K = 0, L = 5, 3, 1$$

while Equations 4.8 and 4.9 yield

$$K = 5, L = 5, 6, 7$$

$$K = 3, L = 3, 4, 5$$

$$K = 1, L = 1, 2, 3$$

so that in both cases $L = 1, 2, 3^2, 4, 5^2, 6, 7$. This new rule is complementary to that of Elliott and the use of the two may be summarized as follows:

If using $\varepsilon_{\max}$ for $\lambda \geq \mu$ and $\varepsilon_{\min}$ for $\lambda < \mu$ then use

$K = \min(\lambda\mu)$ rule.

If using $\varepsilon_{\min}$ for $\lambda \geq \mu$ and $\varepsilon_{\max}$ for $\lambda < \mu$ then use

$K = \max(\lambda\mu)$ rule.

If one chooses always to project from $\varepsilon = \varepsilon_{\max}$ as was done in (25) then for $\lambda < \mu$ one must use the $K = \max(\lambda\mu)$ rule while for $\lambda \geq \mu$ one must use the $K = \min(\lambda\mu)$ rule. On the other hand, if one chooses always to project from $\varepsilon = \varepsilon_{\min}$ corresponding to the highest weight Gel'fand state then for $\lambda < \mu$ one must use $K = \min(\lambda\mu)$ rule while for $\lambda \geq \mu$ one must use the $K = \max(\lambda\mu)$ rule. For these two cases we summarize:

Projection from $\varepsilon = \varepsilon_{\max}$

For both $\lambda \geq \mu$ and $\lambda < \mu$:

$$K = \mu, \mu-2, \dots, 1 \text{ or } 0;$$

$$L = K, K+1, \dots, K+\lambda, K \neq 0,$$

$$L = \lambda, \lambda-2, \dots, 1 \text{ or } 0, K = 0. \quad (4.10)$$

Projection from $\varepsilon = \varepsilon_{\min}$

For both $\lambda \geq \mu$ and $\lambda < \mu$:

$$K = \lambda, \lambda-2, \dots, 1 \text{ or } 0;$$

$$L = K, K+1, \dots, K+\mu, K \neq 0,$$

$$L = \mu, \mu-2, \dots, 1 \text{ or } 0, K = 0. \quad (4.11)$$

The $\varepsilon = \varepsilon_{\max}$ and $\varepsilon = \varepsilon_{\min}$ conditions used above clearly refer to the 3-axis in view of the definitions of Equations 2.5a and 2.7c. For projection from the other extremal states one must simply apply the same rules to the appropriately permuted cartesian axis labels. In addition, notice that the rules of Equations 4.10 and 4.11 may be obtained from one another by simply interchanging λ and μ . The simplicity of this result propagates throughout the remaining developments of this thesis and for this reason, unless otherwise specified, we shall be concerned only with the canonical $U(3)$ $R(3)$ projected states as defined by Equation 4.5 together with the rule of Equation 4.11. It is to be emphasized, however, that the techniques are equally applicable to the $U(3) \supset R(3)$ states obtained by projection from any of the extremal $U(3) \supset U(2) \times U(1)$ basis states.

The proof given by Elliott that the $R(3)$ projected basis defined by Equation 4.5 span the space of the $[H_\alpha]$ IR of $U(3)$ is made by reductio ad absurdum. It consists of showing

that the hypothesis that there exists a function $|H_\alpha; L'M'\rangle$ belonging to the $[H_\alpha]$ IR of $U(3)$ and linearly independent of the $|H_\alpha; KLM\rangle$ given by Equation 4.5 together with the rule of Equation 4.11 necessarily leads to the conclusion that an integral of the form

$$\int d\tau \langle H_\alpha; L'M' | P | H_\alpha \rangle$$

vanishes identically with P being an arbitrary $U(3)$ operator and the integration being carried out over the $U(3)$ group manifold. Since any of the $|H_{\alpha\beta}\rangle$ may be expressed in the form $P|H_\alpha\rangle$, the vanishing of this integral leads to the conclusion that the $|H_\alpha; L'M'\rangle$ does not belong to the $[H_\alpha]$ IR of $U(3)$. If however there exists an L' in the $[H_\alpha]$ IR of $U(3)$ which is not represented by at least one of the $R(3)$ projected basis states given by Equation 4.5 together with the rule of Equation 4.11 then the result is trivially true and inconclusive. It is therefore necessary, as pointed out by Williams and Pursey (38) for the case of the rotation group in five dimensions, to show that those values of L predicted to be missing from the $[H_\alpha]$ IR of $U(3)$ by the rule of Equation 4.11 are indeed absent. This is done by means of degeneracy diagrams. Before proceeding, however, we shall first demonstrate that the rule of Equation 4.11 does in fact give the correct $SU(3)$ dimension formula.

Let $d(K)$ denote the number of states for fixed K .

Then if $K \neq 0$

$$\begin{aligned} d(K) &= \sum_{j=0}^{\mu} [2(K+j) + 1] \\ &= (\mu+1)(2K+\mu+1). \end{aligned}$$

If $K = 0$, for either μ even or μ odd,

$$d(0) = (\mu+1)(\lambda+1)/2.$$

Denoting the dimensionality of the $(\lambda\mu)$ IR of $SU(3)$ by $D(\lambda\mu)$ we then have for λ even that

$$\begin{aligned} D(\lambda\mu) &= \sum_{\substack{K=2 \\ \text{even}}}^{\lambda} d(K) + d(0) \\ &= (\lambda+1)(\mu+1)(\lambda+\mu+2)/2 \end{aligned} \tag{4.12}$$

which is the correct dimension formula (39) equivalent to the result given by Equation 3.26 for $U(3)$. For λ odd we similarly find that

$$\begin{aligned} D(\lambda\mu) &= \sum_{\substack{k=1 \\ \text{odd}}}^{\lambda} d(K) \\ &= (\lambda+1)(\mu+1)(\lambda+\mu+2)/2. \end{aligned}$$

We now proceed to complete the Elliott proof by proving the following LEMMA: Any angular momentum L which is not allowed by the rules of Equations 4.10 and 4.11 applied to an IR of $SU(3)$ is entirely absent from that IR.

First of all, one sees that either rule predicts that $L \leq \lambda + \mu$. For the special cases $(\lambda, 0)$ or $(0, \mu)$ the rules predict $L = \lambda, \lambda - 2, \dots, 1$ or 0 and $L = \mu, \mu - 2, \dots, 1$ or 0 , respectively. For these cases, we shall demonstrate the Lemma directly. The other possibilities are summarized in Table 1 in which it is assumed that neither λ nor μ is zero.

A degeneracy diagram for $SU(3)$ is a rectangular lattice of points (q, p) with p and q defined by Equations 2.13 and 2.14. In Figure 1 the coordinate axes corresponding to $2\Lambda = \mu + p - q$ and $-\varepsilon = 3(p + q) - (2\lambda + \mu)$ are superimposed on the p and q axes with the coordinates of the boundary points of the lattice labeled by both their $[q, p]$ and their $(-\varepsilon, 2\Lambda)$ values. We have also indicated the equations of the boundary lines in terms of the 2Λ and $-\varepsilon$ coordinates. In passing, note that $-\varepsilon$ is three times the hypercharge (40). Each point represents a state labeled by the corresponding values of ε and Λ and is $2\Lambda + 1$ fold degenerate.

We shall be interested in the number of occurrences of a value of $\Lambda_0 \geq 0$ in an IR $(\lambda\mu)$ when Λ_0 is interpreted as the eigenvalue of Λ_2 . Then the number of occurrences of $2\Lambda_0 = K = L_0$ may be used to determine whether or not a particular value of L occurs in a manner very similar to the well-known derivation of the $R(3)$ Clebsch-Gordan series (20). Denote by $d_{(\lambda\mu)}(L_0)$ the number of occurrences of a particular value of L_0 in the IR $(\lambda\mu)$. Then a particular L occurs if

and only if

$$n_{(\lambda\mu)}(L) \equiv d_{(\lambda\mu)}(L) - d_{(\lambda\mu)}(L+1) \geq 1 .$$

Indeed, $n_{(\lambda\mu)}(L)$ is just the number of times the angular momentum L occurs in the IR $(\lambda\mu)$. To determine $d_{(\lambda\mu)}(L_0)$ one need merely count the total number of points in alternate rows with constant 2Λ from the minimum allowable value of $2\Lambda = 2\Lambda_0 = L_0$ upward toward the maximum allowable value of $2\Lambda = \lambda + \mu$. For example, consider Figure 2 which gives the degeneracy diagram for the IR $(7,5)$. Table 2 lists the values of $d_{(\lambda\mu)}(L_0)$. The maximum value of L is 12 and this occurs once. Similarly $L = 11$ also occurs once, $L = 10$ twice, etc.

From the general degeneracy diagram of Figure 1 note that $2\Lambda_0 = L_0$ has a maximum value of $\lambda + \mu$. Therefore $L_{\max} = \lambda + \mu$, or $L \leq \lambda + \mu$. Consider the special cases of $(\lambda, 0)$ or $(0, \mu)$ for which the rectangle degenerates to a straight line. For $(\lambda, 0)$ it is clear that $L = \lambda, \lambda - 2, \dots, 1$ or 0 and similarly the values of L in the $(0, \mu)$ are $L = \mu, \mu - 2, \dots, 1$ or 0 . We are therefore now faced with consideration of the three special cases summarized in Table 1.

It is not difficult to convince oneself that

$$d_{(\lambda\mu)}(0) = \left[\frac{\lambda + \mu + \lambda\mu + 2}{2} \right] ,$$

$$d_{(\lambda\mu)}(1) = \left[\frac{\lambda + \mu + \lambda\mu + 1}{2} \right]$$

and

$$d_{(\lambda\mu)}(2) = \left[\frac{\lambda + \mu + \lambda\mu}{2} \right] ,$$

where $[]$ denotes largest integer contained in the argument. Consider then $n_{(\lambda\mu)}(0)$. Unless both λ and μ are even this is clearly zero. Therefore in cases (i) and (iii) $L = 0$ cannot occur. Finally consider $n_{(\lambda\mu)}(1)$ and note that this is zero only if λ and μ are both even so that in case (ii) $L = 1$ does not occur. All angular momentum L predicted to be missing from an IR of $SU(3)$ by the rules of Equations 4.10 and 4.11 are therefore indeed absent.

4.2. Transformation brackets

For many of the remaining developments in this thesis it will prove to be convenient to abbreviate the notation introduced in Sections 3.1 through 4.1 by replacing $H_{\alpha\beta}$ by G and H_{α} by G_{HW} . Sums on the allowable $H_{\alpha\beta}$ as given by Equation 3.12 for a given IR $[H_{\alpha}]$ of $U(3)$ will simply be indicated as a sum on G . In addition, whenever it is necessary to specify $H_1, H_2, H_3, H_{12}, H_{22}$ and H_{11} explicitly these will be replaced by h_1, h_2, h_3, s, t and r respectively. It will also prove to be convenient to replace Λ by j and Λ_0 interpreted as the eigenvalue of Λ_3 by m . Furthermore,

we shall feel free to suppress redundant $U(3)$ IR labels H_α which occur in Gel'fand basis vectors. That is,

$$\begin{aligned}
 |G\rangle &= |H_{\alpha\beta}\rangle \\
 &= \left| \begin{array}{ccc} H_1 & H_2 & H_3 \\ & H_{12} & H_{22} \\ & & H_{11} \end{array} \right\rangle \\
 &= \left| \begin{array}{ccc} h_1 & h_2 & h_3 \\ & s & t \\ & & r \end{array} \right\rangle \\
 &= \left| \begin{array}{ccc} s & & t \\ & r & \end{array} \right\rangle .
 \end{aligned} \tag{4.13}$$

Since the Gel'fand basis vectors $|G\rangle$ for a given IR of $U(3)$ form an orthonormal set which spans the representation space, an arbitrary $U(3) \supset R(3)$ basis vector $|G;KLM\rangle$ belonging to the IR may be expanded in terms of the $|G\rangle$ as

$$|G;KLM\rangle = \sum_{G'} \langle G' | G;KLM \rangle |G'\rangle , \tag{4.14}$$

where it is to be understood that $h'_\alpha = h_\alpha$. The $\langle G' | G;KLM \rangle$ in Equation 4.14 are the transformation brackets between the Gel'fand $U(3) \supset U(2) \times U(1)$ scheme and the $U(3) \supset R(3)$ scheme introduced by Elliott. The inverse of the transformation is only guaranteed to exist if the $|G;KLM\rangle$ are

$U(3) \supset R(3)$ basis vectors obtained from extremal $U(3) \supset U(2) \times U(1)$ basis vectors together with the appropriate rule for determining the allowable K and L values. We shall derive an expression for the general transformation brackets $\langle G' | G; KLM \rangle$; those for the canonical $U(3) \supset R(3)$ basis vectors of Equation 4.5 follow as a special case.

From Equation 4.2 we have

$$\langle G' | G; KLM \rangle = \int d\Omega D_{MK}^{L*}(\Omega) \langle G' | R(\Omega) | G \rangle \quad (4.15)$$

so that the problem reduces to one of finding an explicit expression for the matrix elements $\langle G' | R(\Omega) | G \rangle$. Following the convention of Rose, $R(\Omega)$ may be written as

$$R(\Omega) = R_3(\theta_1) R_2(\theta_2) R_3(\theta_3) \quad (4.16)$$

where the θ_α are Euler angles and the $R_\alpha(\theta)$ are given by

$$R_\alpha(\theta) = e^{-i\theta L_\alpha} \quad (4.17)$$

The L_α are the cartesian components of the physical orbital angular momentum given by Equation 2.4. Since any cartesian component of the angular momentum can be related to L_3 by the appropriate axis permutation we may write

$$L_2 = (23)(-L_3)(23) \quad (4.18)$$

or by using Equation 2.9b

$$L_2 = (23)(-2\Lambda_2)(23) . \quad (4.19)$$

Thus the rotation operator $R(\Omega)$ may be written in terms of the Λ_2 as

$$R(\Omega) = S(2\theta_1)(23)S(-2\theta_2)(23)S(2\theta_3) \quad (4.20)$$

where

$$S(\varnothing) = e^{-i\varnothing\Lambda_2} . \quad (4.21)$$

Using the fact that the basis for an IR of $U(2)$ and $SU(2)$ differ by at most a phase factor and the fact that the Gel'fand basis vectors diagonalize the Λ_3 component of Λ we have from the isomorphism of $SU(2)$ and $R(3)$ that

$$\left\langle \begin{matrix} s' & t' \\ & r' \end{matrix} \middle| S(\varnothing) \middle| \begin{matrix} s & t \\ & r \end{matrix} \right\rangle = \delta_{s's} \delta_{t't} d_{m'm}^j(\varnothing) ; \quad (4.22)$$

$$j = (s-t)/2, \quad m = r-(s-t)/2, \quad m' = r' - (s'+t')/2 .$$

Here we have used the convention of Rose (41) for the rotation matrix $d_{m'm}^j(\varnothing)$. In addition, using the result given by Chacón and Moshinsky (42) for the matrix elements of the transposition (23) in the Gel'fand basis in terms of Racah (43) coefficients we have that

$$\left\langle \begin{matrix} s' & t' \\ & r' \end{matrix} \middle| (23) \middle| \begin{matrix} s & t \\ & r \end{matrix} \right\rangle = (-1)^{h_3} \delta_{r'r} \delta_{s'+t'-r', h-s-t} \\ \times \sqrt{(2j'+1)(2j+1)} W(abcd; ef); \quad (4.23)$$

$$a = (h_1 + h_2 - s' - t')/2, \quad b = (h_2 + r - t - t')/2, \quad c = (s + s' - h_2 - r)/2,$$

$$d = (s' + t' - h_3 - r)/2, \quad e = (h_1 + r - t - s')/2, \quad f = (h_1 + r - s - t')/2,$$

and where $h = h_1 + h_2 + h_3$ is the total number of quanta. From these results it then follows that

$$\begin{aligned} \left\langle \begin{matrix} s' & t' \\ r' \end{matrix} \middle| R(\Omega) \middle| \begin{matrix} s & t \\ r \end{matrix} \right\rangle &= \sum_{\substack{\sigma\tau \\ \rho\rho'}} \left\langle \begin{matrix} s' & t' \\ r' \end{matrix} \middle| S(2\theta_1) \middle| \begin{matrix} s' & t' \\ \rho' \end{matrix} \right\rangle \\ &\times \left\langle \begin{matrix} s' & t' \\ \rho' \end{matrix} \middle| (23) \middle| \begin{matrix} \sigma & \tau \\ \rho' \end{matrix} \right\rangle \left\langle \begin{matrix} \sigma & \tau \\ \rho' \end{matrix} \middle| S(-2\theta_2) \middle| \begin{matrix} \sigma & \tau \\ \rho \end{matrix} \right\rangle \\ &\times \left\langle \begin{matrix} \sigma & \tau \\ \rho \end{matrix} \middle| (23) \middle| \begin{matrix} s & t \\ \rho \end{matrix} \right\rangle \left\langle \begin{matrix} s & t \\ \rho \end{matrix} \middle| R(2\theta_3) \middle| \begin{matrix} s & t \\ r \end{matrix} \right\rangle \end{aligned} \quad (4.24)$$

or

$$\begin{aligned} \left\langle \begin{matrix} s' & t' \\ r' \end{matrix} \middle| R(\Omega) \middle| \begin{matrix} s & t \\ r \end{matrix} \right\rangle &= \sqrt{2j'+1)(2j+1)} \sum_{\rho'} d_{m'\mu'}^{j'}(2\theta_1) \\ &\times \sum_{\rho} d_{\mu m}^j(2\theta_3) \sum_{\sigma\tau} (2k+1) d_{\nu'\nu}^k(-2\theta_2) W(a'b'c'd'; e'f') \\ &\times W(abcd; ef) \delta_{s'+t'-\rho', h-\sigma-\tau} \delta_{\sigma+\tau-\rho, h-s-t}; \end{aligned} \quad (4.25)$$

$$\begin{aligned}
j' &= (s'-t')/2, & m' &= r' - (s'+t')/2, & \mu' &= \rho' - (s'+t')/2, \\
j &= (s-t)/2, & m &= r - (s+t)/2, & \mu &= \rho - (s+t)/2, \\
k &= (\sigma-t)/2, & v &= \rho - (\sigma+\tau)/2, & v' &= \rho' - (\sigma+\tau)/2, \\
a' &= (h_1+h_2-s'-t')/2, & b' &= (h_2+\rho'-\tau-t')/2, & c' &= (\sigma+s'-h_2-\rho')/2, \\
d' &= (s'+t'-h_3-\rho')/2, & e' &= (h_1+\rho'-\tau-s')/2, & f' &= (h_1+\rho'-\sigma-t')/2, \\
a &= (h_1+h_2-\sigma-\tau)/2, & b &= (h_2+\rho-t-\tau)/2, & c &= (s+\sigma-h_2-\rho)/2, \\
d &= (\sigma+\tau-h_3-\rho)/2, & e &= (h_1+\rho-t-\sigma)/2, & f &= (h_1+\rho-s-\tau)/2.
\end{aligned}$$

The transformation brackets of Equation 4.15 are then given by

$$\begin{aligned}
\langle G' | G; KLM \rangle &= \sqrt{(2j'+1)(2j+1)} \sum_{\rho'} I_1(m'j'\mu'M) \\
&\times \sum_{\rho} I_1(\mu jmK) \sum_{\sigma\tau} (2k+1) I_2(v'kvMLK) W(a'b'c'd'; e'f') \\
&\times W(abcd; ef) \delta_{s'+t'-\rho', h-\sigma-\tau} \delta_{\sigma+\tau-\rho, h-s-t} \quad (4.26)
\end{aligned}$$

where we define

$$I_1(m'jml) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi d_{m',m}^j(2\phi) e^{il\phi} \quad (4.27)$$

and

$$I_2(m'jmn'ln) \equiv \frac{1}{2} \int_0^\pi \sin\phi d\phi d_{m',m}^j(-2\phi) d_{n',n}^l(\phi) \quad (4.28)$$

The evaluation of the integrals I_1 and I_2 is performed in Appendices A and B respectively. Since $U(3) \supset SU(3) \supset R(3)$ these transformation brackets are of course invariant under the unimodular restriction. In addition, if one chooses to perform the projection of Equation 4.2 on the state

$|G\rangle \equiv i^{-w_2} |G\rangle$, where $|G\rangle$ is the Gel'fand basis state defined by Chacón and Moshinsky (42) and $w_2 = h-s-t$, it can be shown that the transformation brackets are real.

These transformation brackets relate the set of non-orthogonal basis vectors $|G;KLM\rangle$ to the set of orthonormal basis vectors $|G\rangle$ and are therefore in general the elements of a nonunitary matrix A . As already pointed out, the inverse expansion of the $|G\rangle$ in terms of the $|G_{HW};KLM\rangle$ exists and the coefficients can be obtained by inverting the appropriate A matrix. The dimensionality of this matrix A is equal to the dimensionality of the IR, $d(\lambda\mu)$, given by Equation 4.12. For our purposes it will prove to be more convenient to consider a nonunitary matrix B the elements of which are the coefficients in the expansion

$$|G;KLM\rangle = \sum_{G'K'} B(GK;G'K') |G';K'LM\rangle \quad (4.29)$$

where again it is to be understood that $h'_\alpha = h_\alpha$. In general these B 's are not uniquely determined since the $|G';K'LM\rangle$ over-span the representation space. One may, however, restrict $G' = G_{HW}$ since the $|G_{HW};K'LM\rangle$ span the

representation space. In this case a unique solution for the B's exists and may be determined from the set of simultaneous equations

$$\langle G_{HW} | G; KLM \rangle = \sum_{K'} B(GK; G_{HW} K') \langle G_{HW} | G_{HW}; K' LM \rangle \quad (4.30)$$

in which the coefficients of the B's are a particularly simple special case of Equation 4.26. Explicitly,

$$\langle G_{HW} | G_{HW}; KLM \rangle = \sum_{\rho} I_1(jjmM) I_1(mjjK) I_2(kkkMLK);$$

$$k = (\rho - h_3)/2, \quad j = (h_1 - h_2)/2, \quad m = \rho - (h_1 + h_2)/2.$$

(4.31)

The dimensionality of the matrix B is equal to the number of allowable K values corresponding to a given L value. Since K resolves the degeneracy in the $U(3) \supset R(3)$ scheme, this must be equal to the total number of occurrences, $n_{(\lambda\mu)}(L)$, of L in the IR.

4.3. Normalization and overlap integrals

The normalization and overlap integrals for the $U(3) \supset R(3)$ basis vectors are simply related to the transformation brackets. This may most easily be seen by considering

$$\begin{aligned}
A_L(G'K'; GK) &\equiv \langle G'; K'LM | G; KLM \rangle \\
&= \int d\Omega D_{MK}^L(\Omega) \langle G' | R^{-1}(\Omega) | G; KLM \rangle \\
&= \sum_{M'} \int d\Omega D_{MK}^L(\Omega) D_{MM'}^{L*}(\Omega) \langle G' | G; KLM' \rangle \\
&= \sum_{M'} (2L+1)^{-1} \delta_{K'M'} \langle G' | G; KLM' \rangle \\
&= (2L+1)^{-1} \langle G' | G; KLM' \rangle.
\end{aligned} \tag{4.32}$$

Notice that the special case of Equation 4.31 is simply $(2L+1)$ times the $A_L(G_{HW}K'; G_{HW}K)$ for the canonical $U(3) \supset R(3)$ basis vectors.

For the canonical $U(3) \supset R(3)$ basis vectors it is perhaps well to note for which values of L the overlap is non-zero. From Equation 4.11 it is clear that there can only be overlap if K' and K differ by an even interger. In addition the $A_L(G_{HW}K'; G_{HW}K)$ must be zero if L lies outside the following range of values

$$L = \max(K'K), \max(K'K) + 1, \dots, \min(K'K) + \mu; \min(K'K) \neq 0,$$

$$L = \mu, \mu-2, \dots, \max(K'K) + 1 \text{ or } \max(K'K), \min(K'K) = 0.$$

For the rule of Equation 4.10 μ must be replaced by λ and the K values appropriately redefined. In either case the same set of L 's emerges. By using the conditions on K' and K and following a logic similar to that of Williams and

Pursey (30) we find that these considerations also yield a formula for the number of occurrences $n_{(\lambda\mu)}(L)$ of a given L in the IR $(\lambda\mu)$. Explicitly

$$n_{(\lambda\mu)}(L) = [(\lambda+\mu-L)/2] + [(\lambda+1-L)/2] + [(\mu+1-L)/2] + 1 \quad (4.33)$$

where $[]$ means the largest integer contained in the argument and is to be interpreted as zero if the argument is negative. In addition to yielding an explicit expression for the dimensionality of the matrix B introduced in Section 4.2, Equation 4.29 this result may also be used to derive the dimension formula of Equation 4.12. Explicitly,

$$D(\lambda\mu) = (\lambda+1)(\mu+1)(\lambda+\mu+2)/2$$

$$= \sum_{L=0}^{\lambda+\mu} (2L+1)n_{(\lambda\mu)}(L) .$$

5. $U(3) \supset R(3)$ TENSORS PROJECTED FROM THE $U(3) \supset U(2) \times U(1)$ TENSORS

When working within the framework of the full invariance group of the Hamiltonian for the isotropic harmonic oscillator it is necessary to consider the $U(3)$ tensorial properties of physical operators. It is again convenient to consider the reduction $U(3) \supset R(3)$ rather than the reduction $U(3) \supset U(2) \times U(1)$ for tensor labeling because the $U(3) \supset R(3)$ reduction involves the physical orbital angular momentum properties of the operators which are so important to their physical interpretation. This section is therefore devoted to a study of $U(3) \supset R(3)$ tensors. First of all, $U(3) \supset R(3)$ tensors are defined in analogy with Equation 4.2 for the $U(3) \supset R(3)$ basis states. Then an explicit form for the $U(3) \supset R(3)$ coupling coefficients is derived and used to give the first steps in the development of a complete $U(3) \supset R(3)$ tensorial algebra.

5.1. Projected tensors

A $U(3) \supset R(3)$ tensor labeling scheme is possible by defining in analogy with Equation 4.2

$$T(G; KLM) \equiv \int d\Omega D_{MK}^{L*}(\Omega) R(\Omega) T(G) R^{-1}(\Omega) , \quad (5.1)$$

where $T(G)$ is a $U(3) \supset U(2) \times U(1)$ tensor defined by

$$[C_{\alpha\beta}, T(G)] = \sum_{G'} \langle G' | C_{\alpha\beta} | G \rangle T(G') . \quad (5.2)$$

In Equation 5.2 the $C_{\alpha\beta}$ are the infinitesimal generators of $U(3)$. The requisite quantum number K resolves the multiplicity in the reduction $U(3) \supset R(3)$ for the generalized spherical tensors in precisely the same manner as found to be true for the basis vectors. In particular, the $T(G_{HW}; KLM)$ with K given by the rule of Equation 4.11 may be considered to be a set of canonical $U(3) \supset R(3)$ projected tensors forming a complete basis for a tensorial representation of the $[H_\alpha]$ IR of $U(3)$. In what follows we shall consider the $T(G; KLM)$.

By simply using the definition of Equation 5.1 it is not difficult to show that the generalized spherical tensors possess the following properties:

$$R(\Omega) T(G; KLM) R^{-1}(\Omega) = \sum_{M'} D_{M'M}^L(\Omega) T(G; KLM') , \quad (5.3)$$

$$[T(G; KLM)]^+ = (-1)^{K-M} T(G; -K, L, -M) \quad (5.4)$$

and

$$T(G; KLM) = \sum_{K'} D_{KK'}^L(\Omega) T'(G; K'LM) \quad (5.5)$$

where

$$T'(G; KLM) = [R^{-1}(\Omega) T(G) R(\Omega)](G; KLM) . \quad (5.6)$$

Equation 5.3 gives the transformation properties of the

generalized spherical tensors under rotations. Equation 5.4 relates the set of Hermitian conjugate generalized spherical tensors to the set of generalized spherical tensors, and Equation 5.5 relates generalized spherical tensors obtained from rotated $U(3) \supset U(2) \times U(1)$ tensors in terms of the generalized spherical tensors of Equation 5.1.

5.2. Coupling coefficients

The $U(3) \supset R(3)$ coupling coefficients are defined by

$$|G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle = \sum_{G_3_{HW} K_3 L_3 M_3 \alpha} |G_3_{HW}; K_3 L_3 M_3\rangle_\alpha \quad (5.7)$$

where α distinguishes the multiple occurrences of the IR G_3 in the reduction of the direct product $G_1 \times G_2$. These may be related to the $SU(3) \supset SU(2) \times U(1)$ coupling coefficients (40) by using the transformation brackets of Section 4.2. Explicitly, if $R(\Omega; \beta)$ is the rotation operator in the space labeled by β , then from the definition of Equation 4.2 we have

$$\begin{aligned} |G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle &= \int d\Omega D_{M_1 K_1}^{L_1*}(\Omega) R(\Omega; 1) |G_1\rangle |G_2; K_2 L_2 M_2\rangle \\ &= \int d\Omega D_{M_1 K_1}^{L_1*}(\Omega) R(\Omega; 1) |G_1\rangle R(\Omega; 2) R^{-1}(\Omega; 2) |G_2; K_2 L_2 M_2\rangle \end{aligned}$$

$$= \sum_{K_2'} \int d\Omega D_{M_1 K_1}^{L_1*}(\Omega) D_{M_2 K_2}^{L_2*}(\Omega) R(\Omega;1) R(\Omega;2) |G_1\rangle |G_2; K_2 L_2 K_2'\rangle. \quad (5.8)$$

From the well-known $R(3)$ Clebsch-Gordan series for the rotation matrices and the fact that $R(\Omega;1)R(\Omega;2) = R(\Omega;3)$, this may be written as

$$|G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle = \sum_{L_3 K_2'} C(L_1 L_2 L_3; M_1 M_2 M_3) \\ \times C(L_1 L_2 L_3; K_1 K_2' K_3') \int d\Omega D_{M_3 K_3}^{L_3*}(\Omega) R(\Omega;3) |G_1\rangle |G_2; K_2 L_2 K_2'\rangle. \quad (5.9)$$

By applying Equation 4.14 we find that the right-hand side of Equation 5.9 becomes

$$\sum_{L_3} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2' K_3') \\ G_2' K_2' \\ \times \langle G_2' | G_2; K_2 L_2 K_2' \rangle \int d\Omega D_{M_3 K_3}^{L_3*}(\Omega) R(\Omega;3) |G_1\rangle |G_2'\rangle. \quad (5.10)$$

Now, if we use the $SU(3) \supset SU(2) \times U(1)$ coupling coefficients to express

$$|G_1\rangle |G_2'\rangle = \sum_{G_3 \alpha} \langle G_3 | G_1; G_2' \rangle_{\alpha} |G_3\rangle_{\alpha} \quad (5.11)$$

together with Equation 4.29 we obtain the final results

$$\begin{aligned}
|G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle &= \sum_{\substack{G_3 L_3 \alpha \\ G_2' K_2'}} C(L_1 L_2 L_3; M_1 M_2 M_3) \\
&\times C(L_1 L_2 L_3; K_1 K_2' K_3') \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3 | G_1; G_2' \rangle_\alpha \\
&\times \int d\Omega \mathcal{D}_{M_3 K_3}^{L_3*}(\Omega) R(\Omega; 3) |G_3\rangle_\alpha \\
&= \sum_{\substack{G_3 L_3 \alpha \\ G_2' K_2'}} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2' K_3') \\
&\times \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3 | G_1; G_2' \rangle_\alpha |G_3; K_3' L_3 M_3\rangle_\alpha \quad (5.12a) \\
&= \sum_{\substack{K_3 L_3 \alpha \\ G_2' K_2' G_3'}} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2' K_3') \\
&\times \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3' | G_1; G_2' \rangle_\alpha B(G_3' K_3'; G_{3HW} K_3) \\
&\times |G_{3HW}; K_3 L_3 M_3\rangle_\alpha \quad (5.12b)
\end{aligned}$$

where it is to be understood that the h_α' of G_3' are equal to the h_α of G_{3HW} .

From this development we see that the $U(3) \supset R(3)$ coupling coefficients are given by

$$\begin{aligned}
& \langle G_{3HW}; K_3 L_3 M_3 | G_1, K_1 L_1 M_1; G_2, K_2 L_2 M_2 \rangle_\alpha \\
&= \sum_{\substack{G_2', K_2' \\ G_3'}} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2' K_3') \\
&\quad \times \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3' | G_1; G_2' \rangle_\alpha B(G_3' K_3'; G_{3HW} K_3) .
\end{aligned}
\tag{5.13}$$

The reduced $U(3) \supset R(3)$ coupling coefficients follow directly as

$$\begin{aligned}
& \langle G_{3HW}; K_3 L_3 || G_1, K_1 L_1; G_2, K_2 L_2 \rangle_\alpha = \sum_{\substack{G_2', K_2' \\ G_3'}} C(L_1 L_2 L_3; K_1 K_2' K_3') \\
&\quad \times \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3' | G_1; G_2' \rangle_\alpha B(G_3' K_3'; G_{3HW} K_3) .
\end{aligned}
\tag{5.14}$$

Since Resnikoff (40) has given a complete general solution for the $SU(3) \supset SU(2) \times U(1)$ coupling coefficients, including a resolution of the multiplicity problem in an orthogonal way, Equations 5.13 and 5.14 yield the complete general solution for the $U(3) \supset R(3)$ coupling coefficients.

Although the $|G_3; K_3 L_3 M_3\rangle$ are a set of nonlinearly independent basis vectors which over-span the representation space so that no unique expansion of the direct product in terms of these basis vectors exists, the particular expansion coefficients of Equation 5.12a are well-defined and for this reason are useful for computational purposes. It is to be emphasized, however, that the expansion coefficients of

Equations 5.13 and 5.14 are unique to within the resolution of the multiplicity problem as given by Resnikoff (40).

Pursey¹ has pointed out that a more symmetric form for the $U(3) \supset R(3)$ coupling coefficients may be given in terms of the product of two transformation brackets by using the unitarity properties of the Clebsch-Gordan coefficients. Explicitly, if we expand

$$\begin{aligned}
 |G_1; K_1 L_1 K_1' \rangle |G_2; K_2 L_2 K_2' \rangle &= \sum_{G_1' G_2'} \langle G_1' | G_1; K_2 L_1 K_1' \rangle \\
 &\times \langle G_2' | G_2; K_2 L_2 K_2' \rangle |G_1' \rangle |G_2' \rangle \\
 &= \sum_{G_3 \alpha} \sum_{G_1' G_2'} \langle G_1' | G_1; K_1 L_1 K_1' \rangle \langle G_2' | G_2; K_2 L_2 K_2' \rangle \\
 &\times \langle G_3 | G_1' G_2' \rangle_\alpha |G_3 \rangle_\alpha
 \end{aligned} \tag{5.15}$$

then

$$\begin{aligned}
 \int d\Omega D_{M_3 K_3}^{L_3*} (\Omega) R(\Omega; 3) |G_1; K_1 L_1 K_1' \rangle |G_2; K_2 L_2 K_2' \rangle \\
 &= \sum_{M_1 M_2} \int d\Omega D_{M_3 K_3}^{L_3*} (\Omega) D_{M_1 K_1}^{L_1} (\Omega) D_{M_2 K_2}^{L_2} (\Omega) |G_1; K_1 L_1 M_1 \rangle \times |G_2; K_2 L_2 M_2 \rangle \\
 &= (2L_3 + 1)^{-1} \sum_{M_1 M_2} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1' K_2' K_3')
 \end{aligned}$$

¹D. L. Pursey, Department of Physics, Iowa State University, Ames, Iowa. $U(3) \supset R(3)$ coupling coefficients. Private communication. 1968.

$$x |G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle \quad (5.16)$$

which from Equation 5.15 is equal to

$$\sum_{G_3 \alpha} \langle G_1' | G_1; K_1 L_1 K_1' \rangle \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3 | G_1'; G_2' \rangle_\alpha \\ G_1' G_2' \\ |G_3; K_3' L_3 M_3\rangle_\alpha \quad (5.17)$$

Multiplying both Equation 5.16 and 5.17 by $C(L_1 L_2 L_3; K_1' K_2' K_3')$ and summing over K_1' and K_2' we find by using the unitarity of the $R(3)$ Clebsch-Gordan coefficients that

$$\sum_{M_1 M_2} C(L_1 L_2 L_3; M_1 M_2 M_3) |G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle \\ = \sum_{G_3 \alpha} (2L_3 + 1) C(L_1 L_2 L_3; K_1' K_2' K_3') \langle G_1' | G_1; K_1 L_1 K_1' \rangle \\ G_1' K_1' G_2' K_2' \\ x \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3 | G_1'; G_2' \rangle_\alpha |G_3; K_3' L_3 M_3\rangle_\alpha \quad (5.18)$$

Multiplying this result by $C(L_1 L_2 L_3; M_1 M_2 M_3)$ and summing over L_3 and once again using the unitarity of the $R(3)$ Clebsch-Gordan coefficients we find the following results analogous to Equations 5.12a and 5.12b

$$|G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle = \sum_{G_3 L_3 \alpha} C(L_1 L_2 L_3; M_1 M_2 M_3) \\ G_1' K_1' G_2' K_2' \\ x C(L_1 L_2 L_3; K_1' K_2' K_3') (2L_3 + 1) \langle G_1' | G_1; K_1 L_1 K_1' \rangle \langle G_2' | G_2; K_2 L_2 K_2' \rangle \\ x \langle G_3 | G_1'; G_2' \rangle_\alpha |G_3; K_3' L_3 M_3\rangle_\alpha \quad (5.19a)$$

$$\begin{aligned}
&= \sum_{\substack{K_3 L_3 \alpha \\ G_1' K_1' G_2' K_2' G_3'}} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1' K_2' K_3') \\
&\quad \times (2L_3+1) \langle G_1' | G_1; K_1 L_1 K_1' \rangle \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3' | G_1'; G_2' \rangle_\alpha \\
&\quad \times B(G_3' K_3'; G_{3HW} K_3) | G_{3HW}; K_3 L_3 M_3 \rangle_\alpha \quad (5.19b)
\end{aligned}$$

where again it is to be understood that h_α' of G_3' must be equal to the h_α of G_{3HW} . An alternative expression for the $U(3) \supset R(3)$ coupling coefficients is then given by

$$\begin{aligned}
\langle G_{3HW}, K_3 L_3 M_3 | G_1, K_1 L_1 M_1; G_2, K_2 L_2 M_2 \rangle_\alpha &= \sum_{\substack{G_1' K_1' \\ G_2' K_2' \\ G_3'}} C(L_1 L_2 L_3; M_1 M_2 M_3) \\
&\quad \times C(L_1 L_2 L_3; K_1' K_2' K_3') (2L_3+1) \langle G_1' | G_1; K_1 L_1 K_1' \rangle \\
&\quad \times \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3' | G_1'; G_2' \rangle_\alpha B(G_3' K_3'; G_{3HW} K_3) \quad (5.20)
\end{aligned}$$

An alternative expression for the reduced $U(3) \supset R(3)$ coupling coefficients follow directly as

$$\begin{aligned}
\langle G_{3HW}, K_3 L_3 || G_1, K_1 L_1; G_2, K_2 L_2 \rangle_\alpha &= \sum_{\substack{G_1' K_1' \\ G_2' K_2' \\ G_3'}} C(L_1 L_2 L_3; K_1' K_2' K_3') \\
&\quad \times (2L_3+1) \langle G_1' | G_1; K_1 L_1 K_1' \rangle \langle G_2' | G_2; K_2 L_2 K_2' \rangle \langle G_3' | G_1'; G_2' \rangle_\alpha \\
&\quad \times B(G_3' K_3'; G_{3HW} K_3) \quad (5.21)
\end{aligned}$$

Although this form for the $U(3) \supset R(3)$ coupling coefficients

is convenient for a study of their symmetry properties, the results of Equations 5.13 and 5.14 are more convenient for computational purposes since they involve fewer sums.

5.3. Tensorial matrix elements

It remains to determine the matrix elements of the generalized spherical tensors in the $U(3) \supset R(3)$ basis. This may be accomplished by using Equation 5.1 to write

$$\begin{aligned} T(G_1; K_1 L_1 M_1) |G_2; K_2 L_2 M_2\rangle &= \int d\Omega D_{M_1 K_1}^{L_1*}(\Omega) R(\Omega) T(G_1) R^{-1}(\Omega) \\ &\times |G_2; K_2 L_2 M_2\rangle \\ &= \sum_{K_2'} \int d\Omega D_{M_1 K_1}^{L_1*}(\Omega) D_{M_2 K_2'}^{L_2*}(\Omega) R(\Omega) T(G_1) |G_2; K_2 L_2 K_2'\rangle. \end{aligned} \quad (5.22)$$

From the well-known $R(3)$ Clebsch-Gordan series for the rotation matrices and the result expressed by Equation 4.14 we obtain

$$\begin{aligned} T(G_1; K_1 L_1 M_1) |G_2; K_2 L_2 M_2\rangle &= \sum_{L_3 K_2'} C(L_1 L_2 L_3; M_1 M_2 M_3) \\ &\times C(L_1 L_2 L_3; K_1 K_2' K_3') \int d\Omega D_{K_3' M_3}^{L_3*}(\Omega) R(\Omega) T(G_1) |G_2; K_2 L_2 K_2'\rangle \\ &= \sum_{L_3 K_2'} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2' K_3') \langle G_2' | G_2; K_2 L_2 K_2'\rangle \\ &\times \int d\Omega D_{K_3' M_3}^{L_3*}(\Omega) R(\Omega) T(G_1) |G_2'\rangle. \end{aligned} \quad (5.23)$$

Now, if we use the $SU(3) \supset SU(2) \times U(1)$ tensorial result

$$\begin{aligned} T(G_1) |G_2'\rangle &= \sum_{G_3\alpha} \langle G_3 | T(G_1) |G_2'\rangle_\alpha |G_3\rangle_\alpha \\ &= \sum_{G_3\alpha} \langle G_3 | G_1; G_2'\rangle_\alpha \langle G_3 || T(G_1) || G_2'\rangle_\alpha |G_3\rangle_\alpha, \end{aligned} \quad (5.24)$$

where $\langle G_3 || T(G_1) || G_2'\rangle_\alpha$ is the reduced $U(3) \supset U(2) \times U(1)$ tensorial matrix element corresponding to the state $|G_3\rangle_\alpha$, we obtain the final results analogous to Equations 5.12a and 5.12b

$$\begin{aligned} T(G_1; K_1 L_1 M_1) |G_2; K_2 L_2 M_2\rangle &= \sum_{\substack{G_3 L_3 \alpha \\ G_2' K_2'}} C(L_1 L_2 L_3; M_1 M_2 M_3) \\ &\quad \times C(L_1 L_2 L_3; K_1 K_2' K_3') \langle G_2' | G_2; K_2 L_2 K_2'\rangle \langle G_3 | G_1; G_2'\rangle_\alpha \\ &\quad \times \langle G_3 || T(G_1) || G_2'\rangle_\alpha \int d\Omega D_{K_3' M_3}^{L_3*}(\Omega) R(\Omega) |G_3\rangle_\alpha \\ &= \sum_{\substack{G_3 L_3 \alpha \\ G_2' K_2'}} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2' K_3') \langle G_2' | G_2; K_2 L_2 K_2'\rangle \\ &\quad \times \langle G_3 | G_1; G_2'\rangle_\alpha \langle G_3 || T(G_1) || G_2'\rangle_\alpha |G_3; K_3' L_3 M_3\rangle_\alpha \quad (5.25a) \\ &= \sum_{\substack{K_3 L_3 \alpha \\ G_2' K_2' G_3'}} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2' K_3') \langle G_2' | G_2; K_2 L_2 K_2'\rangle \\ &\quad \times \langle G_3' | G_1; G_2'\rangle_\alpha \langle G_3' || T(G_1) || G_2'\rangle_\alpha B(G_3' K_3'; G_{3HW}^K) |G_{3HW}^{K_3' L_3 M_3}\rangle_\alpha. \end{aligned} \quad (5.25b)$$

The matrix elements of the generalized spherical tensors are then given by

$$\begin{aligned}
 & \langle G_3; K_3 L_3 M_3 | T(G_1; K_1 L_1 M_1) | G_2; K_2 L_2 M_2 \rangle_\alpha = \\
 & \sum_{\substack{G_2' K_2' \\ G_3'' K_3''}} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2' K_3'') \langle G_2' | G_2; K_2 L_2 K_2' \rangle \\
 & \times \langle G_3'' | G_1; G_2' \rangle_\alpha \langle G_3'' || T(G_1) || G_2' \rangle_\alpha B(G_3'' K_3'; G_{3HW} K_3'') \\
 & \times \langle G_3; K_3 L_3 M_3 | G_{3HW}; K_3'' L_3 M_3 \rangle_\alpha . \tag{5.26}
 \end{aligned}$$

By using Equation 5.13 together with the result that

$\langle G_3'' || T(G_1) || G_2' \rangle_\alpha$ depends only upon the h_α'' , h_α' and h_α we find that

$$\begin{aligned}
 & \langle G_3; K_3 L_3 M_3 | T(G_1; K_1 L_1 M_1) | G_2; K_2 L_2 M_2 \rangle_\alpha \\
 & = \langle G_3 || T(G_1) || G_2 \rangle_\alpha \sum_{K_3''} \langle G_{3HW}; K_3'' L_3 M_3 | G_1, K_1 L_1 M_1; G_2, K_2 L_2 M_2 \rangle_\alpha \\
 & \times \langle G_3; K_3 L_3 M_3 | G_{3HW}; K_3'' L_3 M_3 \rangle_\alpha . \tag{5.27}
 \end{aligned}$$

The reduced matrix elements of the generalized spherical tensors follow from Equation 5.14 and are

$$\begin{aligned}
& \langle G_3; K_3 L_3 || T(G_1; K_1 L_1) || G_2; K_2 L_2 \rangle_\alpha \\
&= \langle G_3 || T(G_1) || G_2 \rangle_\alpha \sum_{K_3''} \langle G_{3HW}; K_3'' L_3 || G_1, K_1 L_1; G_2, K_2 L_2 \rangle_\alpha \\
&\times \langle G_3; K_3 L_3 | G_{3HW}; K_3'' L_3 \rangle_\alpha, \tag{5.28}
\end{aligned}$$

where we have made use of the fact that the normalization and overlap integrals defined by Equation 4.32 are independent of the M projection.

6. DEFORMED $U(3) \supset R(3)$ AND $U(3) \supset U(2) \times U(1)$ BASES

In this section the invariance of the fundamental commutators to a canonical transformation is used as a basis for developing a group theoretical generalization of the results of Sections 2 through 5. The physical significance of the generalization obtained is considered.

6.1. Canonical transformation

A necessary and sufficient condition for a transformation between generalized coordinates and their conjugate momenta to be canonical is that the fundamental commutators be invariant under the transformation (44). As a simple example, consider the one-dimensional harmonic oscillator

$$H(P, Q) = P^2/(2m) + m\omega^2 Q^2/2 \quad (6.1)$$

with

$$[P, Q] = i\hbar, [Q, Q] = [P, P] = 0. \quad (6.2)$$

Define

$$Q' = (1/\sqrt{2})(\sqrt{m\omega} Q - iP/\sqrt{m\omega}), \quad (6.3a)$$

$$P' = (1/\sqrt{2})(P/\sqrt{m\omega} - i\sqrt{m\omega} Q). \quad (6.3b)$$

Then by direct substitution into Equation 6.1 and 6.2 we have that

$$H(P, Q) = K(P', Q') = i\omega Q'P' + \hbar\omega/2 \quad (6.4)$$

with

$$[Q', P'] = i\hbar, [Q', Q'] = [P', P'] = 0 . \quad (6.5)$$

Equation 6.5 proves that the transformation defined by Equations 6.3 is canonical. Identifying $A^+ = Q'/\sqrt{\hbar}$ and $A = iP'/\sqrt{\hbar}$ we recognize in addition that this is the well-known transformation which takes one from the coordinate-momentum representation to the creation-annihilation operator representation for the oscillator. Explicitly,

$$H = \hbar\omega(A^+A + \frac{1}{2}) . \quad (6.6)$$

For the case of n particles moving in an m -dimensional anisotropic harmonic oscillator potential we have

$$H(P, Q) = \sum_{\alpha=1}^m \sum_{s=1}^n [(P_{\alpha}^s)^2 / (2m) + m\omega_{\alpha}^2 (Q_{\alpha}^s)^2 / 2] \quad (6.7)$$

with

$$[Q_{\alpha}^s, P_{\beta}^t] = i\hbar\delta_{\alpha\beta}^{st}, [Q_{\alpha}^s, Q_{\beta}^t] = [P_{\alpha}^s, P_{\beta}^t] = 0 . \quad (6.8)$$

Rather than rewriting this in the form

$$H(P, Q) = H_0(P, Q) + H_1(P, Q) \quad (6.9)$$

with

$$H_0(P, Q) = \sum_{\alpha=1}^m \sum_{s=1}^n [(P_{\alpha}^s)^2 / (2m) + m\omega_0^2 (Q_{\alpha}^s)^2 / 2]$$

and

$$H_1(P, Q) = \sum_{\alpha=1}^m \sum_{s=1}^n m(\omega_{\alpha}^2 - \omega_0^2) (Q_{\alpha}^s)^2 / 2$$

so as to reduce the problem to that of an isotropic harmonic oscillator with an anisotropy H_1 add as a perturbation, we define in analogy with Equation 6.3 the canonical transformation

$$Q_{\alpha}'^s = (1/\sqrt{2})(\sqrt{m\omega_{\alpha}} Q_{\alpha}^s - iP_{\alpha}^s/\sqrt{m\omega_{\alpha}}) , \quad (6.10a)$$

$$P_{\alpha}'^s = (1/\sqrt{2})(P_{\alpha}^s/\sqrt{m\omega_{\alpha}} - i\sqrt{m\omega_{\alpha}} Q_{\alpha}^s) . \quad (6.10b)$$

By direct substitution into Equations 6.7 and 6.8 we then have that

$$H(P, Q) = K(P', Q') = \sum_{\alpha=1}^m \sum_{s=1}^n (i\omega_{\alpha} Q_{\alpha}'^s P_{\alpha}'^s + \hbar\omega_{\alpha}/2) \quad (6.11)$$

with

$$[Q_{\alpha}'^s, P_{\beta}'^t] = i\hbar\delta_{\alpha\beta}^{st}, [Q_{\alpha}'^s, Q_{\beta}'^t] = [P_{\alpha}'^s, P_{\beta}'^t] = 0 . \quad (6.12)$$

Again identifying $a_{\alpha}^{s+} = Q_{\alpha}'^s/\sqrt{\hbar}$ and $a_{\alpha}^s = iP_{\alpha}'^s/\sqrt{\hbar}$ we recognize this to be the creation-annihilation operator representation for the anisotropic harmonic oscillator. Explicitly,

$$H = \sum_{\alpha=1}^m \sum_{s=1}^n \hbar\omega_{\alpha} (a_{\alpha}^{s+} a_{\alpha}^s + \frac{1}{2}) . \quad (6.13)$$

Dropping the additive constant term

$$\sum_{\alpha=1}^m \hbar \omega_{\alpha} (n/2)$$

and introducing dimensionless variables by setting $\hbar$, m and $\omega_0 = \omega_{\alpha}/\varepsilon_{\alpha}$ equal to one this reduces to the form

$$H = \sum_{\alpha=1}^m \sum_{s=1}^n \varepsilon_{\alpha} a_{\alpha}^{s+} a_{\alpha}^s \quad (6.14)$$

for the reduced Hamiltonian. If $m = 3$ this is the appropriate anisotropic harmonic oscillator generalization of Equation 2.3. If the $\varepsilon_{\alpha} = 1$ for $1 \leq \alpha \leq m$ this result reduces to that of the isotropic harmonic oscillator.

The canonical transformation defined by Equations 6.10 is basic for the reduction of Equation 6.7 to that of Equation 6.14. It corresponds to an anisotropic scale change for the cartesian coordinates. This may most easily be seen by defining

$$q_{\alpha}^s = \sqrt{\varepsilon_{\alpha}} Q_{\alpha}^s, \quad (6.15a)$$

$$p_{\alpha}^s = P_{\alpha}^s / \sqrt{\varepsilon_{\alpha}}. \quad (6.15b)$$

Then the transformation becomes

$$Q_{\alpha}^{'s} = (1/\sqrt{2}) (\sqrt{m\omega_0} q_{\alpha}^s - i p_{\alpha}^s / \sqrt{m\omega_0}),$$

$$P_{\alpha}^{'s} = (1/\sqrt{2}) (p_{\alpha}^s / \sqrt{m\omega_0} - i \sqrt{m\omega_0} q_{\alpha}^s).$$

obtainable from Equations 6.10 directly by setting $\varepsilon_\alpha = 1$.

Equation 6.10 also yield a powerful tool with which to study the anisotropic harmonic oscillator. It is

$$[a_\alpha^{s+}, a_\beta^t] = \delta_{\alpha\beta}^{st}, \quad [a_\alpha^s, a_\beta^t] = [a_\alpha^{s+}, a_\beta^{t+}] = 0 \quad (6.16)$$

obtainable from Equation 6.12 together with the definition of the a's. Explicitly this result says that the deformed creation and annihilation operators satisfy precisely the same commutation relations as do the nondeformed creation and annihilation operators introduced in Equation 2.2. Therefore, any group theoretical results for the isotropic harmonic oscillator which depends only upon the commutation relations of the A's must also be applicable to the anisotropic harmonic oscillator. This result is basic to the development of our deformed potential many-particle theory.

6.2. Irreducible representations

In the remaining sections of this thesis we shall adopt the convention that the lower case form for a quantity introduced in Sections 2 through 5 as referring to the isotropic harmonic oscillator, refers to the anisotropic harmonic oscillator analogue of the quantity. For example, the a_α^{s+} and a_α^s introduced in Section 6.1 refer respectively to the anisotropic oscillator analogue of the A_α^{s+} and A_α^s introduced in Section 2.2.

Employing this convention, consider the operators $c_{\alpha\beta}^{st}$, $c_{\alpha\beta}$ and c^{st} defined by

$$c_{\alpha\beta}^{st} = a_{\alpha}^{s+} a_{\beta}^t, \quad (6.17)$$

$$c_{\alpha\beta} = \sum_{s=1}^n c_{\alpha\beta}^{ss} \quad (6.18)$$

and

$$c^{st} = \sum_{\alpha=1}^m c_{\alpha\alpha}^{st}. \quad (6.19)$$

From Equation 6.13 we see that these operators satisfy the commutation relations

$$[c_{\alpha\beta}^{st}, c_{\rho\sigma}^{uv}] = \delta_{\beta\rho}^{tu} c_{\alpha\sigma}^{sv} - \delta_{\sigma\alpha}^{vs} c_{\rho\beta}^{ut}, \quad (6.20)$$

$$[c_{\alpha\beta}, c_{\rho\sigma}] = \delta_{\beta\rho} c_{\alpha\sigma} - \delta_{\sigma\alpha} c_{\rho\beta} \quad (6.21)$$

and

$$[c^{st}, c^{uv}] = \delta^{tu} c^{sv} - \delta^{vs} c^{ut} \quad (6.22)$$

respectively. In addition, we have that

$$[c_{\alpha\beta}, c^{st}] = 0. \quad (6.23)$$

Therefore, in analogy with the results found for the $c_{\alpha\beta}^{st}$, $c_{\alpha\beta}$ and c^{st} in Section 3.1 these operators may also be interpreted as the infinitesimal generators of the unitary groups in the reduction $U(mn) \supset U(m) \times U(n)$. In addition,

the results of Section 3.2 are applicable for a Hilbert space representation of the Gel'fand basis vectors expressed in terms of the a_{α}^{s+} operating on the $|o\rangle$ defined by

$$a_{\alpha}^s |o\rangle = 0, \quad 1 \leq \alpha \leq m, \quad 1 \leq s \leq n. \quad (6.24)$$

Such states will be labeled by $|h_{\alpha\beta}\rangle$ or equivalently by $|g\rangle$. Since they are obtainable from the results given in Section 3.2 by simply replacing $H_{\alpha\beta}$ by $h_{\alpha\beta}$ and A_{α}^{s+} by a_{α}^{s+} , a development similar to the one given in that section would be redundant. For this reason we shall not consider the explicit construction of the $|h_{\alpha\beta}\rangle$. Notice, however, that the $|h_{\alpha\beta}\rangle$ are not equal to the $|H_{\alpha\beta}\rangle$. And in particular, notice that the $|o\rangle$ defined by Equation 6.24 is not equal to the $|O\rangle$ defined by Equation 3.17. The explicit relationship between the $|h_{\alpha\beta}\rangle$ and the $|H_{\alpha\beta}\rangle$ will be investigated further in Section 7. Let it suffice here to reiterate that although they are group theoretically equivalent they are not equal.

Outside of the definition of Equation 4.2, the results of Section 4 are purely group theoretical, depending only upon the labeling scheme involved. Therefore, if we define in analogy with Equation 4.2

$$|g;klm\rangle \equiv \int d\Omega D_{mk}^{l*}(\Omega) r(\Omega) |g\rangle \quad (6.25)$$

the results of that section are applicable to these deformed $U(3) \supset R(3)$ projected basis states. In Equation 6.25 $r(\Omega)$ is the $R(3)$ rotation operator with infinitesimal generators

$$l_{\alpha} = -i \sum_{\beta, \gamma=1}^3 \epsilon_{\alpha\beta\gamma} l_{\beta\gamma} \quad (6.26)$$

satisfying

$$[l_{\alpha}, l_{\beta}] = i l_{\gamma}, (\alpha\beta\gamma \text{ cyclic}) . \quad (6.27)$$

Notice that the l_{α} in Equation 6.26 are not the components of the physical orbital angular momentum L_{α} of Equation 2.4. Their relationship will be discussed in Section 7. The states of Equation 6.25 are however well-defined and do possess simple mathematical properties. Specifically, they are eigenstates of l^2 and l_3 . The quantum number k may in turn be given precisely the same interpretation as the K of Section 4 and is to be given by either the rule of Equation 4.10 or Equation 4.11. The transformation brackets of Section 4.2 and the normalization and overlaps of Section 4.3 are also applicable to these deformed $U(3) \supset R(3)$ projected basis states without any modification.

In a similar manner it is possible to generalize the results of Section 5 by defining in analogy with Equation 5.1

$$T(g; klm) \equiv \int d\Omega D_{mk}^{l*}(\Omega) r(\Omega) T(g) r^{-1}(\Omega) . \quad (6.28)$$

Again it is true that the l does not refer to the physical orbital angular momentum. Nevertheless, the tensors are well-defined and possess simple mathematical properties. Specifically, the $U(3) \supset R(3)$ coupling coefficients of Section 5.2 and the tensorial results of Section 5.3 are applicable to these deformed $U(3) \supset R(3)$ projected tensors without any modification.

6.3. Physical significance

Although the $|h_{\alpha\beta}\rangle$ introduced in Section 6.2 diagonalize the physical Hamiltonian H given by Equation 6.14, H is not a constant for all the $|h_{\alpha\beta}\rangle$ of a given IR of $U(3)$. This is because H is a measure of the total energy of a system and for the deformed oscillator, phonons oscillating in different cartesian directions may carry different energy quanta. The generalization of H for the nondeformed oscillator,

$$h = \sum_{\alpha=1}^3 c_{\alpha\alpha} , \quad (6.29)$$

is a constant for all the $|h_{\alpha\beta}\rangle$ of a given IR of $U(3)$. This is because it is a measure of the total number of phonons and not a measure of the total energy of the system. For example, suppose $\varepsilon_1 = \varepsilon_2 = 2\varepsilon_3$. Then for a single particle carrying two phonons of excitation H may be respectively 2, 3 or 4 corresponding to 2, 1 or 0 phonons of excitation

in the 3-direction. On the other hand, h is always equal to 2. Group theoretically this result may be stated quite simply; h is a Casimir operator for the deformed IR of $U(3)$ and H is merely a linear combination of the diagonalized infinitesimal generators. In the limit of small deformation these two approach one another. In fact by construction we have the more general result

$$\lim_{\substack{\epsilon \rightarrow 1 \\ \alpha}} |h_{\alpha\beta}\rangle = |H_{\alpha\beta}\rangle. \quad (6.30)$$

The group theoretical concept of a weight structure introduced in Section 3.3 can only be equated to the physical concept of an oscillator shell structure for small deformation. This may be seen from Figure 3 where we have plotted the relative energy levels for a single particle moving in an axially symmetric oscillator potential as a function of the oscillator deformation. In the plot we have required the product $\epsilon_1 \epsilon_2 \epsilon_3 = 1$; that is, we have set $\epsilon_1 = \epsilon_2 = e^b$ and $\epsilon_3 = e^{-2b}$. This is the so-called constant volume constraint (45) derived from the incompressibility of nuclear matter. For small b , say $b < 0.1$ a shell-like structure is seen to persist. For $b > 0.1$, however, the splitting of the levels which are degenerate at $b = 0$ becomes comparable to $\hbar\omega$ and a shell structure can no longer be assigned. Explicitly, for the case of $\epsilon_1 = \epsilon_2 = 2\epsilon_3$ considered above we see that the weight structure $\bar{u} = (2, 0, 0)$

only dictates two phonons of excitation on a single particle and clearly not a particular shell structure since the energy may be 2, 3 or 4 $\hbar\omega$. Conversely, if an energy of $2\hbar\omega$ is specified then the single particle configurations in terms of weight structures could be either $\bar{u} = (2,0,0)$ as above or $\bar{u} = (1,0,0)$ corresponding to one phonon of excitation in the 1 or 2 direction. Regardless of the deformation, however, the weight structure concept is significant in that the reduction $U(n) \supset K(n) \supset S(n)$ of Equation 3.23 leads to states of definite particle permutation symmetry.

Although the $|g;klm\rangle$ introduced in Equation 6.3 are obtained from the $|g\rangle$ these states do not diagonalize the physical Hamiltonian H given by Equation 6.14. This may most easily be seen by recognizing that although $[l_\alpha, h] = 0$ as for the isotropic oscillator, $[l_\alpha, H] \neq 0$. These states do however diagonalize l^2 and l_3 , but again these are not the physical angular momentum operators L^2 and L_3 . That is, the $|g;klm\rangle$ states diagonalize neither the physical Hamiltonian H nor the physical angular momentum operators L^2 and L_3 . For this reason they possess no obvious physical significance; they are merely the group theoretical generalization of the $|G;KLM\rangle$,

$$\lim_{\epsilon_\alpha \rightarrow 1} |g;klm\rangle = |G;KLM\rangle \quad (6.31)$$

Both sets of states, the "unphysical" $|g;klm\rangle$ and the "physical" $|G;KLM\rangle$, form a complete set and therefore either forms an adequate basis with which to study nuclear structure. Which basis is most useful is determined by the number of states required in the expansion of the nuclear wavefunction. Nuclear structure physicists have preferentially chosen the $|G;KLM\rangle$ because of the physical significance of the individual states. The $|g;klm\rangle$ should not however be excluded; they may well be the most suitable choice. This is particularly true in light of the fact that many nuclei are known to be deformed. In the next sections of this thesis we shall investigate this possibility.

7. DEFORMED BASES EXPANDED IN TERMS OF THE NONDEFORMED BASES

This section is devoted to the tedious task of expanding to first order in the deformation the deformed bases in terms of the nondeformed bases. To carry out such a task it is necessary to first of all expand the rotation operator $r(\Omega)$ introduced in Equation 6.25 in terms of the rotation operator $R(\Omega)$ introduced in Equation 2.15. This is done in Section 7.1. It is also necessary to expand the $|g\rangle$ of Section 6 in terms of the $|G\rangle$ of Section 3 and in particular the $|o\rangle$ defined by Equation 6.24 in terms of the $|O\rangle$ defined by Equation 3.17. This is done in Section 7.2. These results are then used in Sections 7.3 and 7.4 to complete the reduction of the deformed $U(3) \supset R(3)$ states in terms of the nondeformed $U(3) \supset R(3)$ states. The results serve as a basis for further study of the extent to which the deformed bases represent particle-hole mixing found to be necessary in nondeformed potential many-particle theory in order to obtain closer agreement with experiment. The results will also serve as a basis for truncating the allowable values for the core angular momentum in the phenomenological theory of Sections 8 and 9.

7.1. Operators

To expand the $r(\Omega)$ in terms of the $R(\Omega)$ first of all consider the fundamental relationships of the A 's, a 's, Q 's, P 's, q 's and p 's to one another. Using dimensionless variables we have from Equations 6.10 and 6.15 that

$$A_{\alpha}^S = (1/\sqrt{2}) (Q_{\alpha}^S + iP_{\alpha}^S) = (1/\sqrt{2}) (q_{\alpha}^S/\sqrt{\epsilon_{\alpha}} + i\sqrt{\epsilon_{\alpha}} p_{\alpha}^S) , \quad (7.1a)$$

$$A_{\alpha}^{S+} = (1/\sqrt{2}) (Q_{\alpha}^S - iP_{\alpha}^S) = (1/\sqrt{2}) (q_{\alpha}^S/\sqrt{\epsilon_{\alpha}} - i\sqrt{\epsilon_{\alpha}} p_{\alpha}^S) \quad (7.1b)$$

and

$$a_{\alpha}^S = (1/\sqrt{2}) (\sqrt{\epsilon_{\alpha}} Q_{\alpha}^S + iP_{\alpha}^S/\sqrt{\epsilon_{\alpha}}) = (1/\sqrt{2}) (q_{\alpha}^S + ip_{\alpha}^S) , \quad (7.2a)$$

$$a_{\alpha}^{S+} = (1/\sqrt{2}) (\sqrt{\epsilon_{\alpha}} Q_{\alpha}^S - iP_{\alpha}^S/\sqrt{\epsilon_{\alpha}}) = (1/\sqrt{2}) (q_{\alpha}^S - ip_{\alpha}^S) . \quad (7.2b)$$

It follows that

$$Q_{\alpha}^S = (1/\sqrt{2}) (A_{\alpha}^S + A_{\alpha}^{S+}) = (1/\sqrt{2\epsilon_{\alpha}}) (a_{\alpha}^S + a_{\alpha}^{S+}) , \quad (7.3a)$$

$$P_{\alpha}^S = (1/i) (1/\sqrt{2}) (A_{\alpha}^S - A_{\alpha}^{S+}) = (1/i) (\sqrt{\epsilon_{\alpha}}/2) (a_{\alpha}^S - a_{\alpha}^{S+}) . \quad (7.3b)$$

Letting $s(\emptyset)$ and $c(\emptyset)$ represent respectively the hyperbolic sine and the hyperbolic cosine of $\emptyset$ and defining $\epsilon_{\alpha} = e^{b_{\alpha}}$ so that the constant volume condition introduced in Section 6 becomes

$$\sum_{\alpha=1}^3 b_{\alpha} = 1 \quad (7.4)$$

we have that

$$A_{\alpha}^S = c(b_{\alpha}/2)a_{\alpha}^S - s(b_{\alpha}/2)a_{\alpha}^{S+}, \quad (7.5a)$$

$$A_{\alpha}^{S+} = c(b_{\alpha}/2)a_{\alpha}^{S+} - s(b_{\alpha}/2)a_{\alpha}^S \quad (7.5b)$$

and

$$a_{\alpha}^S = c(b_{\alpha}/2)A_{\alpha}^S + s(b_{\alpha}/2)A_{\alpha}^{S+}, \quad (7.6a)$$

$$a_{\alpha}^{S+} = c(b_{\alpha}/2)A_{\alpha}^{S+} + s(b_{\alpha}/2)A_{\alpha}^S. \quad (7.6b)$$

Notice that either of these expansions may be obtained from the other by simply interchanging upper and lower case operators and changing b_{α} to $-b_{\alpha}$. This symmetry is related to the fact that oscillator anisotropy corresponds to coordinate deformation and hence is relative to the system with respect to which the anisotropy is measured. For example, if with respect to system A, system B is oblatelly deformed then with respect to system B, system A is prolately deformed. This symmetry allows the inverse of any expansion given in the following sections to be obtained by simply interchanging upper and lower case operators and replacing b_{α} by $-b_{\alpha}$. To avoid redundancy we will therefore simply expand lower case operators in terms of upper case operators. If the inverse expansion is required it is to be understood that this symmetry must be applied to the equation used as a reference.

Furthermore, since definitions given for upper case operators in terms of the A's will apply equally well for the lower case analogue in terms of the a's, the lower case analogue will not be given explicitly. Again it is to be understood that this interchange of upper and lower case operators must be applied to the equation used as a reference.

It will also prove to be convenient to have expressions for the expansions of the C's in terms of the c's. For this we define

$$C_{\alpha\beta}^{+st} \equiv A_{\alpha}^{s+} A_{\beta}^{t+} + (1/2)\delta_{\alpha\beta}^{st}, \quad C_{\alpha\beta}^{-st} \equiv A_{\alpha}^{s-} A_{\beta}^{t-} - (1/2)\delta_{\alpha\beta}^{st} \quad (7.7a)$$

$$C_{\alpha\beta}^{+} \equiv \sum_{s=1}^n C_{\alpha\beta}^{+ss}, \quad C_{\alpha\beta}^{-} \equiv \sum_{s=1}^n C_{\alpha\beta}^{-ss} \quad (7.7b)$$

$$C^{+st} \equiv \sum_{\alpha=1}^m C_{\alpha\alpha}^{+st}, \quad C^{-st} \equiv \sum_{\alpha=1}^m C_{\alpha\alpha}^{-st} \quad (7.7c)$$

Then by direct substitution we have

$$\begin{aligned} c_{\alpha\beta}^{st} &= c(b_{\alpha}/2)c(b_{\beta}/2)C_{\alpha\beta}^{st} + c(b_{\alpha}/2)s(b_{\beta}/2)C_{\alpha\beta}^{+st} \\ &\quad + s(b_{\alpha}/2)c(b_{\beta}/2)C_{\alpha\beta}^{-st} + s(b_{\alpha}/2)s(b_{\beta}/2)(C_{\beta\alpha}^{ts} + \delta_{\alpha\beta}^{st}) \end{aligned} \quad (7.8a)$$

$$\begin{aligned} c_{\alpha\beta} &= c(b_{\alpha}/2)c(b_{\beta}/2)C_{\alpha\beta} + c(b_{\alpha}/2)s(b_{\beta}/2)C_{\alpha\beta}^{+} \\ &\quad + s(b_{\alpha}/2)c(b_{\beta}/2)C_{\alpha\beta}^{-st} + s(b_{\alpha}/2)s(b_{\beta}/2)(C_{\beta\alpha} + \delta_{\alpha\beta}) \end{aligned} \quad (7.8b)$$

$$\begin{aligned}
c^{st} = c^{st} + (1/2) \sum_{\alpha=1}^m [s(b_{\alpha})(c_{\alpha\alpha}^{+st} + c_{\alpha\alpha}^{-st}) \\
+ (c(b_{\alpha}) - 1)(c_{\alpha\alpha}^{ts} + c_{\alpha\alpha}^{st} + \delta^{st})] . \quad (7.8c)
\end{aligned}$$

In deriving these expressions we have made use of trigonometric identities for the hyperbolic sine and cosine functions (46).

Now we shall apply these results to the angular momentum operators. Adopting the summation convention we have from Equation 7.8b that

$$\begin{aligned}
l_{\alpha} &= -i\epsilon_{\alpha\beta\gamma}c_{\beta\gamma} \\
&= -i\epsilon_{\alpha\beta\gamma}\{[c(b_{\beta}/2)c(b_{\gamma}/2) - s(b_{\beta}/2)s(b_{\gamma}/2)]c_{\beta\gamma} \\
&\quad - c(b_{\beta}/2)s(b_{\gamma}/2)(c_{\gamma\beta}^{-} - c_{\beta\gamma}^{+})\} \\
&= -i\epsilon_{\alpha\beta\gamma}\{c[(b_{\beta}-b_{\gamma})/2]c_{\beta\gamma} \\
&\quad + s[(b_{\beta}-b_{\gamma})/2](1/2)(c_{\gamma\beta}^{-} - c_{\beta\gamma}^{+})\} \\
&= c[(b_{\beta}-b_{\gamma})/2]L_{\alpha} + s[(b_{\beta}-b_{\gamma})/2]N_{\alpha} , (\alpha\beta\gamma \text{ cyclic}) . \quad (7.9)
\end{aligned}$$

We have exploited the fact that $S_{\alpha} = \epsilon_{\alpha\beta\gamma}T_{\beta\gamma}$ is zero if $T_{\beta\gamma}$ is symmetric and $2T_{\beta\gamma}$ if $T_{\beta\gamma}$ is antisymmetric. Furthermore,

$$N_{\alpha} \equiv -i\delta_{\alpha\beta\gamma}(1/2)(c_{\gamma\beta}^{-} - c_{\beta\gamma}^{+}) \quad (7.10)$$

with $\delta_{\alpha\beta\gamma} = |\epsilon_{\alpha\beta\gamma}|$.

We now address ourselves to finite rotations. For this it is convenient to define

$$B_{\alpha}^s \equiv i p_{\alpha}^s, \quad (7.11a)$$

$$B_{\alpha}^{s+} \equiv Q_{\alpha}^s \quad (7.11b)$$

so that

$$[B_{\alpha}^s, B_{\beta}^{t+}] = \delta_{\alpha\beta}^{st}. \quad (7.12)$$

Then in analogy with the C's of Section 3 and the H, L's and Q's of Section 2 we define

$$D_{\alpha\beta}^{st} \equiv B_{\alpha}^{s+} B_{\beta}^t, \quad (7.13a)$$

$$D_{\alpha\beta} \equiv \sum_{s=1}^n D_{\alpha\beta}^{ss}, \quad (7.13b)$$

$$D^{st} \equiv \sum_{\alpha=1}^3 D_{\alpha\alpha}^{st}, \quad (7.13c)$$

$$K \equiv \sum_{\alpha=1}^3 D_{\alpha\alpha}, \quad (7.14)$$

$$L_{\alpha} \equiv -i\epsilon_{\alpha\beta\gamma} D_{\beta\gamma}, \quad (7.15)$$

$$P_0 \equiv 2D_{33} - D_{11} - D_{22}, \quad (7.16a)$$

$$P_{\pm 1} \equiv \mp \sqrt{3/2} [(D_{13} + D_{31}) \pm i(D_{23} + D_{32})], \quad (7.16b)$$

$$P_{\pm 2} \equiv \sqrt{3/2} [(D_{11} - D_{22}) \pm i(D_{12} + D_{21})] . \quad (7.16c)$$

In light of Equation 7.12 the results of the previous sections are equally applicable to these newly defined variables and their lower case analogues. This is simply a manifestation of the fact that the D's of Equations 7.13 are respectively the infinitesimal generators of the general linear groups in the chain $GL(mn) \supset GL(m) \times GL(n)$ (47). From Equations 7.1, 7.7 and 7.13 we have that

$$D_{\alpha\beta} + D_{\beta\alpha} = C_{\alpha\beta}^- - C_{\beta\alpha}^+ . \quad (7.17)$$

Therefore, Equation 7.10 may be written as

$$\begin{aligned} N_{\alpha} &= -i\delta_{\alpha\beta\gamma} (1/2) (D_{\beta\gamma} + D_{\gamma\beta}) \\ &= -i\delta_{\alpha\beta\gamma} D_{\beta\gamma} . \end{aligned} \quad (7.18)$$

We now specialize to the case of axial symmetry with $b_1 = b_2 = b$ and $b_3 = -2b$. We then have that

$$l_1 = c(3b/2)L_1 + s(3b/2)N_1 , \quad (7.19a)$$

$$l_2 = c(3b/2)L_2 - s(3b/2)N_2 , \quad (7.19b)$$

$$l_3 = L_3 .$$

Defining $x = -i\phi c(3b/2)$ and $y = +i\phi s(3b/2)$ the rotation operators of Equation 4.17 become

$$r_1(\emptyset) = e^{xL_1 - yN_1} , \quad (7.20a)$$

$$r_2(\emptyset) = e^{xL_2 + yN_2} , \quad (7.20b)$$

$$r_3(\emptyset) = e^{-i\emptyset L_3} . \quad (7.20c)$$

From Equation 4.16 we have that $r(\Omega) = r_3(\theta_1)r_2(\theta_2)r_3(\theta_3)$.

To expand $r(\Omega)$ in terms of $R(\Omega)$ it is therefore necessary to expand $r_2(\emptyset)$ in terms of $R_2(\emptyset)$. To do this, define

$$M_1 \equiv (-i/2)(D_{31} - D_{13}) , \quad (7.21a)$$

$$M_2 \equiv (1/2)(D_{33} - D_{11}) , \quad (7.21b)$$

$$M_3 \equiv (1/2)(D_{31} + D_{13}) , \quad (7.21c)$$

$$K_2 \equiv 4M_2 , \quad (7.22a)$$

$$L_2 \equiv 2M_1 , \quad (7.22b)$$

$$N_2 \equiv -2iM_3 . \quad (7.22c)$$

The M_α 's satisfy the $SU(2)$ commutation relations

$$[M_\alpha, M_\beta] = iM_\gamma , \quad (\alpha\beta\gamma \text{ cyclic}) \quad (7.23)$$

and therefore

$$[N_2, L_2] = K_2 , \quad (7.24a)$$

$$[K_2, L_2] = 4N_2 \quad (7.24b)$$

Defining the n -fold multicommutator of u with v by

$$(uv^n) \equiv [[\dots[[u, v], v], \dots], v] \quad (7.25)$$

we have by mathematical induction that

$$(N_2, L_2^{2n}) = 2^{2n} N_2 \quad (7.26a)$$

$$(N_2, L_2^{2n+1}) = 2^{2n} K_2 \quad (7.26b)$$

Hausdorff's theorem (48) then says that for $y < 1$

$$r_2(\emptyset) = e^{xL_2} [1 + yf(N_2, xL_2) + \dots] \quad (7.27a)$$

$$= [1 + yg(N_2, xL_2) + \dots] e^{xL_2} \quad (7.27b)$$

where

$$f(N_2, xL_2) = N_2 + (x/2!)(N_2, L_2) + (x^2/3!)(N_2, L_2^2) + \dots$$

$$g(N_2, xL_2) = f(N_2, -xL_2) \quad .$$

Considering only $f(N_2, xL_2)$ we have from Equations 7.26 that

$$f(N_2, xL_2) = [s(2x)/(2x)]N_2 + [(c(2x) - 1)/(4x)]K_2 \quad .$$

For b small so that $x = -i\emptyset$ and $y = i(3b/2)\emptyset$ we find that this reduces to

$$f(N_2, xL_2) = [(\sin\emptyset \cos\emptyset)/\emptyset]N_2 - i[(\sin^2\emptyset)/(2\emptyset)]K_2$$

yielding

$$yf(N_2, xL_2) = [i(3b/2)(\sin\theta \cos\theta)N_2 + (3b/4)(\sin^2\theta)K_2] . \quad (7.28)$$

Using Equations 7.16, 7.21 and 7.22 one may write this in terms of the P's as

$$yf(N_2, xL_2) = -\sqrt{3/8} b[\sin\theta \cos\theta (P_{+1} - P_{-1}) + (1/2)\sin^2\theta (P_{+2} + P_{-2} - \sqrt{6} P_0)] . \quad (7.29)$$

To complete the expansion of $r(\Omega)$ in terms of $R(\Omega)$ we use Equation 7.27a and the fact that

$$e^{i\theta_3 L_3} P_q e^{-i\theta_3 L_3} = P_q e^{iq\theta_3}$$

to write for small b

$$r(\Omega) = R(\Omega) \left\{ 1 - \sqrt{3/8} b[\sin\theta \cos\theta (e^{i\theta_3} P_{+1} - e^{-i\theta_3} P_{-1}) + (1/2)\sin^2\theta (e^{2i\theta_3} P_{+2} + e^{-2i\theta_3} P_{-2} - \sqrt{6} P_0)] \right\} .$$

Then by using the definition of the rotation matrices given by Rose (41) and the explicit formulae for the $d_{\mu\mu}^2$, (θ_2) given for example by Brink and Satchler (49) we obtain the form

$$\begin{aligned}
r(\Omega) &= R(\Omega) \{ 1 + (b/2) [(1 - D_{00}^{2*}(\Omega)) P_0 - D_{01}^{2*}(\Omega) P_{+1} \\
&\quad - D_{0-1}^{2*}(\Omega) P_{-1} - D_{02}^{2*}(\Omega) P_{+2} - D_{0-2}^{2*}(\Omega) P_{-2}] \} \\
&= R(\Omega) \{ 1 + (b/2) [P_0 - \sum_m P_m D_{0m}^{2*}(\Omega)] \} . \quad (7.30a)
\end{aligned}$$

In a similar fashion using Equation 7.27b one finds

$$\begin{aligned}
r(\Omega) &= \{ 1 - (b/2) [(1 - D_{00}^{2*}(\Omega) P_0 + D_{-10}^{2*}(\Omega) P_{+1} + D_{10}^{2*}(\Omega) P_{-1} \\
&\quad - D_{-20}^{2*}(\Omega) P_{+2} - D_{20}^{2*}(\Omega) P_{-2}] \} R(\Omega) \\
&= \{ 1 - (b/2) [P_0 - \sum_m (-1)^M P_{-M} D_{M0}^{2*}(\Omega)] \} R(\Omega) . \quad (7.30b)
\end{aligned}$$

These results may be put into the form

$$r(\Omega) = R(\Omega) \{ 1 + (b/2) [P_0 - R^{-1}(\Omega) P_0 R(\Omega)] \} , \quad (7.31a)$$

$$r(\Omega) = \{ 1 - (b/2) [P_0 - R(\Omega) P_0 R^{-1}(\Omega)] \} R(\Omega) . \quad (7.31b)$$

7.2. $U(3) \supset U(2) \times U(1)$ basis

As a first step in the expansion to lowest order in the deformation of the $|g\rangle$ in terms of the $|G\rangle$ consider the case of a single particle in a one dimensional oscillator, $m = n = 1$. In terms of Hermite polynomials we have

$$|n\rangle = \frac{1}{\sqrt{\epsilon/\pi}} \sqrt{1/(2^n n!)} \exp[-\epsilon Q^2/2] H_n(\sqrt{\epsilon} Q) , \quad (7.32a)$$

$$|N\rangle = \frac{4}{\pi} \sqrt{1/(2^n n!)} \exp[-Q^2/2] H_N(Q) . \quad (7.32b)$$

Therefore in the expansion

$$|n\rangle = \sum_N \langle N|n\rangle |N\rangle \quad (7.33)$$

the expansion coefficients $\langle N|n\rangle$ are given by

$$\langle N|n\rangle = \frac{4}{\pi} \sqrt{\epsilon} \sqrt{1/(\pi 2^{N+n} N! n!)} \int_{-\infty}^{\infty} dQ \exp[-(1+\epsilon)Q^2/2] H_N(Q) H_n(\sqrt{\epsilon}Q) . \quad (7.34)$$

The evaluation of the integral in Equation 7.34 is performed in Appendix C. For $\epsilon = e^b$ and α and β equal to zero or one the result is

$$\begin{aligned} \langle 2N+\alpha | 2n+\beta \rangle &= \delta_{\alpha\beta} (-1)^{N+n} \sqrt{(2N+\alpha)! (2n+\alpha)! / c(b/2)} \\ &\times 1/(c^\alpha (b/2) 2^{N+n} N! n!) F(1/2+\alpha; -N, -n; 1/2+\alpha, 1/2+\alpha; Z_1, Z_2) \end{aligned} \quad (7.35)$$

where $Z_1 = [\sqrt{\epsilon} c(b/2)]^{-1}$ and $Z_2 = \sqrt{\epsilon} [c(b/2)]^{-1}$ and F is the second Appell function as given for example by Rainville (50). For the special case of the deformed vacuum Equation 7.35 reduces to

$$\langle 2N|o\rangle = (-1)^N [\sqrt{(2N)! / c(b/2)} / (2^N N!)] [t(b/2)]^N \quad (7.36)$$

with $t(b/2) = s(b/2)/c(b/2)$. For a first order theory in the deformation we then find that Equation 7.33 together

with Equation 7.36 yields

$$\begin{aligned} |0\rangle &= |0\rangle - b/(2\sqrt{2}) |2\rangle \\ &= [1 - (b/4)c^+] |0\rangle . \end{aligned} \quad (7.37)$$

In general

$$|n\rangle = \sqrt{1/n!} (a^+)^n |0\rangle . \quad (7.38)$$

But by using the fact that $[(a^+)^n, (b/4)c^+] = 0$ to first order in b we may use Equation 7.37 to write this in the form

$$|n\rangle = \sqrt{1/n!} [1 - (b/4)c^+] (a^+)^n |0\rangle . \quad (7.39)$$

By mathematical induction we have

$$\begin{aligned} (a^+)^n |0\rangle &= [(A^+)^n + (b/4)n(n-1)(A^+)^{n-2}] |0\rangle \\ &= [1 + (b/4)c^-] (A^+)^n |0\rangle . \end{aligned} \quad (7.40)$$

Equation 7.39 may therefore be further simplified to

$$\begin{aligned} |n\rangle &= [1 + (b/4)(c^- - c^+)] |N\rangle \\ &= [1 + (b/2)D] |N\rangle \end{aligned} \quad (7.41)$$

with $n = N$. Here we have used the result of Equation 7.17 to express $c^- - c^+ = 2D$.

Now consider the expansion to first order in the deformation of the $|g\rangle$ in terms of the $|G\rangle$. By definition

$$|o\rangle = \prod_{\alpha=1}^m \prod_{s=1}^n |o\rangle_{\alpha}^s .$$

Applying the result of Equation 7.37 to this expression we have that

$$\begin{aligned} |o\rangle &= \prod_{\alpha=1}^m \prod_{s=1}^n [1 - (b_{\alpha}/4) c_{\alpha\alpha}^{+ss}] |o\rangle_{\alpha}^s \\ &= [1 - \sum_{\alpha=1}^m \sum_{s=1}^n (b_{\alpha}/4) c_{\alpha\alpha}^{+ss}] |o\rangle \\ &= [1 - (b/4) o^+] |o\rangle \end{aligned} \quad (7.42)$$

in analogy with Equation 7.37. Here we have set

$$\begin{aligned} o^+ &= \sum_{\alpha=1}^m \sum_{s=1}^n (b_{\alpha}/b) c_{\alpha\alpha}^{+ss} \\ &= \sum_{\alpha=1}^m (b_{\alpha}/b) c_{\alpha\alpha}^+ . \end{aligned} \quad (7.43)$$

In general

$$|g\rangle = f(a^+) |o\rangle . \quad (7.44)$$

But in analogy with Equation 7.39 we may use the fact that $[f(a^+), (b/4) o^+] = 0$ to first order in b together with Equation 7.42 to write this in the form

$$|g\rangle = [1 - (b/4) o^+] f(a^+) |o\rangle . \quad (7.45)$$

The function $f(a^+)$ is a homogeneous polynomial in the a_{α}^{s+} .

Applying the result of Equation 7.40 to an arbitrary term in $f(a^+)$ we see that

$$\begin{aligned}
 \prod_{\alpha=1}^m \prod_{s=1}^n (a_{\alpha}^{s+})^{n_{\alpha}^s} |0\rangle &= \prod_{\alpha=1}^m \prod_{s=1}^n [1 + (b_{\alpha}/4) C_{\alpha\alpha}^{-ss}] (A_{\alpha}^{s+})^{n_{\alpha}^s} |0\rangle \\
 &= [1 + \sum_{\alpha=1}^m \sum_{s=1}^n (b_{\alpha}/4) C_{\alpha\alpha}^{-ss}] \prod_{\alpha=1}^m \prod_{s=1}^n (A_{\alpha}^{s+})^{n_{\alpha}^s} |0\rangle \\
 &= [1 + (b/4) O^-] \prod_{\alpha=1}^m \prod_{s=1}^n (A_{\alpha}^{s+})^{n_{\alpha}^s} |0\rangle \tag{7.46}
 \end{aligned}$$

where

$$\begin{aligned}
 O^- &= \sum_{\alpha=1}^m \sum_{s=1}^n (b_{\alpha}/b) C_{\alpha\alpha}^{-ss} \\
 &= \sum_{\alpha=1}^m (b_{\alpha}/b) C_{\alpha\alpha}^- . \tag{7.47}
 \end{aligned}$$

Therefore it must be true that

$$f(a^+) |0\rangle = [1 + (b/4) O^-] f(A^+) |0\rangle . \tag{7.48}$$

Using this result we may further simplify Equation 7.45 to

$$\begin{aligned}
 |g\rangle &= [1 + (b/4) (O^- - O^+)] |G\rangle \\
 &= [1 + (b/2) D] |G\rangle \tag{7.49}
 \end{aligned}$$

with $g = G$. Here we have used the result of Equation 7.17 to write

$$\begin{aligned}
D &= \sum_{\alpha=1}^m (b_{\alpha}/b) (c_{\alpha\alpha}^{-} - c_{\alpha\alpha}^{+}) \\
&= 2 \sum_{\alpha=1}^m (b_{\alpha}/b) D_{\alpha\alpha} .
\end{aligned} \tag{7.50}$$

For $m = 3$ with axial symmetry Equation 7.49 reduces to simply

$$|g\rangle = [1 - (b/2)P_0]|G\rangle \tag{7.51}$$

with P_0 given by Equation 7.16a.

7.3. Mixed bases

In this section we shall adhere strictly to the convention that upper and lower case symbols refer respectively to nondeformed and deformed quantities. For example, the state labeled by $|g;KLM\rangle$ refers to the state obtained by application of the nondeformed angular momentum projection operator on the deformed cartesian basis state $|g\rangle$. Explicitly

$$|g;KLM\rangle \equiv \int d\Omega D_{MK}^{L*}(\Omega) R(\Omega) |g\rangle \tag{7.52}$$

Furthermore, we shall indicate equality of the upper and lower case labels by letting $g = G$, $k = K$, $l = L$ and $m = M$. Using these conventions and the results of Sections 7.1 and 7.2 it is then a simple matter to give an expression for the expansion to first order in the deformation of the $|g;KLM\rangle$ and the $|G;KLM\rangle$. To see this we use Equation 7.51 to write

Equation 7.52 in the form

$$|g;KLM\rangle = |G;KLM\rangle - (b/2) \int d\Omega D_{MK}^{L*}(\Omega) R(\Omega) P_0 |G\rangle . \quad (7.53)$$

Writing

$$\begin{aligned} R(\Omega) P_0 &= R(\Omega) P_0 R^{-1}(\Omega) R(\Omega) \\ &= \sum_{M'} D_{M'0}^2(\Omega) P_{M'} R(\Omega) \\ &= \sum_{M'} (-1)^{M'} D_{M'0}^{2*}(\Omega) P_{-M'} R(\Omega) \end{aligned}$$

and using the $R(3)$ Clebsch-Gordan series for rotation matrices we see that Equation 7.53 becomes

$$\begin{aligned} |g;KLM\rangle &= |G;KLM\rangle - (b/2) \sum_{M'} (-1)^{M'} P_{-M'} \\ &\quad \times \int d\Omega D_{MK}^{L*}(\Omega) D_{M'0}^{2*}(\Omega) R(\Omega) |G\rangle \\ &= |G;KLM\rangle - (b/2) \sum_{L'M'} (-1)^{M'} P_{-M'} C(L2L'; MM', M+M') \\ &\quad \times C(L2L'; KOK) |G;KL', M+M'\rangle \end{aligned} \quad (7.54)$$

In a similar fashion by applying the results of Equation 7.30b we have that

$$\begin{aligned} |g;KLM\rangle &= [1 - (b/2) P_0] |g;klm\rangle + (b/2) \sum_{m'} (-1)^{m'} P_{-m'} \\ &\quad \times \int d\Omega D_{mk}^{1*}(\Omega) D_{m'0}^{2*}(\Omega) r(\Omega) |g\rangle \end{aligned}$$

$$\begin{aligned}
&= [1 - (b/2)P_0] |g; klm\rangle + (b/2) \sum_{l'm'} (-1)^{m'} P_{-m'} \\
&\quad \times C(121'; mm', m+m') C(121'; k0k) |g; kl', m+m'\rangle \quad (7.55)
\end{aligned}$$

Applying the results of Section 5 then completes the expansion.

7.4. $U(3) \supset R(3)$ basis

To obtain the expansion of the $|g; klm\rangle$ in terms of the $|G; KLM\rangle$ we must apply the results of both Equations 7.31b and 7.51. In particular, again letting $g = G$, $k = K$, $l = L$ and $m = M$ we have

$$\begin{aligned}
|g; klm\rangle &= \int d\Omega D_{mk}^{1*}(\Omega) r(\Omega) |g\rangle \\
&= \int d\Omega D_{mk}^{1*}(\Omega) \{1 - (b/2)[P_0 - R(\Omega)P_0R^{-1}(\Omega)]\} R(\Omega) \\
&\quad \times \{[1 - (b/2)P_0]R^{-1}(\Omega)R(\Omega)\} |G\rangle \\
&= \int d\Omega D_{mk}^{1*}(\Omega) [1 - (b/2)P_0] R(\Omega) |G\rangle \\
&= [1 - (b/2)P_0] |G; KLM\rangle \quad (7.56)
\end{aligned}$$

Apply the results of Section 5 again completes the expansion.

Notice that the expansions of Equations 7.51, 7.54, 7.55 and 7.56 all involve only the second rank spherical tensors

P_μ defined by Equations 7.16. These correspond to $L = 2$ and from Equation 7.17 are seen to correspond to linear combination of the $C_{\alpha\beta}^- - C_{\beta\alpha}^+$. The P_μ are therefore linear combinations of $(2,0)$ and $(0,2)$ $SU(3)$ tensors. Although we shall not consider their tensorial properties here, Biedenharn and Louck (32) have given the mathematical tools to do so. Physically, however, we now see explicitly that a particular deformed basis state corresponds to the same nondeformed state plus a particular particle-hole mixture of the same parity dictated by the deformation. Such particle-hole mixing has been used within the framework of nondeformed potential many-particle theory to obtain theoretical results in closer agreement with experiment. The deformed potential many-particle states already include such mixing and possess in addition simple state labels. The fact that these states are indeed physically significant remains to be shown. We shall do this in Section 9 for O^{18} .

8. PHENOMENOLOGY

A first step in testing to see whether or not the deformed potential many-particle states are in fact physically significant is to apply the phenomenological technique introduced by Nilsson (16) for the case of a single particle moving in a deformed potential to the case of one or more particles moving in a deformed potential. That is, to consider the motion of one or more particles moving in an axially symmetric deformed potential, describable in terms of the deformed potential many-particle states, coupled to that of a deformed core which itself is allowed to rotate adiabatically. This section is therefore devoted to first of all describing the Hamiltonian for such a system and then obtaining the appropriate overall wavefunctions. A diagonalization technique for the Hamiltonian in the basis obtained is also presented.

8.1. Hamiltonian

The Hamiltonian for a physical system described by collective and particle coordinates may be written in the form

$$H = H_c + H_p \quad (8.1)$$

where H_c is the collective Hamiltonian and H_p is the particle Hamiltonian. For a system of particles moving in a

potential created by a deformed core which itself is allowed to rotate adiabatically, H_c is simply the rigid rotor Hamiltonian for the core given in terms of body-fixed coordinates by

$$H_c = \sum_{\alpha=1}^3 L_{\alpha}^2 / (2\mathcal{I}_{\alpha}) \quad (8.2)$$

and H_p is the deformed particle Hamiltonian plus an appropriate particle-particle interaction,

$$H_p = \sum_{\alpha=1}^3 \varepsilon_{\alpha} c_{\alpha\alpha} + V. \quad (8.3)$$

This is the strong coupling limit (51) and the usual approach used is to set $\bar{I} = \bar{L} + \bar{j}$ and write Equation 8.2 in the form

$$\begin{aligned} H_c &= \sum_{\alpha=1}^3 (I_{\alpha} - j_{\alpha})^2 / (2\mathcal{I}_{\alpha}) \\ &= \sum_{\alpha=1}^3 (I_{\alpha}^2 + j_{\alpha}^2 - 2I_{\alpha}j_{\alpha}) / (2\mathcal{I}_{\alpha}) \end{aligned}$$

with $\bar{I}$, $\bar{L}$ and $\bar{j}$ being the total, core and particle angular momentum respectively. Then in the limit of axial symmetry so that j_3 , the projection of $\bar{j}$ on the body-fixed symmetry axis, is a constant of the motion and $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I} \neq \mathcal{I}_3$, H can be written in the form

$$H = H_I + H_{RPC} + H_0 \quad (8.4)$$

with

$$H_I = (I^2 - I_3^2 - j_3^2)/(2\ell) + (I_3 - j_3)^2/(2\ell_3) ,$$

$$H_{RPC} = (I_1 j_1 + I_2 j_2)/(\ell) ,$$

$$H_0 = H_p + (j^2)/(2\ell) .$$

H_I is called the I-rotor for the system, H_{RPC} the rotational-particle-coupling term and H_0 the new particle Hamiltonian.

The Hamiltonian given by Equation 8.4 is then diagonalized in the appropriately symmetrized basis (51),

$$| |K|IM|m\rangle = \left[\frac{2I + 1}{16\pi^2 (1 + \delta_{K0} \delta_{m0})} \right]^{1/2} \\ \times [D_{MK}^{I*}(\Omega)\chi_m + (-1)^{I+j+K+m} D_{M-K}^{I*}(\Omega)\chi_{-m}] , \quad (8.5)$$

treating H_{RPC} as a perturbation. Here the $D_{MK}^{I*}(\Omega)$ are the rotation matrices of Rose (41) and the χ 's are the eigenstates of H_0 . Notice that L is not a constant of the motion in this basis. This is because $\bar{L}$ and $\bar{j}$ are not explicitly coupled up to $\bar{I}$. The basis is suitable for the Hamiltonian $H = H_I + H_0$. The part of the Hamiltonian constraining $\bar{I} = \bar{L} + \bar{j}$ is H_{RPC} and this has been thrown into the residual interaction. Working within this basis one is therefore forced to allow L to take on all values consistent with $|I-j| \leq L \leq I+j$.

In reality however, the core Hamiltonian represents the motion of a closed shell nuclear system constrained to move in its ground state. Since the core is deformed we know from the formalism of Sections 6 and 7 together with the nondeformed analogue that this is the deformed 0^+ ground state. Furthermore, we also know from Section 7 that this deformed 0^+ ground state corresponds to $L = 0^+$ and $L = 2^+$ to first order in the deformation. Allowing the core to have higher values of L corresponds to treating the core to a higher degree of approximation in the deformation. Since one is usually content with treating the particles outside the core to first order in the deformation to be consistent one must allow only the $L = 0^+$ and $L = 2^+$ states of the core to mix in. As has been pointed out, this is not possible in the basis given by Equation 8.5. It is necessary to explicitly couple $\bar{L}$ and $\bar{j}$ to get $\bar{I}$ if the values of L are to be truncated. This we will do in the next section.

8.2. Coupled basis

The appropriate wavefunctions for describing the rotation of a deformed nuclear core are those of the symmetric top given in normalized form by (52)

$$|KLM\rangle = [(2L+1)/(8\pi^2)]^{1/2} D_{MK}^{L*}(\Omega) . \quad (8.6)$$

As in Equation 8.5 the $D_{MK}^{L*}(\Omega)$ are the rotation matrices

given by Rose (41), L is the total angular momentum associated with the rotation, M refers to the space-fixed z -axis projection of $\bar{L}$ and K refers to the body-fixed 3-axis projection of $\bar{L}$. For axial symmetry about the body-fixed 3-axis these functions are eigenstates of the core Hamiltonian. However, because in this case the Hamiltonian is invariant to π rotations about the principle axes these wavefunctions must be appropriately symmetrized. To demonstrate the symmetrization technique let Π_α be the operator for a π rotation about the α -axis. These three operators together with the identity form a representation of the point group D_2 (53). Of the four symmetry types for D_2 , only the A type symmetry has an $L=0$ state and since all even-even nuclei have $L=0$ ground states we shall adopt this symmetry type for the wavefunctions; that is, we require

$$\Pi_\alpha |K|LM\rangle = |K|LM\rangle, \quad 1 \leq \alpha \leq 3. \quad (8.7)$$

Since $\Pi_\alpha \Pi_\beta = \Pi_\gamma$, $\alpha \neq \beta \neq \gamma$, we need only consider the effect of two of the Π_α 's on the states $|KLM\rangle$; we shall choose to consider Π_2 and Π_3 . Using the properties of the rotation matrices we have that

$$\Pi_2 |KLM\rangle = e^{-i\pi(L-K)} |-KLM\rangle, \quad (8.8a)$$

$$\Pi_3 |KLM\rangle = e^{-i\pi K} |KLM\rangle. \quad (8.8b)$$

Consider then

$$||K|LM\rangle = N[|KLM\rangle + \alpha|-KLM\rangle] .$$

We apply Π_2 and Π_3 to obtain

$$\Pi_2 ||K|LM\rangle = N\alpha e^{-i\pi(L+K)} [|KLM\rangle + (1/\alpha)e^{2i\pi K}|-KLM\rangle] ,$$

$$\Pi_3 ||K|LM\rangle = Ne^{-i\pi K} [|KLM\rangle + \alpha e^{2i\pi K}|-KLM\rangle] .$$

From Equation 8.7 we then see that K must be an even integer and that $\alpha = e^{i\pi L} = (-1)^L$. The appropriately symmetrized wavefunctions are therefore given by

$$||K|LM\rangle = N[|KLM\rangle + (-1)^L|-KLM\rangle]$$

with K an even integer and the obvious requirement that if K is zero L must be even. To determine the normalization N we take the inner product to obtain

$$1 = N^2 2(1 + \delta_{KO})$$

so that

$$N = 1/\sqrt{2(1+\delta_{KO})} .$$

The final form for the normalized and symmetrized wavefunctions is then

$$||K|LM\rangle = 1/\sqrt{2(1+\delta_{KO})} [|KLM\rangle + (-1)^L|-KLM\rangle] . \quad (8.9)$$

For arbitrary core deformation the eigenstates of the core

Hamiltonian are simply linear combinations of these functions with L and M fixed.

To the core states given by Equation 8.6 we must couple states describing the motion of the particles. Since the particles are moving in the potential created by the core the appropriate particle states are the $|g;klm\rangle$ of Section 6 with the b_α dictated by the core deformation. To these we must couple the conjugate spin-isospin states $|\tilde{g};sm\rangle$ to obtain states of the form $|gklsjm\rangle$ (22). From Equations 7.19 it is seen that l is not the physical orbital angular momentum of the particles so this is not the usual l - s coupling used in nuclear physics. However, for a first order theory in the deformation we see from Equation 7.56 that l may be considered to be the physical orbital angular momentum of the particles insofar as the energy matrix is concerned. We shall therefore use the $|gklsjm\rangle$ as if they are states of total physical angular momentum j for the particles with projection m on the body-fixed 3-axis.

The product states,

$$|gklsjm\rangle_B |KLM\rangle, \quad (8.10)$$

with $|KLM\rangle$ given by Equation 8.1 and $|gklsjm\rangle_B$ the particle wavefunction in the body-fixed system form a complete basis for the composite system of the core plus particles. However, such a basis does not exploit the fact that the total

angular momentum I and its projection M on the space-fixed z -axis must be constants of the motion. Furthermore, notice that for axial symmetry both $|m|$ and $|K|$ must remain good quantum numbers (16). Figure 3 describes this situation diagrammatically. Appropriately coupled wavefunctions may be obtained by coupling in the space-fixed system the $|KLM\rangle$ of the core to the $|gklsjm\rangle_S$ of the particle given by

$$\begin{aligned} |gklsjm\rangle_S &= R^{-1}(\Omega) |gklsjm\rangle_B \\ &= \sum_{m'} D_{mm'}^{j*}(\Omega) |gklsjm'\rangle_B \end{aligned} \quad (8.11)$$

in which Ω specifies the Euler angles of the body-fixed system with respect to the space-fixed system. That is (54),

$$|gklsj;KL;IM\rangle = \sum_m C(jLI;m,M-m,M) |gklsjm\rangle_S |KLM-m\rangle. \quad (8.12)$$

This result may be put into the form of Equation 8.10 by using Equations 8.6 and 8.11 together with the $R(3)$ Clebsch-Gordan series for the rotation matrices and the symmetry and unitarity properties of the $R(3)$ Clebsch-Gordan coefficients.

$$\begin{aligned} |gklsj;KL;IM\rangle &= \sum_{mm'} C(jIL;m,M-m,M) |gklsjm'\rangle_B \\ &\times \left[(2L+1)/(8\pi^2) \right]^{1/2} D_{mm'}^{j*}(\Omega) D_{M-mK}^{L*}(\Omega) \end{aligned}$$

$$\begin{aligned}
&= \sum_{I'mm'} C(jLI;m,M-m,M) C(jLI';m,M-m,M) C(jLI';m'K,m'+K) \\
&\quad \times [(2L+1)/(8\pi^2)]^{1/2} |gklsjm\rangle_B D_{Mm'+K}^{I'*}(\Omega) \\
&= \sum_m C(jLI;m,K,m+K) [(2L+1)/(2I+1)]^{1/2} \\
&\quad \times |gklsjm\rangle_B |m+K,IM\rangle \\
&= \sum_m (-1)^{j-m} C(jIL;m,-m-K,-K) |gklsjm\rangle_B |m+K,IM\rangle . \quad (8.13)
\end{aligned}$$

For the case of axial symmetry about the body-fixed 3-axis the Hamiltonian of the system is again invariant to π rotations about the three body-fixed axes. The wavefunctions of Equation 8.13 must therefore be appropriately symmetrized. To do this we need in addition to Equations 8.8 the fact that

$$\Pi_2 |gklsjm\rangle_B = e^{-i\pi(j-m)} |gklsj-m\rangle_B , \quad (8.14a)$$

$$\Pi_3 |gklsjm\rangle_B = e^{-i\pi m} |gklsjm\rangle_B . \quad (8.14b)$$

Consider then

$$\begin{aligned}
|gklsj|m|;|m+K|IM\rangle &= N[|gklsjm\rangle_B |m+K,IM\rangle \\
&\quad + \alpha |gklsj-m\rangle_B |-m-K,IM\rangle .
\end{aligned}$$

Again requiring that the wavefunctions transform according to the A type symmetry of the point group D_2 we find that $K + 2m$ must be an even integer and that $\alpha = e^{i\pi(I+j)} = (-1)^{I+j}$

with $I + j$ an integer. Furthermore, if both m and K are zero $I + j$ must be an even integer. We also find that $N = \sqrt{2(1+\delta_{m0}\delta_{K0})}$ so that the final form for the normalized and symmetrized wavefunctions is

$$\begin{aligned} |gklsj|m|; |m+K|IM\rangle = & 1/\sqrt{2(1+\delta_{m0}\delta_{K0})} [|gklsjm\rangle_B |m+K, IM\rangle \\ & + (-1)^{I+j} |gklsj-m\rangle_B |-m-K, IM\rangle. \end{aligned} \quad (8.15)$$

This basis diagonalizes the following operators with their corresponding eigenvalues:

$$I^2 \sim I(I+1)$$

$$I_z \sim M$$

$$I_3^2 \sim (m+K)^2$$

$$j^2 \sim j(j+1)$$

$$j_3^2 \sim (m)^2$$

$$s^2 \sim s(s+1)$$

$$l^2 \sim l(l+1)$$

$$\text{Casimir operators for } U(3) \sim [h_\alpha].$$

For arbitrary deformation the eigenstates of the Hamiltonian will be linear combinations of these functions with I and M fixed.

Now consider the symmetrization of the coupled wavefunctions of Equation 8.13. We need the fact that

$$\Pi_2 |gklsj; KL; IM\rangle = (-1)^{j+L-I} e^{-i\pi(I-j-K)} |gklsj; -KL; IM\rangle , \quad (8.16a)$$

$$\Pi_3 |gklsj; KL; IM\rangle = e^{-i\pi(K+2m)} |gklsj; KL; IM\rangle . \quad (8.16b)$$

Consider then

$$|gklsj; |K|L; IM\rangle = N[|gklsj; KL; IM\rangle + \alpha |gklsj; -KL; IM\rangle] .$$

Once again requiring the wavefunctions to transform according to the A type symmetry of the point group D_2 we find that $K + 2m$ must be an even integer and that

$$\alpha = e^{i\pi(I+j)} (-1)^{I-j-L} = (-1)^{2I-L}$$

with $I + j$ an integer. Furthermore, if K is zero then $2I - L$ must be an even integer. We also find that $N = 1/\sqrt{2(1+\delta_{K0})}$ so that the final form for the normalized and symmetrized wavefunctions is

$$\begin{aligned} |gklsj; |K|L; IM\rangle &= 1/\sqrt{2(1+\delta_{K0})} [|gklsj; KL; IM\rangle \\ &+ (-1)^{2I-L} |gklsj; -KL; IM\rangle] . \end{aligned} \quad (8.17)$$

A somewhat more convenient form for these coupled wavefunctions may be obtained by expressing them in terms of the $|gklsj| m |; |m+K| IM\rangle$ of Equation 8.15. Explicitly,

$$\begin{aligned}
|gklsj; |K|L; IM\rangle &= 1/\sqrt{(1+\delta_{K0})} \sum_m (-1)^{j-m} C(jIL; m, -m-K, -K) \\
&\times \sqrt{1+\delta_{K0}\delta_{m0}} |gklsj|m|; |m+K|IM\rangle .
\end{aligned} \tag{8.18}$$

This basis diagonalizes the following operators with their corresponding eigenvalues:

$$I^2 \sim I(I+1)$$

$$I_z \sim M$$

$$L^2 \sim L(L+1)$$

$$L_3^2 \sim (K)^2$$

$$j^2 \sim j(j+1)$$

$$s^2 \sim s(s+1)$$

$$l^2 \sim l(l+1)$$

$$\text{Casimir operators for } U(3) \sim [h_\alpha] .$$

Again for arbitrary deformation the eigenstates of the Hamiltonian will be linear combinations of these functions with I and M fixed. Notice that since a quantum mechanical body cannot rotate about a symmetry axis one normally chooses $K = 0$ in Equations 8.15 and 8.18 (54). As may be seen from Equation 8.15 we may then restrict $m \geq 0$ with the constraint that if m is zero $I + j$ must be

even. In this limit Equation 8.18 reduces to

$$\begin{aligned}
 |gklsj;OL;IM\rangle &= 1/\sqrt{2} \sum_m (-1)^{j-m} C(jIL;m,-m,0) \\
 &\times \sqrt{1+\delta_{m0}} |gklsj|m|;|m|IM\rangle \\
 &= \sum_{m \geq 0} (-1)^{j-m} \sqrt{2/(1+\delta_{m0})} C(jIL;M,-m,0) \\
 &\times |gklsj|m|;|m|IM\rangle .
 \end{aligned} \tag{8.19}$$

To be sure both the basis given by Equation 8.15 and the basis given by Equation 8.18 are complete and therefore must yield the same results for the energy matrix diagonalization if the complete bases are used. However, as pointed out in Section 7.1, to be consistent in working to only first order in the deformation L must be truncated at $L = 0$ and $L = 2$. Only the basis given by Equation 8.18 which explicitly demonstrates L as a good quantum number allows this to be done. Therefore the $|gklsj;|K|L;IM\rangle$ with $L = 0$ and $L = 2$ form the appropriately coupled basis in terms of which to diagonalize the Hamiltonian for the system.

8.3. Matrix diagonalization

In this section we shall consider the diagonalization of the Hamiltonian H given by Equation 8.1 together with Equation 8.2 and 8.3. Only the special case of axial symmetry with $\mathcal{L}_3 \ll \mathcal{L} = \mathcal{L}_1 = \mathcal{L}_2$ and $\varepsilon_1 = \varepsilon_2 = e^b$, $\varepsilon_3 = e^{-2b}$

will be treated. The appropriate wavefunctions are given by Equation 8.19 in which K taken to be zero reflects the fact that $\mathcal{L}_3$ is small compared to $\mathcal{L}$. The diagonalization procedure is complicated by the fact pointed out in Section 1 and again in Section 4.3 that the particle wavefunctions are not necessarily orthogonal on the k quantum number. It is therefore necessary to consider the diagonalization of H in a nonorthogonal basis. In general this leads to the eigenvalue problem for non-Hermitian matrices, a nontrivial problem (55). However, for the $U(3) \supset R(3)$ states it is possible to avoid this difficulty by using the results of Section 4.3. The simplification is based upon the fact that Equation 4.32 together with Equation 4.26, 4.27 and 4.28 allows one to determine explicitly the elements in the overlap matrix and hence the fact that the matrix so formed is itself Hermitian. To demonstrate the simplification explicitly, consider the model problem of nonorthogonal states $|\beta\rangle$ whose overlap matrix $O_{\beta'\beta} = \langle\beta'| \beta\rangle$ is known to be Hermitian. Expand these states $|\beta\rangle$ in terms of some orthonormal basis $|\alpha\rangle$ spanning the representation space; that is,

$$|\beta\rangle = \sum_{\alpha} T_{\alpha\beta} |\alpha\rangle ,$$

$$\langle\beta| = \sum_{\alpha} T_{\beta\alpha}^+ \langle\alpha| .$$

Then form the Hermitian overlap matrix

$$O = T^+ T .$$

Let S be the unitary matrix which diagonalizes O ,

$$S^+ O S = E ,$$

and define

$$R_{\alpha\beta} = S_{\alpha\beta} / \sqrt{E_{\beta}} .$$

Then

$$R^+ O R = I = (T^+)^{-1} O T^{-1}$$

and we may choose

$$T = R^{-1}$$

or

$$R = T^{-1} .$$

From this we then have

$$|\alpha\rangle = \sum_{\beta} R_{\beta\alpha} |\beta\rangle .$$

It follows that

$$H(\alpha) = R^+ H(\beta) R$$

where $H(\alpha)$ is the energy matrix in the $|\alpha\rangle$ basis and hence Hermitian and $H(\beta)$ is the energy matrix in the $|\beta\rangle$ basis and in general non-Hermitian. To be sure this is equivalent to a Gram-Schmidt orthogonalization process but the technique

is somewhat simpler to use computationally since matrix multiplications and inversions are standard procedures.

It is to be emphasized that this procedure requires the overlap matrix be known and that it be Hermitian. As pointed out in Section 4.3 the overlap matrix for the $U(3)$ $R(3)$ basis states may be obtained from the transformation brackets. In fact, using the phase convention indicated in Section 4.2 following Equation 4.28 the overlap matrix is seen to be real Hermitian and hence real symmetric. It can therefore be diagonalized by the standard Jacobi routine for real symmetric matrices (56). Therefore, although the $|g;klm\rangle$ states are in general nonorthogonal on the quantum number k , this simple procedure allow one to solve for the energy matrix between such states in a simple and direct manner.

To complete the diagonalization then we must first of all obtain the overlap matrix elements for the wavefunctions given by Equation 8.19. To do this consider the overlap matrix elements for the $|gklsj|m|;|m|IM\rangle$ given by Equation 8.15. Using the fact that

$$|gklsjm\rangle_B = \sum_{m'} C(1s_j; m', m-m', m) |g;klm'\rangle_B |\tilde{g}sm-m'\rangle_B$$

together with the unitarity of the $R(3)$ Clebsch-Gordan coefficients we have that

$$\begin{aligned}
& \langle gk'lsj|m|; |m|IM|gklsj|m|; |m|IM\rangle \\
&= 1/[2(1+\delta_{m0})] \{ [\langle gk'lsjm|_B \langle mIM| \\
&\quad + (-1)^{I+j} \langle gk'lsj-m|_B \langle -mIM|] [|gklsjm\rangle_B |mIM\rangle \\
&\quad + (-1)^{I+j} |gklsj-m\rangle_B |mIM\rangle] \} \\
&= \sum_{m'} C^2(ls j; m', m-m', m) \langle g; k' l m' | g; k l m' \rangle \\
&= A(gk'; gk) \tag{8.20}
\end{aligned}$$

with $A(gk'; gk)$ given by Equation 4.32. It then follows that the overlap matrix elements for the wavefunctions given by Equation 8.19 are

$$\begin{aligned}
\langle gk'lsj; OL; IM | gklsj; OL; IM \rangle &= \sum_{\substack{m' \geq 0 \\ m \geq 0}} 2/\sqrt{(1+\delta_{m',0})(1+\delta_{m,0})} \\
&\times (-1)^{2j-m-m'} C(jIL; m, -m, 0) C(jIL; m', -m', 0) \\
&\times \langle gk'lsj|m'|j|m'|IM|gklsj|m|j|m|IM\rangle \\
&= A(gk'; gk) \sum_{m \geq 0} 2/(1+\delta_{m,0}) C^2(jIL; m, -m, 0) \\
&= A(gk'; gk). \tag{8.21}
\end{aligned}$$

Next consider the matrix elements of the collective Hamiltonian. Since

$$\begin{aligned}
H_C |gklsj; OL; IM\rangle &= [(L^2 - L_3^2)/(2\ell) + L_3^2/(2\ell_3)] \\
&\times |gklsj; OL; IM\rangle \\
&= L(L+1)/(2\ell) |gklsj; OL; IM\rangle
\end{aligned}$$

we have quite simply that

$$\begin{aligned}
\langle gk'lsj; OL; IM | H_C | gklsj; OL; IM \rangle \\
= L(L+1)/(2\ell) A(gk'; gk)
\end{aligned} \tag{8.22}$$

with $A(gk'; gk)$ again given by Equation 4.32.

Finally we must consider the matrix elements of the particle Hamiltonian. This is a somewhat more difficult task because as pointed out in Section 6.3 H_p is not diagonal in the $|g; klm\rangle$ basis. To find the matrix elements of H_p it is therefore necessary to apply the results of Section 4.2. Explicitly, for the particle Hamiltonian given by $H_p = \sum_{\alpha} \epsilon_{\alpha} C_{\alpha\alpha}$ we have that

$$\begin{aligned}
\langle gk'l'm | H_p | gklm \rangle &= \sum_{g'g''} \langle g' | g; klm \rangle \langle g''; gk'l'm \rangle^* \langle g'' | H_p | g' \rangle \\
&= \sum_{g'} \langle g' | g; klm \rangle \langle g' | g; k'l'm \rangle^* (\epsilon_{\alpha} w'_{\alpha})
\end{aligned} \tag{8.23}$$

Here the $\langle g' | g; klm \rangle$ are the transformation brackets given by Equation 4.14 and the w'_{α} are the components of the weight vector for the state $|g'\rangle$ given explicitly in terms of the $h_{\alpha\beta}$ by Equation 3.14. We may now use this result to find the matrix elements of H_p between the states given by

Equation 8.15.

$$\begin{aligned}
& \langle gk'l'sj'|m|; |m|IM|H_p|gklsj|m|; |m|IM\rangle \\
&= 1/[2(1+\delta_{m0})] \{ [\langle gk'l'sj'm|_B \langle mIM| \\
&\quad + (-1)^{I+j} \langle gk'l'j'-m|_B \langle -mIM|] H_p [|gklsjm\rangle_B |mIM\rangle \\
&\quad + (-1)^{I+j} |gklsj-m\rangle_B | -mIM\rangle] \} \\
&= \langle gk'l'sj'm|H_p|gklsjm\rangle \\
&= \sum_{m'} C(l'sj'; m', m-m', m) C(lsj; m', m-m', m) \\
&\quad \times \langle gk'l'm'|H_p|gklm'\rangle . \tag{8.24}
\end{aligned}$$

In deriving this result we have made use of Equation 8.14a to relate $|gklsj-m\rangle_B$ to $|gklsjm\rangle_B$. Using this result together with Equation 8.19 we finally have

$$\begin{aligned}
& \langle gk'l'sj'; OL; IM|H_p|gklsj; OL; IM\rangle \\
&= (-1)^{j'+j''} \sum_{m \geq 0} 2/(1+\delta_{m0}) \\
&\quad \times C(j'IL'; m, -m, 0) C(jIL; m, -m, 0) \\
&\quad \times \langle gk'l'sj'|m|; |m|IM|H_p|gklsj|m|; |m|IM\rangle . \tag{8.25}
\end{aligned}$$

Inserting the results of Equation 8.23 into Equation 8.24 and this in turn into Equation 8.25 yields the matrix elements of H_p in the basis given by Equation 8.19.

We have therefore found the $H(\beta)$ and the overlap matrix from which R and R^+ may be found by diagonalization. Diagonalizing $H(\alpha) = R^+ H(\beta) R$ then yields the energy eigenvalues and the corresponding eigenvectors. Notice that the results of Equations 8.24 and 8.25 are completely general. Equation 8.23 however applies only to the deformed particle Hamiltonian. This result must be generalized for H_p given by Equation 8.3.

9. AN EXAMPLE: OXYGEN 18

In this section we shall test the formalism developed in the previous sections by applying the results to the $A = 18$ system treating O^{18} as having two particles moving in an axially symmetric deformed potential created by the deformed O^{16} core which is allowed to rotate adiabatically. The angular momentum of the core is restricted to $L = 0$ and $L = 2$ to be consistent with a first order theory in the deformation for the particles.

9.1. Hamiltonian and coupled basis

The Hamiltonian which we shall diagonalize is given explicitly by Equation 8.1 with

$$H_c = L^2 / (2\mathcal{I}) \quad (9.1a)$$

and

$$H_p = \hbar\omega [e^b (c_{11} + c_{22}) + e^{-2b} (c_{33})] - V_0 \delta_{j0} \quad (9.1b)$$

In units of the $\hbar\omega$ these reduce to

$$H_c = RL^2 \quad (9.2a)$$

and

$$H_p = e^b (c_{11} + c_{22}) + e^{-2b} (c_{33}) - P \delta_{j0} \quad (9.2b)$$

where R is the rotational parameter, P the pairing parameter

and b the deformation parameter. The term $V_0 \delta_{j0}$ in Equation 9.1b is a pairing interaction acting only within particle states having $j = 0$. In a multipole expansion of a particle-particle interaction it corresponds to the short range effects of the multipoles having $l > 2$ (57). The long range effects associated with the $l = 2$ term in such an expansion promote collective effects and hence produce deformation (5, 22, 57); this we have already included. Since the two produce opposite effects on the energy levels of a system one expects a larger V_0 to be required for the deformed potential many-particle states than for the comparable non-deformed potential many-particle theory. We shall see that this is indeed found to be the case for O^{18} .

As pointed out in Section 8.2 the appropriate basis to use if one is to truncate the allowable values of the core angular momentum is the one given by Equation 8.19. Since the Hamiltonian we are considering is the type given by Equation 8.1 together with Equations 8.2 and 8.3 the formalism presented in Section 8.3 is applicable. However, it is necessary to consider the decoupling of the particle orbital angular momentum and the particle spin-isospin coordinates. In the general case of n particles this is not a trivial task (22). However, for the case of two particles it is straightforward. In fact, most of the work has already been done in Section 3.4 where we saw that the $[4]$ and $[2,2]$ IR of

U(3) correspond to symmetric space wavefunctions and the [3,1] IR of U(3) corresponds to antisymmetric space wavefunctions. The overall antisymmetry requirement of the Pauli principle then requires antisymmetric spin-isospin functions to be associated with the [4] and [2,2] IR of U(3) and symmetric functions to be associated with the [3,1] IR of U(3). For the low-lying states in O^{18} the isospin must be equal to one and therefore $s = 0$ for the [4] and [2,2] IR of U(3) and $s = 1$ for the [3,2] IR of U(3). That is, the appropriate wavefunctions for O^{18} are given by

$$|[4]k101;OL;IM\rangle, \quad (9.3a)$$

$$|[3,1]k11j;OL;IM\rangle, \quad (9.3b)$$

$$|[2,2]k101;OL;IM\rangle \quad (9.3c)$$

with k and l given by the rule of Equation 4.11 and $L = 0$ or $L = 2$. The angular momenta j and I may take on the values $|1-s| \leq j \leq 1+s$ and $|L-j| \leq I \leq L+j$ respectively. Notice that for these IR of U(3) there are no multiple occurrences of l so that the k label is redundant and the states are orthonormal. That is, standard diagonalization procedures are applicable.

9.2. Results

Although this section is devoted to reporting the results of diagonalizing the Hamiltonian given by Equation 8.1 together with Equations 9.2, first of all consider

Figure 5 which is given for completeness. It shows the energy levels as a function of deformation for the diagonalization of the two particle Hamiltonian H_p given by Equation 9.2b with $P = 0$ in the deformed IR [4], [3,1] and [2,2] of $U(3)$. The curves are simply $(n_1 + n_2)e^b + (n_3)e^{-2b}$ with $0 \leq n_\alpha \leq 4$ indicating the number of quanta in the α direction. These are the levels which when appropriately summed and truncated couple to $L = 0$ or $L = 2$ to give the states of good total orbital angular momentum indicated in Equations 9.3.

The results of the diagonalization of the phenomenological Hamiltonian H given by Equation 8.1 together with Equations 9.2 for O^{18} are indicated in Figures 6 through 11 where we have plotted the energy eigenvalues as a function of the deformation, $-0.08 \leq b \leq 0.08$, systematically varying the rotational parameter R and the pairing parameter P from graph to graph. Fourteen of the sixteen levels given belong to the [4] IR of $U(3)$. The other two levels corresponding to the second 0^+ and the third 2^+ level in the experimental spectrum (58) given in Figure 12b belong to the [3,1] IR of $U(3)$. These we have indicated by broken curves. The base of these two levels has been adjusted so as to yield a best fit at the values of the parameters b , R and P determined from a best fit for the levels of the [4] IR. This corresponds to accepting the hypothesis of Elliott

(5, 22) that the more symmetric an IR of $U(3)$ is the lower it will lie in energy. The scaling factor must be proportional to some function of the Casimir operators for the group. One such function has been shown by Elliott to correspond to a particle quadrupole-quadrupole interaction (22). We shall not consider this further except to reiterate that the base of the two levels taken from the $[3,1]$ IR of $U(3)$ has been adjusted so as to yield a best fit at the values of the parameters b , R and P determined from a best fit for the levels of the $[4]$ IR.

Specifically consider Figures 6, 7 and 8 where we have fixed $P = 0.2$ and have varied R from $R = 0.015$ to $R = 0.025$ in steps of 0.005 . Notice that at zero deformation as R increases, the ground state 0^+ level and the 2^+ and 4^+ levels degenerate at $E/\hbar\omega = 4$ remain fixed. These levels correspond to $L = 0$. The remaining levels correspond to $L = 2$ and are shifted upward by the factor $6 \times (R_{\text{final}} - R_{\text{initial}})$ in accordance with Equation 9.2a. At nonzero deformation a corresponding change in the splitting of the levels can be detected. In general, at a fixed value of nonzero deformation increasing R shifts all the levels upward. The upper levels are shifted to a lesser degree at $b \neq 0$ than at $b = 0$ because the deformation allows the $L = 0$ and $L = 2$ levels to mix. The strength of the mixing determines the magnitude of the shifts and is dependent upon the levels

involved. A particularly clear example of this is the second 2^+ and the first 4^+ levels for positive deformation. The 2^+ is pulled upward more strongly with increasing R than is the 4^+ .

Next consider Figures 9 and 10 together with Figure 7. Here we have fixed $R = 0.020$ and have varied P from $P = 0.15$ to $P = 0.25$ in steps of 0.05 . At zero deformation as P increases the only levels affected are the ground state 0^+ and the first 2^+ state. These correspond to the $j = 0$ states for $L = 0$ and $L = 2$ respectively. For nonzero deformation a corresponding change can be detected. In general at a fixed value of nonzero deformation increasing P shifts all the levels downward. The ground state 0^+ and the first 2^+ state are shifted to a lesser degree at $b \neq 0$ than at $b = 0$. This is because the deformation allows the levels which are shifted at $b = 0$ and levels which are not shifted at $b = 0$ to mix. Again note that the magnitude of the shifts is dependent upon the levels involved. This may be most clearly seen for the second 2^+ level and the first 4^+ level. With increased P at fixed b these levels move more closely together.

By varying the parameters b , P and R in a systematic manner the final values of $b = 0.021$, $P = 0.222$ and $R = 0.0216$ were obtained as giving a best overall fit to the experimental spectrum. Figure 11 is a plot similar to

those of Figures 6 through 10 at these values for P and R. The value of $b = 0.021$ is indicated by the vertical line marked final value. Figure 12c is a bar graph of the same results multiplied by the scale factor $\hbar\omega = 16$ MeV to which the fit obtained was insensitive. That is, a comparable fit could have been obtained with $\hbar\omega$ in the range $12 \leq \hbar\omega \leq 20$ MeV. It is to be noted that a least squares fit has not been carried out. We have simply obtained the results given by assigning a tolerance of 0 - 3.5% on the first five levels and 3.5 - 7.0% on the remaining levels. This is consistent with the fact that experimentally the lower levels are known more accurately than are the upper levels. Indeed, had we neglected the levels above 5 MeV we could have given a perfect fit to the lowest five experimental levels. It is to be emphasized that the second 0^+ level and the third 2^+ level have been obtained from the states of the $[3,1]$ IR of $U(3)$ given by Equation 9.3b. The other levels have been obtained from the states of the $[4]$ IR of $U(3)$ given by Equation 9.3a.

The final values obtained for the parameters P, R and $\hbar\omega$ are amenable to simple physical interpretations. The value of the pairing energy parameter is 3.55 MeV. This is 1 - 1.5 MeV larger than the usual pairing energy parameter but as pointed out in Section 9.1 this is indicative of the fact that we have incorporated deformation into our theory. The value of the rotational energy parameter is 0.346 MeV.

Simply applying the empirical $L(L+1)$ rule (51) to the first excited state of O^{18} implies a rotational energy parameter of 0.33 MeV. The energy parameter of 16 MeV can be compared to the empirical $41A^{-1/3}$ rule (51) which yields a value of 15.8 MeV for O^{18} . To be sure these do not prove the correctness of the theoretical results but it is reassuring to know the results obtained are not in disagreement with such simple physical interpretations.

The bar graph of Figure 12 gives in addition to the experimental spectrum and the spectrum obtained from the deformed $U(3)$ calculation, the spectrum obtained by de Llano, Mello, Chacón and Flores (59) for a nondeformed theoretical $U(3)$ calculation taking into account the quadrupole-quadrupole, pairing and spin-orbit interactions. The striking contrast in these results compared with those obtained for the deformed $U(3)$ states would seem to indicate that the deformed $U(3)$ states are indeed physically significant!

10. SUMMARY AND CONCLUSION

The basic aim of this thesis has been to extend the ideas introduced by Nilsson (16) and Newton (17) for the case of a single particle moving in a deformed potential to the case of one or more particles moving in a deformed potential. This we have done by first of all establishing the relationship between the $U(3) \supset R(3)$ state labeling scheme introduced by Elliott (5) and the Gel'fand $U(3) \supset U(2) \times U(1)$ state labeling scheme introduced by Nagel and Moshinsky (12). The establishment of this relationship then allowed a rule complementary to the one given by Elliott (5) for the quantum K required in the reduction $U(3) \supset R(3)$ to be given and the use of this K in state labeling to be clarified. An explicit expression for the transformation brackets between the two state labeling schemes was then derived and the results related to the normalization and overlap integrals for the projected states. The transformation brackets were then used to obtain an expression for both the $U(3) \supset R(3)$ coupling coefficients and $U(3) \supset R(3)$ projected tensor matrix elements.

Having established the necessary mathematical tools we then made use of the invariance of the fundamental commutators of the infinitesimal generators of $U(3)$ to a

canonical transformation in order to generalize the results to a deformed potential many-particle theory. An expansion of the deformed potential many-particle states in terms of the nondeformed potential many-particle states was then made. The results were used as a basis for truncating the allowable values of the angular momentum of the core in a phenomenological description of nucleons moving in a deformed potential created by a deformed core which itself was allowed to rotate adiabatically. An application to the $A = 18$ system was made, treating O^{18} as having two particles moving in a deformed potential created by the O^{16} core. The results obtained from the calculations indicated that the deformed potential many-particle states are in fact physically significant in the sense that the eigenvalue spectrum that resulted was in excellent agreement with the experimental spectrum.

In each stage of the development we attempted to present as general a discussion as possible, considering limiting cases only when mathematical complexity became prohibitive. Nevertheless, there remains several areas in which additional research effort may be profitably applied. The first of these was presented in Section 3.3 and involves the group theoretical problem of reducing $U(n)$ with respect to $S(n)$. Additional labels must be obtained to distinguish multiple occurrences of a given IR of $S(n)$ occurring in a

given IR of $U(n)$. As indicated in Section 3.3 the problem is not trivial but its general solution is a prerequisite to solving the problem of n particles moving in a harmonic oscillator potential.

An explicit form for the transformation brackets between the $U(3) \supset R(3)$ and the $U(3) \supset U(2) \times U(1)$ state labeling schemes was derived in Section 4.2. However, in programming the results for computational purposes there appeared to be a high degree of symmetry among the coefficients within an IR of $U(3)$. Furthermore, most of the coefficients were found to be zero with the nonzero coefficients being simple ratios of integers and the square roots of simple ratios of integers. This would seem to suggest that a thorough study of the symmetry properties of the transformation brackets would lead to a reduction in the complexity of the expression given by Equation 4.26 and perhaps even a reduction to a simple algebraic expression.

Another group theoretical problem which merits consideration is that of $U(m)$ fractional parentage coefficients. This is intimately associated with the more general problem of $U(m)$ irreducible tensors introduced in Section 5.3 for the case of $U(3)$. These must be considered if one is to investigate particle-particle interactions known to be of importance in nuclear structure calculations

from l-s and j-j coupling shell model results. In fact, the whole area of inner and outer tensorial products promises to be a most interesting study. To be sure such a study has been initiated by Biedenharn and Louck (23) but there exists the possibility of expanding their results into a generalized Schwinger (10) and Bargmann (11) algebra.

In expanding the deformed potential many-particle states in terms of the nondeformed potential many-particle states the operators $D_{\alpha\beta}$ introduced in Section 7.1 were identified as the infinitesimal generators of the general linear group in three dimensions. Furthermore, Equation 7.23 demonstrated that in the reduction $GL(3) \supset GL(2) \times GL(1)$ one of the infinitesimal generators of $GL(2)$ was equivalent to an angular momentum operator. It therefore appears that the general linear groups play an important role in the expansion of the deformed potential many-particle states in terms of the nondeformed potential many-particle states. Since ultimately one would like to have such an expansion to all orders in the deformation an investigation of this role played by the general linear groups merits consideration. The expansion itself is of course required to determine the strength and the precise manner in which particle-hole interactions enter into the deformed potential many-particle theory. In fact, even for the first order expansion given in Section 7.3 it would be well to complete the

reduction and compare the results in a specific case with particle-hole calculations for the nondeformed potential many-particle theory.

As stated at the beginning of Section 8 the phenomenological theory presented is admittedly a simple first attempt to test the deformed potential many-particle states. Such things as nonaxially symmetric deformations, vibrations of the neutron-proton fluid, rotation-vibration couplings and sophisticated particle-particle interactions have not been included. The results of the simple theory for the case of O^{18} indicate that such amplifications of the theory merit consideration. Even in the simple case of O^{18} additional testing of the deformed potential many-particle states may be obtained by calculating electromagnetic transition probabilities between various energy levels. Furthermore, as a glance at the table of nuclides will indicate, O^{18} is not the only system with which to test the theory!

11. APPENDIX A

To evaluate the integral I_1 defined by Equation 4.27 consider

$$I(p, q, 1) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi (\cos\phi)^p (\sin\phi)^q e^{il\phi} . \quad (A.1)$$

By applying Cauchy's residue theorem one can obtain the result

$$I(p, q, 1) = \frac{(-i)^q}{2^{p+q}} \sum_n (-1)^n \begin{pmatrix} p \\ s-n \end{pmatrix} \begin{pmatrix} q \\ n \end{pmatrix} \quad (A.2)$$

where $s = (p+q+1)/2$ must be integral for I to be nonzero.

Then by using the result given by Edmonds (60) for the

$d_{m'm}^{j(\text{Edmonds})}(-\phi) = d_{m'm}^{j(\text{Rose})}(\phi)$ one obtains

$$\begin{aligned} I_1(m'jml) &= (-1)^{j-m} \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!} \right]^{1/2} \sum_n (-1)^n \begin{pmatrix} j-m \\ n \end{pmatrix} \begin{pmatrix} j+m \\ j-m'-n \end{pmatrix} \\ &\times \frac{1}{2\pi} \int_0^{2\pi} d\phi (\cos\phi)^{2n+m+m'} (\sin\phi)^{2j-m-m'-2n} e^{il\phi} \\ &= (-1)^{j-m} \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!} \right]^{1/2} \sum_n (-1)^n \begin{pmatrix} j-m \\ n \end{pmatrix} \begin{pmatrix} j+m \\ j-m'-n \end{pmatrix} \\ &\times I(2n+m+m', 2j-m-m'-2n, 1) . \end{aligned} \quad (A.3)$$

12. APPENDIX B

To evaluate the integral I_2 defined by Equation 4.28 consider

$$I(p, q) \equiv \frac{1}{2} \int_0^{\pi/2} d\phi (\cos\phi)^p (\sin\phi)^q. \quad (B.1)$$

In terms of gamma functions one has (46)

$$I(p, q) = \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{q+1}{2})}{4 \Gamma(\frac{p+q}{2} + 1)}. \quad (B.2)$$

Then by once again using the results given by Edmonds (60) one obtains

$$\begin{aligned} I_2(m' j m n' l n) &= (-1)^{j-m'+l-n} \left[\frac{(j+m')! (j-m')! (l+n')! (l-n')!}{(j+m)! (j-m)! (l+n)! (l-n)!} \right]^{1/2} \\ &\times \sum_s (-1)^s \begin{pmatrix} j-m \\ s \end{pmatrix} \begin{pmatrix} j+m \\ j-m'-s \end{pmatrix} \sum_t (-1)^t \begin{pmatrix} l-n \\ t \end{pmatrix} \begin{pmatrix} l+n \\ l-n'-t \end{pmatrix} \\ &\times \frac{1}{2} \int_0^{\pi} d\phi (\cos\phi)^{2s+m+m'} (\sin\phi)^{2j-m-m'-2s+1} \\ &\times (\cos\phi/2)^{2t+n+n'} (\sin\phi/2)^{2l-n-n'-2t} \end{aligned}$$

$$= (-1)^{j-m'+l-n} 2^{2j-m-m'+2} \left[\frac{(j+m')!(j-m')!(l+n')!(l-n')!}{(j+m)!(j-m)!(l+n)!(l-n)!} \right]^{1/2}$$

$$\times \sum_s (-1)^s 2^{-2s} \binom{j-m}{s} \binom{j+m}{j-m'-s} \sum_t (-1)^t \binom{l-n}{t} \binom{l+n}{l-n'-t}$$

$$\times \sum_r (-1)^r \binom{2s+m+m'}{r} I[2(j+s+t-r)+m+m'+n+n'+1 ,$$

$$2(j-s+l-t+r)-m-m'-n-n'+1] . \quad (B.3)$$

13. APPENDIX C

To evaluate the integral $\langle N|n \rangle$ defined by Equation 7.34 consider

$$I(\alpha, r) \equiv \int_{-\infty}^{\infty} dx \exp[-\alpha^2 x^2] x^r$$

$$= \begin{cases} \sqrt{\pi} (1/2, r/2) / \alpha^{r+1}, & r \text{ even} \\ 0 & r \text{ odd} \end{cases}. \quad (C.1)$$

The Appell symbol (61) (a, k) is given by

$$(a, k) = a(a+1)\dots(a+k-1).$$

Using the polynomial expansion for the Hermite polynomials as given for example by Lebedev (62) and Equation C.1 we have for α and β equal to zero or one

$$\begin{aligned} \langle 2N+\alpha | 2n+\beta \rangle &= (-1)^{N+n} \frac{1}{\sqrt{\pi}} \frac{2^{2N+2n+\alpha+\beta} (2N+\alpha)! (2n+\beta)! c(b/2)}{\sqrt{\pi} 2^{2N+2n+\alpha+\beta} (1/2+\alpha, N) (1/2+\beta, n) [c(b/2)]^{(\alpha+\beta)/2}} \\ &\times \sum_{p=0}^N \sum_{q=0}^n \frac{((1+\alpha+\beta)/2, p+q) (-N, p) (-n, q) \sqrt{\pi} \epsilon^q}{((1+\alpha+\beta)/2, p) ((1+\alpha+\beta)/2, q) p! q! [\sqrt{\epsilon} c(b/2)]^{p+q}} \delta_{\alpha\beta} \\ &= \delta_{\alpha\beta} (-1)^{N+n} 2^{N+n} (1/2+\alpha, N) (1/2+\alpha, n) \sqrt{1/[(2N+\alpha)! (2n+\alpha)! c(b/2)]} \\ &\times [1/c(b/2)]^\alpha \\ &\sum_{p=0}^N \sum_{q=0}^n \frac{(1/2+\alpha, p+q) (-N, p) (-n, q)}{(1/2+\alpha, p) (1/2+\alpha, q) p! q!} [\sqrt{\epsilon} c(b/2)]^{-p} [c(b/2)/\sqrt{\epsilon}]^{-q} \end{aligned}$$

$$\begin{aligned}
&= \delta_{\alpha\beta} (-1)^{N+n} \sqrt{(2N+\alpha)! (2n+\alpha)! / c(b/2)} / (c^\alpha (b/2) 2^{N+n} N! n!) \\
&\quad \times F(1/2+\alpha; -N, -n; 1/2+\alpha, 1/2+\alpha; z_1, z_2) . \tag{C.2}
\end{aligned}$$

In Equation C.2 we have set

$$\varepsilon = e^b$$

$$z_1 = [\sqrt{\varepsilon} c(b/2)]^{-1}$$

and

$$z_2 = \sqrt{\varepsilon} [c(b/2)]^{-1} .$$

14. APPENDIX D TABLES

Table 1. Values of L missing from the $IR(\lambda_\mu)$ of $SU(3)$ when neither λ nor μ are zero

Case	λ	μ	Missing values of L
(i)	Odd	Odd	0
(ii)	Even	Even	1
(iii)	Odd	Even	0
	Even	Odd	

Table 2. Number of occurrences of L in the $IR(7,5)$ of $SU(3)$

L_0	$d_{(\lambda_\mu)}(L_0)$	$n_{(\lambda_\mu)}(L_0)$
12	1	1
11	2	1
10	4	2
9	6	2
8	9	3
7	12	3
6	15	3
5	18	3
4	20	2
3	22	2
2	23	1
1	24	1
0	24	0

APPENDIX E FIGURES

Figure 2. Degeneracy diagram for the IR $(7,5)$ of $SU(3)$. $L = 1, 2, 3^2, 4^2, 5^3, 6^3, 7^3, 8^3, 9^2, 10^2, 11, 12$.

Figure 3. Energy levels as a function of deformation for a single particle moving in an axially symmetric deformed harmonic oscillator potential. The equations of the lines are given by $m[\exp(b)] + n[\exp(-2b)]$ for $m + n = 1, 2$ and 3 .

Figure 4. Angular momentum coupling relations for a deformed core plus particle Hamiltonian.

$E/\hbar\omega$

Figure 5. Energy levels as a function of deformation for two particles moving in an axially symmetric deformed harmonic oscillator potential. The equations of the lines are given by $m[\exp(b)] + n[\exp(-2b)]$ for $m + n = 4$.

Figure 6. Energy levels as a function of deformation for the core plus two particle Hamiltonian of Oxygen 18. The angular momentum of the core has been restricted to $L=0$ and $L=2$. The deformed particle states belonging to the $[4]$ (solid curves) and the $[3,1]$ (broken curves) IR of $U(3)$ have been used.

Figure 7. Energy levels as a function of deformation for the core plus two particle Hamiltonian of Oxygen 18. The angular momentum of the core has been restricted to $L=0$ and $L=2$. The deformed particle states belonging to the [4] (solid curves) and the [3,1] (broken curves) IR of $U(3)$ have been used.

Figure 8. Energy levels as a function of deformation for the core plus two particle Hamiltonian of Oxygen 18. The angular momentum of the core has been restricted to $L=0$ and $L=2$. The deformed particle states belonging to the $[4]$ (solid curves) and the $[3,1]$ (broken curves) IR of $U(3)$ have been used.

Figure 9. Energy levels as a function of deformation for the core plus two particle Hamiltonian of Oxygen 18. The angular momentum of the core has been restricted to $L=0$ and $L=2$. The deformed particle states belonging to the [4] (solid curves) and the [3,1] (broken curves) IR of $U(3)$ have been used.

Figure 10. Energy levels as a function of deformation for the core plus two particle Hamiltonian of Oxygen 18. The angular momentum of the core has been restricted to $L=0$ and $L=2$. The deformed particle states belonging to the $[4]$ (solid curves) and the $[3,1]$ (broken curves) IR of $U(3)$ have been used.

Figure 11. Energy levels as a function of deformation for the core plus two particle Hamiltonian of Oxygen 18. The angular momentum of the core has been restricted to $L=0$ and $L=2$. The deformed particle states belonging to the [4] (solid curves) and the [3,1] (broken curves) IR of $U(3)$ have been used.

Figure 12. Energy eigenvalue spectrum determined by a nondeformed $U(3)$ calculation (a), experimentally (b) and by a deformed $U(3)$ calculation (c). Results of (a) are taken from the work of de Llano, Mello, Chacón and Flores (59). In (c) the second 0^+ and the third 2^+ states belong to the $[3,1]$ IR of $U(3)$ and the other states belong to the $[4]$ IR of $U(3)$.

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