FIELD CALCULATION ALGORITHM FOR GENERAL BEAM DISTRIBUTIONS*

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This note is a summary of the algorithm used in our recent paper on the coherent beam-beam interaction.¹ The goal of the work was to study coherent beam-beam interactions, and the development needed was to avoid using the expression for the beam-beam generated fields that assumes a Gaussian beam distribution.² Because of our interest in round beams the algorithm was developed for use with beams that are approximately round.³ We believe there is a similar approach for flat beams, but the Fourier series discussed below probably isn't the right basis for expansion.

The ultrarelativistic limit is assumed; the beam produced fields are calculated by Lorentz transforming to the rest frame of a bunch and calculating the electrostatic field there. In that frame the beam density is treated as having no longitudinal variation. A two-dimensional circular mesh is constructed. Particles are cast onto that mesh and fields calculated at points on the mesh. The mesh has step size Δr in radius and $\Delta\theta$ in azimuth. For most of our work

$$\Delta r = \frac{(\sigma_{x0} + \sigma_{y0})}{2 \times 10}, \quad \Delta\theta = \frac{2\pi}{16} \quad (1)$$

where σ_{x0} and σ_{y0} are the nominal horizontal and vertical sizes, respectively. Define $r_n = n\Delta r$ ($n = 0, \dots, N$) and $\phi_k = k\Delta\theta$ ($k = 0, \dots, K-1$). We usually used $N = 100$ to avoid overflowing the mesh. Particles are accumulated at points (r_n, ϕ_k) . Radial electric fields are calculated at points $(r_n + \Delta r/2, \phi_k)$, and azimuthal electric fields at points $(r_n, \phi_k + \Delta\theta/2)$.

First test particles are cast onto the mesh. A test particle with coordinates (r, θ) falls in the range $r_n \leq r < r_{n+1}$ and $\phi_k \leq \theta < \phi_{k+1}$, is apportioned to the adjacent mesh points in the following fractions

$$\begin{aligned} f_{n,k} &= \frac{(\phi_{k+1} - \theta)}{\Delta\theta} \times \frac{r}{4r_{n+1}} \left[3 - \frac{r}{r_{n+1}} \right] \\ f_{n,k+1} &= \frac{(\theta - \phi_k)}{\Delta\theta} \times \frac{r}{4r_{n+1}} \left[3 - \frac{r}{r_{n+1}} \right] \\ f_{n+1,k} &= \frac{(\phi_{k+1} - \theta)}{\Delta\theta} \times \left[1 - \frac{r}{4r_{n+1}} \left[3 - \frac{r}{r_{n+1}} \right] \right] \end{aligned} \quad (2)$$

1. S. Krishnagopal and R. Siemann, "The Coherent Beam-Beam Interaction", submitted to the 1991 PAC.

2. Bassetti and Erskine

3. Earlier work restricted to round beams in (S. Krishnagopal, PhD Thesis, Cornell University, Jan, 1991) showed the same phenomena as this more general algorithm, and our motivation was to study whether restricting beams to be round was critical.

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OCT 30 1991

$$f_{n+1,k+1} = \frac{(\theta - \phi_k)}{\Delta\theta} \times \left[1 - \frac{r}{4r_{n+1}} \left[3 - \frac{r}{r_{n+1}} \right] \right]$$

The azimuthal weights are just proportional to the distance from ϕ_k and ϕ_{k+1} . The radial weights come from the following argument based on Gauss' Law. Assume that P particles uniformly populated the region $r_n \leq r < r_{n+1}$, and choose a weighting function of the form

$$w_n(r) = a_0 + a_1 r + a_2 r^2 \quad (3)$$

where $w_n(r)$ is the fraction assigned to the mesh at r_n . The coefficients a_0 , a_1 , and a_2 are to be such that the radial field at $r_n + \Delta r/2$ is the same as it would be if the region were occupied by a constant, uniform charge density $\lambda = P/[\pi(r_{n+1}^2 - r_n^2)]$. From Gauss' Law that is true if the total charge enclosed out to $r_n + \Delta r/2$ is the same. Equating the two equations for charge gives

$$\lambda \pi [(r_n + \Delta r/2)^2 - r_n^2] = \sum_{p=1}^P (a_0 + a_1 r_p + a_2 r_p^2) = P(a_0 + a_1 \langle r \rangle + a_2 \langle r^2 \rangle) \quad (4)$$

This equation can be solved by equating the coefficients multiplying Δr , $(\Delta r)^2$, and $(\Delta r)^3$ on the right and left hand sides. The result is the weights given in eq (2) above. When $r \gg \Delta r$, roughly half the charge is placed at $r = r_n$ and half at $r = r_{n+1}$, and for $r \leq \Delta r$ it assigns 50% more charge to $r = 0$ than a uniform distribution would.

Here the possible purposes of this document become mixed-up. We could write about what we did as a record or what we should have done as a guide for future work. The description above is the historical record; i.e. it is what we did do.

The weighting in r evolved as we were learning about the effects of statistical fluctuations on field calculations and results, and it is what we used in our first papers on coherent beam-beam effects. It was left in the code after better ways to control fluctuations were found; these are described below. The justifications for the validity are the comparisons presented in the figures in this paper and the fact that our coherent beam-beam results were tested and found to be insensitive to Δr for $\Delta r = (\sigma_{x0} + \sigma_{y0})/20$ (the nominal), $(\sigma_{x0} + \sigma_{y0})/10$, and $(\sigma_{x0} + \sigma_{y0})/40$. It is our intention to return to the standard "area weighting" as described in Birdsall and Langdon;⁴ however, this has not been tested at this time. If that were done the apportioning would be done as

$$f_{n,k} = \frac{(\phi_{k+1} - \theta)}{\Delta\theta} \times \left\{ \frac{r_{n+1}}{r_{n+1} - r_n} - \frac{r^2}{2(r_{n+1} - r_n)^2} \right\}$$

4. C. K. Birdsall and A. B. Langdon, Plasma Physics via Computer Simulation (New York, McGraw-Hill, 1985).

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$$f_{n,k+1} = \frac{(\phi - \phi_k)}{\Delta\phi} \times \left\{ \frac{r_{n+1}^2}{r_{n+1}^2 - r_n^2} - \frac{r_n^2}{r_{n+1}^2 - r_n^2} \right\} \quad (2')$$

$$f_{n+1,k} = \frac{(\phi_{k+1} - \phi)}{\Delta\phi} \times \left\{ \frac{r_{n+1}^2}{r_{n+1}^2 - r_n^2} - \frac{r_n^2}{r_{n+1}^2 - r_n^2} \right\}$$

$$f_{n+1,k+1} = \frac{(\phi - \phi_k)}{\Delta\phi} \times \left\{ \frac{r_{n+1}^2}{r_{n+1}^2 - r_n^2} - \frac{r_n^2}{r_{n+1}^2 - r_n^2} \right\}$$

An array $m(n,k)$ containing the numbers of test particles accumulated at (r_n, ϕ_k) results when all the test particles have been cast onto the mesh. An array $M(n) = \sum_k m(n,k)$ is constructed, also. The approach to calculating the fields is:

- 1) Use the $M(n)$'s and Gauss' Law to calculate the radial field produced by the average charge. This is the field that a round beam would produce.
- 2) Fourier analyze $m(n,k)$ and use the Green's function for Poisson's equation in polar coordinates⁵ to calculate the radial and azimuthal electric fields produced by the departures from round.

There are statistical fluctuations in the $m(n,k)$'s and $M(n)$'s, and learning to handle these fluctuations was the most time consuming part of developing this algorithm. The rms error of $M(n)$ is approximately $\sqrt{M(n)}$,⁶ and the statistical weight of $M(n)$ in a fit is $1/\sigma^2 = 1/M(n)$. The average charge at radius r is approximated by

$$\langle Q(r) \rangle = \exp(-r^2/2\sigma_0^2) \sum_{j=0}^J c_j p_j(r) \quad (5)$$

where $\sigma_0^2 = 1/2(\sigma_{x0}^2 + \sigma_{y0}^2)$, the c_j and $p_j(r)$ are orthogonal polynomials ($p_j(r)$ is of degree j) determined by least squares fitting of $M(n)$ using Forsythe's method,⁷ and the number of terms in the series, J , is determined by the requirement $\chi^2/\text{degree-of-freedom} = 1$. Usually charges did not cover the full mesh; of the N radial bins, roughly the last $N/2$ were empty. The fit extended out to and included the first empty bin only. Forsythe's method is a way of smoothing data without the restriction of equal weights. The polynomials are determined by the weights and the values of r_n 's, and the coefficients by

5. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, (New York, McGraw-Hill, 1953), p. 1188.
6. The correlation between the $M(n)$'s introduced by the charge casting procedure are ignored.
7. L. G. Kelly, Handbook of Numerical Methods and Applications (Reading, Mass: Addison-Wesley, 1967), p. 68

$M(n)$, the weights, and the r_n 's. The polynomials are orthogonal, so they can be added one at a time without refitting, and the number of polynomials can be increased until the χ^2 condition is satisfied. The IMSL routine RFORP does the fitting, and IMSL routine RCASP is used to evaluate the results of the fit.⁸ Once $\langle Q(r) \rangle$ has been determined, the radial electric field is

$$E_r(r + \Delta r/2) = \frac{1}{2\pi\epsilon_0} \frac{1}{(r_n + \Delta r/2)} \sum_{l=0}^n \langle Q(r_l) \rangle \quad (6)$$

An example of a fit for a Gaussian bunch is in figure 1, and figure 2 shows the average charge for a bunch undergoing coherent oscillations. That takes care of the average charge.

Figure 1: The fit of the average charge of a Gaussian bunch represented by 10^4 test particles with $\sigma_{y0} = \sigma_{x0}/2$. $\sigma_{x0} = 54.77 \mu\text{m}$.

Figure 2: The average charge for a bunch undergoing coherent oscillations.

The $m(n,k)$'s are Fourier analyzed using IMSL FFT routines,⁹ and the resulting Fourier coefficients are fit with Forsythe's method. The errors of the Fourier coefficients are estimated as follows. The real part of the FFT is

$$\text{Re}(n,s) = \sum_{k=0}^{K-1} m(n,k) \cos(2\pi s k / K) \quad (7)$$

Approximating the $m(n,k)$ as uncorrelated, the uncertainty of $\text{Re}(n,s)$ is

$$\sigma_{\text{Re}(n,s)} = \left[\sum_{k=0}^{K-1} m(n,k)^2 \cos^2(2\pi s k / K) \right]^{1/2} \quad (8)$$

Making the approximations $m(n,k) = M(n)/K$ and $\langle \cos^2(\theta) \rangle = 1/2$ gives

$$\sigma_{\text{Re}(n,s)} = \sqrt{M(n)/2} \quad (9)$$

By the same argument the error on the imaginary part is

$$\sigma_{\text{Im}(n,s)} = \sqrt{M(n)/2} \quad (10)$$

The weights used in the least squares fits of the real and imaginary parts are $1/\sigma_{\text{Re}}^2$ and $1/\sigma_{\text{Im}}^2$, respectively.

9. IMSL MATH/Library, IMSL Inc, Houston, Texas.

10. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, (New York, McGraw-Hill, 1953), p. 1188.

The least squares fitting is done separately for the real and imaginary parts of the positive frequency components of the FFT, $s = 1, \dots, K/2$. The negative frequency components are treated by taking the complex conjugate of the corresponding positive frequency components. The equations that follow are written for the real part only; the imaginary part is treated in the same way. First, the χ^2 for consistency with zero is calculated

$$\chi^2 = \sum_{n=1}^N \frac{[\text{Re}(n,s)]^2}{M(n)/2} \quad (11)$$

If $\chi^2/\text{degree-of-freedom} = 1$, the component is set equal to zero, $\text{Re}(n,s) = 0$. If inconsistent with zero, the component is approximated by

$$\langle \text{Re}(r,s) \rangle = \exp(-r^2/2\sigma_0^2) \sum_{j=0}^J d_j q_j(r) \quad (12)$$

The comments following eq. (5) hold here also.

Morse and Feshbach give the Green's function for Poisson's equation in polar coordinates.¹⁰ The line charge at (r_0, ϕ_0) can be written as

$$\lambda(r_0, \phi_0) = \sum_{s=-\infty}^{\infty} \lambda_s(r_0) \exp(is\phi_0) \quad (13)$$

In our case $\lambda_s(r_0) = \langle \text{Re}(r_0, s) \rangle + i \langle \text{Im}(r_0, s) \rangle$ from equation (12) and the analogous one for $\langle \text{Im}(r, s) \rangle$, and the series extends from $-K/2$ to $K/2$ not including $s = 0$

$$\lambda(r, \phi_k) = \sum_{s=-K/2}^{K/2} \langle \text{Re}(n, s) \rangle \cos(s\phi_k) + i \langle \text{Im}(n, s) \rangle \sin(s\phi_k) \quad (14)$$

Using the Green's function the azimuthally varying electric fields are

$$E_r = \frac{1}{4\pi\epsilon_0} \sum_{s \neq 0} e^{is\phi} \left\{ \int_0^r \frac{|\lambda_s(r_0)|}{r_0} dr_0 - \int_r^\infty \frac{|\lambda_s(r_0)|}{r_0} dr_0 \right\} \quad (15)$$

$$E_\phi = \frac{-i}{4\pi\epsilon_0} \sum_{s \neq 0} \frac{s}{|s|} e^{is\phi} \left\{ \int_0^r \frac{|\lambda_s(r_0)|}{r_0} dr_0 + \int_r^\infty \frac{|\lambda_s(r_0)|}{r_0} dr_0 \right\}$$

The expression for E_r is evaluated at $r = r_n + \Delta r/2$, $\phi = \phi_k$, and the expression for E_ϕ at $r = r_n$, $\phi = \phi_k + \Delta\phi/2$. The resultant field values are linearly interpolated to give the field at the positions of test particles in the other beam.

The routine can be tested by randomly generating test particles with a Gaussian distribution and comparing the fields calculated with the fields from Bassetti and Erskine. Figures 3 and 4 show such comparisons.

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Figure 3: The vertical electric field at $x = 0$, $x = 50 \text{ } \mu\text{m}$ for a bunch with $\sigma_{y0} = \sigma_{x0}/2$, $\sigma_{x0} = 54.77 \text{ } \mu\text{m}$. There are two lines overlaid; one is from this numerical field calculation and the other from Bassetti and Erskine.

Figure 4: The horizontal electric field at $x = 0$, $x = 20 \text{ } \mu\text{m}$ for a bunch with $\sigma_{y0} = \sigma_{x0}/2$, $\sigma_{x0} = 54.77 \text{ } \mu\text{m}$. There are two lines overlaid; one is from this numerical field calculation and the other from Bassetti and Erskine.

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