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GUIDING CENTER SIMULATIONS OF STRONG ION BEAMS WITH APPLICATIONS TO THE COUNTERSTREAMING ION TORUS

MASTER

Carol Tull
(Ph.D. thesis)

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This thesis is dedicated to
my husband and to my father

ABSTRACT

In the proposed Counterstreaming Ion Torus (CIT) steady state rather than pulsed operation may be possible if all of the plasma power density is provided by neutral beam injection. After the neutral beams have penetrated the magnetic field, strong ion beam currents are produced. A major concern with the relatively strong counterstreaming ion currents is the effect of the beam self-magnetic fields on the macroscopic equilibrium of the system. Pinching and self focusing of the individual beams may occur, or the repulsive interaction of the two oppositely directed beam currents may destroy the equilibrium entirely. We investigate this macroscopic behavior of the ion beams with a guiding center plasma particle simulation model and we describe a model we have developed to simulate steady state behavior in an ideal CIT configuration.

We consider the guiding center motion of the ions in the magnetostatic limit. In this approximation, the guiding center equations of motion are coupled with self-consistent solutions of Ampere's law, and the self-consistent electric fields are neglected. We describe the details of the numerical solution of these coupled equations. In previous studies with this model, radial pinching and self-focusing of ion beam currents have been observed, even for beam currents which are much lower than those predicted for the CIT reactor. We investigate this phenomenon with simulations of a linearly propagating plasma column in a cylinder. Our test runs with this configuration show that

radial pinching does occur and is accompanied by unstable growth of the beam self-magnetic fields. We describe an extensive set of simulations to study this instability. The results of these simulations show that the instability is numerical rather than physical. We have developed both smoothing algorithms and a modified integration scheme in order to reduce the growth of numerical instabilities. Using the new solution methods, we study the confinement of counterstreaming ion currents for a range of parameters suggested for the CIT. Our analysis of the beam motion shows that equilibria in major radius do exist for these parameters.

In an ideal CIT system, the dominant collisional mechanism is the Fokker-Planck slowing down of the ions on an electron background. In this system it is predicted that the two counterstreaming ion species (T^+ and D^+) will produce a significant amount of fusion reactions during a collisional slowing down period on the electrons. When the hot ions are cooled, they are removed from the system and replaced by more hot ions. Ions are also lost when they reach the limiter of the system. For steady balance these two loss mechanisms are balanced by further injection of the hot species into the system.

We have applied the guiding center model to study this system. We present the details of a guiding center expansion in which the Fokker-Planck slowing down is included, and we derive the modifications to the particle energy and magnetic moment. We discuss the modeling of the system, and we present a method for simulating particle loss and reinjection. We report

simulation results for two interesting CIT configurations. In both cases the D^+ species reaches a steady state balance but, because of limited computer time, a steady state balance is not quite achieved for the T^+ species. In both configurations the results show an increased loss rate (because of limiter losses) over the loss rate predicted for ions removed as a result of electron drag only. We present quantitative steady state results and energy injection rates necessary to maintain steady state balance.

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I. INTRODUCTION

1.1 Motivation

As a result of recent advances in the development of neutral beam injectors, beam injection may play an extremely important role as a fuel and heat source for magnetically confined thermonuclear plasmas.^{1,2} Two proposed toroidal systems employing neutral beam injection as a fuel source are the Two Component Torus (TCT) and the Counterstreaming Ion Torus (CIT). Both systems are based on the tokamak confinement principle. In TCT operation, the beams provide both plasma heating and fueling, while the confining toroidal current is maintained by an externally imposed inductive electric field. On the other hand, the distinctive characteristics of CIT operation are:

1. Counterstreaming ions are produced by neutral beams injected parallel and antiparallel to the toroidal magnetic axis. The head-on collisions of these counterstreaming ions produce the fusion reactions which generate power.
2. After a start-up phase,^{1,3,4} the confining toroidal plasma current is carried entirely by the counterstreaming ions.
3. The ratio of ion to electron energy density ($\pi = \epsilon_i / \epsilon_e$) can be very large whereas in typical tokamaks this ratio is approximately unity.⁵

One of the most significant advantages of the CIT system is the possibility of steady state operation.

A major concern with the relatively strong counterstreaming ion beams in the CIT is the effect of the beam self fields on the macroscopic equilibrium of the system. Pinching and self focusing of the individual beams may occur or the repulsive interaction of the two oppositely directed beam currents may destroy the equilibrium entirely. Both of these phenomena were observed to some degree in previous TCT studies^{6,7} where the beam strengths were relatively weak compared to those of the CIT. The purpose of this thesis is to further study this macroscopic behavior and, in addition, to model the approach to steady state in the CIT.

1.2 Background

1.2.1 Magnetic Confinement in Toroidal Systems

Magnetic confinement schemes are classified as either "open" confinement systems in which the magnetic field lines leave the plasma volume (e.g., magnetic mirrors) or "closed" confinement systems in which the magnetic field lines remain in the plasma region (e.g., tokamaks). In toroidal systems (tokamaks) the plasma exists in a doughnut shaped region containing helically shaped field lines. The cylindrical coordinate system shown in Figure 1.1 is used to describe toroidal plasmas. In this system the e_ϕ direction is referred to as the toroidal direction. The $\phi = \text{constant}$ surface defines a poloidal plane. The distance from the toroidal axis to the center of the plasma is the major radius and the radius of the plasma cross section in the poloidal plane is the minor radius.

Toroidal configurations in which the plasma pressure is

- (a) Cylindrical co-ordinate system describing toroidal plasmas. The plasma exists in the shaded region.

- (b) Co-ordinate system in the poloidal plane. R_0 and r_0 are the major and minor toroidal radii.

Figure 1.1 Plasma in a toroidal geometry.

balanced by a confining magnetic field pressure are shown to exist in several analytical and numerical studies.⁸⁻¹² The minor radius of the plasma is maintained by a poloidal magnetic field component generated by a current in the toroidal direction. This balances the outward-directed pressure of the plasma. Two methods have been used to produce the confining toroidal plasma current. In the standard tokamak configuration a toroidal current is produced inductively in the plasma as the secondary current in a transformer. In other configurations, such as the levitron, physical coils are placed within the plasma volume. A third method, proposed for the CIT, would employ beam driven toroidal confinement currents. This concept was first introduced by Ohkawa³ and has recently been analyzed by Connor and Cordey.⁴ A discussion of beam driven currents in the CIT reactor is presented in Chapter V.

The major radius of the toroidal plasma will tend to expand in the absence of external constraints. This is because there is a net $\underline{J} \times \underline{B}$ force directed outward along the plasma radius, where $\underline{B}$ is the magnetic field due to current flowing on the opposite side of the torus. In practice motion along the major radius has been inhibited by the image currents produced in the copper shell surrounding the torus. More recently the major radius positioning has been accomplished by adding external coils to produce a vertical component of the magnetic field. The coil currents are adjusted to produce a balancing Lorentz force directed radially inward.

In summary, toroidal plasma confinement consists of an equilibrium in both major and minor plasma radii. The magnetic field is composed of an externally generated toroidal field component and a poloidal field component resulting either from the induced toroidal plasma current or from the beam driven toroidal current. Without a copper conducting shell, an additional vertical field component is required for equilibrium in the plasma major radius. Although there has been no analysis presented here on the relative strength of the field components, the stability of a tokamak equilibrium requires

$$B_{\phi} \gg B_{\theta}$$

where B_{θ} is the component of the field in the poloidal plane.

1.2.2 Fusion Reactor Considerations

The four reactions most often considered for thermonuclear power generation are

The reaction rates, $\langle \sigma v \rangle$, are computed by averaging the product of the relative velocity of the reactants and the fusion cross

section over appropriate velocity distribution functions

$$\langle \sigma v \rangle = \frac{\iint d\mathbf{v}_1 d\mathbf{v}_2 |\mathbf{v}_1 - \mathbf{v}_2| \sigma f(\mathbf{v}_1) f(\mathbf{v}_2)}{\iint d\mathbf{v}_1 d\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2)}$$

The reaction rates for the four reactions are computed in reference 13 assuming an isotropic Maxwellian distribution function for the reacting ions. These calculations are reproduced in Figure 1.2. Reaction 3 is advantageous from the point of view of both the reactivity and the energy released per reaction. The disadvantage is the use of tritium fuel.

The reactivity of the CIT reactor is enhanced over that in Figure 1.2 by the head-on collisions between the reactants. This is shown in Figure 1.3 taken from reference 1. The point A in this figure is the maximum reactivity for a 125 keV D^+ beam injected into a cold T^+ plasma. The point B corresponds to the peak reactivity of an isotropic D-T plasma (at a temperature of about 60 keV) in Figure 1.2. The reactivity for counterstreaming ion distributions exceeds the point B for a considerable range of relative velocities.

The power density produced in fusion reactions is

$$P_f = n_1 n_2 \langle \sigma v \rangle W_f$$

where n_1 and n_2 are the reactant densities and W_f is the released fusion energy per reaction. The performance of the reactor is measured in terms of Q , defined as

$$Q_{CIT} = \frac{P_f}{P_b}$$

where P_b is the injected beam power density. The key to main-

Figure 1.2 Plots of fusion reaction rate parameters, $\langle nv \rangle$ for D-T, D- ^3He , D-D (total), and p- ^6Li reactions as obtained by averaging over a Maxwellian distribution.

Figure 1.3 Fusion reactivity of D and T displaced-Maxwellian distributions, each of temperature T_b , with relative displacement velocity v_r . The maximum reactivity of a DT beam-target system ($W_{\text{beam}} \approx 125 \text{ keV}$) is denoted by A. The maximum reactivity of a thermonuclear DT plasma ($T_i = 60 \text{ keV}$) is denoted by B.

taining high Q in the CIT is the removal of cooled beam ions (referred to as warm ions) so as to avoid the buildup of a background plasma. Unless these warm ions are removed from the system, the ion velocity distributions will not maintain the two distinct counterstreaming velocity groups because of charge exchange between fast neutrals and the warm ions. This reduces the reacting population for a given total plasma pressure. Other physical processes such as collisions between the hot ions and warm ions, collisions between impurity ions and hot ions, and the retrapping of warm neutrals will also produce this same result. Recent Q calculations⁵ including a warm ion species show a degradation of the reactor Q by as much as 60% or 70% below the ideal value. Although this limits the applicability of the reactor as a stand alone power generator, there are other near term uses, as discussed in reference 1, where this mode of operation may be advantageous.

1.3 Approach

The purpose of numerical modeling of plasmas is to simulate phenomena which are not calculable analytically. A recent review of numerous simulation codes which have been developed to model both present experimental and proposed fusion reactors is presented in reference 14. The plasma models generally fall into five categories:

1. Kinetic models -- both Vlasov and Fokker-Planck models.

2. Particle orbit calculations -- single particle orbits and self-consistent macroparticle models.
3. Equilibrium calculations -- time independent magnetohydrodynamics (MHD).
4. Time dependent MHD.
5. Hybrid models.

The hybrid simulations employ models from more than one of the first four categories.

The selection of an appropriate model for the plasma depends on the time scale of the physical phenomena being simulated and on the type of information required. The wide range of characteristic time scales in a typical CIT is illustrated in Table 1.2. These were derived from the system parameters listed in Table 1.1. Since we are studying magnetostatic beam confinement, the high frequency electrostatic and electromagnetic phenomena ($\omega > \omega_{pe}$) are not modeled. The underlying assumptions in this approximation are:

1. A low frequency limit of Maxwell's equations is taken, and in particular the displacement current is ignored in Ampere's law.
2. Space charge neutralization of the ion beams is assumed so that self-consistent solutions of Poisson's equation are not necessary.

Similarly, we use the guiding center approximation to describe the beam motion because the details of the fast time scale cyclotron motion are unimportant. The other important physical effect which we study is the self magnetic field interactions. Therefore,

Table 1.1. Input parameters for the calculations in
Table 1.2.

Beam energies

$$E^{D^+} = 80 \text{ keV} \rightarrow v_{\phi} = 2.76 \times 10^8 \text{ cm/sec}$$

$$E^{T^+} = 120 \text{ keV} \rightarrow v_{\phi} = 2.76 \times 10^8 \text{ cm/sec}$$

Toroidal parameters

Major radius:	$R_0 = 375 \text{ cm}$
Minor radius:	$r_0 = 75 \text{ cm}$
Limiter radius:	$r_L = 75 \text{ cm}$
Field:	$B_{\phi} = 50 \text{ kGauss at } R_0$
Safety factor:	$q = 3 \text{ at limiter radius}$

Electron parameters

Density:	$n_e = 10^{14} \text{ cm}^{-3}$
Temperature:	$T_e = 10 \text{ keV}$

Coulomb logarithm

$$\ln \Lambda = 20$$

Table 1.2. Characteristic time scales in a CIT plasma.

τ_{pe}	Electron plasma period	$\left(\frac{m_e}{4\pi n e^2}\right)^{1/2}$	5.6×10^{-12} sec
τ_{ci}	Ion cyclotron period	$\frac{m_i c}{e B}$	4.1×10^{-9} sec (D^+) 6.1×10^{-9} sec (T^+)
τ_{mhd}	Hydromagnetic time scale	L / V_{Alf} for $V_{Alf}^2 = \frac{\beta^2}{8\pi m_i n}$	0.135 μ sec
τ_{ϕ}	Toroidal orbit period	$2\pi R_0 = \frac{2\pi R_0}{V_{\phi}} = \frac{2\pi R_0}{\left(\frac{2}{m_i} E_i\right)^{1/2}}$	8.5 μ sec (D^+)
τ_{θ}	Poloidal orbit period	$q 2\pi / 2\pi$ for $q = \frac{B_0}{B_{\phi}} \frac{R_0}{R_0}$	25.5 μ sec (limiter) 12.8 μ sec (center)
τ_{drift}	Drift (curvature) time	$\frac{e R_0 B_{\phi} \kappa_0}{e E_i}$	117.1 μ sec (D^+) 175.6 μ sec (T^+)
t_{drag}	Collisional electron drag time	$\frac{M_i T_e^{3/2} \times 10^{13}}{\ln \Lambda n_e}$ for M_i in AMU	298.6 msec (D^+) 447.5 msec (T^+)

the model must include a calculation of the magnetic fields arising from the ion current sources (i.e., the self-consistent magnetic fields). For these reasons, we choose a self-consistent macroparticle model to simulate the macroscopic motion of the CIT beams.

1.3.1 Self-Consistent Macroparticle Models

It is impossible to compute individual orbits for approximately 10^{20} or 10^{21} beam particles. Instead, in our model, the plasma is represented by a limited number of macroparticles* (typically 5000). These macroparticles freely overlap and pass through one another. Each macroparticle is an equal portion of the plasma density and represents 10^{16} or 10^{17} individual plasma particles. A spatial grid must be introduced for the calculation of the self fields. The macroparticles are a grid cell in size, and the position of the particle is assumed to be at the center of the macroparticle (see Figure 1.4). In the force calculations the fields are interpolated from the spatial grid to the particle positions. Similarly, in the self-consistent field calculations the sources are accumulated on the grid by interpolation from the particle positions. The force used in the orbit calculations is the point particle force with the charge replaced by the charge density of the macroparticle. The effect of this replacement is that the force disagrees with the correct point

*Macroparticles are also called superparticles or finite-size particles by many researchers.

Figure 1.4 A finite-size particle moving under the influence of fields known on a spatial grid.

particle force for interaction distances, λ , which are shorter than the particle length. One advantage of using a spatial grid is that these short range interactions with $\lambda < \Delta X$ are not present. Thus, it is valid to use macroparticles to simulate interactions with $\lambda > \Delta X$. As a further justification, it has been shown that the physical properties of the plasma are correctly modeled. For example, macroparticle simulations have been used to reproduce the linear plasma dispersion relationships. This is discussed at length in the references.¹⁵⁻²⁶

There are several types of finite-size particle simulations currently used in plasma research. The class of finite-size particle simulations in which the orbit equations are coupled only with self-consistent solutions of Poisson's equation are referred to as electrostatic simulations. The electromagnetic finite-size particle simulations couple self-consistent solutions of the full set of Maxwell's equations with the equations of motion. Both of these types of simulations are used to model high frequency ($\omega_o > \omega_{pe}$) and short length scale ($L_o \sim \lambda_{De}$) physical phenomena. In our magnetostatic particle simulations, the orbit equations are coupled with self-consistent solutions of a low frequency limit of Ampere's law, i.e., the displacement current is neglected.

The notion of a macroparticle was first introduced by Vlasov²¹ in the early 1960's. The first macroparticle models were the electrostatic simulations of Dawson¹⁶, Birdsall, et al.,¹⁷ and Buneman. This work was later extended to full electromagnetic simulations and involved several researchers

including Langdon,²⁹ Byers,²⁸ Okuda,²⁷ and Nielson.¹⁸ An extensive bibliography of the applications of finite-size particle simulations is given in reference 15.

1.3.2 The Guiding Center Approximation

The guiding center approximation is used when charged particles are moving in strong magnetic fields with weak gradients. The field gradient scale lengths are assumed long compared to the ion Larmor radius. The motion can then be analyzed in two parts as shown in Figure 1.5: the fast time scale cyclotron motion ($\Omega_{ci}^{-1} \sim 10^{-9}$ s) and the slow time scale motion of the guiding center ($\tau_{GC} \sim \tau_p \sim 10^{-5}$ s where τ_p is the period for motion in the poloidal plane). To derive the guiding center approximation, the Lorentz force on the particles is expanded using the ratio of the two characteristic time scales as a smallness parameter, $\epsilon = 1/(\Omega_{ci}^{-1} \tau_p)$. Averaging the expanded force over a cyclotron period eliminates the cyclotron motion. An algebraic expression for the velocity of the guiding center, $\underline{v}_{GC}$, can be derived from the averaged results.

The Fokker-Planck collisional drag of the ions in an electron background is important in the CIT steady state. We have incorporated collisional drag into the guiding center approximation by adding a small collisional momentum loss term before expanding the force equation. We solve for the collisional drag modifications to both the energy and the magnetic moment. The details of this expansion including the necessary physical approximations are presented in Chapter II.

Figure 1.5 Total particle position divided into the motion of the guiding center and the cyclotron motion around the guiding center.

1.3.3 Advancement of Self-Consistent Orbits

The advancement of the guiding center position in time requires the solution of an ordinary differential equation of the form

$$\frac{d}{dt} \underline{R}_{gc} = \underline{V}_{gc}(\underline{R}_{gc}, \underline{\beta})$$

For the case of strong beam currents the self magnetic fields of the beams are an important correction to the total field. Therefore, the magnetic fields are updated during the particle orbit calculations with self-consistent solutions of Ampere's law

$$\nabla \times \underline{\beta} = \frac{4\pi}{c} \underline{J}_{TOTAL}$$

where

$$\underline{J}_{TOTAL} = \underline{J}_{COILS} + \underline{J}_{BEAMS}$$

The computational steps in a self-consistent orbit calculation are shown in Figure 1.6. The spatial evolution of the beam particles described by the previous equations cannot be solved analytically because of the complicated relationship between the particle evolution and the self magnetic fields. A few years ago T. H. Johnson⁷ wrote a computer code, GUIDON, to solve these equations on the CDC-6600. We have converted this code to the CDC-7600, and we have made extensive additions to the physical model and numerical algorithms. Some of the major additions to the code are:

1. Analytical modeling of the electron parameters including the current density and the Ohmic heating electric field.

Figure 1.6. Steps in a self-consistent orbit calculation.

2. Modeling of several beam deposition profiles.
3. Development of a statistical method for reinjecting particles with a specified density profile.
4. Modification of the integration scheme and addition of smoothing algorithms for the source terms.
5. Incorporation of collisional slowing down into the integration scheme.
6. Block processing of the particles from the CDC-7600 large core memory to the small core memory.
7. Development of beam moment diagnostics and post processed diagnostics for the CIT steady simulations.
8. Extension of the code to allow for ion species with different masses.

We refer to this new version of the code as GUIDON77.

The details of the numerical techniques and solution approximations are presented in Chapter III. The physical model is discussed in Chapters II and V.

1.4 Content and Outline of Thesis

There are three major problems which are addressed with the simulations presented in this thesis. First we have applied the guiding center model to study the confinement of collisionless counterstreaming ion currents in a CIT reactor. A second problem which we consider is the guiding center simulation of the long time behavior of collisional beams and the existence of a steady state in the CIT reactor. The third problem is a study of the

rapid radial pinching and unstable growth of self-magnetic fields which we observed in our earliest simulations of counterstreaming ion currents.

The purpose of the collisionless CIT simulations is to determine whether the counterstreaming ion beams remain confined. Previous guiding center simulations³¹ of a TCT reactor in which counterstreaming beams were injected into a target plasma system resulted in large particle losses from the system. This lack of beam confinement can occur for several reasons. Both the mutual repulsion of the oppositely directed ion currents and the imbalance of the $\underline{J} \times \underline{B}$ forces resulting from the toroidal geometry produce motion along the major radius which leads to particle losses. Another aspect of beam confinement is the guiding center drift motion. In the geometry of the CIT reactor the curvature drift can cause significant motion in the axial direction. Since all three of these phenomena which determine the beam confinement properties occur on a drift time scale (τ_{drift} in Table 1.2), we have neglected the collisional phenomena in these simulations.

Our first simulations of typical counterstreaming ion currents (see Table 1.1 for parameters) resulted in unstable growth of the beam self fields. We chose to make further studies of this instability with simulations of a single linearly propagating beam in a cylinder. This eliminates the effects of the toroidal geometry and the repulsive interaction between the counterstreaming ion currents. In a finite-size particle model instabilities can develop both as a result of the violation of a physical stability limit and as a result of the interpolation of

fields to particle positions or particle quantities interpolated to grid positions. We describe a series of simulations to test the nature of the instability in Chapter IV. Both physical variables and numerical algorithms are varied in these test cases. The development of the numerical algorithms used in these simulations are discussed at length in Chapter III.

Collisional phenomena are important in determining the long time behavior in a CIT. The dominant collisional process for the system described in Table 1.2 is the slowing down of the hot ions on the electrons (electron drag). In Chapter III we present a development of the guiding center equations in which we have included a momentum loss term for the collisional slowing down. A statistical method for adding Fokker-Planck angular scattering to the guiding center formulation is being developed but is not included in the CIT simulations presented in Chapter V. The existence of a steady state in the CIT with a favorable Q requires that warm ions be lost from the system at some cutoff energy (on the order of twice the electron temperature) to avoid the buildup of an undesirable background plasma. That is, the system parameters are chosen so that the physical processes which result in warm ion losses occur on a very short time scale compared to the slowing down time of the hot ions on the electrons. The three processes which are expected to promote rapid warm ion losses are instability-induced diffusion, charge exchange with low energy neutrals, and enhanced diffusion resulting from trapped particle orbits in the ripples in the toroidal magnetic field. These processes are discussed at length in

reference 1. A second ion loss mechanism results from the ion drift motion along open magnetic flux surfaces to the limiter of the system. At steady state, the total ion losses (resulting from both warm ion losses and the drift motion of the particles to the limiter) must be balanced by further energetic beam injection. In Chapter V we apply the collisional guiding center model to determine the existence and properties of a steady state operation. These simulations model the "ideal" reactor in which there is no warm ion population. Therefore, no warm ion reactions are included. We present these collisional simulations for two CIT configurations of current interest.

The following is an outline of the remaining chapters in this thesis:

1. The physical modeling of the plasma is presented in Chapter II. A formal two time expansion for the motion of charged particles in strong magnetic fields is developed and includes the velocity dependent collisional corrections for Fokker-Planck slowing down.
2. Chapter III is devoted to a description of the numerical methods and illustration of the diagnostics.
3. Chapter IV presents the test simulations of a linearly propagating beam. The details of the numerical modifications and test runs are reported.
4. Chapter V presents a further discussion of the CIT reactor and the modeling of the physical parameters for the reactor. Both collisionless beam confinement

studies and methods for simulating collisional beams are discussed. The validity and limitations of the steady state model are discussed and simulations results are presented for two interesting CIT configurations.

5. Chapter VI summarizes the discussions and conclusions in the previous chapters. A description of future work and extensions to this model is also presented.

II. PHYSICAL MODEL

2.1 Introduction

The physical model of the beam-plasma system consists of a guiding center description for the beams and an equilibrium fluid model for the background plasma. In the first sections of this chapter we derive the guiding center equations. We follow the development in reference 37, but we have included a collision term. Collisions in thermonuclear plasmas represent a weak effect and are modeled with the Fokker-Planck approximation. The purpose of our development is to show the ordering of the terms and to derive the proper collisional modifications to the guiding center motion.

Two terms result from the Fokker-Planck approximation of the collision operator: the Fokker-Planck slowing down and the velocity space diffusion (angular scattering). For the CIT parameters we have considered, the slowing down term is dominant and the angular scattering is negligible. Collisional slowing down involves particle momentum changes and particle energy changes. These collisional momentum changes are included as a correction term to the Lorentz force equation. The result of the guiding center expansion of the force equation shows that collisional drag requires modification to both the particle energy and magnetic moment. In the remaining sections of the chapter we discuss the self-consistent field equations and the model for the vacuum magnetic fields.

2.2 Derivation of the Guiding Center Equations

Using a Two Time Expansion Method

2.2.1 The Expansion and Convergence

We derive the guiding center equations in this section using a two time expansion. A two time expansion is a method for analyzing quasiperiodic motion in which the deviations from exact periodicity occur on a long time scale compared to the oscillation period. To illustrate this, consider the nonrelativistic motion of a charged particle in the presence of a uniform, time independent magnetic field. The acceleration of the particle is given by

$$\ddot{\underline{r}} = \frac{q}{mc} \underline{v} \times \underline{B} \quad (2.1)$$

The solution of this equation is given by

$$\underline{r} = \underline{R} + \underline{\rho} \quad (2.2)$$

$$\underline{R} = \underline{R}(0) + \hat{b} \hat{b} \cdot \dot{\underline{R}}(0) \quad (2.3)$$

$$\underline{\rho} = \rho_L(0) \{ \cos(\Omega_c + \phi)t \hat{e}_x + \sin(\Omega_c + \phi)t \hat{e}_y \}, \quad (2.4)$$

where

$\underline{R}(0)$ = initial particle position

$$= \underline{R}(0) + \underline{\rho}(0)$$

ρ_L = Larmor radius

Ω_c = cyclotron frequency

$\hat{e}_x, \hat{e}_y$ = cartesian unit vectors in a plane perpendicular to the magnetic field, $\underline{B}$

$$\hat{b} = \underline{B}/B$$

ϕ = initial phase angle.

The resulting helical motion around the magnetic field lines is shown in Figure 2.1. In the plane perpendicular to the field lines, the motion is exactly periodic since the cyclotron frequency and the Larmor radius are both constants in time.

Suppose we consider perturbations to this exactly periodic (cyclotron) motion by introducing weak spatial inhomogeneities or slow explicit time variations in the magnetic field or by applying a weak electric field. The perturbations to the oscillatory motion are small provided the following conditions are maintained:

1. gradient scale lengths for the magnetic field are large compared to the Larmor radius and
2. explicit time variations in the fields are small during a cyclotron period.

In addition, we assume the parallel electric field (along the direction of $\hat{b}$) is not too strong so that a nonrelativistic formulation can be used. Perturbations of this nature affect the cyclotron motion by causing variations in both the Larmor radius and cyclotron frequency which occur over several oscillation periods.

In order to analyze the perturbed motion, we introduce two time scales into the problem. The fast time scale, denoted by t , is characteristic of the cyclotron motion. The slow time scale, denoted by $\tilde{t}$, is characteristic of the variations in the

Figure 2.1. Helical particle orbit in a constant magnetic field.

Figure 2.2. Particle position analyzed in terms of the instantaneous Larmor radius and the guiding center position.

fields and thus, in the Larmor radius and the cyclotron frequency. Since the variations in these functions may be important on both time scales, this leads naturally to using the ratio of the change in the slow time variable to the change in the fast time variable as a small parameter in expansions. As an example of a two time expansion, the instantaneous position in the cyclotron orbit is a function of both the fast and slow time variables,

$$\underline{\rho}(t, \tilde{t}) = \rho_L(\tilde{t}) \left[\cos(\Omega_c(\tilde{t}) + \phi) \hat{e}_x + \sin(\Omega_c(\tilde{t}) + \phi) \hat{e}_y \right].$$

Notice that the Larmor radius, ρ_L , is a function of the slow time variable only. On the other hand, the oscillating part of $\underline{\rho}$ depends primarily on the fast time variable. An expansion in time of this function yields

$$\delta \underline{\rho} = \underline{\rho}(t + \Delta t) - \underline{\rho}(t) = \left\{ \frac{\partial \underline{\rho}}{\partial \tilde{t}} + \epsilon \frac{\partial \underline{\rho}}{\partial t} \right\} \Delta t + \dots,$$

where

$$\epsilon = \frac{\partial \tilde{t}}{\partial t}.$$

Similarly, an expansion of the cyclotron frequency, which changes only on the slow time scale, can be written as

$$\delta \Omega_c = \epsilon \frac{\partial \Omega_c}{\partial t} \Delta t.$$

In applying a two time expansion technique to charged particle motion, the particle position in equation (2.2) is expanded in time, and substituted into the force equation. Such an expansion is asymptotic in nature. The convergence of

an asymptotic expansion³⁵⁻³⁷ is stated as follows. Suppose that ϵ is a small parameter in the expansion,

$$\underline{u}(\epsilon) = \underline{u}_0 + \epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + \epsilon^3 \underline{u}_3 + \dots \quad (2.5)$$

Let S_N be the sequence of partial sums formed from the series expansion,

$$S_N = \underline{u}_0 + \epsilon \underline{u}_1 + \dots + \epsilon^N \underline{u}_N. \quad (2.6)$$

then the asymptotic expansion is convergent for

$$\lim_{\epsilon \rightarrow 0} \frac{\underline{u} - S_N}{\epsilon^N} = 0. \quad (2.7)$$

For any fixed time, t , it can be shown that for nonzero ϵ there is an optimum number of terms in the expansion which minimizes the truncation error. Additional terms do not increase the accuracy, and may make it worse.

2.2.2 Expansion of the Force Equation

The force on a charged particle, including the Fokker-Planck collisional slowing down term, is

$$\underline{F} = m \frac{d^2 \underline{r}}{dt^2} = q \underline{E} + \frac{q}{c} \underline{v} \times \underline{B} - m \gamma_0 \underline{v}. \quad (2.8)$$

The last term has the functional form

$$\gamma_0 = \gamma_0(|v|). \quad (2.9)$$

and considers collisions with all of the background charged species including electrons;

$$\gamma_D = \sum_j \gamma_{Dj} \quad \text{for } j = e, i_1, i_2, \dots, i_N \quad (2.10)$$

It is useful to define dimensionless variables for the expansion:

$$\underline{x}^* = \frac{\underline{x}}{\rho_L} \quad (2.11)$$

where

ρ_L = characteristic Larmor radius

$$t^* = \Omega t \quad \text{where} \quad \Omega = \frac{qB}{mc} \quad (2.12)$$

$$\gamma^* = \frac{\gamma_D}{\Omega} \quad (2.13)$$

$$\underline{v}^* = \frac{d\underline{x}^*}{dt^*} \quad (2.14)$$

$$\Omega^* = \frac{\Omega}{|\Omega|} \quad (2.15)$$

$$\underline{a}^* = \frac{q \underline{E}}{m \rho_L \Omega^2} . \quad (2.16)$$

The force equation in dimensionless form becomes

$$\frac{d^2 \underline{x}^*}{dt^{*2}} = \underline{a}^* + \underline{v}^* \times \underline{\Omega}^* - \gamma^* \underline{v}^* . \quad (2.17)$$

For notational convenience the starred notation (*) is dropped in the remainder of this section. All of the following equations are dimensionless unless otherwise stated in the text.

Consider a two time expansion with a small parameter, ϵ_1 , defined as

$$\epsilon_1 = \frac{1}{\Omega_c \tau_0} , \quad (2.18)$$

where T_0 is a characteristic (slow) time scale for the guiding center motion, e.g. a poloidal orbit period. The expansion parameter is also defined in terms of the changes in the slow time variable with respect to the fast time variable:

$$\epsilon_1 = \frac{d\tilde{t}}{dt}. \quad (2.19)$$

It is useful to define a second small parameter, ϵ_2 , as

$$\epsilon_2 = \gamma_D T_0. \quad (2.20)$$

This is a small parameter when the drag time, γ_D^{-1} , is long compared to the characteristic times for the guiding center motion (such as the poloidal orbit periods). With the definitions (2.18) and (2.20), equation (2.15) becomes

$$\frac{d^2 \underline{R}}{dt^2} = \underline{a} + \underline{v} \times \underline{\Omega} - \epsilon_1 \epsilon_2 \underline{v}. \quad (2.21)$$

Next consider the ordering of this last term. In the usual guiding center expansion, the first order drifts, $O(\epsilon_1)$, are retained and the second order terms, $O(\epsilon_1^2)$, are neglected. The collision term, $O(\epsilon_1 \epsilon_2)$, is small compared to these first order terms when $T_0 \ll \gamma_D^{-1}$ so that

$$\epsilon_2 \ll 1$$

and

$$\epsilon_1 \epsilon_2 \ll \epsilon_1. \quad (2.22)$$

On the other hand, for longer characteristic times, the collisional corrections are comparable to the first order guiding

center terms and cannot be neglected; that is for $T_0 \sim \gamma_D^{-1}$.
then

$$\varepsilon_1 \varepsilon_2 = \frac{\gamma_0}{\Omega_c} \ll 1$$

$$\varepsilon_1 = \frac{1}{T_0 \Omega_c} \approx \frac{\gamma_0}{\Omega_c} = \varepsilon_1 \varepsilon_2$$

$$\varepsilon_1^2 \sim \left(\frac{\gamma_0}{\Omega_c} \right)^2 \ll \varepsilon_1 \varepsilon_2.$$

Thus, in this limit it is appropriate to include the collisional terms while ignoring second order drift corrections.

The next step is to expand the particle position and to substitute the result into the fields and force equation. The position of the particle is the sum of the guiding center position, $\underline{R}$, and the instantaneous Larmor radius, $\underline{\rho}$ (Figure 2.2). The Larmor radius is perpendicular to the magnetic field lines and functionally depends on both time variables,

$$\underline{\rho} = \underline{\rho}(t, \tilde{t}) \quad (2.23)$$

$$\underline{R} = \underline{R}(\tilde{t}). \quad (2.24)$$

The expanded position is represented as

$$\underline{r} = \frac{\underline{R}}{\varepsilon_1^n} + \underline{\rho}. \quad (2.25)$$

It is easy to verify that the choice $n = 1$ gives the correct balancing of terms for the zero order periodic motion. For convenience a normalized auxiliary fast time variable is defined as

$$\theta(t) = \Omega t \quad \text{for} \quad 0 \leq \theta \leq 1 \quad (2.26)$$

The frequency of the oscillations, ν , is assumed to be a function of the slow time variable only,

$$\frac{d\theta}{dt} = \nu(\tilde{t}) \quad (2.27)$$

and

$$\theta(t) = \int_0^1 dt \nu(\tilde{t}) = \varepsilon_1 \int_0^{\tilde{t}} \nu(\tilde{t}) d\tilde{t}. \quad (2.28)$$

The functional form of $\nu(\tilde{t})$ will be determined by applying the periodicity constraint for the zero order fast time scale motion.

An expanded form of the left-hand side of equation (2.21) is obtained by taking time derivatives of equation (2.25):

$$\underline{y} = \frac{d\underline{h}}{dt} = \dot{\underline{R}} + \nu \underline{f}_\theta + \varepsilon_1 \underline{f}_{\tilde{t}} \quad (2.29)$$

where

$$\dot{\underline{R}} = \frac{d}{d\tilde{t}} \underline{R} \quad (2.30)$$

$$\underline{f}_\theta = \frac{d\underline{f}}{d\theta} \quad (2.31)$$

$$\underline{f}_{\tilde{t}} = \frac{d\underline{f}}{d\tilde{t}}. \quad (2.32)$$

In the remainder of this analysis, the dot ($\dot{}$) above a variable will indicate a derivative with respect to the slow time variable. Taking another derivative of equation (2.29) yields

$$\frac{d^2 \underline{h}}{dt^2} = \nu^2 \underline{f}_{\theta\theta} + \varepsilon_1 \left\{ \ddot{\underline{R}} + 2\nu \underline{f}_{\theta\tilde{t}} + \dot{\nu} \underline{f}_\theta \right\} + \varepsilon_1^2 \ddot{\underline{f}}. \quad (2.33)$$

The expansions for the fields are given by

$$\underline{a}\left(\frac{\underline{R}}{\epsilon_1} + \underline{\rho}\right) = \underline{a} + \epsilon_1 \underline{\rho} \cdot \frac{\partial \underline{a}}{\partial \underline{R}} + \epsilon_1^2 \underline{\rho} \underline{\rho} : \frac{\partial^2 \underline{a}}{\partial \underline{R} \partial \underline{R}} + \dots \quad (2.34)$$

and similarly

$$\underline{n}\left(\frac{\underline{R}}{\epsilon_1} + \underline{\rho}\right) = \underline{n} + \epsilon_1 \underline{\rho} \cdot \frac{\partial \underline{n}}{\partial \underline{R}} + \epsilon_1^2 \underline{\rho} \underline{\rho} : \frac{\partial^2 \underline{n}}{\partial \underline{R} \partial \underline{R}} + \dots \quad (2.35)$$

Substituting the expansions into the force equation yields

$$\begin{aligned} \nu^2 \underline{p}_{\theta\theta} + \epsilon_1 \{ \ddot{\underline{R}} + 2\nu \underline{p}_{\theta\tilde{z}} + \dot{\nu} \underline{p}_{\theta} \} + \epsilon_1^2 \ddot{\underline{\rho}} = \\ \underline{a} + (\dot{\underline{R}} + \nu \underline{p}_{\theta}) \times \underline{n} + \epsilon_1 \left\{ \underline{\rho} \cdot \frac{\partial \underline{a}}{\partial \underline{R}} + \right. \\ \left. (\dot{\underline{R}} + \nu \underline{p}_{\theta}) \times \underline{\rho} \cdot \frac{\partial \underline{n}}{\partial \underline{R}} + \underline{p}_{\tilde{z}} \times \underline{n} - \right. \\ \left. \epsilon_2 (\underline{R}_{\tilde{z}} + \nu \underline{p}_{\theta}) \right\} + O(\epsilon_1^2). \end{aligned} \quad (2.36)$$

There is a consistency condition on the parallel component of the electric field. Since $\underline{\rho}$ and all of its derivatives are, by definition, in the plane perpendicular to $\underline{B}$, the parallel component of the force equation is

$$\begin{aligned} \epsilon_1 \{ \ddot{\underline{R}} + 2\nu \underline{p}_{\theta\tilde{z}} \} \cdot \hat{\underline{b}} = a_{\parallel} + \epsilon_1 \hat{\underline{b}} \cdot \left\{ \underline{\rho} \cdot \frac{\partial \underline{a}}{\partial \underline{R}} + \right. \\ \left. (\dot{\underline{R}} + \nu \underline{p}_{\theta}) \times \underline{\rho} \cdot \frac{\partial \underline{n}}{\partial \underline{R}} - \epsilon_2 (\dot{\underline{R}} + \nu \underline{p}_{\theta}) \right\} + O(\epsilon_1^2) \end{aligned} \quad (2.37)$$

where

$$\hat{\underline{b}} = \frac{\underline{n}}{n} \quad (2.38)$$

$$a_{\parallel} = \underline{a} \cdot \hat{b}. \quad (2.39)$$

Thus, $\underline{a}_{\parallel}$ must be at most an order ϵ_1^1 quantity.

The force equation for the guiding center motion, $\underline{F}_G$, is found by averaging the total force, $\underline{F}_T$, over the cyclotron period. Let $\langle f \rangle$ denote the average of a given function f over the cyclotron period; then

$$\underline{F}_G = \langle \underline{F}_T \rangle = \int_0^1 d\theta \underline{F}_T. \quad (2.40)$$

Performing this average over equation (2.36) yields

$$\begin{aligned} \underline{F}_G = \epsilon_1 \ddot{\underline{R}} &= \underline{a} + \dot{\underline{R}} \times \underline{\Omega} + \epsilon_1 \langle \underline{p}_\theta \times \underline{p} \cdot \frac{\partial \underline{\Omega}}{\partial \underline{R}} \rangle \\ &\quad - \epsilon_1 \epsilon_2 \dot{\underline{R}} + O(\epsilon_1^2). \end{aligned} \quad (2.41)$$

The force equation for the cyclotron motion, $\underline{F}_{cyc}$, is

$$\underline{F}_{cyc} = \underline{F}_T - \langle \underline{F}_T \rangle = \underline{F}_T - \underline{F}_G. \quad (2.42)$$

Subtracting equation (2.41) from equation (2.36) yields

$$\begin{aligned} \underline{F}_{cyc} &= \nu^2 \underline{p}_{\theta\theta} + \epsilon_1 \{ \underline{p}_\theta \dot{\underline{V}} + 2\nu \underline{p}_{\dot{\theta}\theta} \} \\ &= \nu \underline{p}_\theta \times \underline{\Omega} + \epsilon_1 \left\{ \underline{p} \cdot \frac{\partial}{\partial \underline{R}} (\underline{a} + \dot{\underline{R}} \times \underline{\Omega}) + \nu \underline{p}_\theta \times \underline{p} \cdot \frac{\partial \underline{\Omega}}{\partial \underline{R}} + \right. \\ &\quad \left. \dot{\underline{p}} \times \underline{\Omega} - \nu \langle \underline{p}_\theta \times \underline{p} \cdot \frac{\partial \underline{\Omega}}{\partial \underline{R}} \rangle - \epsilon_2 \nu \underline{p}_\theta \right\} + O(\epsilon_1^2). \end{aligned} \quad (2.43a)$$

Regrouping terms by powers of ϵ_1 gives the result

$$\begin{aligned}
\frac{\partial}{\partial \theta} (\nu^2 \underline{p}_\theta - \nu \underline{p} \times \underline{\Omega}) &= \varepsilon_1 \left\{ \underline{p} \cdot \frac{\partial}{\partial \underline{R}} (\underline{e} + \dot{\underline{R}} \times \underline{\Omega}) + \nu \underline{p}_\theta \times (\underline{p} \cdot \frac{\partial \underline{\Omega}}{\partial \underline{R}}) - \right. \\
&\quad \left. \nu \langle \underline{p}_\theta \times \underline{p} \cdot \frac{\partial \underline{\Omega}}{\partial \underline{R}} \rangle + \dot{\underline{p}} \times \underline{\Omega} - \dot{\nu} \underline{p}_\theta - 2\nu \underline{p} \tilde{\varepsilon}_\theta - \right. \\
&\quad \left. \varepsilon_2 \nu \underline{p}_\theta \right\} + O(\varepsilon_1^2). \quad (2.43b)
\end{aligned}$$

2.2.3 Equation of Motion on the Fast Time Scale

The zero order terms in equation (2.43b) are

$$\frac{\partial}{\partial \theta} (\nu^2 \underline{p}_\theta - \nu \underline{p} \times \underline{\Omega}) = 0. \quad (2.44)$$

An integration over θ yields at most a constant of integration, $\underline{\xi}$, which is in general a function of $\tilde{t}$.

$$\int \frac{\partial}{\partial \theta} (\nu^2 \underline{p}_\theta - \nu \underline{p} \times \underline{\Omega}) = \underline{\xi}(\tilde{t}). \quad (2.45)$$

Applying the periodicity constraint for the zero order motion requires $\underline{\xi}(\tilde{t}) = 0$. Therefore,

$$\nu^2 \underline{p}_\theta - \nu \underline{p} \times \underline{\Omega} = 0. \quad (2.46)$$

This equation can be integrated in a local coordinate system with the z axis along the magnetic field as shown in Figure 2.3. Performing the integration gives the result

$$\underline{p} = \hat{p}(\tilde{t}) \left\{ \sin\left(\frac{\Omega}{\nu} \theta + \phi_0\right) \hat{e}_x + \cos\left(\frac{\Omega}{\nu} \theta + \phi_0\right) \hat{e}_y \right\}, \quad (2.47)$$

where ϕ_0 is a phase factor determined from the initial conditions. The periodicity constraint must be applied to this equation, giving the functional form of ν :

Figure 2.3. Local coordinate system with the Z direction along the magnetic field line.

$$\mathcal{V}(\tilde{t}) = \frac{\Omega}{2\pi}. \quad (2.47a)$$

2.2.4 Guiding Center Equations

Although the consistency of the approximation limits the size of $\underline{a}_{\parallel}$, there is no condition on the size of $\underline{a}_{\perp}$ at this point. To simplify the following analysis, it is assumed that the total electric field is at most a first order quantity.* This simplification is made because there are no strong electric fields in our particular applications. Consequently, there is no zero order $\underline{E} \times \underline{B}$ drift appearing in the results. The limitations on the results are discussed as they are derived. We write the electric field as

$$\underline{a} = \epsilon_1 (\underline{a}_{\parallel} + \underline{a}_{\perp}) = \epsilon_1 \underline{a}_1. \quad (2.48)$$

Then the guiding center force equation becomes

$$\epsilon_1 \ddot{\underline{R}} = \epsilon_1 \underline{a}_1 + \dot{\underline{R}} \times \underline{\Omega} - \epsilon_1 \epsilon_2 \dot{\underline{R}} + \epsilon_1 \mathcal{V} \left\langle \underline{p}_0 \times \underline{p} \cdot \frac{\partial \underline{a}}{\partial \underline{R}} \right\rangle + O(\epsilon_1^2). \quad (2.49)$$

The average of the term

$$\left\langle \underline{p}_0 \times \underline{p} \cdot \frac{\partial \underline{a}}{\partial \underline{R}} \right\rangle$$

is performed using equation (2.47) for $\underline{a}$ and its derivatives giving the result

*Zero order electric fields are included in the analysis in references 37, 38, and 39.

$$\begin{aligned} \varepsilon_1 \ddot{\underline{R}} = \varepsilon_1 \underline{a}_1 + \dot{\underline{R}} \times \underline{\Omega} - \varepsilon_1 \varepsilon_2 \dot{\underline{R}} - \\ \varepsilon_1 \pi \nu \hat{\underline{p}}^2 \hat{\underline{b}} \times \frac{\partial \underline{\Omega}}{\partial \underline{R}} + O(\varepsilon_1^2). \end{aligned} \quad (2.50)$$

This equation can be solved by expanding the guiding center velocity with

$$\dot{\underline{R}} = \dot{\underline{R}}_0 + \varepsilon_1 \dot{\underline{R}}_1 + O(\varepsilon_1^2). \quad (2.51)$$

The zero order part of the force equation becomes

$$\dot{\underline{R}}_0 \times \underline{\Omega} = 0 \quad (2.52)$$

or

$$\dot{\underline{R}}_0 = \dot{\underline{R}}_0 \cdot \hat{\underline{b}} = v_{||} \hat{\underline{b}}. \quad (2.53)$$

The first order part of the force becomes

$$\ddot{\underline{R}}_0 = \underline{a}_1 + \underline{R}_1 \times \underline{\Omega} - \pi \nu \hat{\underline{p}}^2 \hat{\underline{b}} \times \frac{\partial \underline{\Omega}}{\partial \underline{R}} - \varepsilon_2 \dot{\underline{R}}_0. \quad (2.54)$$

The parallel component of this equation is obtained by taking the dot product with $\hat{\underline{b}}$, yielding

$$\dot{v}_{||} = \ddot{\underline{R}}_0 \cdot \hat{\underline{b}} = a_{||} - \pi \nu \hat{\underline{p}}^2 \frac{\partial \underline{\Omega}}{\partial s} \cdot \hat{\underline{b}} - \varepsilon_2 \dot{v}_{||}, \quad (2.55)$$

where $\partial/\partial s$ is a gradient in the direction of the magnetic field. This expression, multiplied by $v_{||}$, is used in the next section in the energy loss calculations. The result is

$$v_{||} \dot{v}_{||} = v_{||} a_{||} - v_{||} \frac{\Omega}{2} \hat{\underline{p}}^2 \frac{\partial \underline{\Omega}}{\partial s} \cdot \hat{\underline{b}} - \varepsilon_2 v_{||}^2. \quad (2.56)$$

Expanding the left-hand side of equation (2.54) using equation (2.53) and taking the cross product with $\hat{\underline{b}}$ gives the

perpendicular equation

$$\begin{aligned} \hat{b} \times \left\{ \dot{v}_{\parallel} \hat{b} + v_{\parallel} \dot{\hat{b}} \right\} = & \hat{b} \times \underline{a}_1 + \Omega \underline{R}_{1\perp} - \frac{\Omega}{2} \hat{r}^2 \hat{b} \times \left(\hat{b} \times \frac{\partial \Omega}{\partial \underline{R}} \right) \\ & - \epsilon_2 \hat{b} \times \dot{\underline{R}}_0. \end{aligned} \quad (2.57)$$

The last term in this equation vanishes by equation (2.53).

The expansion for $\dot{\hat{b}}$ is

$$\dot{\hat{b}} = \frac{\partial \hat{b}}{\partial t} + \underline{v} \cdot \frac{\partial \hat{b}}{\partial \underline{R}} = \frac{\partial \hat{b}}{\partial t} + v_{\parallel} \frac{\partial \hat{b}}{\partial s} + O(\epsilon_1). \quad (2.58)$$

Solving equations (2.57) and (2.58) for the perpendicular drift velocity gives the result

$$\begin{aligned} \dot{\underline{R}}_{1\perp} = \frac{\hat{b}}{\Omega} \times \left\{ \frac{\Omega}{2} \hat{r}^2 \hat{b} \times \frac{\partial \Omega}{\partial \underline{R}} - \underline{a}_1 + \right. \\ \left. v_{\parallel} \frac{\partial \hat{b}}{\partial t} + v_{\parallel}^2 \frac{\partial^2 \hat{b}}{\partial s^2} \right\}. \end{aligned} \quad (2.59)$$

Combining equations (2.53) and (2.59), the guiding center velocity is given by

$$\dot{\underline{R}}_1 = v_{\parallel} \hat{b} + \epsilon_1 \dot{\underline{R}}_{1\perp}, \quad (2.60)$$

where $\dot{\underline{R}}_1$ is the usual perpendicular drift velocity.

Notice that the drift velocity is not modified by the addition of collisions. This is a result of limiting the perpendicular electric fields to be a first order quantity. For zero order perpendicular electric fields the last term in equation (2.57) does not vanish and equation (2.59) for the drift velocity includes an additional collision term. In the applications considered here, there are no strong electric fields and, hence, the usual collisionless guiding center velocity is

left unmodified. The collisional modifications to the energy and magnetic moment are considered next.

2.2.5 Collisional Corrections to the Total Energy and Magnetic Moment

The collisional corrections to the energy and magnetic moment are evaluated by considering the average of the dot product of the force and the velocity

$$\frac{dW}{dt} = \langle \underline{v} \cdot \underline{F}_T \rangle = \left(\frac{dW}{dt} \right)_0 + \left(\frac{dW}{dt} \right)_c, \quad (2.61)$$

where $(dW/dt)_0$ is the power loss (gain) in the collisionless case and $(dW/dt)_c$ is the power loss (gain) as a result of collisions. The dot product is calculated separately for each part of the motion using

$$\begin{aligned} \langle \underline{v} \cdot \underline{F} \rangle &= \langle \underline{v} \cdot \underline{F}_G \rangle + \langle \underline{v} \cdot \underline{F}_{ce} \rangle \\ &= \underline{F}_G \cdot \langle \underline{v} \rangle + \langle \underline{v} \cdot \underline{F}_{ce} \rangle. \end{aligned} \quad (2.62)$$

First consider the guiding center contribution,

$$\begin{aligned} \langle \underline{v} \cdot \underline{F}_G \rangle &= \langle (\dot{\underline{R}}_0 + \underline{v} \underline{F}_\theta) + \epsilon_1 (\underline{F}_T + \dot{\underline{R}}_{1\perp}) \rangle \cdot \epsilon_1 \ddot{\underline{R}} + O(\epsilon_1^2) \\ &= \epsilon_1 \dot{\underline{R}}_0 \cdot \ddot{\underline{R}}_0 + \epsilon_1 \underline{v} \ddot{\underline{R}}_0 \langle \underline{F}_\theta \rangle + O(\epsilon_1^2) \\ &= \epsilon_1 \dot{\underline{R}}_0 \cdot \ddot{\underline{R}}_0 + O(\epsilon_1^2). \end{aligned} \quad (2.63)$$

This can be rewritten explicitly in terms of the fields using equation (2.56) to give the result

$$\langle \underline{v} \cdot \underline{F}_G \rangle = \varepsilon_1 \left(v_{||} a_{||} - v_{||} \frac{n}{2} \hat{p}^2 \frac{\partial n}{\partial s} - \varepsilon_2 v_{||}^2 \right) + O(\varepsilon_1^2). \quad (2.64)$$

The dot product of the velocity with the cyclotron force is

$$\langle \underline{v} \cdot \underline{F}_{cyC} \rangle = \langle (R_o + v \underline{p}_o) \cdot \underline{F}_{cyC} \rangle + \varepsilon_1 \langle (\hat{R}_{1\perp} + \underline{p}_\perp) \cdot \underline{F}_{cyC} \rangle + O(\varepsilon_1^2). \quad (2.65)$$

This expression is expanded and computed in Appendix I, giving the result

$$\langle \underline{v} \cdot \underline{F}_{cyC} \rangle = \varepsilon_1 \frac{d}{dt} \left(\frac{n}{2} \hat{p}^2 \right). \quad (2.66)$$

Combining equations (2.64) and (2.66) yields

$$\langle \underline{v} \cdot \underline{F}_T \rangle = \varepsilon_1 \left\{ v_{||} a_{||} - v_{||} \frac{n}{2} \hat{p}^2 \frac{\partial n}{\partial s} \right\} - \varepsilon_1 \varepsilon_2 v_{||}^2 + \varepsilon_1 \frac{d}{dt} \left(\frac{n}{2} \hat{p}^2 \right) + O(\varepsilon_1^2). \quad (2.67)$$

The time derivative in the last term is expanded as

$$\frac{d}{dt} \left(\frac{n}{2} \hat{p}^2 \right) = \frac{n}{2} \frac{d}{dt} \left(\frac{n}{2} \hat{p}^2 \right) + \frac{n}{2} \hat{p}^2 \frac{d}{dt} n. \quad (2.68)$$

Furthermore, the total time derivative of the field is

$$\frac{d}{dt} n = \frac{\partial n}{\partial t} + \underline{v} \cdot \frac{\partial n}{\partial \underline{R}} \quad (2.69)$$

$$\frac{d}{dt} n = \frac{\partial n}{\partial t} + v_{||} \frac{\partial n}{\partial s} + O(\varepsilon_1) \quad (2.70)$$

yielding

$$\frac{d}{dt} \frac{n^2}{2} \hat{p}^2 = n \frac{d}{dt} \left(\frac{n}{2} \hat{p}^2 \right) + \frac{n}{2} \hat{p}^2 \frac{d}{dt} n +$$

$$v_{||} \frac{n}{2} \hat{p}^2 \frac{\partial n}{\partial s} + O(\varepsilon_1). \quad (2.71)$$

In the applications considered in Chapter V there is no explicit time dependence in the magnetic field. Combining equations (2.68) and (2.71) with this assumption yields

$$\frac{d}{dt} W = \varepsilon_1 \left\{ a_{||} v_{||} - \varepsilon_2 v_{||} + n \frac{d}{dt} \frac{n}{2} \hat{p}^2 \right\}$$

$$+ O(\varepsilon_1^2). \quad (2.72)$$

Before continuing further, it is useful to rewrite equation (2.72) in dimensional form. In terms of dimensionless variables the magnetic moment is given by

$$\mu = \frac{m}{2} \frac{v_{\perp}^2}{\beta} = \frac{q}{2c} \hat{p}^2 n. \quad (2.73)$$

Taking the product with the magnetic field yields

$$\mu \beta = \frac{n^2}{2} \hat{p}^2. \quad (2.74)$$

Therefore, the dimensional form of equation (2.72) is

$$\frac{d}{dt} W = q v_{||} E_{||} - \gamma_D m v_{||}^2 + \beta \frac{d}{dt} \mu. \quad (2.75)$$

In computing the energy changes in the cyclotron motion, only the left-hand side of the force equation was used to give

$$\langle \underline{v} \cdot \underline{F}_{cyc} \rangle^{rhs} = \epsilon_1 \frac{d}{d\tilde{t}} \left(\frac{\Omega^2 \hat{p}^2}{2} \right) = \epsilon_1 \left\{ -\frac{\Omega^2}{2} \frac{d}{d\tilde{t}} \hat{p}^2 + \Omega \frac{d}{d\tilde{t}} \Omega \hat{p}^2 \right\} \quad (2.76)$$

Performing the average over the right-hand side of the cyclotron force equation (Appendix I) yields

$$\begin{aligned} \langle \underline{v} \cdot \underline{F}_{cyc} \rangle^{rhs} = & -\epsilon_1 \Omega^2 \frac{d}{d\tilde{t}} \frac{\hat{p}^2}{2} - \epsilon_1 \epsilon_2 \hat{p}^2 \Omega^2 \\ & + \epsilon_1 \Omega \frac{d}{d\tilde{t}} \left(\frac{\Omega}{2} \hat{p}^2 \right). \end{aligned} \quad (2.77)$$

Equating these two expressions yields

$$\Omega \frac{d}{d\tilde{t}} \left(\frac{\Omega}{2} \hat{p}^2 \right) = -\epsilon_2 \hat{p}^2 \Omega^2. \quad (2.78)$$

Expressing this equation in dimensional form yields

$$\frac{d}{dt} \mu = -2 \gamma_D \mu \quad (2.79)$$

Substituting equation (2.79) into equation (2.75) eliminates the magnetic moment and gives the desired expression for the time rate of change of the energy,

$$\frac{dW}{d\tilde{t}} = q v_{||} E - 2 \gamma_D \frac{m}{2} (v_{||}^2 + v_{\perp}^2) \quad (2.80)$$

and

$$\left(\frac{dW}{d\tilde{t}} \right)_c = -2 \gamma_D W. \quad (2.81)$$

2.3 Self-Consistent Field Calculations

In a self-consistent calculation the fields are updated during the particle orbit calculations to include the effects of the spatial redistribution of the sources. In addition to

particle source terms (beams), the background plasma sources must also be included self-consistently.

Suppose L_0 and ω_0 are the characteristic length and frequency in our application. The following assumptions are made in the physical modeling of the self-consistent fields of the beams and the background plasma:

1. long spatial length scales compared to the Debye length, $L_0 \gg \lambda_D$
2. low frequencies compared to the cyclotron and plasma frequencies, $\omega_0 \ll \Omega_c, \omega_p$
3. exact macroscopic and microscopic space charge neutralization
4. negligible poloidal currents.

Since space charge neutralization of the ion beams is assumed, self-consistent solutions of Poisson's equation are not included in the field calculations. Furthermore, with the low frequency assumption, the displacement current in Ampere's law can be neglected. The remaining field equation to be solved self-consistently is

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} \quad (2.82)$$

where

$$\underline{J} = \{ J_\phi(\text{plasma}) + J_\phi(\text{beams}) \} \hat{e}_\phi. \quad (2.83)$$

$J_\phi(\text{plasma})$ is the contribution from the background plasma and is obtained from the pressure balance relation,

$$\nabla p = \frac{J_{\phi}}{c} (\text{plasma}) \times \underline{B}. \quad (2.84)$$

The solutions of equation (2.82) are most easily interpreted in terms of a magnetic flux function, ψ , defined by

$$\underline{B} = \frac{\nabla \psi \times \hat{e}_{\phi}}{R}. \quad (2.85)$$

The magnetic flux function is analogous to the stream function in incompressible hydrodynamics since it arises from the vanishing divergence of $\underline{B}$. The relationship between ψ and the only surviving component of the vector potential, A_{ϕ} , is

$$\psi = R A_{\phi}, \quad (2.86)$$

where

$$\underline{B} = \nabla \times \underline{A} \quad (2.87)$$

$$\nabla \cdot \underline{A} = 0 \quad (2.88)$$

$$\nabla^2 \underline{A} = -\frac{4\pi}{c} \underline{J}. \quad (2.89)$$

Expanding equation (2.89) in cylindrical coordinates and substituting equation (2.86) gives the equation for the magnetic flux function,

$$\frac{1}{R} \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) = -\frac{4\pi}{c} J_{\phi} \quad (2.90)$$

which is solved numerically.

2.4 Vacuum Field Model

Experimentally the toroidal field varies approximately inversely with radial position. The functional form used in the

simulations is

$$\beta_{\phi} = \frac{I}{5R} . \quad (2.91)$$

where

B_{ϕ} = toroidal field in Gauss

I = current in Amperes

R = distance from the Z axis in centimeters.

A uniform vertical field is assumed. The strength of the field determines the major radius, R_0 , of the beam-plasma system and is derived from the formula^{9,40}

$$\beta_z (\text{Gauss}) = \frac{I_T}{10 R_0} \left\{ \ln 8A + \frac{\ell_i - 3}{2} \right\} , \quad (2.92)$$

where

I_T = total toroidal current in Amperes

$A = R_0/r_0$ = aspect ratio

ℓ_i = internal inductance per length .

There is also a term proportional to the ratio of the plasma pressure to the magnetic field energy density which is not included in equation (2.92).

The strength of the vertical field is related to the boundary conditions imposed on the system. The two types of boundary conditions which may be used in the present model are:

1. perfectly conducting rectangular walls
2. far field boundary conditions (no conducting walls).

In the first case, image currents are set up in the walls of the conductor and the field due to these image currents directly affects the positioning of the beams. In the second type of boundary condition, only the applied vertical field affects the positioning of the beams. This is the case for which equation (2.92) is applicable. The validity of equation (2.92) is discussed in Chapter V.

III. NUMERICAL METHODS

3.1 Introduction

We present numerical algorithms and solution methods for self-consistent guiding center motion. In our code, GUIDON77, we have added an algorithm for time centering the integrator, we have included collisional slowing down of the particles, and we have implemented several smoothing methods. The time centering of the integrator and the method for including collisional corrections are discussed in the first sections. Smoothing algorithms and their properties in Fourier k space are studied in the next sections. We have selected the best smoothing scheme from those studied for the numerical test runs in the next chapter.

Because of the huge amount of physical data generated in the orbit calculations, efficient and informative diagnostics are essential. We have developed several diagnostics to aid in interpreting the results of the CIT simulations. These diagnostics, along with some of the original diagnostics in GUIDON, are discussed in the last sections of the chapter.

3.2 Solution Methods

The self-consistent orbit calculations proceed in three steps (Figure 3.1):

1. The particle positions are advanced in time by numerically integrating the velocities over a finite time interval, Δt .
2. Collisional corrections to the motion are performed

Figure 3.1. Procedure for self consistent orbit calculations.

by numerically advancing the magnetic moment and energy equations over Δt .

3. Particle source terms are evaluated and self-consistent fields are updated by numerically solving Ampere's law for the magnetic flux function.

The guiding center velocities are recomputed with the updated fields and the three steps are repeated until the maximum simulation time is reached.

3.2.1 Particle Orbit Advancement

The particle positions satisfy ordinary differential equations of the form

$$\frac{d\mathbf{R}_i}{dt} = \mathbf{V}_i(\mathbf{R}_i, \mathbf{B}) \quad \text{for } i = 1, 2, 3, \dots, N. \quad (3.1)$$

where N is the number of simulation particles (macroparticles) and $\mathbf{V}_i$ is the guiding center velocity. The guiding center velocity (Eq. 2.59) was derived using the expansion developed in Chapter II. The numerical solutions of these equations are obtained using a predictor-corrector method.

Predictor step

$$\mathbf{R}_i^{n+1*} = \mathbf{R}_i^{n-1} + 2\Delta t \mathbf{V}_i^n \quad (3.2a)$$

$$\mathbf{V}_i^{n+1*} = f(\mathbf{R}_i^{n+1*}, \mathbf{B}^n) \quad (3.2b)$$

Corrector step

$$\mathbf{R}_i^{n+1} = \mathbf{R}_i^n + \Delta t \frac{1}{2} (\mathbf{V}_i^n + \mathbf{V}_i^{n+1*}) \quad (3.3a)$$

$$\mathbf{V}_i^{n+1} = f(\mathbf{R}_i^{n+1}, \mathbf{B}^n) \quad (3.3b)$$

The time step in this scheme is limited by the size of the truncation errors and the errors induced by the straight line approximations to the curved particle trajectory. The non-linearity in the physical phenomena may also require reduced time steps. Test simulation runs are always performed to determine the maximum time step for a given physical problem.

A time centered difference scheme is second order accurate. The leap frog scheme for updating the positions in each of the above steps is a time centered scheme with respect to the position and velocity variables. However, there is some loss of time centering because the magnetic field in the guiding center velocity is evaluated at the old time step. In the tests presented in the next chapter, the lack of time centered fields is shown to add to the nonphysical pinching of the beam. To center the fields, we have added a calculation of the fields between the predictor and corrector steps, as shown in Figure 3.2. Test runs with this new method are discussed in Section 4.2.6.

3.2.2 Collisional Modifications to the Guiding Center Motion

The collisional corrections to the guiding center motion are performed each time step after the advancement of the particle positions. In Chapter II we showed that it is necessary to correct the guiding center motion of particles for collisions by correcting both the energy and the magnetic moment. Refer to Eqs. (2.79) and (2.81). We consider the slowing down of ions on a Maxwellian electron background. The ordering of the ion velocity with respect to the electron thermal velocity is

Figure 3.2. Procedure for updating the fields between predictor and corrector steps.

$$V_i \ll V_{the} . \quad (3.4)$$

In this approximation, the ion collisional energy losses are

$$\frac{dW_i}{dt} = - \frac{2(W - W_e)}{\tau_{de}}, \quad (3.5)$$

where

$$\frac{1}{\tau_{de}} = \frac{4}{3} \frac{\sqrt{\pi}}{2} e^4 \frac{Z_i^2 n_e \ln \Lambda \sqrt{m_e}}{m_i (k T_e)^{3/2}} = \frac{1.06 \times 10^{-13} n_e \ln \Lambda}{m_i T_e^{3/2}}, \quad (3.6)$$

and W_e is the electron energy. In Eq. (3.6) the mass is in AMU, the temperature is in keV and the density is in cm^{-3} . The Coulomb logarithm, $\ln \Lambda$, is defined as³³

$$\ln \Lambda = \ln(2/\chi_{\min}), \quad (3.7)$$

where $\chi_{\min}$ is the minimum scattering angle in the center of mass system. As shown in Figure 3.3 there are collisional energy losses both perpendicular and parallel to the magnetic field lines. When angular scattering is neglected, the pitch angle, α , remains constant and the changes in the magnetic moment can be written as

$$\frac{d\mu}{dt} = - \frac{2(\mu - \mu_{oe})}{\tau_{oe}}, \quad (3.8)$$

where

$$B\mu_{oe} = \frac{2}{3} W_e . \quad (3.9)$$

The electron drag time appearing in the expressions for the energy losses and the changes in the magnetic moment depends explicitly on both the electron temperature and electron density. There is also an implicit density dependence in the $\ln \Lambda$ term.

V^0 = velocity before collisional slowing down

V^n = velocity after collisional slowing down

α = pitch angle

$V_{\perp}^0, V_{\perp}^n$ = perpendicular velocity before and after collisional slowing down

Figure 3.3. The velocity vectors before and after collisional slowing down.

In our simulations, $\ln \Lambda$ is held constant. Invoking a quasi-neutrality condition using the ion particles to compute an electron density results in large noise levels in the computed values of the slowing down time. For this reason, the electron density is modeled analytically. Three electron density profiles have been programmed:

constant density profile

$$n_e = \begin{cases} n_{e0} & \text{for } r \leq r_{e0} \\ 0 & \text{for } r > r_{e0} \end{cases} \quad (3.10)$$

parabolic density profile

$$n_e = n_{e0} \left(1 - \frac{r^2}{r_{e0}^2} \right) \quad (3.11)$$

exponential density profile

$$n_e = n_{e0} \left\{ .9 \exp\left(\frac{16r^2}{r_{e0}^2} \right) + .1 \left(1 - \frac{r}{r_{e0}} \right) \right\} \quad (3.12)$$

where

R_{e0} = major radius of electrons

r_{e0} = minor radius of electrons

r = distance along minor radius

n_{e0} = central electron density.

The last two profile shapes are motivated by experimental results in previous toroidal systems. Since $\ln \Lambda$ is a very slowly varying function, it is assumed constant and read as input into the problem. The electron temperature is also modeled analytically with a profile shape given by

$$T_e = T_{eo}(R_{eo}) \left\{ 1 - C_1 \frac{n^2}{n_{eo}^2} \right\}, \quad (3.13)$$

for

C_1 = input parameter.

The parameter C_1 is determined empirically from experimental toroidal data and is currently set to $C_1 = .97$.

The finite difference form of the energy equation (3.5) is

$$W_i^{n+1} = W_i^n \exp\left(\frac{-2\Delta t}{\tau_{pe}}\right) + W_e \left[1 - \exp\left(\frac{-2\Delta t}{\tau_{pe}}\right) \right]. \quad (3.14)$$

Since the pitch angle α is unchanged as a result of collisional slowing down, the magnetic moment can be advanced from the computed values of the energy,

$$\mu_i^{n+1} = \mu_i^n \frac{W_i^{n+1}}{W_i^n}. \quad (3.15)$$

3.2.3 Self-Consistent Field Calculations

The self-consistent field equation which must be solved numerically for the magnetic flux function is

$$-\frac{1}{R} \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) = \frac{4\pi}{c} J_\phi, \quad (3.16)$$

where

$$J_\phi = J_\phi(\text{fluid}) + J_\phi(\text{particles}). \quad (3.17)$$

Since the differential operator is linear, a fast method for solving this equation is to use fast Fourier transforms in the z direction and finite differences in the R direction. Fast Fourier transform methods are discussed in reference 41 and the

application to Eq. (3.16) is described in reference 31. The current density and flux function are evaluated on a grid staggered with respect to the magnetic field component grid (Figure 3.4). This increases the accuracy of the magnetic field component evaluation by centering the first derivatives:

$$\beta_R(i,j) = -\frac{1}{R_i \Delta z} \left\{ \frac{1}{2} (\psi_{i+\frac{1}{2}} + \psi_{i-\frac{1}{2}})_{j+\frac{1}{2}} - \frac{1}{2} (\psi_{i+\frac{1}{2}} + \psi_{i-\frac{1}{2}})_{j-\frac{1}{2}} \right\} \quad (3.18)$$

$$\beta_z(i,j) = \frac{1}{R_i \Delta R} \left\{ \frac{1}{2} (\psi_{j+\frac{1}{2}} + \psi_{j-\frac{1}{2}})_{i+\frac{1}{2}} - \frac{1}{2} (\psi_{j+\frac{1}{2}} + \psi_{j-\frac{1}{2}})_{i-\frac{1}{2}} \right\}. \quad (3.19)$$

In some physical problems in which the fluid pressure is a function of both ψ and B , a staggered spatial grid may not be a good choice. The accuracy gained in field evaluation is lost in the evaluation of the pressure function. In these more general applications other methods for computing the fields, such as spline fitting, may be desirable.

There are two contributions to the source term in equation (3.16): the particle source term and a fluid source term. In the TCT applications, the fluid source term is nonlinear,

$$\mathcal{J}_\phi(\text{fluid}) = \mathcal{J}_\phi(\psi). \quad (3.20)$$

Johnson⁷ developed an iterative scheme to handle nonlinearities of this type. In the CIT simulations, the electron fluid current density is modeled analytically as a function of the independent variables only. Therefore, nonlinear iteration is unnecessary.

The particle source terms are evaluated by accumulating the particle currents attributed to each mesh point. Since

Figure 3.4. Staggered grid for B and ψ calculations.

radiative effects are ignored, the superparticle charge density is assumed constant and spread over a cell width in each direction. A bilinear interpolation scheme (area weighting¹⁷) is used to weight the charge density among the four nearest grid points as shown in Figure 3.5.

3.3 Smoothing Algorithms

The result of simulating a plasma with a limited number of superparticles is that fluctuations are produced in a particle quantity which is interpolated onto a grid. These fluctuations may cause unphysical wave motion which can grow large enough to distort the physical phenomena being simulated. Increasing the number of simulation particles is not an efficient solution to this problem since the fluctuation levels are a very slowly decreasing function of the number of simulation particles¹⁷. The most practical solution to this problem is to apply smoothing to the grid quantities. Two other compelling reasons for smoothing noisy grid quantities are:

1. The Fourier components of the function which are not well represented on the grid can be selectively damped.
2. The aliasing of short wavelength phenomena onto longer wavelength modes can be minimized.

One method for smoothing grid quantities is to replace the value of the grid function at a grid point i with a weighted average of the value at i and the neighboring grid points. In this section several examples of this form of smoothing are

Contribution of the particle charge to individual grid points:

- To i, j
- To $i + 1, j$
- To $i + 1, j - 1$
- To $i, j - 1$

Figure 3.5. Interpolation of a super particle to a discrete grid.

considered starting from the simplest case of one dimensional smoothing. Although smoothing in real space is generally not as convenient as smoothing in Fourier $\underline{k}$ space, for the cases considered in this thesis (i.e., $\Delta x = \Delta y$), the adjustment of weighting factors for groups of grid points does provide sufficient flexibility for changing the shape of the amplitudes of the modes in Fourier $\underline{k}$ space.

3.3.1 Smoothing in One Dimension

Consider a function f whose values are known on a discrete spatial mesh in one dimension as shown in Figure 3.6. The smoothed value of the function at a grid point i is given by

$$\bar{f}_i = \sum_j \alpha_j f_j, \quad (3.21)$$

where the summation over j includes the point i and a selected set of neighboring grid points. The α_j are weighting factors for these grid points and are determined by imposing conditions on the size of selected modes in Fourier $\underline{k}$ space. In all of the smoothing schemes discussed in this thesis, equal weighting is given to points which are at equal distances from the grid point i . Thus, a three point scheme involving the two nearest neighboring grid points is

$$\bar{f}_i = \alpha_1 f_i + \alpha_2 (f_{i+1} + f_{i-1}). \quad (3.22)$$

The finite Fourier series for the function f is given by⁴¹

$$f(x_i) = \sum_{k_1=0}^{N-1} A_{k_1} e^{-\frac{2\pi i k_1 x_i}{L}} = \sum_{k=0}^{\frac{2\pi(N-1)}{L}} A_k e^{-ikx_i} \quad (3.23)$$

where $\Sigma \rightarrow$ summation over k for $k = 0, \frac{2\pi}{L}, \frac{4\pi}{L}, \dots, \frac{2\pi(N-1)}{L}$ and

Figure 3.6. One dimensional spatial mesh.

Figure 3.7. Representation of the shortest wave lengths on the grid. The circled points are the points of the waveform evaluated on the grid.

$$A_k = \frac{1}{L} \sum_j f(x_j) e^{ikx_j}. \quad (3.24)$$

Evaluating the function at the other two grid points yields

$$f_{i+1} = \sum_k A_k e^{-ik(x_i + \Delta x)} \quad (3.25)$$

$$f_{i-1} = \sum_k A_k e^{-ik(x_i - \Delta x)}. \quad (3.26)$$

Combining these equations gives the series form of the smoothed function,

$$\bar{f}_i = \sum_k (\alpha_1 + 2\alpha_2 \cos k\Delta x) A_k e^{-ikx_i}. \quad (3.27)$$

Thus the mode amplitudes of the smoothed function are

$$\bar{A}_k = (\alpha_1 + 2\alpha_2 \cos k\Delta x) A_k. \quad (3.28)$$

A normalized mode amplitude function is used in the following analysis and is defined as

$$\tilde{f}_k = \frac{\bar{A}_k}{A_k} \quad \text{for } k = 0, \frac{2\pi}{L}, \frac{4\pi}{L}, \dots, \frac{2\pi}{L}(N-1) \quad (3.29)$$

A constraint which must be applied to the smoothing algorithms is that long wavelength phenomena (which are well represented on the grid) remain unaffected by the smoothing. Thus, for the longest wavelengths, $k\Delta x = 0$, the normalized amplitude of the smoothed function must be unity:

$$\tilde{f}(k\Delta x|_0) = \frac{\bar{A}_0}{A_0} = 1. \quad (3.30)$$

Applying this constraint to equation (3.28) yields

$$1 = \alpha_1 + 2\alpha_2. \quad (3.31)$$

Next consider the representation of short wavelengths on the grid as shown in Figure 3.7. Obviously the shortest wavelength which is represented on the grid is $\lambda_{\min} = 2 \Delta x$ or $k_{\max} = \pi/\Delta x$. Since it is desirable to represent physical phenomena with a minimum of about ten points per wavelength, then it is beneficial to damp or eliminate modes for $k \Delta x \geq 2\pi/9$. In the example in equation (3.27) the α_j can be chosen to eliminate the shortest wavelength:

$$\tilde{f}(k\Delta x|_{\pi}) = \alpha_1 - 2\alpha_2 = 0, \quad (3.32)$$

yielding

$$\alpha_1 = 2\alpha_2. \quad (3.33)$$

The solutions of equations (3.30) and (3.33) yield the smoothing scheme:

$$\alpha_1 = \frac{1}{2} \quad (3.34)$$

$$\alpha_2 = \frac{1}{4} \quad (3.35)$$

$$\bar{f}_i = \frac{1}{2} f_i + \frac{1}{4} (f_{i+1} + f_{i-1}) \quad (3.36)$$

$$\tilde{f}(k) = \frac{1}{2} (1 + \cos k\Delta x). \quad (3.37)$$

The normalized amplitudes are plotted in Figure 3.8. Notice that the longest wavelength modes are least damped, and the shortest wavelengths are heavily damped as desired.

This smoothing algorithm can also be written in terms of a finite difference operator:

$$\bar{f}_i = (1 + \alpha_3 \delta_i^2) f_i, \quad (3.38)$$

where δ_i^2 is the second order centered difference operator

Figure 3.8. Normalized mode amplitude function for the smoothing algorithm in equation (3.36).

Figure 3.9. A comparison of the mode amplitude functions for the smoothing algorithms in equations (3.36) and (3.41).

Figure 3.10. Normalized mode amplitude function for the smoothing scheme in Eq. (3.43).

$$\delta_i^2 f_i = f_{i+1} - 2f_i + f_{i-1}, \quad (3.39)$$

and

$$\alpha_3 = \frac{\Delta x^2}{4} \quad (3.40)$$

This suggests that additional smoothing of the function may result by applying this finite difference operation another time, giving a smoothed function of the form

$$\begin{aligned} \bar{f}_i &= (1 + \alpha_3 \delta_i^2)^2 f_i \\ &= \frac{3}{8} f_i + \frac{1}{16} (f_{i+2} + f_{i-2}) + \frac{1}{4} (f_{i+1} + f_{i-1}) \end{aligned} \quad (3.41)$$

$$\tilde{f}(k) = \frac{3}{8} + \frac{1}{2} \cos k\Delta x + \frac{1}{8} \cos 2k\Delta x. \quad (3.42)$$

The normalized amplitudes of this algorithm are compared to the previous case in Figure 3.9. Although this additional damping of the short wavelengths may be a requirement in some simulations, it should be recognized that wavelengths shorter than $\lambda \leq 8\lambda_{\min}$ are being damped. Therefore, physical wavelengths of interest must be chosen equal to or longer than about sixteen grid spacings.

This last algorithm can be derived from a more general set of five point schemes involving four points equally spaced at either Δx or $2\Delta x$ from the grid point i ,

$$f_i = \alpha_1 + \frac{\alpha_2}{2} (f_{i+1} + f_{i-1}) + \frac{\alpha_3}{2} (f_{i-2} + f_{i+2}) \quad (3.43)$$

$$\tilde{f}(k) = \alpha_1 + \alpha_2 \cos k\Delta x + \alpha_3 \cos 2k\Delta x. \quad (3.44)$$

The condition on long wavelengths yields

$$1 = \alpha_1 + \alpha_2 + \alpha_3 . \quad (3.45)$$

Requiring the amplitudes of the shortest wavelengths to vanish yields

$$\alpha_1 = \alpha_2 - \alpha_3 . \quad (3.46)$$

The solution of equations (3.45) and (3.46) gives the results,

$$\alpha_2 = \frac{1}{2} . \quad (3.47)$$

This gives a smoothed function of the form,

$$\tilde{f}(k) = \frac{1}{2} - \alpha_3 + \frac{1}{2} \cos k\Delta x + \alpha_3 \cos 2k\Delta x . \quad (3.48)$$

At this point α_3 is arbitrary; however, there is a zero at $k\Delta x = \pi$ and f is positive at $k\Delta x = 0$. If f is allowed to become negative in the interval $0 \leq k\Delta x \leq \pi$, phase errors are introduced into the modes. To prevent this, the condition $\tilde{f}' < 0$ in the interval will give a function which is always positive but continuously decreasing to zero at $k\Delta x = \pi$. This requires that

$$|\alpha_3| \leq \frac{1}{8} . \quad (3.49)$$

The parameter α_3 can be varied over this range to give the possible shapes of $\tilde{f}(k)$ as shown in Figure 3.10. The choice $\alpha_3 = 1/8$ reduces to the scheme in equation (3.42).

One other question which must be considered in developing these numerical schemes is the magnitude of short wavelength phenomena, $\lambda < 2\Delta x$, which are not represented on the grid. Although the unsmoothed amplitudes of these modes are generally very small compared to the physical phenomena being simulated, these short wavelength modes can affect the simulations by

reappearing on the grid as longer wavelengths. The inaccurate representation of short wavelengths as long wavelengths is referred to as aliasing and is strictly a result of using a spatial grid. The aliasing of $\lambda = \frac{4}{3} \Delta x$ and $\lambda = \frac{4}{5} \Delta x$ modes into the $\lambda = 4 \Delta x$ mode is illustrated in Figure 3.11. Aliasing can be analyzed by dividing $k \Delta x$ into zones of length π , the Brillouin zones¹⁷ (Figure 3.12). Then it is easy to see the aliasing of the high k modes. All modes with $\cos(n\pi + \delta)$ alias into the $\cos(\pi - \delta)$ mode since

$$\cos(n\pi + \delta) = (-1)^n \cos \delta \quad (3.50)$$

and

$$\cos(\pi - \delta) = -\cos(\delta) . \quad (3.51)$$

Furthermore, each mode in a given zone aliases not only into the $n = 1$ zone, but also into all other zones below the given zone as shown in Figure 3.12c. Therefore, smoothing either in real space or Fourier space should be selected so that there is no possibility of amplification of these modes. The damping of modes outside the first Brillouin zone for the schemes in equations (3.37) and (3.42) is shown in Figure 3.13.

3.3.2 Smoothing in Two Dimensions

Smoothing in two dimensions becomes much more complicated because waves can propagate at angles to the axes. In the methods discussed here, the grid cell is assumed to be square, $\Delta x = \Delta y = h$. One possible two dimensional scheme motivated by the one dimensional schemes in equations (3.38) and (3.41) is

$$\bar{f}_{ij} = (1 + \alpha_1 \delta_{ij}^2) f_{ij} , \quad (3.52)$$

a) Aliasing of the $\frac{4 \Delta x}{3}$ wave into the $4 \Delta x$ wave on the grid

b) Aliasing of the $\frac{4 \Delta x}{5}$ mode onto the $4 \Delta x$ mode on the grid.

Figure 3.11. Aliasing of short wavelengths onto long wavelengths on a grid.

Figure 3.12 a) Division of $k \Delta x$ space into n Brillouin zones. Figure 3.12 b) Aliasing of modes in the $n = 2$ zone into the $n = 1$ zone. Figure 3.12 c) Aliasing of modes in the higher Brillouin zones into the $n = 1$ and $n = 2$ zones.

a) Three Brillouin zones for equation (3.37)

b) Three Brillouin zones for equation (3.42)

Figure 3.13. Examples of the normalized mode amplitude function for three Brillouin zones.

where

$$\delta_{ij}^2 = \delta_i^2 + \delta_j^2. \quad (3.53)$$

The δ_i^2 and δ_j^2 are centered second order difference operators in the x and y directions. Expanding the difference operators gives the algorithm,

$$\bar{f}_{ij} = (1 - \frac{4\alpha_1}{h^2})f_{ij} + \frac{\alpha_1}{h^2} \{ (f_{i+1,j} + f_{i-1,j}) + (f_{i,j+1} + f_{i,j-1}) \} \quad (3.54)$$

The five points in the scheme are shown in Figure 3.14a. The Fourier series representation of the function $f(x,y)$ is given by

$$f(x_i, y_j) = \sum_{k_1} \sum_{k_2} A_{k_1, k_2} e^{-i \frac{2\pi}{L} (k_1 x + k_2 y)}, \quad (3.55)$$

where

$$\underline{k} = k_x \hat{e}_x + k_y \hat{e}_y = k (\cos \theta \hat{e}_x + \sin \theta \hat{e}_y)$$

and $k_x = \frac{2\pi}{L} k_1$

$$k_y = \frac{2\pi}{L} k_2.$$

The Fourier analysis of the smoothing algorithm follows through the same steps as the one dimensional analysis. The result is

$$\tilde{f}(k) = (1 - \frac{4\alpha_1}{h^2}) + \frac{2\alpha_1}{h^2} (\cos k_x h + \cos k_y h). \quad (3.56)$$

The shortest wavelengths along either axis ($\theta = 0, \pi/2$) can be forced to vanish with the choice $\alpha_1 = h^2/4$ yielding

$$\bar{f}_{ij} = \frac{1}{4} \{ f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} \} \quad (3.57)$$

$$\tilde{f}(k) = \frac{1}{2} \{ \cos k_x h + \cos k_y h \} \quad (3.58)$$

a) Mesh points for the smoothing scheme in Eq. (3.54).

b) Mesh points for the smoothing scheme in Eq. (3.68).

c) Mesh points for the smoothing scheme in Eq. (3.70).

Figure 3.14. Grid points used in the two dimensional smoothing algorithms.

This seemingly innocuous averaging over four adjacent grid points has some very poor properties in k space. The $\theta = \text{constant}$ planes for the $n = 1$ zone are plotted in Figure 3.15 for $\theta = 0, \pi/16, \pi/8$, and $\pi/4$. Off axis large phase errors ($\tilde{f} < 0$) are introduced. This really is not a surprising result since the smoothing does not involve any contribution from the original point (i, j) .

Schemes involving these five points can be developed using the general form

$$f_{ij} = \alpha_1 f_{ij} + \alpha_2 \{ (f_{i+1,j} + f_{i-1,j}) + (f_{i,j+1} + f_{i,j-1}) \} \quad (3.59)$$

$$\tilde{f}(k) = \alpha_1 + 2\alpha_2 \{ \cos k_x h + \cos k_y h \}. \quad (3.60)$$

The long wavelength normalization condition yields

$$1 = \alpha_1 + 4\alpha_2, \quad (3.61)$$

and the normalized mode function becomes

$$\tilde{f}(k) = (1 - 4\alpha_2) + 2\alpha_2 (\cos k_x h + \cos k_y h). \quad (3.62)$$

Often the α_1 in these schemes are determined by the wave propagation at $\theta = 0$ and $\theta = \pi/4$. The mode functions at these two angles are

$$\tilde{f}_0 = (1 - 4\alpha_2) + 2\alpha_2 (\cos kh + 1) \quad (3.63)$$

$$\tilde{f}_{\frac{\pi}{4}} = (1 - 4\alpha_2) + 4\alpha_2 \cos \frac{\sqrt{2}}{2} kh. \quad (3.64)$$

These are linear functions in α_2 and are plotted in Figure 3.16 for the shortest wavelength. To prevent amplification, α_2 must be positive. Requiring $\tilde{f} \geq 0$ to avoid phase errors gives the range for α_2 :

$$0 < \alpha_2 < .1557. \quad (3.65)$$

Figure 3.15. Normalized mode function for $\theta = 0$, $\pi/16$, $\pi/8$, and $\pi/4$ for the scheme in equation (3.57).

Figure 3.16. Normalized mode function for $\theta = 0$, $\pi/4$ as a function of α_2 for the shortest wavelengths, $k \Delta x = \pi$.

In this scheme the amplitude of the shortest wavelength mode decreases as θ changes from zero to $\pi/4$. The mode function is plotted in Figure 3.17 for different θ and a selection of α_2 values. At the optimal value of $\alpha_2 = .1557$, the short wavelength amplitudes are $\tilde{f}_0(\pi) = .377$ and $\tilde{f}_{\pi/4}(\pi) \cong 0$. Additional points are required in the scheme to further minimize these short wavelength modes. More points are added in the next two examples.

Damping can be increased by applying a difference operator of the form used in equation (3.51) twice,

$$\bar{f}_{ij} = (1 + \alpha_1 \delta_{ij}^2)^2 f_{ij}, \quad (3.66)$$

where

$$\alpha_1 = \frac{h^2}{4}. \quad (3.67)$$

An expansion of this equation gives

$$\begin{aligned} \bar{f}_{ij} = & \frac{1}{4} f_{ij} + \frac{1}{8} (f_{i+1,j+1} + f_{i-1,j-1} + f_{i-1,j+1} + f_{i+1,j-1}) + \\ & \frac{1}{16} (f_{i,j-2} + f_{i,j+2} + f_{i-2,j} + f_{i+2,j}) \end{aligned} \quad (3.68)$$

$$\tilde{f}(k) = \frac{1}{4} + \frac{1}{2} \cos k_x h \cos k_y h + \frac{1}{8} \{ \cos 2k_x h + \cos 2k_y h \} \quad (3.69)$$

The nine grid points involved in the scheme are illustrated in Figure 3.14b and the amplitudes of the modes are shown in Figure 3.18. As expected, there is more damping with this scheme. However, the long wavelengths are damped slightly too much and the short wavelengths are not damped quite enough near $\theta = \pi/4$. A more symmetric array of grid points can be expected to give better properties for the smoothed function.

Figure 3.17. Normalized mode amplitude function for the five point smoothing schemes in equation (3.59).

Figure 3.18. Normalized mode amplitude function for the smoothing scheme in equation (3.18).

The last example to be considered here involves an array of eight points in a square region around the point to be smoothed (Figure 3.14c). The assumed form of the smoothing algorithm is

$$\begin{aligned} \bar{f}_{ij} = & \alpha_1 f_{ij} + \alpha_2 (f_{i+1,j+1} + f_{i-1,j+1} + f_{i+1,j-1} + f_{i-1,j-1}) + \\ & \alpha_3 (f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}) \end{aligned} \quad (3.70)$$

$$\begin{aligned} \tilde{f}(k) = & \alpha_1 + 2\alpha_2 [\cos k_x \Delta x + \cos k_y \Delta y] + \\ & 4\alpha_3 \cos k_x \Delta x \cos k_y \Delta y. \end{aligned} \quad (3.71)$$

The long wavelength condition requires,

$$1 = \alpha_1 + 4\alpha_2 + 4\alpha_3. \quad (3.72)$$

Eliminating the shortest wavelength on axis requires,

$$\alpha_1 = 4\alpha_3. \quad (3.73)$$

Combining the two conditions gives the result,

$$\alpha_2 = \frac{1}{4} - 2\alpha_3 \quad (3.74)$$

$$\begin{aligned} \tilde{f}(k) = & 4\alpha_3 + \left(\frac{1}{2} - 4\alpha_3\right)(\cos k_x h + \cos k_y h) + \\ & 4\alpha_3 \cos k_x h \cos k_y h. \end{aligned} \quad (3.75)$$

Requiring $\tilde{f} \geq 0$ for the shortest wavelengths at $\theta = \pi/4$ gives the range of α_3 :

$$.05873 \leq \alpha_3 < 1. \quad (3.76)$$

The special case of this set of algorithms most often used in two dimensional smoothing is

$$\alpha_3 = \frac{1}{16}; \quad \alpha_2 = \frac{1}{8}; \quad \alpha_1 = \frac{1}{4} \quad (3.77)$$

$$\bar{f}_{ij} = \frac{1}{4} + \frac{1}{8} \{ f_{i+1,j+1} + f_{i+1,j-1} + f_{i-1,j+1} + f_{i-1,j-1} \} + \frac{1}{16} \{ f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} \}. \quad (3.78)$$

The amplitudes of the modes are plotted in Figure 3.19.

All of the schemes discussed previously are related to a two dimensional Fourier cosine series of the mode amplitude function,

$$\tilde{f}(k) = \sum_{k_x} \sum_{k_y} (a_{k_x k_y} \cos k_x h \cos k_y h) \quad (3.79)$$

A given set of schemes is generated by a truncation of this series and equating the coefficients involving grid points which are equidistant from the point being smoothed. The distance of grid points from this point is

$$d^2 = (k_x h)^2 + (k_y h)^2 \quad \text{for}$$

$$k_x = 0, \frac{2\pi}{L_x}, \dots, \frac{2\pi}{L_x}(N_x-1); \quad k_y = 0, \frac{2\pi}{L_y}, \dots, \frac{2\pi}{L_y}(N_y-1) \quad (3.80)$$

Equidistant grid points off the coordinate axes give rise to the cosine product terms,

$$(\cos k_x h \cos k_y h) \quad \text{for } k_x, k_y \neq 0. \quad (3.81)$$

Similarly, equidistant grid points along the coordinate axes involve only one cosine factor,

$$\cos k_x h \quad \text{for } k_x \neq 0; k_y = 0, \quad (3.82)$$

or

$$\cos k_y h \quad \text{for } k_x = 0; k_y \neq 0. \quad (3.83)$$

Figure 3.19. Normalized mode amplitude function for the smoothing scheme in equation (3.78).

Thus, smoothing algorithms can be developed by selecting grid points in position space and truncating the series with only those terms. The coefficients are adjusted to select the desired shape of the mode amplitudes as illustrated with the previous examples.

In summary, we have studied several two dimensional algorithms for smoothing a grid function in real space. We have found that the best method for developing algorithms is to replace the grid function by a weighted average of the function at a selection of neighboring grid points. The weight factors can be varied to adjust the amplitude of the Fourier modes. We have chosen the algorithm in Eq. (3.77) for the test cases in Chapter IV because it minimizes the shortest wavelength modes at all angles.

3.4 Particle Initialization and Reinjection

In this section we discuss our new method for initializing the particle positions from a specified beam deposition profile. The profiles which can be selected are:

1/r density profile

$$n_b(r) = n_0 \{ r_b/r - 1 \}$$

for

(3.84)

$$n_0 = \frac{N_b}{\pi R_b^2}$$

empirical density profile for TCT beams

$$n_b(r) = n_o \left\{ .9 \exp\left(-16 \frac{r^2}{r_b^2}\right) + .1 \left(1 - \frac{r}{r_b}\right) \right\}$$

$$\text{for } n_o = \frac{N_b}{\pi r_b^2 (8.96 \times 10^{-2})} \quad (3.85)$$

constant density profile

$$n_b(r) = \begin{cases} n_o & \text{for } r \leq r_b \\ 0 & \text{for } r > r_b \end{cases}$$

$$\text{for } n_o = \frac{N_b}{\pi r_b^2} \quad (3.86)$$

parabolic density loading

$$n_b(r) = n_o \left(1 - \frac{r^2}{r_b^2}\right)$$

$$\text{for } n_o = \frac{2 N_b}{\pi r_b^2} \quad (3.87)$$

The method for loading particles into the beam deposition profile is as follows: the minor radius of the beam is divided into radial shells (see Figure 3.20a). The number of particles in each radial interval is computed from the specified density profile with

$$N(r_i \leq r \leq r_{i+1}) = 2\pi \int_{r_i}^{r_{i+1}} n_b(r) r dr. \quad (3.88)$$

Within a given interval the angle and radial increment are randomized. In order to conserve particles properly in

a) The beam cross section is divided into radial shells for the Monte Carlo loading.

b) The distribution function is sampled to determine the radial position, r_k . The angle θ is randomized.

Figure 3.20. Monte-Carlo loading of particles over the beam deposition radius.

converting from floating point to integer numbers, the number of particles computed for each shell is compared to the cumulative number of particles and modified by one, if required.

In our steady state model (Chapter V), particle losses are balanced by reinjection. We have used a Monte Carlo method to reload individual particles which have been lost from the system. The probability of loading a particle in the i th radial shell is computed from

$$p_i(n_i \leq n \leq n_{i+1}) = \frac{1}{N_b} \int_0^{2\pi} d\theta \int_{n_i}^{n_{i+1}} n_b n dr. \quad (3.89)$$

The cumulative distribution function is constructed as shown in Figure 3.20b from

$$P(n_k) = \sum_{i=0}^{k-1} p_i. \quad (3.90)$$

This distribution function is sampled to determine the radial shell and linearly interpolated to determine the radial increment within the shell. The angle is randomized.

3.5 Diagnostics

The diagnostics in the code are of two types:

1. Instantaneous plots of particle and flux variables generated at regular time intervals.
2. Time history data saved on disk files and plotted afterwards with post processing codes.

The five types instantaneous plots are shown in Figure 3.21.

The physical variables plotted are:

a) Velocity space plot

b) Particle current density versus radius at $Z = 0$

c) Poloidal flux contours

d) Poloidal flux function versus radius at $Z = 0$

e) Particle positions in the poloidal plane

Figure 3.21. Examples of graphical output from GUIDON77.

1. The Z and R components of the guiding center velocity are represented as a point in the V_Z vs V_R plane for every simulation particle (Figure 3.21a).
2. The particle current density is plotted as a function of R at Z = 0.0 (Figure 3.21b).
3. Flux contours are plotted in the poloidal plane (Z vs R) as shown in Figure 3.21c.
4. The flux function is plotted as a function of R at Z = 0.0 (Figure 3.21d).
5. The particle positions are plotted in the poloidal plane (Z vs R) and magnetic field lines are superimposed on this plot (Figure 3.21e).

In addition to these instantaneous plots, the trace of the portion of a test particle orbit computed since the last graphical output is superimposed on both the velocity space and the flux contour plots. The particle orbit is represented as a series of dots on the flux plots whereas the orbit points are connected with straight line segments on the velocity space plots. Any particle in either beam can be selected as the test particle to be followed during the run. Another version of GUIDON77 without self-field calculations has been used to calculate single particle orbits in fixed magnetic fields. These single particle orbits provide a useful check on the algorithms in the code. The periods can be compared with the calculated toroidal and poloidal periods given by

$$\tau_T = \frac{2\pi R_o}{V_\phi}, \quad \text{where} \quad V_\phi = \sqrt{\frac{2}{m_i} W} \quad (3.91)$$

$$\tau_p = g \tau_T, \quad \text{where} \quad g = \frac{B_o R_o}{B_\phi R_o}. \quad (3.92)$$

One of the more important post processed diagnostics is produced by taking moments of the beam current density function. For the CIT studies the smoothed beam current density is assumed to be a parabolic profile of the form,

$$J_\phi(r) = \frac{2I_b}{\pi r_b^2} \left\{ 1 - \left(\frac{r}{r_b} \right)^2 \right\}, \quad (3.93)$$

where

$$r^2 = (R - R_b)^2 + (Z - Z_b)^2. \quad (3.94)$$

The parameters for this profile (R_b, Z_b, I_b, r_b) are determined by taking moments of the actual simulated particle current density. The moments are given by

$$I_b = \iint dR dZ J_\phi \rightarrow \Delta R \Delta Z \sum_i \sum_j J_\phi(ij) \quad (3.95)$$

$$I_b R_b = \iint R J_\phi dR dZ \rightarrow \Delta R \Delta Z \sum_i \sum_j R_i J_\phi(ij) \quad (3.96)$$

$$I_b Z_b = \iint Z J_\phi dR dZ \rightarrow \Delta R \Delta Z \sum_i \sum_j Z_j J_\phi(ij) \quad (3.97)$$

$$\begin{aligned} \frac{1}{3} I_b r_b^2 &= \iint \left\{ (R - R_b)^2 + (Z - Z_b)^2 \right\} J_\phi dR dZ \\ &\rightarrow \sum_i \sum_j \Delta R \Delta Z \left\{ (R_i - R_b)^2 + (Z_j - Z_b)^2 \right\} J_\phi(ij). \end{aligned} \quad (3.98)$$

The average beam position (R_b, Z_b) is computed and saved at each time step. The orbit of the average beam position is

a) Mean beam position

b) Beam current as a function of time

c) Vertical field as a function of time

d) Particle loss plot

Figure 3.22. Diagnostics produced with post processing codes.

reconstructed with a post processing code (Figure 3.22a). This diagnostic has been particularly useful in designing the vertical centering field to center the beams and minimize particle losses. The other set of diagnostics developed to analyze the evolution of the CIT beams to steady state are shown in Figure 3.22 and include the following:

1. Time history plots of the total beam current and individual beam currents (Figure 3.22b).
2. Time history plot of the magnitude of the vertical centering field (Figure 3.22c).

In our steady state simulations, particle losses are computed at each time step for both beams. One particle loss mechanism is the loss of particles which have drifted to the limiter. The other loss mechanism is the removal of cooled ions from the system. The code computes both the total losses and the limiter losses at every time step. The loss rates can be computed by constructing the cumulative losses as a function of time as shown in Figure 3.22d. The slope of the straight line drawn through this plot is an estimate of the loss rate. Applications of these diagnostics to the steady state model are discussed in Chapter V.

IV. NUMERICAL STUDIES OF SELF PINCHING OF STRONG ION BEAMS

4.1 Introduction

In the proposed Counterstreaming Ion Torus (CIT) the ion beams drive currents larger than 1 megampere (MA). In our first simulations of typical CIT beams, the beam self fields became unstable on a relatively short time scale ($t \lesssim 250 \mu s$). This instability in the fields was accompanied by clumping in velocity space and resulted in strong pinching and collapse of the beams. The series of test runs presented in this chapter show that this instability is numerical rather than physical. The application of smoothing algorithms to the mesh quantities and the updating of the fields between the predictor-corrector steps significantly reduce self pinching. Although the pinching is not necessarily eliminated with these modifications, the onset of the instability is delayed for times long compared to the simulation times in the current applications.

4.2 Pinching of a Linearly Propagating Beam

We chose to study beam pinching phenomena with a single linearly propagating beam for several reasons. The large curvature drift present in toroidal systems is eliminated in a linear system. By using a single beam rather than counterstreaming beams there is no physical repulsion as for the case of counterstreaming beams. Finally, we have chosen to simulate the total current with particles because the most severe numerical noise is associated with the limited number of particles in

the system rather than the discretization of a fluid quantity onto a grid.

4.2.1 The Standard Case

A linearly propagating beam can be simulated with GUIDON77 by choosing a large aspect ratio, $A = R_0/r_0$, and adjusting the toroidal field current to give the desired toroidal field strength at the center of the beam. In runs 15 and 16A through 16U, referred to in the following tables, the aspect ratio is 10^3 . The beam current is set to 1 MA to match the net beam driven current in one of the CIT configurations. In problem 16 a toroidal field strength of 50 kGauss resulted in the onset of the instability at $t \sim 450 \mu s$. Approximately 30 minutes of computer time is required for running test simulations out to $450 \mu s$. For practical test simulations the beam pinching time must be decreased. This is accomplished by reducing the toroidal field strength relative to the poloidal field strength. In run 16A the toroidal field strength is half of the value in run 16. The remaining input parameters in these runs are listed in Table 4.1. The onset of the instability occurs at $t \sim 80 \mu s$ in run 16A. Otherwise there is no qualitative difference in the pinching between this run and run 16. Subsequent runs, based on the standard run 16A, are discussed in the next section and are summarized in Table 4.2.

4.2.2 Beam Pinching in the Standard Case

The initialization of run 16A is shown in Figure 4.1. Although the particle current density is quite noisy, the resulting flux contours are smooth. The evolution of the current

Table 4.1. Input parameters for the standard cases 16A and 16.

Toroidal parameters

Major radius: $R_0 = 75,000$ cm
 Limiter radius: $r_L = 75$ cm
 Toroidal field: $B_\phi = 25$ kGauss on axis (16A)
 $= 50$ kGauss on axis (16)
 Aspect ratio: $A = 10^3$
 Vertical field: $B_Z = 0$

Injected beam

Ion species: D^+
 Energy: 80 keV
 Current: 1 MA
 Injection angle: 5°
 Major radius: $R_b = 75,000$ cm
 Minor radius: $r_b = 75,000$ cm
 Total number of
 particles: 1.06×10^{22} particles

Computational parameters

Time step: $\Delta t = 0.2$ μ sec
 Radial interval: 74,800 cm to 75,200 cm
 Axial interval: -200 cm to +200 cm
 Grid spacings: $\Delta R = \Delta Z = 6.25$ cm
 Number of simulation
 particles: 2500

Figure 4.1. The standard case 16A at $t = 0$.

density is shown in Figure 4.2. A spike begins to form at 50 μ s and by 80 μ s the central peak is an order of magnitude larger than the initial peak current density. The velocity space plots during this same time are shown in Figure 4.3. A ring is formed in velocity space at the time beam pinching occurs. The particles which were originally in the center of the beam in position space filled the low velocity interior of this ring formation. By 90 μ s these particles have formed a high velocity clump (Figure 4.3h) and a strongly pinched current filament at the center of the beam (Figure 4.4). This is a result of the rapid growth of the beam self fields.

Another characteristic of the beam pinching is the unstable spiraling of the particle orbits. A further examination of other test particle orbits in these runs shows that particles spiral both in and out from the center of the beam. This spiraling can be caused by the lack of a perpendicular beam pressure or by inaccuracies in the numerical algorithms. Both of these are tested in the remainder of this chapter.

In order to compare the growth of the instability in the test runs, a pinch time is estimated for each case. This result is included in Table 4.2 for each run. The pinch time is approximated as the time at which the following three conditions occur:

1. Clumping and ring formation in velocity space
2. Inaccurate numerical test particle orbits
3. Large, highly pinched peaks in the particle current density.

a) At $t = 20 \mu\text{sec}$ b) At $t = 40 \mu\text{sec}$ c) At $t = 50 \mu\text{sec}$ d) At $t = 60 \mu\text{sec}$ e) At $t = 70 \mu\text{sec}$ f) At $t = 80 \mu\text{sec}$

Figure 4.2. Growth of the current density in the standard case 16A.

a) At $t = 20 \mu\text{sec}$ b) At $t = 30 \mu\text{sec}$ c) At $t = 40 \mu\text{sec}$ d) At $t = 50 \mu\text{sec}$

Figure 4.3. Test particle orbit in velocity space in the standard case 16A.

e) At $t = 60 \mu\text{sec}$ f) At $t = 70 \mu\text{sec}$ g) At $t = 80 \mu\text{sec}$ h) At $t = 90 \mu\text{sec}$

Figure 4.3. (cont'd) Test particle orbit in velocity space in the standard case 16A.

Figure 4.4. Particle position plots after pinching has occurred.

Thus, the pinch time defines the end of the initial moderate growth and the onset of beam collapse and filamentation.

4.3 Test Cases

4.3.1 Time Step Tests (16C and 16D) and Variation of Physical Parameters (16B and 16M)

The time step, $\Delta t = .2 \mu s$, is halved and quartered in runs 16C and 16D. Comparing these runs with 16A shows that the growth trend of the peak current density is greatly reduced with the smaller time steps. The time at which pinching occurs in the two runs is increased to approximately $200 \mu s$ and $340 \mu s$, respectively.

Two physical variables which may also affect the results in the standard case are the injection angle and the drift velocity contributions. Increasing the injection angle to 30° (16B) effectively reduces the beam self fields since poloidal currents are neglected. As expected, the pinching is somewhat delayed to $\tau_p \simeq 100 \mu s$. In run 16M the curvature and grad B drifts are not computed in the velocity calculations in order to check the assumption that the toroidal geometry effects are negligible. The results in this run are identical to those in run 16A.

In summary, we have found that varying the injection angle and not computing the drift velocity contributions to the beam motion do not significantly change the beam pinching. On the other hand, decreasing the time step greatly increases the pinch time. Since the growth rate of a physical instability cannot depend on the time step in a numerical calculation, these runs

Table 4.2. Test cases for a linearly propagating plasma in a cylinder.

Test case	Run	Parameters	τ_p and other comments
B_ϕ tests	16	$B_\phi = 50$ kG	Pinches at ~ 450 μ sec
	16A	$B_\phi = 25$ kG	Pinches at ~ 80 μ sec
Injection angle tests	16A	$\chi = 5^\circ$	$\tau_p \sim 80$ μ sec
	16B	$\chi = 30^\circ$	$\tau_p \sim 100$ μ sec
Δt tests	16A	$\Delta t = 0.2$ μ sec	$\tau_p \sim 80$ μ sec
	16C	$\Delta t = 0.1$ μ sec	$\tau_p \gtrsim 200$ μ sec
	16D	$\Delta t = 0.05$ μ sec	$\tau_p \gtrsim 340$ μ sec
Smoothing tests	16A	No smoothing	$\tau_p \sim 80$ μ sec
	16F	$S(\psi)$	$\tau_p \sim 180$ to 190 μ sec
	16G	$S(\psi + J)$	$\tau_p \gtrsim 230$ μ sec
	16H	$S(J)$	$\tau_p \sim 180$ to 190 μ sec
Field iterations	16A	No field iterations	$\tau_p \sim 80$ μ sec
	16I	5 iterations between Δt	$\tau_p \sim 80$ μ sec
Drift contributions	16A	$V_{\text{DRIFT}} \neq 0$	$\tau_p \sim 80$ μ sec
	16M	$V_{\text{DRIFT}} = 0$	$\tau_p \sim 80$ μ sec

Table 4.2 (cont'd)

Test case	Run	Parameters	τ_p and other comments
Increased number of particles	16G	2500 particles; $S(\psi + J)$	$\tau_p \sim 230 \mu\text{sec}$
	16S	5000 particles and $S(\psi + J)$	Pinching beyond $200 \mu\text{sec}$; J_{max} approx. equivalent to 16G but with lower fluctuation levels
Field updating between leap frog steps	16G	$S(\psi + J)$ only	$\tau_p \sim 230 \mu\text{sec}$
	16U	Field updating and $S(\psi + J)$	$\tau_p \sim 230 \mu\text{sec}$; equivalent to 16G for these one beam tests. However, field updating greatly improves the results for 2 beam cases.

indicate that the instability is numerical rather than physical. Rather than reducing the time step in the simulations, more practical methods for increasing numerical stability are considered in the next sections.

4.3.2 Smoothing Tests (16F,16G,16H) and Field

Iteration Tests (16I)

The original version of GUIDON did not use smoothing of either the sources or the fields. The smoothing algorithm used in these test runs is given by equation (3.78). The algorithm is applied in three different forms in runs 16F, 16G, and 16H. The following notation is used in Table 4.2:

$S(J)$: smoothing of current density only

$S(\psi)$: smoothing of fluxes only

$S(\psi + J)$: smoothing of both fluxes and current density.

The evolution of the smoothed and unsmoothed current density in run 16H is compared in Figure 4.5. The beneficial effects of smoothing are apparent from the damping out and smoothing of the peaks in the current density. There is little difference in the results when either the fluxes or the currents are smoothed (16F,16H). The pinch time is about 190 μ s in both runs. Smoothing of both quantities together improves the results by extending the pinch time to about 230 μ s. These test runs show that smoothing increases the beam stability more effectively than halving the time step. Furthermore, the added computer time for the smoothing operation is negligible.

Figure 4.5. Comparison of the smoothed and unsmoothed current density at $t = 10 \mu\text{sec}$, $30 \mu\text{sec}$ and $50 \mu\text{sec}$.

Another important benefit of smoothing is the greatly reduced tendency for particle orbits to spiral. The test particle orbit for run 16H is shown in Figure 4.6. This increased stability in the particle orbits occurs in all three of the smoothing test cases.

Performing iterations on the fields following the particle orbit integration is another relatively inexpensive method which may allow for longer time steps. The field iteration method used in run 16I is illustrated in Figure 4.7. The results in 16I are very similar to the standard case, showing that field iterations at the end of each time step are unnecessary.

4.3.3 Increasing the Number of Simulation Particles (16S)

The adequacy of the mesh spacing is tested by increasing the number of simulation particles since the noise properties are determined by the number of simulation particles per grid cell. The number of particles is doubled and both the fluxes and the current density are smoothed in run 16S. The initialization of this run is shown in Figure 4.8. The noise in the particle current density is reduced, as expected. Nevertheless, as the run is continued, there is very little difference in the pinching of the beam in 16S from that in run 16G. Doubling the number of particles appears to have a very weak effect compared to smoothing the sources and fields. This is a fortunate result since doubling the number of simulation particles not only doubles the computer time but also doubles the number of particle storage locations (11 locations per particle).

a) Velocity space at $t = 10 \mu\text{sec}$ b) Flux configuration at $t = 10 \mu\text{sec}$ c) Velocity space at $t = 20 \mu\text{sec}$ d) Flux configuration at $t = 20 \mu\text{sec}$ e) Velocity space at $t = 30 \mu\text{sec}$ f) Flux configuration at $t = 30 \mu\text{sec}$

Figure. 4.6. Test particle orbit for run 16H.

g) Velocity space at $t = 40 \mu\text{sec}$ h) Flux configuration at $t = 40 \mu\text{sec}$ i) Velocity space at $t = 50 \mu\text{sec}$ j) Flux configuration at $t = 40 \mu\text{sec}$

Figure 4.6. (cont'd) Test particle orbit for run 16H.

Figure 4.7. Field iteration scheme.

a) Velocity space

b) Particle current density versus radius

c) Flux configuration

d) Particle positions in poloidal plane

Figure 4.8. Run 16S with 5000 particles at $t = 0$.

4.3.4 Field Updating Between Leap Frog Steps (16U)

The method for updating the fields between the predictor-corrector steps is discussed in Section 3.2.1 and shown in Figure 3.2. This procedure is tested in run 16U. Both the fluxes and sources are smoothed in this test case. The result is that there is no improvement over test case 16G.

Although field updating is not necessary for this one beam test case, several other runs for counterstreaming ion beams show that this procedure is essential for strong counterstreaming currents. Some of these two beam results are shown in Figure 4.9. The flux configuration is significantly affected by this additional updating of the fields. This method is used for all of the two beam runs discussed in Chapter V.

4.4 Summary

The pinching of the plasma column is a numerical rather than physical instability. Table 4.3 summarizes the relative strengths of the numerical algorithms and physical parameters in preventing beam pinching. Numerical modifications, and in particular smoothing, greatly extends the time for stable beam motion. Smoothing and field updating methods together are at least as important as halving the time step. In addition, there is increased stability in the particle orbits when the sources and fluxes are smoothed. These algorithms have been permanently added to the coding in GUIDON77.

Figure 4.9 Effects of smoothing and field updating for the case of two counterstreaming beams.

Table 4.3. Relative strength of numerical and physical variables on the beam pinching.

Parameter tested	Effect on beam pinching	Related runs
B_ϕ tests	Very strong	16,16A
Smoothing tests	Strong	16F,16G,16H
Δt tests	Strong	16C and 16D
Field updating	Weak*	16U
Injection angle	Weak	16B
Increased number of particles	Weak	16S
Field iterations	Weak	16I
Drift contributions	Negligible	16M

* Field updating has a strong effect on the pinching in the case of 2 counterstreaming beams.

V. SIMULATIONS OF THE COUNTERSTREAMING ION TORUS FUSION REACTOR

5.1 Introduction

In this chapter we describe the simulation model for the CIT fusion reactor. We discuss two separate types of simulations. In one set of simulations the drift motion of the beams is computed and collisions are neglected. We refer to these as collisionless beam runs. In the other type of simulation we include the collisional drag of the hot ion species on the background electrons.

We use collisionless beam simulations to study the mutual repulsion of the counterstreaming beams. The spatial confinement of the beams not only depends on the beam repulsion but also on the strength of the vertical centering field. We have developed a method for adjusting the vertical field in our simulations to insure optimal centering of the beams. Our investigations over a range of CIT beam currents and energies show that the beams can be confined and collisionless beam equilibria do exist.

The purpose of the collisional beam simulations is to determine if a steady state balance can be reached between hot ion injection and the two particle loss mechanisms in ideal CIT operation. One type of particle loss results from the injection of particles onto open flux contours. These particles follow the contours out to the limiter. The other type of loss is the removal of ions which have cooled down because of collisional slowing down on the electron background (electron drag). In

the ideal CIT system it is assumed that these cooled ion species are removed from the system so that ion-ion collisional interactions are not important. We have run steady state simulations for two interesting CIT configurations. Loss rates are computed for both types of loss mechanisms and we compare these results to predicted electron drag losses. Beam energies and currents at steady state and injection currents necessary to maintain steady state balance are calculated from the simulated results.

5.2 CIT Reactor Operation

We briefly describe the reactor operation in this section. A complete analysis of the reactor is presented in reference 1.

A confined deuterium plasma is required to trap the injected neutral beams in the start-up phase of the CIT reactor. This plasma is generated by the usual gas filling and Ohmic heating method. Deuterium and tritium neutral beams are tangentially injected into the system in opposite directions along the magnetic field lines as shown in Figure 5.1. After ionization and trapping by the background plasma, the deuterons and tritons form two velocity groups with a relative displacement velocity along the magnetic field lines (Figure 5.2). Thus, the beam character of the counterstreaming particles is retained. These ions rapidly form this type of thermalized distribution because the self equilibration time for the ions is short compared to the interaction times with the electrons and the other ion species.

Figure 5.1. Geometry of tangential injection into a torus.

Figure 5.2. Counterstreaming ion velocity distributions parallel to the toroidal magnetic field. In this case the two distributions have the same density, temperature, and total energy.

After the start-up phase the CIT plasma configuration is obtained by the use of diverter coils which remove the background ions (cold ions), leaving only energetic counterstreaming ions in the system. The continued trapping of injected neutrals by the energetic ion species is accomplished with the following ionization processes:

1. Impact ionization with ions moving in the opposite direction
2. Electron ionization
3. Charge exchange with ions moving in the same direction.

It is assumed in this ideal CIT operation that all of the cold ions can be removed from the system.

The dominant collisional interaction of the ionized beams, when impurities are neglected, is electron drag. This results in an energy flow from the hot ion species to the electrons which maintains the electron temperature.* It is essential that decelerated beam ions (warm ions) are removed from the system before ion-ion collisional interactions become important. Ion-ion collisions would divert some of the hot ion energy away from the electrons. The mechanisms for warm ion removal are discussed at length in reference 1. Removed warm ion species are replaced by hot ions resulting from further high energy neutral

*If the electron temperature is allowed to drop the slowing down period of the ions is decreased. This in turn reduces the number of fusion reactions which occur before ion removal.

injection. At steady state the warm ion losses are balanced by equivalent gains in the hot ion species.

In the usual toroidal system the confining electron plasma current results from the induced toroidal electric field. The electron acceleration by the electric field is balanced by the resistive momentum losses on the ions so that there is no net force on the electrons in the toroidal direction. That is,

$$0 \leftarrow n_e m_e \dot{v}_e = e n_e E_\phi - \gamma_{ie} m_i n_i (v_e - v_i), \quad (5.1)$$

where γ_{ie} is the ion-electron collision frequency (inverse drag time). Ohkawa³ proposed that electron force balance can also be achieved by using a high energy ion beam, eliminating the need for the induced toroidal electric field. In his analysis a low energy, high current beam is also injected so that the toroidal momentum remains zero. In this case the electron force balance becomes

$$0 \leftarrow m_h n_h \gamma_{he} (v_h - v_e) - m_l \gamma_{le} (v_e + v_l) n_l \quad (5.2)$$

where the subscripts h and l denote the high energy and low energy ion beams, respectively. Ohkawa shows that force balance is achieved with the proper choice of charge and charge to mass ratios of the beams. Connor and Cordey⁴ consider the case of two counterstreaming beams and describe the electrons with a Fokker-Planck equation. They also include toroidal effects which yield a trapped electron contribution to the current density. They find the result,

$$J_{\phi} = J_b \left\{ 1 - \frac{1}{Z_{eff}} (1 - A_1 \epsilon^{1/2}) \right\}$$

$$Z_{eff} = \sum_i \frac{n_i Z_i^2}{n_e}, \quad (5.3)$$

where

- J_b = total ion current density
- Z_{eff} = the effective charge of the plasma
- $A_1 \epsilon^{1/2}$ = contribution to the current from trapped electron effects
- ϵ = inverse aspect ratio
- A_1 = constant.

This expression is also valid if both beams are high energy, high current beams.

5.3 Simulation Model

The motion of the CIT beam ions is simulated on two time scales. On the fast time scale the simulations follow the self consistent guiding center motion of counterstreaming collisionless ions. Electron drag on the ions is ignored. A second type of simulation follows the guiding center motion, but also includes the slow time scale drag. These latter simulations model the steady state in the CIT reactor.

5.3.1 Collisionless Beam Simulations

The ions are assumed to be fully distributed over a specified beam cross section. The beam density profiles are modeled analytically with the parabolic form given by

$$n_b = n_{b0} \left(1 - \frac{\lambda^2}{R_b^2} \right)$$

$$\lambda^2 = (R - R_b)^2 + (z - z_b)^2, \quad (5.4)$$

where

R_b, r_b = beam major and minor radii

n_{b0} = central beam density at R_b .

The loading of the simulation particles over the beam cross section is described in Section 3.4.

In the experiments a limiter is placed within the toroidal volume to remove ions which are drifting to the walls. In the simulations the limiter is modeled as a circle of radius r_L centered at the major radius of the torus. Particles which drift beyond this radius are assumed to be lost at the limiter. The beam current is maintained in these collisionless runs by replacing the limiter losses by newly injected ions.

Losses at the limiter are a result of improper beam centering by the vertical magnetic field and the repulsion of each beam by the selffields of the other beam. The collisionless model is used to determine whether counterstreaming ion beam equilibria can be found.

The simulations are performed for two different CIT configurations. In one case the beam currents are chosen so that there is no net beam driven current. The confining poloidal magnetic field is generated entirely by electron current in a background Ohmic heated plasma. The net current density is

$$J_\phi = J_{OH}, \quad (5.5a)$$

where

J_{OH} = electron plasma current density.

The electron current density is modeled as a parabolic function of the minor radius as given by Eq. (5.9). The induced toroidal electric field is computed from the electron current density as follows:

$$E_{OH} = \sigma^{-1} J_{OH} \quad (5.5b)$$

$$\sigma^{-1} = .68027 \sqrt{2\pi m_e} e^2 T_e^{-1/2} \ln \Lambda, \quad (5.5c)$$

where

σ^{-1} = plasma resistivity

E_{OH} = induced toroidal electric field

$\ln \Lambda$ = Coulomb logarithm.

We have neglected the action of the induced electric field on the ions in our simulations.

In the other configuration the current density is entirely beam driven. The net current density, as discussed in Section 5.2, is given by

$$J_\phi = J_b \left\{ 1 - \frac{1}{Z_{eff}} (1 - \epsilon^{1/2} A_1) \right\}. \quad (5.6)$$

In all of our simulations we have assumed an effective Z of 1.5 and we have neglected the trapped electron contribution to the current. Although simulations are not discussed for the case in which both beam driven and background currents are present, this more general case has been included in the coding. The net toroidal current in this case is

$$J_\phi = J_{OH} + J_b \left\{ 1 - \frac{1}{Z_{eff}} (1 - \epsilon^{1/2} A_1) \right\}. \quad (5.7)$$

Electrons in the system form a charge neutralizing background. For a dense electron background, it is assumed that the initial electron temperature, density, and toroidal current density profiles are maintained throughout the evolution of the system. These electron parameters are modeled as parabolic functions in the minor radius:

$$n_e = n_{e0} \left(1 - \frac{r^2}{r_e^2} \right) \quad (5.8)$$

$$J_{e\phi} = J_{0\phi} \left(1 - \frac{r^2}{r_e^2} \right) \quad (5.9)$$

$$T_e = T_{e0} \left(1 - c_1 \frac{r^2}{r_e^2} \right), \quad (5.10)$$

where

r = radius in poloidal plane

r_e = electron minor radius

$J_{0\phi}$ = central electron current density (when present)
at the major electron radius

n_{e0} = central electron density

T_{e0} = central electron temperature.

Experimentally the toroidal field is approximately inversely proportional to the radial position. Therefore, a $1/R$ variation is assumed in the simulations. The vertical centering field is uniform in space, but its strength may vary in time. The magnitude of the field necessary to center either an electron current or beam driven current must be determined with test simulations. This is discussed in Sections 5.4 and 5.5.

5.3.2 Steady State Simulations

In the collisional beam simulations a steady state is reached when particle losses, primarily due to warm ion removal, are balanced by particle reinjection. An energy cutoff is used to simulate warm ion removal. Ions decelerated (by electron drag) below the selected energy cutoff are removed and replaced by reinjected hot ions. The Monte Carlo method for particle reinjection is discussed in Section 3.4.

The simulated particle loss rates can be compared to the analytical electron drag loss rates for each species. The average particle lifetime τ_L is defined by:

$$\frac{dN}{dt} = -\frac{N_0}{\tau_L}, \quad (5.11)$$

where

N_0 = total number of ions of the given species.

Analytically, the particle life time due to electron drag τ_{LA} is computed by integrating the energy loss equation from the injection energy to the cutoff energy,

$$\int_{W_I}^{W_c} \frac{dW}{(W - W_e)} = - \int_0^{\tau_{LA}} \frac{dt}{\tau_D} \quad (5.12)$$

yielding

$$\tau_{LA} = \tau_D \ln \left(\frac{W_I - W_e}{W_c - W_e} \right), \quad (5.13)$$

where

W_I = injection energy

W_e = electron energy

W_c = cutoff energy.

Knowing the particle life time, the average energy during the slowing down time can be computed from

$$\langle W \rangle = \frac{1}{\tau_{LA}} \int_0^{\tau_{LA}} W(t) dt, \quad (5.14)$$

where

$$W(t) = W_e + (W_I - W_e) e^{-t/\tau_D}. \quad (5.15)$$

Evaluating this integral gives the result,

$$\langle W \rangle = \frac{W_e}{\tau_{LA}} + \frac{(W_I - W_c)}{\ln\left(\frac{W_I - W_e}{W_c - W_e}\right)}. \quad (5.16)$$

In the simulations the cutoff energy is selected as about twice the electron energy, $W_c > 2W_e$. Numerical Fokker-Planck calculations have shown that at this energy the energy losses to background ions become comparable to the energy losses to the electrons.⁴²

Physically the electron drag time is several orders of magnitude longer than the simulation time step and poloidal orbit periods for the guiding center motion. For practical simulations this necessitates the enhancement of the collision frequencies. The simulated drag times for the beams are reduced with

$$\tau'_{Di} = \frac{\tau_{Di}}{\gamma_D} \quad , \quad (5.17)$$

where

γ_D = enhancement factor for the collision frequencies

τ'_{Di} = reduced electron drag time used in the simulations

τ_{Di} = physical electron drag time.

The important assumption which is made in reducing the drag time in this manner is that the short time scale collisionless guiding center motion remains unaffected by artificially strong collisions. This requires that the drag times are at least several times longer than the poloidal orbit periods.

The total loss rate is the sum of the limiter loss rate and the drag loss rate. That is,

$$\left(\frac{dN}{dt} \right)_{TOT} = \left(\frac{dN}{dt} \right)_{LIM} + \left(\frac{dN}{dt} \right)_{DRAG} .$$

In the code we compute both the number of particles lost at the limiter and the number lost by low energy ion removal (drag losses). We plot the cumulative losses of both kinds as a function of simulation time. The drag loss rate is the slope of a straight line constructed through the cumulative drag loss plot, provided the time scale has been converted to a physical time scale by multiplying by the enhancement factor as given in Eq. (5.17). This raises a question about the proper scaling for the limiter loss rates. There are two types of limiter losses: those associated with hot ion reinjection onto open magnetic flux surfaces (occurring on a long time scale) and those resulting from short time scale collisionless beam drift motion.

The first kind of limiter loss must be scaled in the same way as the drag losses. We assume, for well centered beams, that the second type of limiter loss is an initial transient and will not affect the long term steady state balance. Therefore, in all of the collisional simulations, the limiter loss rates are scaled down with the enhancement factor.

5.4 Testing the Simulation Model

The numerical properties of collisionless counterstreaming beams are tested in the first collisionless runs discussed in this section. Next the strength of the vertical field is determined with a series of one beam runs in which the beam carries the same current as the net beam driven current in two beam problems. The fixed parameters in all of these runs are listed in Table 5.1.

Finally several steady state simulations based on the vertical field tests are studied for a range of CIT parameters. The results of these test runs lead to one revision of this simulation model as discussed in Section 5.5.

5.4.1 Numerical Tests

The purpose of these numerical tests is to determine the beam currents which can be reasonably simulated with the numerical modifications to the basic code discussed in Chapters III and IV. The fixed parameters for the runs are listed in Table 5.1.

In the first set of test runs listed in Table 5.2 the total current is beam driven (TWOB200A, TWOB201A, TWOB202A). The

Table 5.1. Fixed parameters for the numerical test runs.

Toroidal parameters

$$R_0 = 375 \text{ cm}$$

$$r_0 = 75 \text{ cm}$$

$$r_L = 75 \text{ cm}$$

$$B_\phi = 50 \text{ kGauss on axis}$$

Injected beams

$$E^{T^+} = 120 \text{ keV (coinjecteD)}$$

$$E^{D^+} = 80 \text{ keV (counterinjecteD)}$$

$$R_b = 375 \text{ cm}$$

$$r_b = 75 \text{ cm}$$

$$Z_{\text{eff}} = 1.5$$

Electrons

$$n_e = 10^{14} \text{ cm}^{-3} \text{ at } R_0$$

$$T_e(R_0) = 10 \text{ keV}$$

Table 5.2. Numerical test runs.

Run	I_{T^+} (MA)	I_{D^+} (MA)	$\frac{n_{T^+}}{n_{D^+}}$	B_z (Gauss)
TWOB200A	1.3	-0.30	4.00	-450
TWOB201A	8.0	-7.00	1.14	-450
TWOB202A	4.67	-3.67	1.27	-450
Beam driven currents ($I_{OH} = 0.$)				
Run	I_{T^+} (MA)	I_{D^+} (MA)	I_{OH} (MA)	B_z (Gauss)
TWOB13	8.3	-8.3	1.0	-1000
TWOB13A	4.2	-4.2	1.0	-1000
Canceling beam currents ($n_{T^+}/n_{D^+} \equiv 1.$)				

beam currents are chosen in each case to drive a net current of 1 MA. The individual beam currents as well as the density ratio are listed for each run. The initial flux contours for the three runs are shown in Figures 5.3 b, d, and f. The flux contours are quite smooth in the weakest case, TWOB200A. However as the beam strengths are increased, wiggles are introduced into the contours. The evolution of run TWOB200A in Figure 5.4 shows that the flux contours and test particle orbits remain smooth. A well defined beam shape is maintained in velocity space as well. The motion of the test particle orbit across flux contours in this figure is a result of time dependent changes in the flux contours.

The strongest case (TWOB201A) is shown at $t = 50 \mu s$ and $100 \mu s$ in Figure 5.5. The growth of the noise in the fields is quite evident in the flux contour plots. Well defined beam shapes in velocity space, such as those in the weak case TWOB-200A, are not present in this run. The test particle orbit is quite erratic and by $100 \mu s$ the numerical approximation to the orbit is very inaccurate (Figure 5.5d).

The results from TWOB202A (Figure 5.6) show that there is definitely noise in the fields but the particle orbits are not as adversely affected as those in TWOB201A. This is a borderline case for most simulations. However, for problems in which only gross properties of the beam motion are being studied, this may be satisfactory.

We ran two test cases with canceling beam currents and a 1 MA electron background current. These two runs, TWOB13 and

a) Run TWOB200A at $t = 0$ b) Run TWOB200A at $t = 0$ c) Run TWOB202A at $t = 0$ d) Run TWOB202A at $t = 0$ e) Run TWOB201A at $t = 0$ f) Run TWOB201A at $t = 0$ Figure 5.3. Runs TWOB200A, TWOB202A, and TWOB201A at $t = 0$.

Figure 5.4. Run TWOB200A at $t = 10 \mu\text{sec}$, and $t = 50 \mu\text{sec}$.

a) Flux configuration at $t = 50 \mu\text{sec}$ b) Velocity space at $t = 50 \mu\text{sec}$ c) Flux configuration at $t = 100 \mu\text{sec}$ d) Velocity space at $t = 100 \mu\text{sec}$ Figure 7.5. Run TWOB201A at $t = 50 \mu\text{sec}$ and $t = 100 \mu\text{sec}$.

Figure 5.6. Run TWOB202A at $t = 10 \mu\text{sec}$, $t = 50 \mu\text{sec}$, and $t = 100 \mu\text{sec}$.

TWOBL3A, are listed in Table 5.2. An examination of the test particle orbit in each case shows very poor behavior for those in TWOBL3 as well as those in TWOBL3A (see Figure 5.7). Since the beam strengths are comparable in TWOBL3A and TWOB202A, we have compared the qualitative behavior in these runs more closely and have found that they do not differ very much. These runs indicate that the presence of the electron current density is not having any smoothing effect on the noise resulting from the noncancelation of the beam currents. We conclude from all five runs that without further numerical tests for the higher beam strengths, the beam currents should be limited to less than 4 MA.

5.4.2 Vertical Field Design for the Collisional Test Runs

An approximate analytical expression for the vertical centering field can be derived by assuming a large aspect ratio and requiring plasma equilibrium in both major and minor radii. The resulting expression for the field strength is⁹

$$B_z = \frac{I}{10R_0} \left\{ \ln\left(\frac{9R_0}{r_0}\right) - \frac{1}{2} \right\}, \quad (5.18)$$

where

I = total current in Amps

B_z = centering field in Gauss.

This model was used as a first estimate for the field required to center two counterstreaming beams when $I_{OH} = 0$. In this case the plasma current in equation (5.18) is replaced by the net current driven by the two beams. Test runs using this

a) Flux configuration at $t = 0$ b) Velocity sapce at $t = 0$ c) Flux configuration at $t = 50 \mu\text{sec}$ d) Velocity space at $t = 50 \mu\text{sec}$ Figure 5.7 Run TWOB13A at $t = 0$ and $t = 50 \mu\text{sec}$.

e) Flux configuration at $t = 0$ g) Flux configuration at $t = 50 \mu\text{sec}$ f) Velocity space at $t = 0$ h) Velocity space at $t = 50 \mu\text{sec}$ Figure 5.7. (cont'd) Run TWOB13A at $t = 0$ and $t = 50 \mu\text{sec}$

model show that the field is almost a factor of two too strong. The reason for this large disagreement is that there are image currents set up on the computational boundary which influence the beam motion. The analytical model assumes there are no conducting boundaries near the plasma.

One method often used in simulations with conducting boundaries is to remove the computational boundary far enough from the plasma region so that the image currents are negligible. In our code we are currently using a 64 x 64 grid. Because of the additional memory locations required it is not practical to consider more mesh points. Thus, any increase in the computational domain has the disadvantage of a correspondingly coarser grid in the plasma region. However, we ran some test simulations with a larger domain. The results showed that the image currents continue to affect the beam motion. Therefore, we have continued to use the smaller domain, and we center the beams on the injection point by adjusting the strength of the vertical field.

Because of the complicated nature of the image currents, the centering field must be designed from simulation test runs. Using the analytical model as a guide, the field is assumed to depend on the total current. Therefore, in the following simulations, the field is designed to center a single beam with the same current as the net current driven in a two beam problem. In these test runs particles which drift to the limiter are replaced via the injection scheme described in Section 3.4. The beam moment diagnostics described in Section 3.5 are used

to determine the centering of the beam for a given field strength. The results of a series of these runs for a fixed current show that there are beam equilibria for a range of centering fields. For the typical equilibrium, after an initial drift, the average beam motion forms a series of approximately closed orbits as shown in the plot in Figure 5.8. In this example the field is obviously too strong since the major radius of the center of the orbits is about 25 cm nearer the toroidal axis than the major radius of the injection point.

For the purposes of determining the vertical field it is unnecessary to run long beam equilibrium simulations. Instead, short tests can be run to determine the direction of the initial beam drift. A series of 50 μ s test runs for a 1 MA current beam resulted in the average beam motion shown in Figure 5.9. A vertical field of 600 Gauss is selected for the collisional test runs with unequal beam currents. In the case of canceling beam currents, the vertical field is designed based on the electron (Ohmic) current and equation (5.18). This assumes that there is a reasonable numerical cancelation of the beam currents.

Based on these runs, the time required for the average beam motion to complete an orbit is approximately 100 μ s. The shortest collisional drag time should be chosen equal to or greater than 100 μ s so that this fast time scale physics is not masked by collisional reinjection. Therefore, an enhancement factor of 3×10^3 is chosen for the collisional test runs.

Figure 5.8. The average beam position shows a settling into closed orbits. Radial and axial distances are measured from the injection point.

Figure 5.9. Average beam position resulting from varying the vertical field in the 50 μsec test cases.

5.4.3 Collisional Test Cases over a Range of CIT Parameters

The set of collisional beam test runs is listed in Table 5.3 and quantitative results are compiled in Table 5.4. The fixed parameters in these runs are the initial net current and beam energies. The initial energies are selected to optimize the relative velocity, $V_R \sim 2V_i$, where V_i is the toroidal velocity of either beam. The beam density ratio is varied in the first three runs. Since the velocities are fixed by the energies, there is a corresponding variation in the total number of particles in the system. Each of these test runs approaches a steady state in a simulation time of 600 μ s as shown in Figures 5.10, 5.11, and 5.12. The deuteron currents reach a reasonable steady state after about four electron drag times. The results show that a similar steady state can be expected for the triton currents in a simulation time of about 300 to 400 μ s longer. The steady state beam currents listed in Table 5.4 are estimated by taking an average of the last maximum and minimum in the currents.

The input parameters for the runs in Table 5.3 were computed from a specified density ratio and an initial net beam current of 1 MA. Since the net steady state current is an operational parameter of the system, it is much more desirable to design the simulations for a specified net steady state current rather than for an initial net current. An estimation procedure for specifying the net steady state current is described in Section 5.5.1.

Table 5.3. Collisional test cases.

Run	Initial currents (MA)			$\frac{n_{T^+}}{n_{D^+}}$	N_{TOT}	B_Z	γ_D
	I_{T^+}	I_{D^+}	I_{OH}				
<u>Tests with unequal beam currents</u>							
TWOB209	1.5	-0.5	0.	3.0	3.18×10^{20}	-600	3000
TWOB210	2.0	-1.0	0.	2.0	4.77×10^{20}	-600	3000
TWOB211	3.0	-2.0	0.	1.5	7.95×10^{20}	-600	3000
<u>Vertical field tests</u>							
TWOB212	2.0	-1.0	0.	2.0	4.77×10^{20}	-300	3000
TWOB213	2.0	-1.0	0.	2.0	4.77×10^{20}	-750	3000
TWOB214	2.0	-1.0	0.	2.0	4.77×10^{20}	-900	3000
TWOB215	2.0	-1.0	0.	2.0	4.77×10^{20}	-450	3000
<u>Enhancement factor tests</u>							
TWOB216	2.0	-1.0	0.	2.0	4.77×10^{20}	-750	3000
TWOB217	2.0	-1.0	0.	2.0	4.77×10^{20}	-750	1500
TWOB218	2.0	-1.0	0.	2.0	4.77×10^{20}	-750	0

Table 5.4. Quantitative results for the collisional test cases.

Run	Steady state currents (MA)			Simulated loss rates		Electron drag loss rates	
	$I_{T^+}^S$	$I_{D^+}^S$	I_{TOT}^S	$\left(\frac{dN}{dt}\right)_{T^+}^S$	$\left(\frac{dN}{dt}\right)_{D^+}^S$	$\left(\frac{dN}{dt}\right)_{T^+}^e$	$\left(\frac{dN}{dt}\right)_{D^+}^e$
TWOB209	1.06	-0.39	0.67	3.21×10^{20}	2.21×10^{20}	2.74×10^{20}	1.82×10^{20}
TWOB210	1.37	-0.77	0.60	5.13×10^{20}	4.77×10^{20}	3.65×10^{20}	3.63×10^{20}
TWOB211	2.20	-1.57	0.63	1.01×10^{21}	1.31×10^{20}	5.47×10^{20}	7.27×10^{20}
TWOB212	1.44	-0.78	0.66	1.25×10^{21}	9.82×10^{20}	3.65×10^{20}	3.63×10^{20}
TWOB213	1.41	-0.79	0.62	3.73×10^{20}	4.29×10^{20}	3.65×10^{20}	3.63×10^{20}
TWOB214	1.42	-0.80	0.62	5.03×10^{20}	5.68×10^{20}	3.65×10^{20}	3.63×10^{20}
TWOB215	1.38	-0.77	0.61	8.16×10^{20}	6.64×10^{20}	3.65×10^{20}	3.63×10^{20}
Limiter loss rates							
TWOB216	1.37	-0.77	0.60	3.73×10^{20}	4.29×10^{20}	4.52×10^{19}	6.00×10^{19}
TWOB217	1.37	-0.79	0.58	4.33×10^{20}	4.82×10^{20}	5.14×10^{19}	8.83×10^{19}
TWOB218	--	--	--	--	--	8.34×10^{23}	3.25×10^{23}

Figure 5.10. Simulation results for TWOB209.

a) Loss plots for both beams

b) D^+ current c) T^+ current

Figure 5.11. Simulation results for TWOB210.

Figure 5.12. Simulation results for TWOB211.

The slope of a straight line drawn through the cumulative loss plots gives simulated loss rates for the beams. The simulated loss rates are compared to the predicted electron drag loss rates in Table 5.4. The results for these first three runs are also plotted in Figure 5.13. The disagreement with the analytical loss rates is largest at the lowest density ratio, n_r , where

$$n_n = \frac{n_{r+}}{n_{d+}} . \quad (5.19)$$

There are two reasons for this: '

1. The self-field interactions are larger at low n_r because each beam carries a larger current.
2. Analytical calculations have shown that a field that centers a coinjected beam does not center a counterinjected beam.⁴⁰ Therefore, as the proportion of counterinjected particles is increased (low n_r), there are more losses at the limiter.

The magnitude of the vertical field is tested in the second set of runs listed in Table 5.3. The remaining parameters in these runs are identical to those in TWOB210. The net steady state current is insensitive to changes in the vertical field, as shown in Figure 5.14. However, the loss rates are significantly affected, as shown in Figure 5.15. For the optimal field of about 750 Gauss in run TWOB213, the simulated triton loss rate agrees very well with the predicted electron drag loss rate. The disagreement for the deuteron beam is caused by the noncentering of a counterinjected beam. These

Figure 5.13. Comparison of simulated and predicted loss rates as a function of the density ratio.

Figure 5.14. Steady state beam currents as a function of the vertical field.

Figure 5.15. Loss rates as a function of the vertical field.

runs show that it is possible to design the field strength so as to minimize particle losses and to obtain close agreement with theoretical loss rates. However, there are other results from run TWOB213 which indicate that this may not be the correct method for choosing the vertical field.

One noteworthy feature of the result from TWOB213 is the shape of the flux contours during the build-up phase. The beam cross section is superimposed on the flux contours shown in Figure 5.16. Particles reinjected onto the open contours (large R) within the beam cross section will follow the open contours to the limiter. Flux configurations such as this occur quite frequently throughout the run. In a real system this would significantly increase the cumulative particle losses at the limiter.

There is a significant correlation between the shape of the contours and the net beam current plotted in Figure 5.17. The worst flux configuration occurs at the minima in the net current. The vertical field at these minima is too strong and causes the open contours. Proper shaping of the contours requires varying the vertical field with the current throughout the simulation as discussed in Section 5.5.3.

The last set of runs in Table 5.3 test the enhancement factor. The field is maintained at the optimized value for the collisional runs. Both the total losses and losses at the limiter are computed in these runs. In the first two runs (TWOB216 and TWOB217) it is unclear whether the quantitative differences in the loss rates are a result of maintaining the

a) At $t = 200 \mu\text{sec}$ b) At $t = 350 \mu\text{sec}$ c) At $t = 400 \mu\text{sec}$ d) At $t = 450 \mu\text{sec}$

Figure 5.16. Flux configurations for TW08213. The 75 cm beam deposition radius is marked.

Figure 5.17 Total current in run TWOB210.

a) T^+ losses at the limiter

b) D^+ losses at the limiter

Figure 5.18 Particle losses at the limiter in run TWOB218.

field constant in time or to artificially large collisional enhancement. In the last run (TWOB218) huge particle losses at the limiter occur for both beams compared to those in TWOB216 and TWOB217, as shown in Figure 5.18. This result can be understood by examining the details of the motion of the beams in run TWOB218. The average beam positions plotted in Figure 5.19 show that both beams experience a large drift toward the outer wall. This motion is decidedly different from the one beam motion in Figure 5.9 used to design the field.

In summary, the conclusions made from these test runs are:

1. A technique must be developed for adjusting the vertical field throughout the run to keep the beams centered.
2. The field must be adjusted on the basis of two-beam test runs rather than with one-beam runs with the same net current.
3. Because of the large transient loss due to the fixed vertical field, it is unclear from these runs that the enhancement factor has been properly selected.

5.5 The Final Steady State Model

We have developed a method for adjusting the vertical field during the simulations. This method is based on linearly fitting the vertical field to the currents. The parameters, beam currents and energies, at the two reference points used in constructing the linear relation are computed using the parameter estimation procedure described in Section 5.5.1. In order

a) Average T^+ position b) Average D^+ position

Figure 5.19. Average beam positions in run TWOB218. The scales measure distances from the injection point.

to determine the corresponding vertical field at the reference points, we study collisionless two beam motion similar to the one-beam equilibria studies in Section 5.4.2. We show that two beam equilibria exist in which the average motion of both beams forms orbits around an "equilibrium" point on the axis. The field can be adjusted so that the "equilibrium" point of one of the beams is close to the injection point and the "equilibrium" point of the other beam is only slightly displaced. Collisional runs using this new method show that the beams remain well centered throughout the run.

5.5.1 Beam Current Parameter Estimates

The beam input parameters to the model are the injection energies, W_0 , the net current at steady state, I_{ss} , and either the density ratio, n_r , or the total number of particles in the system, N_T . Suppose the density ratio of the beams is specified,

$$n_r = \frac{n_{T+}}{n_{D+}} = \frac{N_{D+}}{N_{T+}}. \quad (5.20)$$

The beam volumes are assumed equal for simplicity. At steady state the individual beam energies can be estimated with equation (5.16). Solving for the toroidal velocities in this equation yields

$$\langle v_\phi \rangle_{T+} = \left(\frac{2}{m_{T+}} \langle W \rangle_{T+} \right)^{1/2} \quad (5.21)$$

$$\langle v_\phi \rangle_{D+} = \left(\frac{2}{m_{D+}} \langle W \rangle_{D+} \right)^{1/2}. \quad (5.22)$$

The current at steady state is the sum of the individual currents,

$$I_{ss} = \frac{e}{2\pi R_b} \left\{ N_{T^+} \langle v_\phi \rangle^{T^+} + N_{D^+} \langle v_\phi \rangle^{D^+} \right\}. \quad (5.23)$$

Equations (5.20) and (5.23) can be solved for the number of particles in each beam in terms of the specified input parameters,

$$N_{D^+} = \frac{I_{ss} (2\pi R_b)/e}{\{ n_n \langle v_\phi \rangle^{T^+} + \langle v_\phi \rangle^{D^+} \}} \quad (5.24)$$

$$N_{T^+} = n_n N_{D^+}. \quad (5.25)$$

The injection currents are estimated using these computed values for the number of particles in each beam. Several runs have shown that estimating the injection currents in this way results in the selected net current at steady state.

5.5.2 Collisionless Equilibrium for Counterstreaming Beams

Collisionless beam motion is studied in runs TWOB218, TWOB219, and TWOB220. The centering fields in these runs are 750, 1000, and 900 Gauss, respectively. The average beam positions are plotted in Figures 5.19, 5.20, and 5.21. A better qualitative picture of the net beam motion is obtained by drawing a rough curve through the detailed helical orbits as shown in Figures 5.22, 5.23, and 5.24. The 900 Gauss centering field in TWOB220 gives the best results for centering both orbits near the injection point. In run TWOB218 the beams are both drifting to the outer wall and do not form an orbit about

Figure 5.20. Average beam positions in run TWOB219 for a 200 μsec simulation. The scales measure distances from the injection point.

Figure 5.21. Average beam positions for run TWOB220. The scales measure the distance from the injection point.

Figure 5.22. Rough beam positions for run TWOB218.

Figure 5.23. Rough beam positions for run TWOB219.

Figure 5.24. Rough beam positions for TWOB220.

the axis. On the other hand, after the initial drift in runs TWOB219 and TWOB220, the average motion of both beams forms orbits around the axis. The orbits are smaller and closer to the injection point in run TWOB220 than those in TWOB219. The cumulative limiter losses for the two runs are compared in Figures 5.25 and 5.26. These results show that the limiter losses can be minimized by centering the beams as close as possible to the injection point. Furthermore, these figures illustrate that, for the proper choice of vertical field, after an initial loss at the limiter there are no further losses resulting from collisionless beam motion.

Particle position plots and the flux contours at the end of the equilibrium runs (TWOB219 and TWOB220) are shown in Figures 5.27 and 5.28. The closed contours in the beam deposition region for both runs insures that there is no reinjection of particles onto open flux contours. Particles reinjected at large R within the beam cross section may be lost as they move to the inner side of the torus because of the circular shape of the limiter. This effect is more pronounced for the case of TWOB219 as shown in the final particle position plots for the two runs compared to the initial beam loading in Figure 5.29.

5.5.3 Varying the Vertical Field Linearly with the Beam Currents

The centering field can be varied during the simulation runs either by assuming a functional relationship between the field strength and beam currents or by constructing a table of

Figure 5.25. T^+ losses at the limiter. Figure 5.26. D^+ losses at the limiter.

Figure 5.27. Run TWOB219 at $t = 550 \mu\text{sec}$.

Figure 5.28. Run TW08220 at $t = 750 \mu\text{sec}$.

Figure 5.29. Particle positions for runs TWOB219 and TWOB220 at $t = 0$.

values using design runs like those in Section 5.5.2. Specifying a field for the entire range of individual beam currents and over the possible combinations of beam currents makes the second alternative quite impractical.

It is reasonable to assume that the required vertical field is linearly related to the current in the circulating beams. Two reference points must be obtained to construct this linear relation. Since the field may depend weakly on beam energies both the energies and currents must be specified at the reference points. The initialization procedure described in Section 5.5.1 estimates these parameters at two points: initialization and steady state. The field is determined at these two points for the runs listed in Table 5.5.

The beam moment diagnostics for some of the runs are shown in Figures 5.30 and 5.31. The scales in the plots measure the distance along the coordinate axes from the injection point. The best field strength for the initial (high energy, high current) configuration (Figure 5.30) appear to be between 1200 and 1300 Gauss. The 1300 Gauss field is preferable because the stronger triton beam is better centered. At the steady state (low energy, low current) reference configuration a 900 Gauss field best centers the triton beam (Figure 5.31).

These reference tests, run for a longer time, verify that the limiter loss rate is minimized if the triton beam is centered on the injection point. Centering of the triton beam is best accomplished by scaling the vertical field strength with the triton current rather than the net current. This scaling

Table 5.5. Runs to determine the field at two reference points for the linear scaling of B_z with beam current.

Run	Field (Gauss)	Run	Field (Gauss)
TWOB231	-1500	TWOB221	-750
TWOB232	-1100	TWOB222	-850
TWOB233	-1300	TWOB224	-875
TWOB234	-1200	TWOB238	-900
		TWOB239	-925
High energies		Average energies	
$W_{T^+} = 120 \text{ keV}$		$W_{T^+} = 61.25 \text{ keV}$	
$W_{D^+} = 80 \text{ keV}$		$W_{D^+} = 49.1 \text{ keV}$	
Initial currents		Steady state currents	
$I_{T^+} = 3.89 \text{ MA}$		$I_{T^+} = 2.4 \text{ MA}$	
$I_{D^+} = -1.97 \text{ MA}$		$I_{D^+} = -1.2 \text{ MA}$

a) T^+ beam; run TWOB231 b) D^+ beam; run TWOB231 c) T^+ beam; run TWOB233 d) D^+ beam; run TWOB233 e) T^+ beam; run TWOB234 f) D^+ beam; run TWOB234

Figure 5.30. Average beam positions for the high energy, initial current runs listed in Table 5.5.

a) T^+ beam, run TWOB238 b) D^+ beam, run TWOB238 c) T^+ beam, run TWOB239 d) D^+ beam, run TWOB239 e) T^+ beam, run TWOB224 f) D^+ beam, run TWOB224

Figure 5.31. Average beam positions for the average energy, steady-state current runs listed in Table 5.5.

is tested with the collisional runs listed in Table 5.6. The field is linearly varied as a function of the triton and net current in runs TWOB240 and TWOB241, respectively. The beam position plots are shown in Figures 5.32 and 5.33 after 100 μ s in each run. Both beams remain much closer to the injection point for scaling on the triton rather than the total current.

Run TWOB240 is continued to 300 μ s. At 225 μ s the field strength is too weak and both beams begin to drift to the outer wall (Figure 5.34). This drift motion occurs as both the triton and the net current approach a minimum (Figure 5.35). Since the minimum in the currents is below the estimated steady state value, a solution to this problem is to set the minimum field to the value designed for the steady state.

This procedure with a field minimum was tested in the steady state run TWOB247 discussed in Section 5.6. The tritons remain fairly well centered around the injection point. The maximum distances along each coordinate axis away from the injection point are about 3 cm in Z and 4 cm in R. The deuteron motion is to the left of the injection point as expected for a counterinjected beam. The motion in the Z direction is at most ± 4 cm.

The net current and vertical field history plots are shown in Figure 5.36. A line is drawn on the net current plot through the steady state value. The arrows point to the times at which the minimum in the field occurs based on the triton current. There is extremely good agreement between these points and the steady state value of the net current. We conclude from these

Table 5.6. Collisional test cases with a linearly varied vertical field.

Run	B_z functional dependence	Slope	Intercept	Minimum vertical field
TWOB240	Linear function of T^+ current	-1.14×10^{-4}	22.76	None
TWOB241	Linear function of total current	-1.17×10^{-4}	-269.00	None
TWOB244	Linear function of T^+ current	-1.14×10^{-4}	22.76	-900 Gauss

a) T^+ beam b) D^+ beamFigure 5.32. Average beam positions in run TWOR240 after $t' = 100 \mu\text{sec}$.

Figure 5.33. Average beam positions in run TWOB241 after $t' = 100 \mu\text{sec}$.

a) T^+ beam b) D^+ beamFigure 5.34. Average beam positions in run TW08240 after $t' = 250 \mu\text{sec}$.

a) T⁺ current b) D⁺ current

c) Total current

Figure 5.35. Beam currents and total current in run TWOB240,

a) Total current

b) Vertical field

Figure 5.36. Total current and vertical field versus time for run TWOB244.

runs that for unequal beam strengths the vertical field strength should be varied linearly with the strong beam current.

5.5.4 Summary and Conclusions

Equilibria are shown to exist for collisionless counter-streaming beams. With a carefully chosen vertical field the limiter losses are minimized and the beams remain well centered at the injection point. A procedure for varying the vertical field for the collisional runs has been developed. The beam centering is maintained with this method, which insures the correct scaling of the limiter loss rates. The enhancement factor is reduced to a value of 2×10^3 for the steady state runs. This allows most of the initial transient losses to occur before any collisional reinjections begin.

There is another more expensive method to initialize a steady run. A start up simulation without enhanced collisions could be run until the limiter losses vanish. This run could then be restarted with enhanced collisions. Since the initial transients should have no effect on the final steady state results, this method is not used for the steady state runs in Section 5.6.

A summary of the final model is presented in Table 5.7.

5.6 Steady State Simulations for Two CIT Configurations

The steady state simulation in run TWOB247 is an unequal beam case with a density ratio of 2 and a collisional enhancement factor of 2×10^3 . The other input parameters for this run are listed in Table 5.8 and quantitative results are

Table 5.7. Summary of the final CIT model.

Toroidal design

Specified inputs:

Major radius, R_0 Minor radius, r_0 Limiter radius, r_L Toroidal field on axis, $B_\phi(R_0)$

Vertical centering field:

Spatially uniform

Case 1. $I_{OH} \neq 0$ (canceling beam currents)

$$B_z = -\frac{I}{10R} \left\{ \ln \frac{8R}{a} - \frac{1}{2} \right\}$$

Case 2. $I_{OH} = 0$. B_z is linearly adjusted on the coinjector current so that the coinjector beam remains centered at the specified beam major radius.

Injected beams

Beam deposition geometry:

Major radii, R_b Minor radii, r_b Electron trapping constant, A_1 Injection energies, E^{T+} or E^{D+} Cutoff energy, E_c Number of beam particles, N_b^{T+} or N_b^{D+} Effective Z , Z_{eff}

Parabolic beam deposition profiles:

$$n_b = n_b(R_b) (1 - r^2/r_b^2)$$

$$\text{where } n_b(R_b) = \frac{2N_b}{\pi r_b^2}$$

Table 5.7 (continued)

Cutoff energy:

Ions slowed down to E_c are reinjected at high energies.

Electrons

Specified inputs:

Total current, I_{OH} Central density, $n_e(R_0)$ Central temperature, $T_e(R_0)$

Parabolic temperature profile:

$$T_e = T_e(R_0) (1 - 0.97 r^2/r_0^2)$$

Parabolic density profile:

$$n_e = n_e(R_0) (1 - r^2/r_0^2)$$

Parabolic current density:

$$J_{OH} = J_{OH}(R_0) (1 - r^2/r_0^2)$$

where

$$J_{OH}(R_0) = \frac{2I_{OH}}{\pi a^2}$$

Total current density

$$J = J_{OH} + (J_{T^+} + J_{D^+}) (1 - \frac{1}{Z_{eff}} + 1.46 \epsilon^{1/2} A_1)$$

Table 5.8. Input parameters for run TWOB247.

Toroidal parameters

Major radius: $R_0 = 375$ cm
 Minor radius: $r_0 = 75$ cm
 Toroidal field: $B_\phi = 50$ kGauss on axis
 Vertical field: max = 1300 Gauss
 min = 900 Gauss

Injected beams

Injection energies: $E^{T^+} = 120$ keV
 $E^{D^+} = 80$ keV
 Cutoff energy: $E_c = 30$ keV
 Total number of particles: $N = 9.27 \times 10^{20}$ particles
 Density ratio: $n_{T^+}/n_{D^+} = 2$
 Major radii: $R_b = 375$ cm
 Minor radii: $r_b = 75$ cm
 Electron drag time: $\tau_S^{D^+} = 298.3 \times 10^{-3}$ sec
 Simulated electron drag time: $(\tau_S^{D^+})' = 150$ μ sec
 Desired steady state current: 1.20 MA

Electrons

Density: $n_e(R_0) = 10^{14}$ cm $^{-3}$
 Temperature: $T_e(R_0) = 10$ keV
 Current: $I_{OH} = 0$.

presented in Table 5.9. Both the individual and net beam currents at steady state are in good agreement with the estimated values. The simulated loss rates are larger than the electron drag rates due to some limiter losses. As a result, the injection currents are larger than the analytical predictions.

The evolution of the particle losses and currents to steady state is shown in Figure 5.37. An excellent steady state balance is attained for the deuterons. The tritons are also approaching a steady state. The average energy of each beam at the end of the run is in very good agreement with analytical predictions. Neither the steady state beam currents or energies appear to be too sensitive to the larger limiter losses in the simulation. The flux contours in the beam deposition region remain closed throughout the run. Contour plots are shown in Figure 5.38 for $t = 0, 1.6$, and 3.2 sec.

The amount of peaking in the energy distribution is an important factor in determining the micro-stability of the beam distributions. The beam energy distributions in the simulation are constructed by dividing the total energy interval, $(W_0 - W_C)$, into 100 equal bins and computing the number of particles in each energy interval. The deuteron histograms are shown in Figure 5.39. Because of the noise* in the histogram a rough curve is drawn through each histogram to show the qualitative

*This noise could be reduced either by using fewer energy bins or by smoothing the results.

Table 5.9. Steady state results and injection rates for run TWOB247.

Steady state currents

	Simulated	Estimated
I_{SS}^{T+}	2.69	2.75
I_{SS}^{D+}	-1.51	-1.50
I_{SS}^{TOT}	1.18	1.25

Injection currents

$$I_{inj}^{T+} = 135 \text{ A}$$

$$I_{inj}^{D+} = 144 \text{ A}$$

Loss rates

$$\left(\frac{dN}{dt}\right)^{T+} = 8.4 \times 10^{20} \text{ particles/sec}$$

$$\left(\frac{dN}{dt}\right)^{D+} = 9.0 \times 10^{20} \text{ particles/sec}$$

Analytical comparisons from electron drag

$$\frac{dN}{dt} = - \frac{N_0}{\tau_{life}}$$

$$\left(\frac{dN}{dt}\right)^{T+} = 7.1 \times 10^{20} \text{ particles/sec; } I_{inj}^{T+} = 113.7 \text{ A}$$

$$\left(\frac{dN}{dt}\right)^{D+} = 7.06 \times 10^{20} \text{ particles/sec; } I_{inj}^{D+} = 113.2 \text{ A}$$

Figure 5.37. Steady-state results for run TWO8247.

Figure 5.38. Flux configurations and beam positions for the steady-state run TWOB247.

a) The energy distribution for D^+ at 3.25 sec

b) Rough drawing of the energy distribution for D^+ at 3.25 sec

Figure 5.39. The deuteron energy distribution at the end of the steady-state run TW08247.

shape of the distribution. The triton energy distribution has not quite reached a steady state (Figure 5.40), but it also shows the peaking of the distribution at low energies.

Another CIT configuration is simulated in run TWOB018 (Table 5.10). This run is designed so that the steady-state beam currents cancel and the net 1 MA current at steady state is carried by the background electrons. Initially the beam currents of approximately 3.0 and 3.3 MA are noncanceling.

The evolution of the system to a steady state is shown in Figure 5.41. Quantitative results are listed in Table 5.11.

The vertical field in this run is held constant and set to the value needed to center the electron current. As a result, there are significant transient particle losses as the beam strengths vary during the approach to steady state. Nevertheless, as the system approaches steady state the flux contours are closed in the beam deposition region because of the nearly complete beam current cancelation. Neither beam has quite reached a steady state at the end of the run (2.2 sec). However, the deuteron beam is almost settled, and the energy distribution is peaked toward lower energies as in the previous case. The energy distributions, flux contours, and other diagnostics are shown at the end of the simulation in Figure 5.42.

a) At $t' = 3.00$ sec b) At $t' = 3.20$ sec

Figure 5.40. The triton energy distribution near the end of the steady-state run TWOB247.

Table 5.10. Input parameters for run TWOB018.

Toroidal parameters

Major radius: $R_0 = 375$ cm
 Minor radius: $r_0 = 75$ cm
 Toroidal field: $B_\phi = 50$ kGauss on axis
 Vertical field: $B_z = -1000$ Gauss

Injected beams

Injection energy: $E^{T+} = 120$ keV (coinjected)
 $E^{D+} = 80$ keV
 Cutoff energy: $E_c = 30$ keV
 Density ratio: $n_r = 1.12$
 Total number of particles: 10^{21}
 Major radii: 375 cm
 Minor radii: 75 cm
 Simulated electron drag time: $(\tau_S^{D+})' \sim 150$ μ sec
 Electron drag time: 298.3×10^{-3} sec
 Desired steady state current: 1.0 MA

Electrons

Density: 10^{14} cm $^{-3}$
 Current: $I_{OH} = 1.0$ MA
 Temperature: 10 keV

a) T^+ current b) D^+ current c) Loss plots for the T^+ beam d) Loss plots for the D^+ beam

Figure 5.41 Steady state results for run TWOB018

e) Total current

f) Average T⁺ energy g) Average D⁺ energy

Figure 5.41 (cont'd) Steady-state results for run TWGB01.

Table 5.11. Steady state results and injection rates for run TWOB018.

Steady state currents

$$I_{SS}^{T^+} = 2.34 \text{ MA}$$

$$I_{SS}^{D^+} = 2.33 \text{ MA}$$

$$I_{SS}^{TOT} = 0.97 \text{ MA}$$

Loss rates

$$\left(\frac{dN}{dt}\right)_{T^+} = 7.20 \times 10^{20} \text{ particles/sec}$$

$$\left(\frac{dN}{dt}\right)_{D^+} = 1.27 \times 10^{21} \text{ particles/sec}$$

Injection currents

$$I_{inj}^{T^+} = 115.3 \text{ A}$$

$$I_{inj}^{D^+} = 203.4 \text{ A}$$

Analytical comparisons from electron drag

$$\left(\frac{dN}{dt}\right)_{T^+} = 6.08 \times 10^{20} \text{ particles/sec; } I_{inj}^{T^+} = 97.4 \text{ A}$$

$$\left(\frac{dN}{dt}\right)_{D^+} = 1.09 \times 10^{21} \text{ particles/sec; } I_{inj}^{D^+} = 174.6 \text{ A}$$

Figure 5.42 Final results at $t = 2.1$ sec for run TWO3018.

VI. CONCLUSIONS AND FUTURE WORK

6.1 Summary and Conclusions

We have presented the details of a guiding center expansion of the Lorentz force equation in which we have included terms describing Fokker-Planck collisional slowing down. In the expansion we assumed no zero order electric fields and no explicit time dependence in the magnetic field. With these assumptions we derived the collisional modifications to the energy and magnetic moment of a charged particle. In the more general case with zero order electric fields, the guiding center velocity must also be corrected for collisions.

We have studied the motion of strong ion currents with a guiding center macroparticle model. Radial pinching has been observed in previous studies of the guiding center motion of strong ion currents. We have investigated this phenomena with simulations of a linearly propagating plasma column in a cylinder. Our test runs with this configuration show that the radial pinching is accompanied by an unstable growth of the beam self-fields. We have described an extensive set of simulations used to study this instability. Both numerical and physical quantities were varied in these simulations. The results of the runs show that the instability is numerical rather than physical. We have developed both smoothing methods and modifications to the integration scheme to solve these numerical problems.

A major question which arises in a toroidal system composed of two strong oppositely-directed current elements is whether the mutual repulsion of the currents will cause a lack of

confinement in major radius. In this configuration it is assumed that ions lost at the limiter of the system are replaced so that the ion currents are maintained. We have studied the confinement of the ion currents for a range of parameters suggested for the counterstreaming ion torus. By analyzing the details of the average beam motion we have shown that the beams can be confined for this parameter range.

We have investigated the possibility of steady state operation in an ideal CIT system. In this system it is predicted that two counterstreaming ion species (T^+ and D^+) will produce a significant amount of fusion reactions during a collisional slowing down period on an electron background. When the hot ions are cooled to an energy minimum, they are removed from the system and replaced by more hot ions. Ions are also lost from this system as a result of following open magnetic flux surfaces to the limiter of the system. For steady state balance, these two loss mechanisms are balanced by injection of the hot species into the system.

We have simulated two interesting configurations for the CIT reactor. In both cases the D^+ species reached a steady state balance. The T^+ beams were approaching a steady state, but for practical reasons the simulations were not run long enough to achieve a true balance for these beams. In both configurations the results have shown an increased loss rate (at the limiter) over the loss rate predicted from an electron drag calculation. We have presented quantitative steady-state results and energy re-injection rates necessary to maintain steady state balance.

6.2 Future Work

There are several areas of further development which are being considered. In our simulations of the ideal CIT reactor we have limited the collisional phenomena to electron drag. In a real system, there will always be background cooled ions and other impurity species. It is important to account for collisions with these ions, and in general, to model collisions with a background of several species. Therefore, the collisional slowing down in the current model should be generalized to include several species and to take into account the velocity dependence of the collision frequency. We are currently investigating a Monte-Carlo method for incorporating pitch angle scattering into the model.

Two other calculations can be added to the present model. A plasma equilibrium balance can be computed for a background plasma, when present. We have also developed a method for computing the safety factor, q , of a beam plasma system. This is currently being tested.

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APPENDIX I

CALCULATION OF ENERGY LOSSES IN THE CYCLOTRON OSCILLATIONS

The power in the cyclotron motion is given by

$$\langle \mathbf{F}_{\text{cyc}} \cdot \mathbf{v} \rangle = \langle \{ (\dot{\mathbf{R}}_0 + \mathbf{v} \cdot \mathbf{p}_0) + \mathcal{E}_1 (\mathbf{R}_{11} + \mathbf{p}_{\tilde{t}}) \} \cdot \mathbf{F}_{\text{cyc}} \rangle \quad (\text{I.1})$$

The term $\mathbf{R}_0 \cdot \mathbf{F}_{\text{cyc}}$ vanishes because the two vectors are perpendicular. Substituting the left-hand side of the force equation (2.43a) and expanding the product yields

$$\begin{aligned} \langle \mathbf{F}_{\text{cyc}} \cdot \mathbf{v} \rangle^{\text{lhs}} = & \langle \mathbf{v} \cdot \mathbf{p}_0 \cdot \mathbf{v}^2 \cdot \mathbf{p}_{00} + \mathcal{E}_1 \mathbf{v} \cdot \mathbf{p}_0 \cdot (\dot{\mathbf{v}} \cdot \mathbf{p}_0 + 2 \mathbf{v} \cdot \mathbf{p}_{\tilde{t}}) \\ & + \mathcal{E}_1 \mathbf{v}^2 \cdot \mathbf{p}_{00} \cdot (\dot{\mathbf{R}}_{11} + \mathbf{p}_{\tilde{t}}) + O(\mathcal{E}_1^2) \rangle \end{aligned} \quad (\text{I.2})$$

This equation requires an evaluation of the derivatives of $\mathbf{p}$. Using equation (2.47), the derivatives are

$$\dot{\mathbf{p}}_0 = -2\pi \hat{b} \times \mathbf{p} \quad (\text{I.3})$$

$$\dot{\mathbf{p}}_{\tilde{t}} = \hat{\mathbf{p}}_{\tilde{t}} \hat{e}_p \quad \text{for} \quad \hat{e}_p = \mathbf{p} / |\mathbf{p}| \quad (\text{I.4})$$

$$\dot{\mathbf{p}}_{00} = -(2\pi)^2 \mathbf{p}. \quad (\text{I.5})$$

Evaluation of each of the $\mathcal{E}_1$ terms individually gives

$$\mathbf{v} \cdot \mathbf{v} \cdot \mathbf{p}_0 \cdot \mathbf{p}_0 = \frac{p_0^2}{2} \frac{d}{dt} v^2 = \frac{p^2}{2} \frac{d}{dt} \Omega^2 \quad (\text{I.6})$$

$$2 \mathbf{v}^2 \cdot \mathbf{p}_0 \cdot \dot{\mathbf{p}}_{00} = v^2 \frac{d}{dt} p_0^2 = \Omega^2 \frac{d}{dt} p^2 \quad (\text{I.7})$$

$$\mathbf{v}^2 \cdot \mathbf{p}_{00} \cdot \dot{\mathbf{p}}_{\tilde{t}} = -(2\pi)^2 \mathbf{v}^2 \cdot \mathbf{p} \cdot \dot{\mathbf{p}}_{\tilde{t}} = -\frac{\Omega^2}{2} \frac{d}{dt} p^2 \quad (\text{I.8})$$

Combining these terms yields

$$\epsilon_1 (\nu \underline{p}_\theta \cdot [\dot{\nu} \underline{p}_\theta + 2 \nu \underline{p}_{\theta\theta}] + \nu^2 \underline{p}_{\theta\theta} \cdot \underline{p}_\theta) = \epsilon_1 \frac{d}{dt} \frac{1}{2} \Omega^2 \rho^2 \quad (I.9)$$

Therefore,

$$\begin{aligned} \langle \underline{V} \cdot \underline{F}_{cyc} \rangle^{rhs} &= \nu^3 \left\langle \frac{d}{dt} \frac{\rho^2}{2} \right\rangle + \epsilon_1 \left\langle \frac{d}{dt} \frac{\Omega^2}{2} \rho^2 \right\rangle \\ &+ \epsilon_1 \langle \underline{p}_{\theta\theta} \rangle \cdot \dot{\underline{R}}_\perp \end{aligned} \quad (I.10)$$

The first and last terms in this equation vanish as a result of the averaging, leaving

$$\langle \underline{V} \cdot \underline{F}_{cyc} \rangle^{rhs} = \epsilon_1 \frac{d}{dt} \frac{\Omega^2}{2} \hat{\rho}^2 \quad (I.11)$$

Next consider $\underline{V} \cdot \underline{F}_{cyc}^{rhs}$. As before the term $\underline{R}_0 \cdot \underline{F}_{cyc}$ vanishes leaving

$$\begin{aligned} \langle \underline{V} \cdot \underline{F}_{cyc} \rangle^{rhs} &= \langle \nu \underline{p}_\theta \cdot \{ \nu \underline{p}_\theta \times \underline{\Omega} + \epsilon_1 \underline{p} \cdot \frac{\partial}{\partial \underline{R}} (\underline{a} + \dot{\underline{R}} \cdot \underline{\Omega}) \} \rangle \\ &+ \epsilon_1 \langle \nu \underline{p}_\theta \cdot \{ \nu \underline{p}_\theta \times \underline{p} \cdot \frac{\partial \underline{\Omega}}{\partial \underline{R}} - \nu \langle \underline{p}_\theta \times \underline{p} \cdot \frac{\partial \underline{\Omega}}{\partial \underline{R}} \rangle \} \rangle \\ &+ \epsilon_1 \langle \nu \underline{p}_\theta \cdot \dot{\underline{p}} \times \underline{\Omega} - \epsilon_2 \nu \underline{p}_\theta \rangle \\ &+ \epsilon_1 \langle (\dot{\underline{p}} + \dot{\underline{R}}_{\perp}) \cdot \nu (\underline{p}_\theta \times \underline{\Omega}) \rangle \end{aligned} \quad (I.12)$$

Evaluating the averages term by term gives

$$\begin{aligned} \nu \langle \underline{p}_\theta \cdot (\underline{p}_\theta \times \underline{\Omega}) \rangle &= \langle \underline{p}_\theta \cdot \left(-\frac{2\pi\nu}{\Omega} \right) \cdot (\underline{\Omega} \times \underline{p}) \times \underline{\Omega} \rangle \\ &= -\epsilon_1 \Omega^2 \frac{d}{dt} \frac{\hat{\rho}^2}{2} \end{aligned} \quad (I.13)$$

$$\nu \dot{\underline{R}}_{\perp} \cdot \langle \underline{p}_\theta \rangle \times \underline{\Omega} = 0 \quad (\text{average vanishes}) \quad (I.14)$$

$$\langle \nabla \underline{p}_\theta \cdot (\nabla \underline{p}_\theta \times \underline{n}) \rangle = 0 \quad (\text{perpendicular vectors}) \quad (\text{I.15})$$

$$\langle \nabla^2 \underline{p}_\theta \cdot (\underline{p}_\theta \times \underline{p} \cdot \frac{\partial \underline{n}}{\partial \underline{R}}) \rangle = 0 \quad (\text{perpendicular vectors}) \quad (\text{I.16})$$

$$\begin{aligned} -\langle \nabla \underline{p}_\theta \cdot \nabla \langle \underline{p}_\theta \times (\underline{p} \cdot \frac{\partial \underline{n}}{\partial \underline{R}}) \rangle \rangle &= \nabla \langle \underline{p}_\theta \rangle \cdot \langle \underline{p}_\theta \times \underline{p} \cdot \frac{\partial \underline{n}}{\partial \underline{R}} \rangle \\ &= 0 \quad (\text{average vanishes}) \end{aligned} \quad (\text{I.17})$$

$$\epsilon_1 \langle \nabla \underline{p}_\theta \cdot (\dot{\underline{R}} \times \underline{n}) \rangle = \epsilon_1 \frac{\Omega^2}{2} \frac{d}{dt} \hat{p}^2 \quad (\text{I.18})$$

$$-\epsilon_1 \epsilon_2 \nabla^2 \langle \underline{p}_\theta \cdot \underline{p}_\theta \rangle = -\epsilon_2 (2\pi)^2 \nabla^2 \hat{p}^2 = -\epsilon_2 \Omega^2 \hat{p}^2 \quad (\text{I.19})$$

The last term to be evaluated is

$$\begin{aligned} \langle \nabla \underline{p}_\theta \cdot \{ \underline{p} \cdot \frac{\partial}{\partial \underline{R}} (\underline{a} + \dot{\underline{R}} \times \underline{n}) \} \rangle \\ = \nabla \langle \underline{p}_\theta \cdot \underline{p} \rangle \cdot \frac{\partial}{\partial \underline{R}} (\underline{a} + \dot{\underline{R}} \times \underline{n}) \end{aligned} \quad (\text{I.20})$$

The first factor on the right-hand side can be rewritten as

$$\nabla \langle \underline{p}_\theta \cdot \underline{p} \rangle = 2\pi \nabla \frac{\hat{p}^2}{2} (\hat{e}_1 \hat{e}_2 - \hat{e}_2 \hat{e}_1) = \frac{\Omega \hat{p}^2}{2} (\hat{e}_1 \hat{e}_2 - \hat{e}_2 \hat{e}_1) \quad (\text{I.21})$$

Taking the contraction product of this tensor with the differential operator gives

$$\begin{aligned} (\hat{e}_1 \hat{e}_2 - \hat{e}_2 \hat{e}_1) \frac{\partial}{\partial \underline{R}} &= -(\hat{e}_1 \times \hat{e}_2) \times \frac{\partial}{\partial \underline{R}} \\ &= -\hat{b} \times \frac{\partial}{\partial \underline{R}} \end{aligned} \quad (\text{I.22})$$

Substituting (I.21) and (I.22) into (I.19) gives

$$\begin{aligned} \langle \nabla \underline{p}_\theta \cdot \{ \underline{p} \cdot \frac{\partial}{\partial \underline{R}} (\underline{a} + \dot{\underline{R}} \times \underline{n}) \} \rangle \\ = -2\pi \hat{p} \left(\hat{b} \times \frac{\partial}{\partial \underline{R}} \right) \cdot (\underline{a} + \dot{\underline{R}} \times \underline{n}) \end{aligned} \quad (\text{I.23})$$

The last term can be rewritten as

$$(\hat{\underline{b}} \times \frac{\partial}{\partial \underline{R}}) \cdot (\dot{\underline{R}} \times \underline{\Omega}) = \left(\frac{\partial}{\partial \underline{R}} \times \{ \dot{\underline{R}} \times \underline{\Omega} \} \right) \cdot \hat{\underline{b}} \quad (\text{I.24})$$

and

$$\begin{aligned} \frac{\partial}{\partial \underline{R}} \times (\dot{\underline{R}} \times \underline{\Omega}) &= \dot{\underline{R}} \left(\frac{\partial}{\partial \underline{R}} \cdot \underline{\Omega} \right) + (\underline{\Omega} \cdot \frac{\partial}{\partial \underline{R}}) \dot{\underline{R}} \\ &\quad - \underline{\Omega} \left(\frac{\partial \dot{\underline{R}}}{\partial \underline{R}} \right) - \left(\dot{\underline{R}} \cdot \frac{\partial}{\partial \underline{R}} \right) \underline{\Omega} \end{aligned} \quad (\text{I.25a})$$

$$\frac{\partial}{\partial \underline{R}} \times (\dot{\underline{R}} \times \underline{\Omega}) = - \left(\dot{\underline{R}} \cdot \frac{\partial}{\partial \underline{R}} \right) \underline{\Omega} \quad (\text{I.25b})$$

The first three terms vanish in equation (I.25a) because $\underline{R}$ and $\dot{\underline{R}}$ are independent variables and the magnetic field has a vanishing divergence. The first term on the right-hand side of equation (I.20) can be rewritten using Faraday's law giving

$$\hat{\underline{b}} \times \left(\frac{\partial}{\partial \underline{R}} \cdot \underline{a} \right) = \left(\frac{\partial}{\partial \underline{R}} \times \underline{a} \right) \cdot \hat{\underline{b}} = \frac{\partial \underline{\Omega}}{\partial \underline{t}} \cdot \hat{\underline{b}} = \dot{\underline{\Omega}} \quad (\text{I.26})$$

Substituting (I.26) and (I.25b) into (I.20) gives

$$\begin{aligned} &\langle \underline{V} \cdot \underline{F}_{cy} \cdot \left(\underline{\hat{e}} \cdot \frac{\partial}{\partial \underline{R}} \{ \underline{a} + \dot{\underline{R}} \times \underline{\Omega} \} \right) \rangle \\ &\quad = - \hat{\rho}^2 (\underline{\Omega} \cdot \dot{\underline{\Omega}} + \underline{\Omega} \cdot \dot{\underline{R}} \cdot \frac{\partial \underline{\Omega}}{\partial \underline{R}}) \cdot \hat{\underline{b}} \\ &\quad = - \underline{\Omega} \cdot \hat{\rho}^2 \cdot \dot{\underline{\Omega}} \end{aligned} \quad (\text{I.27})$$

Collecting terms gives the result

$$\begin{aligned} \langle \underline{V} \cdot \underline{F}_{cy} \rangle^{rhs} &= - \epsilon_1 \underline{\Omega}^2 \frac{d}{dt} \left(\frac{\hat{\rho}^2}{2} \right) + \epsilon_1 \frac{\underline{\Omega}}{2} \hat{\rho}^2 \dot{\underline{\Omega}} \\ &\quad - \epsilon_1 \epsilon_2 \underline{\Omega}^2 \hat{\rho}^2 + \underline{\Omega}^2 \frac{d}{dt} \frac{\hat{\rho}^2}{2} \end{aligned} \quad (\text{I.28})$$

This can be rewritten using the following derivative expansion

$$\Omega \frac{d}{dt} \left(\frac{\Omega \hat{p}^2}{2} \right) = -\Omega^2 \frac{d}{dt} \left(\frac{\hat{p}^2}{2} \right) + \frac{\Omega}{2} \hat{p}^2 \dot{\Omega} \quad (I.29)$$

Substitution into equation (I.28) gives the final result

$$\begin{aligned} \langle \underline{Y} \cdot \underline{E}_{eye} \rangle^{hs} = & -\epsilon_1 \Omega^2 \frac{d}{dt} \left(\frac{\hat{p}^2}{2} \right) - \epsilon_1 \epsilon_2 \hat{p}^2 \Omega \\ & + \epsilon_1 \Omega \frac{d}{dt} \left(\frac{\Omega \hat{p}^2}{2} \right) \end{aligned} \quad (I.30)$$

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