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Transformations from Barycentric
to Laboratory Coordinates for
Two- and Three-Body Scattering Events

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by

William B. Maier II

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TRANSFORMATIONS FROM BARYCENTRIC TO LABORATORY
COORDINATES FOR TWO- AND THREE-BODY SCATTERING EVENTS

by

William B. Maier II

ABSTRACT

Formulas which transform differential cross sections from the barycentric, or center-of-mass, coordinate system to the laboratory coordinate system are derived and presented. Methods for obtaining the laboratory kinetic-energy distributions of reaction products from the differential cross sections, σ' , in the barycentric system are given. These discussions pertain to two types of reactions: those, such as

which produce two particles and those, such as

which produce three particles, where Q is the energy released by the reaction. These two reactions are treated explicitly for two simple forms of σ' , namely $\sigma' \propto 1/\sin \theta^*$ and $\sigma' = \text{constant}$, where θ^* is the scattering angle in the barycentric system; for these cases we plot the distributions of the reaction products over kinetic energy and scattering angle in the laboratory coordinate system.

A few incidental results are mentioned below. For a given value of Q , the laboratory kinetic energy of a particle produced by reaction (1) is analytically determined by the laboratory scattering angle, θ ; particles produced by reaction (2) and scattered at an angle θ have a distribution of laboratory kinetic energies that depends on σ' . If the cross section for a reaction which produces three bodies is proportional to the volume of momentum space available to the products, then the final kinetic energy tends to be equipartitioned among the reaction products, provided that the products' masses are not too different. If one of the products, b_3 , of reaction (2) is much less massive than the others, then the separation of b_1 and b_2 approximates a two-particle disintegration in which the total kinetic energy is not fixed. Simultaneous measurements of the laboratory kinetic energy and angular distributions of one type of secondary particle produced in reaction (1) are sufficient to determine the dependence of σ' on θ^* and Q .

PREFACE

A few preliminary remarks concerning this report seem to be in order. The results are derived by conceptually simple manipulations of well-known formulas. These results are not represented as being entirely new, and very little effort has been made to correlate them with earlier work, although a considerable amount of previous work does exist.¹⁻⁶ The derivations are detailed, even when the result is already available, because the author feels that the convenience of a relatively complete collection of results and derivations justifies some repetition of earlier work. The author's concern in this work is that of one experimentally investigating atomic and molecular scattering, but the results may be of interest to persons working in other fields.

I. INTRODUCTION

An increasing effort is being directed toward understanding atomic and molecular scattering processes. Reactions, such as

and

where a , X , b , Y , b_1 , b_2 , and b_3 are particles and Q is the energy released by the reaction, are of importance in many fields and therefore are studied extensively. In most cases the differential cross sections for such processes are not known. Often the efficiency with which a particular reaction product is collected in an experiment depends sensitively upon the differential cross section in the laboratory coordinate system (LCS); an understanding of the differential cross section in LCS would therefore aid in estimating the collection efficiency of reaction products in a given experiment. Moreover, a theoretical understanding of atomic and molecular

scattering processes must encompass an understanding of their differential cross sections; tests of theoretically predicted differential cross sections must be made in LCS. The purposes of this work are to collect some results which give insight into the forms of differential cross sections in LCS and to treat the problem of transforming differential cross sections from the barycentric (center-of-mass) coordinate system (BCS) to LCS.

We shall transform solid angle from BCS to LCS for reaction (1) and show that for reaction (2) the problem cannot be so simply stated. We shall treat the problem of transforming differential cross sections from BCS to LCS for reactions (1) and (2) and give explicit formulas for a few simple cases. We shall examine the kinetic energy distributions of secondary particles produced by reactions (1) and (2) for a few cases. We shall examine the differences between reaction (1), which produces two secondary particles, and reaction (2), which produces three secondary particles.

The results presented in the following sections are based on classical, nonrelativistic arguments and are derived, essentially, from conservation of energy and momentum; specifically, this assumption implies that the particles' "rest masses" are not zero. Some of the results are useful only when total mass is conserved; the reader should be able to discern such cases quickly. Unless otherwise mentioned, only a single value of Q is considered for a given reaction; more general results can be obtained by summing appropriately over Q (see Appendix C). It is further assumed, unless otherwise noted, that the differential cross section in BCS depends only on θ^* , defined in Fig. 1; these results apply, for example, to central-potential scattering.

A glossary of symbols and terms is given in Appendix A.

Fig. 1. A two-body disintegration. (a) Scattering event in LCS. The target particle, X , is at rest. The velocity vectors lie in a single plane. (b) Scattering event in BCS. The velocity vectors lie in a single plane. $\vec{v}'_a$ is parallel to $\vec{v}_X$, and $\vec{v}'_b$ is parallel to $\vec{v}_Y$.

II. THE TWO-BODY PROBLEM

A. General

Consider a reaction

$$a + X \rightarrow b + Y + Q, \quad (1)$$

where

$a \equiv$ projectile

$X \equiv$ target particle

$b \equiv$ observed secondary particle

$Y \equiv$ unobserved secondary particle $\equiv$ recoil particle

$Q \equiv$ energy released in the reaction ($Q < 0$ for an endothermic reaction) .

We shall refer to this type of reaction, or any other which produces two secondary particles, as a two-body disintegration. Let

$\vec{v}_b \equiv$ velocity of particle b in LCS

$\vec{p}_b \equiv$ momentum of particle b in LCS

$E_b \equiv$ kinetic energy of particle b in LCS ,

etc. (see Fig. 1) .

Then the velocity, $\vec{v}_c$, of the barycentric coordinate system is^{7,8}

$$\vec{v}_c = \frac{m_a \vec{v}_a + m_X \vec{v}_X}{m_a + m_X} = \frac{m_b \vec{v}_b + m_Y \vec{v}_Y}{m_b + m_Y} \quad (3)$$

The energy E_o available for reaction in BCS is

$$E_o = E_a + E_X - \frac{1}{2} (m_a + m_X) v_c^2 = \frac{m_a m_X (\vec{v}_a - \vec{v}_X)^2}{2(m_a + m_X)} \quad (4)$$

We shall need another parameter,

$$\gamma = v_c / v'_b, \quad (5)$$

where $\vec{v}'_b \equiv$ velocity of particle b in BCS,

$$v'_b \equiv |\vec{v}'_b|$$

It can be shown^{7,8} that

$$\tan \theta = \frac{\sin \theta^*}{\cos \theta^* + \gamma} \quad 0 \leq \theta \leq \pi, \quad (6)$$

where

$\theta^* \equiv$ scattering angle of b in BCS ,

$\theta \equiv$ scattering angle of b in LCS .

Since the kinetic energy, E'_b , of particle b in BCS is given by

$$E'_b = \frac{m_Y}{m_Y + m_b} (E_o + Q) ,$$

$$\gamma = v_c \sqrt{\frac{(m_Y + m_b)m_b}{2(E_0 + Q)m_Y}} \quad (7)$$

Equations (5) and (7) are general expressions and apply, as examples, to reaction (1) or to the spontaneous disintegration of an excited particle, A,

$$A \rightarrow b + Y + Q \quad (8)$$

where $\vec{v}_c = \vec{v}_A$.

Often the target particle, X, may be taken as stationary, and, in such cases, Eqs. (3) and (4) become⁷

$$\vec{v}_c = \frac{m_a \vec{v}_a}{m_a + m_X} \quad (9)$$

$$E_0 = \frac{1}{2} \mu_{aX} v_a^2 = \frac{\mu_{aX}}{m_a} E_a \quad (10)$$

$$\gamma = \left\{ \frac{m_a m_b E_a (m_Y + m_b)}{[m_Y(m_X + m_a)Q + m_Y m_X E_a][m_X + m_a]} \right\}^{1/2} \quad (11)$$

where $\mu_{aX} = m_a m_X / (m_a + m_X)$ is the reduced mass of a and X. Or, with conservation of mass,

$$\gamma = \left[\frac{m_a m_b E_a}{(M - m_b)(Mq + m_X E_a)} \right]^{1/2} \quad (12)$$

where $M = m_X + m_a = m_Y + m_b$ is the total mass.

The results of Section II depend explicitly on only θ and γ . Thus, if the value of γ is known for a two-body disintegration, the formulas which are derived can be used, whether the reaction be Eq. (1), Eq. (8), or something else. In fact, if γ is known for a certain secondary particle, b, produced by any process, the results of this section apply. It should be noted that whenever $\gamma = 0$ the barycentric and laboratory coordinate systems are identical.

B. Transformation of Solid Angle from the Barycentric to the Laboratory Coordinate System

The transformation of solid angle from LCS to BCS is straightforward⁷ and results in

$$d\Omega = \frac{1 + \gamma \cos \theta^*}{(1 + 2\gamma \cos \theta^* + \gamma^2)^{3/2}} d\Omega^* \quad 0 \leq \theta^* \leq \pi$$

where

$d\Omega \equiv$ solid angle in LCS,

$d\Omega^* \equiv$ solid angle in BCS.

Care must be exercised in the application of this formula when $\gamma > 1$.

The transformation of solid angle from BCS to LCS is also straightforward, but the resulting expressions depend upon whether $\gamma \geq 1$ or $\gamma < 1$. From Eq. (6),

$$\gamma \sin \theta = \sin(\theta^* - \theta)$$

If $\gamma < 1$, then $|\gamma \sin \theta| < 1$ for all θ , and

$$\theta^* = \theta + \arcsin(\gamma \sin \theta) \quad 0 \leq \theta \leq \pi \quad (13)$$

If $\gamma \geq 1$, then, since $|\gamma \sin \theta| \leq 1$,

$$0 \leq \theta \leq \theta_m = \arcsin(1/\gamma) \quad , \quad (14)$$

and there are two solutions of Eq. (6):

$$\theta_+^* = \theta + \arcsin(\gamma \sin \theta) \quad 0 \leq \theta^* \leq \frac{\pi}{2} + \theta_m \quad , \quad (15)$$

and

$$\theta_-^* = \pi + \theta - \arcsin(\gamma \sin \theta) \quad \frac{\pi}{2} + \theta_m \leq \theta^* \leq \pi \quad . \quad (16)$$

(See Fig. 2.)

For $\gamma < 1$,

$$d\theta^* = \frac{(\sqrt{1 - \gamma^2 \sin^2 \theta} + \gamma \cos \theta)}{\sqrt{1 - \gamma^2 \sin^2 \theta}} d\theta \quad , \quad (17)$$

and

$$\sin \theta^* = \sin \theta (\sqrt{1 - \gamma^2 \sin^2 \theta} + \gamma \cos \theta) \quad , \quad (18)$$

so

$$d\Omega^* = f(\theta) d\Omega \quad 0 \leq \theta \leq \pi \quad , \quad (19)$$

and

$$f(\theta) = \frac{(\sqrt{1 - \gamma^2 \sin^2 \theta} + \gamma \cos \theta)^2}{\sqrt{1 - \gamma^2 \sin^2 \theta}} \quad . \quad (20)$$

For $\gamma \geq 1$,

$$d\theta^* = \frac{(\sqrt{1 - \gamma^2 \sin^2 \theta} \pm \gamma \cos \theta)}{\sqrt{1 - \gamma^2 \sin^2 \theta}} d\theta \quad , \quad (21)$$

and

Fig. 2. (a) Vector addition of center-of-mass velocity in LCS, $\vec{v}_c$, to observed particle velocity in BCS, $\vec{v}_b$, for $\gamma > 1$. For a given value of $\theta < \theta_m = \arcsin(1/\gamma)$, θ^* has two values, θ_+^* and θ_-^* , corresponding to two values for each of $\vec{v}_b'$ and $\vec{v}_b$, but to only one value of $\vec{v}_b'$. (b) Representation of $d\theta^*/d\theta$ for $\gamma > 1$. Near $\theta = \theta_m$, a small change in θ corresponds to a much larger change in θ^* , and $d\theta^*/d\theta \rightarrow \infty$ as $\theta \rightarrow \theta_m$.

$$\sin \theta^* = \sin \theta \left(\gamma \cos \theta \pm \sqrt{1 - \gamma^2 \sin^2 \theta} \right) \quad (22)$$

Since the plus and minus signs in Eqs. (21) and (22) belong together,

$$\begin{aligned} d\Omega^* &= f_+(\theta) d\Omega & 0 \leq \theta^* \leq \frac{\pi}{2} + \theta_m \\ &= f_-(\theta) d\Omega & \theta_m + \frac{\pi}{2} \leq \theta^* \leq \pi \end{aligned} \quad (23)$$

$$f_{\pm}(\theta) = \frac{\left(\gamma \cos \theta \pm \sqrt{1 - \gamma^2 \sin^2 \theta} \right)^2}{\sqrt{1 - \gamma^2 \sin^2 \theta}} \quad 0 \leq \theta \leq \theta_m \quad (24)$$

The parameter γ is constant for a given scattering event.

C. Transformations of Differential Cross Sections from BCS to LCS

1. General Discussion

Suppose that the differential scattering cross section in BCS is $\sigma'(\theta^*)$. For $\gamma < 1$,

$$\sigma'(\theta^*) d\Omega^* = \sigma'(\theta^*(\theta)) f(\theta) d\Omega \quad 0 \leq \theta \leq \pi, \quad (25)$$

with $\theta^*(\theta)$ from Eq. (13) and $f(\theta)$ from Eq. (20). For $\gamma \geq 1$,

$$\sigma'(\theta^*) d\Omega^* = \left[\sigma'(\theta_+^*(\theta)) f_+(\theta) + \sigma'(\theta_-^*(\theta)) f_-(\theta) \right] d\Omega \quad 0 \leq \theta \leq \theta_m. \quad (26)$$

θ_m , $\theta_{\pm}^*(\theta)$, and $f_{\pm}(\theta)$ are given by Eqs. (14) to (16) and (24); $\theta_+^*(\theta)$ implies that for $0 \leq \theta^* \leq \theta_m + \pi/2$ $\theta^* = \theta_+^* = \theta + \arcsin(\gamma \sin \theta)$, and $\theta_-^*(\theta)$ implies that for $\theta_m + \pi/2 \leq \theta^* \leq \pi$, $\theta^* = \theta_-^* = \pi + \theta - \arcsin(\gamma \sin \theta)$.

2. Two Simple Examples

Let us consider two particularly simple examples, which have very different angular dependences in BCS.

Suppose that scattering is isotropic in BCS, so that if $\sigma'(\theta^*)$ is normalized to one unit of area for particles scattered into 4π sr, then

$$\sigma'(\theta^*) = \frac{1}{4\pi} \text{ cm}^2/\text{sr}, \quad (27)$$

if the unit of area is cm^2 . Equations (25) and (26) become

$$\sigma'(\theta^*) d\Omega^* = g(\theta) d\theta, \quad (28)$$

where

$$g(\theta) = \begin{cases} \frac{\left(\sqrt{1 - \gamma^2 \sin^2 \theta} + \gamma \cos \theta \right)^2 \sin \theta}{2\sqrt{1 - \gamma^2 \sin^2 \theta}} & 0 \leq \theta \leq \pi \text{ for } \gamma < 1 \\ \frac{(1 + \gamma^2 \cos 2\theta) \sin \theta}{\sqrt{1 - \gamma^2 \sin^2 \theta}} & 0 \leq \theta \leq \theta_m \text{ for } \gamma \geq 1 \\ 0 & \theta_m \leq \theta \text{ for } \gamma \geq 1 \end{cases} \quad (29)$$

$\theta_m = \arcsin(1/\gamma)$.

$g(\theta) d\theta$ is the probability that a particle scatters at θ into $d\theta$ in LCS. Equation (25) can be rewritten

$$\sigma'(\theta^*) d\Omega^* = 4\pi \sigma'(\theta^*(\theta)) g(\theta) d\theta \quad \gamma < 1, \quad (30)$$

and Eq. (26) becomes

$$\sigma'(\theta^*) d\Omega^* = 4\pi \left[\sigma'(\theta_+^*(\theta)) g_+(\theta) + \sigma'(\theta_-^*(\theta)) g_-(\theta) \right] d\theta \quad \gamma \geq 1, \quad (31)$$

$$g_{\pm}(\theta) = \frac{(\sqrt{1 - \gamma^2 \sin^2 \theta} \pm \gamma \cos \theta)^2 \sin \theta}{2\sqrt{1 - \gamma^2 \sin^2 \theta}}. \quad (32)$$

Notice that for $\gamma \geq 1$

$$g_+(\theta) + g_-(\theta) = g(\theta)$$

and that

$$\int_0^{\theta_0} g(\theta) d\theta = 1 \quad \theta_0 = \begin{cases} \pi & \text{for } \gamma < 1 \\ \theta_m & \text{for } \gamma \geq 1 \end{cases}$$

The functions g , g_+ , and g_- are plotted in Figs. 3 through 6 for a few values of γ . For $\gamma > 1$, infinities occur at $\theta = \theta_m = \arcsin(1/\gamma)$; the physical reason for these infinities can be inferred from Fig. 2, where it is observed that as $\theta \rightarrow \theta_m$, $d\theta^*/d\theta \rightarrow \infty$. Values for g , g_+ , and g_- are tabulated in Appendix B for selected values of γ and θ .

Suppose that "glory" scattering^{9,10} obtains in BCS; i.e., suppose that $\sigma'(\theta^*) \propto 1/\sin \theta^*$. Normalize $\sigma'(\theta^*)$ to one unit of area for particles scattered into 4π sr; then

$$\sigma'(\theta^*) = \frac{1}{2\pi^2 \sin \theta^*}. \quad (33)$$

Fig. 3. Plot of $g(\theta)$ vs θ for various values of $\gamma \leq 1$. Isotropic, two-body disintegration. Curve K corresponds to the assumed barycentric angular distribution. Compare with Fig. 18.

Fig. 4. Plot of $g(\theta)$ vs θ for various values of $\gamma \geq 1$. Isotropic, two-body disintegration. Compare with Fig. 19.

Fig. 5. Plot of $g_+(\theta)$ vs θ for various values of $\gamma \geq 1$.

Fig. 6. Plot of $g_-(\theta)$ vs θ for various values of $\gamma \geq 1$.

Equations (25) and (26), or Eqs. (30) and (31), become

$$\sigma'(\theta^*) d\Omega^* = S(\theta) d\theta, \quad (34)$$

where

$$S(\theta) = \begin{cases} \frac{\sqrt{1 - \gamma \sin^2 \theta + \gamma \cos \theta}}{\pi \sqrt{1 - \gamma^2 \sin^2 \theta}} & 0 \leq \theta \leq \pi \quad \text{for } \gamma < 1 \\ \frac{2\gamma \cos \theta}{\pi \sqrt{1 - \gamma^2 \sin^2 \theta}} & 0 \leq \theta \leq \theta_m \quad \text{for } \gamma \geq 1 \\ 0 & \theta_m \leq \theta \quad \text{for } \gamma \geq 1 \end{cases} \quad (35)$$

$S(\theta)d\theta$ is the probability that a particle scatters at an angle θ into $d\theta$ in LCS. The function $S(\theta)$ is plotted in Figs. 7 and 8 for a few values of γ . Infinities occur at $\theta = \theta_m$ for $\gamma > 1$; refer to Fig. 2 for the reason. For all of the curves,

$$\int_0^{\theta_0} S(\theta) d\theta = 1 \quad \theta_0 = \begin{cases} \pi & \gamma < 1 \\ \theta_m & \gamma \geq 1 \end{cases}.$$

Notice that if Eq. (34) is rewritten as

$$\sigma'(\theta^*) d\Omega^* = s(\theta) d\Omega,$$

then

$$s(\theta) \propto S(\theta)/\sin \theta;$$

$s(\theta)$ always has an infinity for forward scattering, in addition to the infinity at $\theta = \theta_m$ for $\gamma > 1$. Values

Fig. 7. Plot of $S(\theta)$ vs θ for various values of $\gamma \leq 1$. Glory, two-body disintegration. Curve K corresponds to the assumed barycentric angular distribution. Compare with Fig. 20.

Fig. 8. Plot of $S(\theta)$ vs θ for various values of $\gamma \geq 1$. Glory, two-body disintegration. Compare with Fig. 21.

for $S(\theta)$ for selected values of γ are tabulated in Appendix B.

Figures 7 and 8 should be compared with Figs. 3 and 4 to see the differences in the laboratory angular distributions caused by the differences in isotropic and glory angular distributions in BCS. In both cases, if

$$\sigma'(\theta^*) d\Omega^* = \zeta(\theta^*) d\theta^* ,$$

then $\zeta(\theta^*)$ is given by curve K for $\gamma = 0$ in the appropriate figure; i.e., whenever $\gamma = 0$, the barycentric and laboratory angular distributions are identical.

D. Energy Distributions of the Products of Two-Body Disintegrations

For some applications it is of interest to know the laboratory kinetic energy of the reaction products. For example, one can obtain information about the value of Q for a reaction by measuring the laboratory kinetic energy of the secondary particles.

In BCS the cross section may be a function of θ^* , E , and Q ; i.e., $\sigma'(\theta^*, E, Q)$, where E is the total energy available to the disintegrating system,

$$E = E_0 + Q \quad . \quad (36)$$

For given Q and E_0 , E'_b has only one value,

$$E'_b = \frac{\mu_{bY}}{m_b} E = \frac{\mu_{bY}}{m_b} (E_0 + Q) = \frac{m_Y E}{m_b + m_Y} \quad , \quad (37)$$

and the probability distribution function, $P(E'_b)$, is

$$P(E'_b) = \delta \left[E'_b - \frac{\mu_{yb}}{m_b} (E_o + Q) \right] ,$$

where $\delta(x) \equiv$ Dirac δ -function.

In LCS there will be a distribution of kinetic energies for particle b, even for fixed Q. The kinetic energy in LCS, E_b , is given by

$$E_b = E'_b (1 + \gamma^2 + 2\gamma \cos \theta^*) \quad (38)$$

Combine Eqs. (6) and (22) and obtain

$$\cos \theta^* = \cos \theta \left(\gamma \cos \theta \pm \sqrt{1 - \gamma^2 \sin^2 \theta} \right) - \gamma , \quad (39)$$

so that

$$E_b = E'_b \left(\gamma \cos \theta \pm \sqrt{1 - \gamma^2 \sin^2 \theta} \right)^2 \quad (40)$$

If $\gamma \leq 1$, use only the plus sign in Eq. (40).

The probability distribution, $P(E_b)$, depends on the functional form of the differential scattering cross section in BCS. Also, as Eq. (40) shows, there is an explicit connection between θ and E_b , so that; given θ , one can predict which values of E_b occur. Two simple examples follow.

Suppose that scattering is isotropic in LCS, i.e., that

$$\sigma'(\theta^*) = \frac{1}{4\pi} .$$

There is more than one way to find the kinetic energy distribution in LCS, $P_1(E_b)$, but the easiest way is to note that with $d\Omega^* = 2\pi \sin \theta^* d\theta^*$,

$$\sigma'(\theta^*) d\Omega^* = \frac{1}{2} \sin \theta^* d\theta^* \quad (41)$$

From Eq. (38)

$$dE_b = - 2 E'_b \gamma \sin \theta^* d\theta^* , \quad (42)$$

so that

$$\sigma'(\theta^*) d\Omega^* = \frac{-1}{4\gamma E'_b} dE_b .$$

The minus sign merely pertains to the order of integration, and

$$P_1(E_b) dE_b = \frac{dE_b}{4\gamma E'_b} , \quad (1 - \gamma)^2 E'_b \leq E_b \leq (1 + \gamma)^2 E'_b \quad (43)$$

$$= 0 \quad \text{otherwise} ,$$

where $P_1(E_b) dE_b \equiv$ probability of finding particle b with energy E_b in dE_b . Rewrite Eq. (43) with

$$\rho \equiv E_b/E'_b \quad (44)$$

$$P_1(\rho) d\rho \equiv P_1(E_b) dE_b = \frac{d\rho}{4\gamma} \quad (1 - \gamma)^2 \leq \rho \leq (1 + \gamma)^2 \quad (45)$$

$$= 0 \quad \text{otherwise}$$

Figure 9 is a plot of $P_1(\rho)$ vs ρ for a few values of γ . A value of ρ determines a value of θ (see Fig. 2), so particles with a certain kinetic energy always are at a predictable value of θ .

Fig. 9. Plot of the kinetic energy distribution, $P_i(\rho)$, of secondary particles in LCS when the scattering in BCS is isotropic. $\int P_i(\rho) d\rho = 1$. Compare with Figs. 10 and 24.

Fig. 10. Plot of the kinetic energy distribution, $P_g(\rho)$, of secondary particles in LCS when glory scattering holds in BCS. $\int P_g(\rho) d\rho = 1$. Compare with Fig. 9.

Suppose that "glory" scattering obtains in BCS.

$$\sigma'(\theta^*) = \frac{1}{2\pi^2 \sin \theta^*},$$

$$\begin{aligned} \sigma'(\theta^*) d\Omega^* &= d\theta^* / \pi \\ &= \frac{-dE_b}{2E'_b \gamma \sin \theta^*} \end{aligned}$$

Since $0 \leq \theta^* \leq \pi$, Eq. (38) is equivalent to

$$\sin \theta^* = \frac{\sqrt{4\gamma^2 - \left(1 + \gamma^2 - \frac{E_b}{E'_b}\right)^2}}{2\gamma} \quad (46)$$

$$\sigma'(\theta^*) d\Omega^* = \frac{-dE_b}{\pi E'_b \sqrt{4\gamma^2 - \left(1 + \gamma^2 - \frac{E_b}{E'_b}\right)^2}}.$$

Thus,

$$\begin{aligned} P_g(E_b) &= \frac{1}{\pi E'_b \sqrt{4\gamma^2 - \left(1 + \gamma^2 - \frac{E_b}{E'_b}\right)^2}} \quad (1 - \gamma)^2 E'_b \leq E_b \leq (1 + \gamma^2) E'_b \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (47)$$

where $P_g(E_b) dE_b \equiv$ probability of finding particle b with energy E_b in dE_b . With $\rho = E_b / E'_b$,

$$P_g(\rho)d\rho \equiv P_g(E_b)dE_b = \frac{d\rho}{\pi \sqrt{4\gamma^2 - (1 + \gamma^2 - \rho)^2}} \quad (1 - \gamma)^2 \leq \rho \leq (1 + \gamma)^2 \quad (48)$$

$$= 0 \quad \text{otherwise}$$

Figure 10 is a plot of $P_g(\rho)$ vs ρ for a few values of γ . Particles with given ρ , i.e., with given kinetic energy, are scattered at one value of θ .

In these two examples, the kinetic energy distribution of particle b in LCS is symmetric around $E_b = E'_b(1 + \gamma^2)$, i.e., around $\rho = (1 + \gamma^2)$. The kinetic energy distribution, $P(E_b)$, in LCS alters if $\sigma'(\theta^*)$ changes, even when there is only one fixed value of Q possible (compare Figs. 9 and 10). Thus, a particular distribution of secondary-particle kinetic energies, $P(E_b)$, is associated with given Q and $\sigma'(\theta^*)$. If $P(E_b)$ is known, then Eq. (38) can be used to obtain $\sigma'(\theta^*)$ uniquely; thus, a measurement of $P(E_b)$ determines $\sigma'(\theta^*)$ implicitly.* In Appendix C we shall treat briefly an important, practical consequence of the result that $P(E_b)$ depends on the value of Q and on $\sigma'(\theta^*)$.

III. THE THREE-BODY PROBLEM

A. General

Suppose that three particles, say b_1 , b_2 , and b_3 separate after interaction. We refer, henceforth, to a reaction which produces three secondary particles as a three-body disintegration. Examples of three-body disintegrations are the reaction shown in Fig. 11,

and the spontaneous decomposition of a body,

where the symbols have been previously defined. Equations (3), (4), (6), (9), and (10) still hold for pertinent situations, where θ is the scattering angle of particle b_1 in LCS (see Fig. 11) and θ^* is the scattering angle of particle b_1 in BCS, defined as the angle between $\vec{P}'_1$ and $\vec{v}_c$. Let

$\vec{P}'_1$ = momentum of particle b_1 in BCS

$\vec{v}'_1$ = velocity of particle b_1 in BCS

E'_1 = kinetic energy of particle b_1 in BCS

$\vec{P}_1$ = momentum of particle b_1 in LCS

$\vec{v}_1$ = velocity of particle b_1 in LCS

E_1 = kinetic energy of particle b_1 in LCS

E = total kinetic energy available to b_1 , b_2 , and b_3 in BCS ,

etc.

In BCS,

$$\vec{P}'_1 + \vec{P}'_2 + \vec{P}'_3 = 0 \quad , \quad (50)$$

$$E'_1 + E'_2 + E'_3 = E \quad . \quad (51)$$

*Suppose that the kinetic energy distribution is determined for secondary particles, b, scattered into a relatively small portion of the allowed solid angle in LCS. The dependence of the total cross section on E_b cannot be uniquely inferred from the dependence on E_b of such an energy distribution (cf. Fig. 9, Fig. 10, and Eq. (40)).

Fig. 11. Three-body disintegration in LCS. The target particle, X, is at rest. The velocities are not necessarily coplanar.

Fig. 12. Momenta of three separating particles in their barycentric coordinate system.

Equation (50) implies that $\vec{P}_1'$, $\vec{P}_2'$, and $\vec{P}_3'$ are coplanar as in Fig. 12. If any two angles, say α and β , between the momenta are specified, then Eqs. (50) and (51) may be solved to give¹¹

$$P_1' = (2 M_0 E)^{1/2} \sin \alpha, \quad (52)$$

$$P_2' = (2 M_0 E)^{1/2} \sin \beta, \quad (53)$$

and

$$P_3' = - (2 M_0 E)^{1/2} \sin (\alpha + \beta), \quad (54)$$

where $P_1' \equiv |\vec{P}_1'|$, etc.

$$M_0 = \frac{m_1 m_2 m_3}{m_1 m_2 \sin^2(\alpha + \beta) + m_1 m_3 \sin^2 \beta + m_2 m_3 \sin^2 \alpha}, \quad (55)$$

and

$$\begin{aligned} 0 &\leq \alpha \leq \pi, \\ 0 &\leq \beta \leq \pi, \\ \pi &\leq \alpha + \beta \leq 2\pi. \end{aligned} \quad (56)$$

Notice that the expressions for P_1' , P_2' , and P_3' become indeterminate if $\vec{P}_1'$, $\vec{P}_2'$, and $\vec{P}_3'$ are colinear, in accordance with the fact that for three bodies moving along a line, there are three unknowns (P_1 , P_2 , P_3) to be determined by the two relations obtained from conservation of energy and conservation of momentum.

Let the direction of $\vec{P}_1'$ in BCS be given by the spherical polar and azimuthal angles, θ^* and φ^* , where θ^* is measured from $\vec{v}_c$. The directions of $\vec{P}_2'$ and $\vec{P}_3'$ are then fixed by the specification of a third angle, χ^* , representing a rotation about $\vec{P}_1'$ of the plane containing $\vec{P}_1'$, $\vec{P}_2'$, and $\vec{P}_3'$. Note that

$$\begin{aligned} 0 &\leq \theta^* \leq \pi, \\ 0 &\leq \varphi^* \leq 2\pi, \\ 0 &\leq \chi^* \leq 2\pi. \end{aligned}$$

The scattering cross section in BCS depends, generally, on α , β , E , θ^* , φ^* , and χ^* ; however, θ^* , φ^* , and χ^* merely give the orientation of the system of separating particles at large distances.

We now compute the density of states, $d\tau$, in momentum space available to a three-body disintegration.

$$d\tau = d^3\vec{P}_1' d^3\vec{P}_2' d^3\vec{P}_3' ,$$

or

$$d\tau = d^3\vec{P}_c d^3\vec{P}_1' d^3\vec{P}_2' , \quad (57)$$

where

$$\vec{P}_c \equiv \text{momentum of the center of mass}$$

$$\vec{P}_c = M\vec{v}_c$$

$$M \equiv m_1 + m_2 + m_3$$

$$m_i \equiv \text{mass of particle } b_i$$

For a particular reaction or disintegration, $\vec{P}_c$ is constant and $d^3\vec{P}_c$ has nothing to do with the dependence of reaction cross section on α, β, E , etc. It is sufficient to consider $d^3\vec{P}_1' d^3\vec{P}_2'$.

$$d^3\vec{P}_1' d^3\vec{P}_2' = \frac{\partial(P_{1x}', P_{1y}', P_{1z}', P_{2x}', P_{2y}', P_{2z}')}{\partial(E, \alpha, \beta, \theta^*, \varphi^*, \chi^*)} dE d\alpha d\beta d\theta^* d\varphi^* d\chi^* \quad (58)$$

After considerable algebra (see Appendix D), one obtains

$$d^3\vec{P}_1' d^3\vec{P}_2' = 4E^2 M_0^3 \sin^2\alpha \sin^2(\alpha + \beta) \sin^2\beta \sin\theta^* dE d\alpha d\beta d\varphi^* d\chi^* d\theta^* \quad (59)$$

M_0 is defined by Eq. (55).*

Consider

$$h(\alpha, \beta) = M_0^3 \sin^2\alpha \sin^2\beta \sin^2(\alpha + \beta) \quad (61)$$

$h(\alpha, \beta)d\alpha d\beta$ is the statistical weight in momentum space associated with α and β . Thus, if the scattering cross section were determined by the density of states in momentum space, then, for fixed E ,

$$\sigma'(\theta^*, \alpha, \beta) \propto h(\alpha, \beta) \quad (62)$$

We shall discuss and use this case later. The distribution function $h(\alpha, \beta)$ has a maximum at the most probable value of α and β . Let us find this maximum.

$$E_1' = \frac{P_1'^2}{2m_1} = \frac{M_0 E}{m_1} \sin^2\alpha ,$$

$$E_2' = \frac{M_0 E}{m_2} \sin^2\beta ,$$

$$E_3' = \frac{M_0 E}{m_2} \sin^2(\alpha + \beta) ,$$

and

$$h = \frac{m_1 m_2 m_3 E_1' E_2' E_3'}{E^3}$$

*For comparison, examine a similar volume of momentum space for a two-body disintegration producing b and Y .

$$\begin{aligned} d^3\vec{P}_b' d^3\vec{P}_Y' &= d^3\vec{P}_c d^3\vec{P}_b' \\ &= d^3\vec{P}_c (2\mu_{bY}E)^{1/2} \sin\theta^* dE d\theta^* d\varphi^* \end{aligned} \quad (60)$$

$$\mu_{bY} = m_b m_Y / (m_b + m_Y)$$

The other symbols have been previously defined.

Since

$$E'_1 + E'_2 + E'_3 = E ,$$

$$h = m_1 m_2 m_3 E'_2 E'_3 (E - E'_2 - E'_3)/E^3 . \quad (63)$$

From

$$\frac{\partial h}{\partial E'_2} = \frac{2h}{E'_2} = 0 ,$$

we find

$$E'_1 = E'_2 = E'_3 = E/3 . \quad (64)$$

Then compute at $E'_1 = E'_2 = E'_3 = E/3$,

$$A \equiv \frac{\partial^2 h}{\partial E'^2_2} = -\frac{2}{3}$$

$$B \equiv \frac{\partial^2 h}{\partial E'_2 \partial E'_3} = -\frac{1}{3}$$

$$C \equiv \frac{\partial^2 h}{\partial E'^2_3} = -\frac{2}{3}$$

$$B^2 - 4AC < 0 .$$

Thus,¹² $h(\alpha, \beta)$ has a relative maximum at angles α and β such that $E'_1 = E'_2 = E'_3 = E/3$, provided that Eq. (50) can be simultaneously satisfied; otherwise, $h(\alpha, \beta)$ has a maximum value at some point on the boundary described by Eq. (56). This result can be stated in a physically meaningful way. The greatest statistical weight is attached to those values of α and β at which available energy is equipartitioned among the three separating particles in BCS, provided that momentum can be simultaneously conserved. These points are illustrated in Figs. 13 and 15, where a normalized form of $h(\alpha, \beta)$ is plotted and called $H(\alpha, \beta)$.

$$H(\alpha, \beta) = \frac{8}{\pi} \left(\frac{M}{m_1 m_2 m_3} \right)^{3/2} M_0^3 \sin^2 \alpha \sin^2 \beta \sin^2(\alpha + \beta) , \quad (65)$$

where

$$M = m_1 + m_2 + m_3 ,$$

and

$$\iint_R H(\alpha, \beta) d\alpha d\beta = 1 ;$$

the region R is defined by Eq. (56) and is bounded in Figs. 13 and 15 by the lines labeled A. $H(\alpha, \beta)$ is multivalued at three points, $(\alpha, \beta) = (0, \pi)$, $(\pi, 0)$, and (π, π) ; this corresponds to the previously mentioned indeterminacy of P'_1 , P'_2 , and P'_3 when the moments are colinear.

If $\sigma'(\theta^*, \alpha, \beta) \propto h(\alpha, \beta) \propto H(\alpha, \beta)$ as in Eq. (62), the most probable values of α and β are those for which energy is partitioned equally among the three particles, provided that Eq. (50) can be simultaneously satisfied.

It may be of interest to note that

$$A d^3 P'_1 d^3 P'_1 = E^2 dE \frac{8}{\pi} \left(\frac{M}{m_1 m_2 m_3} \right)^{3/2} M_0^3 \sin^2 \alpha \sin^2 \beta \sin^2(\alpha + \beta) \frac{\sin \theta^* d\theta^* d\phi^* d\chi^*}{8\pi^2}$$

Fig. 13. (a) Contours of $H(\alpha, \beta)$ for $m_1 = m_2 = m_3 = 1$. The most probable values of α and β are 120° .
 (b) Contours of $H(\alpha, \beta)$ for $m_1 = m_2 = 1$, $m_3 = 2$.
 (c) Contours of $H(\alpha, \beta)$ for $m_1 = 1$, $m_2 = 2$, $m_3 = 1$.
 The maximum value of $H(\alpha, \beta)$ in each case is close to the value denoted by G. The contours depend on the masses through the ratios $m_1:m_2:m_3$.

Fig. 14. Plots of $F = F(f_1, f_2)$ for different sets of masses. Parts (a), (b), and (c) of Fig. 13 map into parts (a), (b), and (c), respectively. Allowed values of f_1 and f_2 are within the ellipses. F depends only on $m_1:m_2:m_3$. F times the area of the ellipse is unity in each case.

Fig. 15. (a) Contours of $H(\alpha, \beta)$ for $m_1 = 1, m_2 = 2, m_3 = 3$.
 (b) Contours of $H(\alpha, \beta)$ for $m_1 = m_2 = 4, m_3 = 1$.
 (c) Contours of $H(\alpha, \beta)$ for $m_1 = m_2 = 1, m_3 = 10$. Note that the maximum value of $H(\alpha, \beta)$ is at $(\alpha, \beta) = (\pi, \pi)$. The maximum value of $H(\alpha, \beta)$ in each case is close to the value denoted by G. The contours depend on the masses through the ratios $m_1:m_2:m_3$.

Fig. 16. Plots of $F = F(f_1, f_2)$ for different sets of masses. Parts (a), (b), and (c) of Fig. 15 map into parts (a), (b), and (c), respectively. Allowed values of f_1 and f_2 are within the ellipses. F depends only on $m_1:m_2:m_3$. F times the area of the ellipse is unity in each case.

is normalized such that

$$A \int_{\alpha} \int_{\beta} \int_{\chi^*} \int_{\phi^*} \int_{\theta^*} d^3\vec{P}_1' d^3\vec{P}_2' = E^2 dE \quad (66)$$

The function $H(\alpha, \beta)$ is somewhat complicated and hard to use. The average value of $\bar{q}$ of some quantity $q(\alpha, \beta)$,

$$\bar{q} = \int_R \int q(\alpha, \beta) H(\alpha, \beta) d\alpha d\beta \quad (67)$$

may be hard to find analytically even when $q(\alpha, \beta)$ is simple.

Consider the parameters f_i , where $f_i \equiv$ fraction of energy E carried away by particle b_i , $i = 1, 2, 3$. Then the kinetic energy of the i th particle in BCS is

$$E_i' = f_i E \quad (68)$$

and

$$f_1 + f_2 + f_3 = 1 \quad (69)$$

Equation (59) can be written as

$$d^3\vec{P}_1' d^3\vec{P}_2' = 4 m_1 m_2 m_3 E^2 (1 - f_1 - f_2) f_1 f_2 \frac{\partial(\alpha, \beta)}{\partial(f_1, f_2)} \sin \theta^* dE df_1 df_2 d\phi^* d\chi^* d\theta^* \quad (70)$$

After some algebraic manipulations (Appendix D), one obtains

$$\frac{\partial(\alpha, \beta)}{\partial(f_1, f_2)} = \frac{1}{4 f_1 f_2 (1 - f_1 - f_2)} \quad (71)$$

$$d^3\vec{P}_1' d^3\vec{P}_2' = m_1 m_2 m_3 E^2 \sin \theta^* dE df_1 df_2 d\phi^* d\chi^* d\theta^* \quad (72)$$

We need to find the allowed values of f_1 and f_2 . Since

$$f_3 \geq 0 \quad ,$$

$$f_1 + f_2 \leq 1 \quad .$$

Also

$$f_1 \geq 0 \quad ,$$

$$f_2 \geq 0 \quad .$$

These conditions define triangles bounded by the lines $f_3 = 0$, $f_1 = 0$, $f_2 = 0$, as shown in Figs. 14 and 16; however, only the values of f_1 and f_2 within the ellipses shown in Figs. 14 and 16 are allowed. Observe that the boundaries of the region of allowed values of f_1 and f_2 must correspond to colinear separations. Then solve

$$P_1' + P_2' + P_3' = 0$$

$$\frac{P_1'^2}{2m_1} + \frac{P_2'^2}{2m_2} + \frac{P_3'^2}{2m_3} = E$$

to obtain the ellipses in Figs. 14 and 16,

$$f_{2\pm} = \frac{m_2 m_3}{(m_2 + m_3)^2} \left[\frac{m_1 f_1}{m_3} + 1 + \frac{m_3}{m_2} - \frac{M}{m_2} f_1 \pm 2 \sqrt{\frac{m_1}{m_3} f_1 \left(1 + \frac{m_3}{m_2} - \frac{M}{m_2} f_1 \right)} \right] \quad (73)$$

The maximum permissible value, f_{1m} , of f_1 is

$$f_{1m} = \frac{m_2 + m_3}{M} = \frac{m_2 + m_3}{m_1 + m_2 + m_3} \quad (74)$$

$$m_1 m_2 m_3 \int_0^{f_{1m}} \int_{f_{2-}}^{f_{2+}} df_2 df_1 = \frac{\pi}{2} \left(\frac{m_1 m_2 m_3}{M} \right)^{3/2} \quad (75)$$

Normalized as in Eq. (66),

$$A d^3\vec{P}_1' d^3\vec{P}_2' = E^2 dE \left[\frac{2}{\pi} \frac{M^{3/2}}{(m_1 m_2 m_3)^{1/2}} df_1 df_2 \right] \frac{\sin \theta^* d\theta^* dp^* d\chi^*}{8\pi^2} \quad (76)$$

The function

$$F \equiv F(f_1, f_2) = \frac{2}{\pi} \frac{M^{3/2}}{(m_1 m_2 m_3)^{1/2}} \quad (77)$$

is the probability distribution for f_1 and f_2 in momentum space. Thus, $F(f_1, f_2) df_1 df_2$ is the probability that the fractions f_1 and f_2 of the energy E carried away by particles b_1 and b_2 is in df_1 and df_2 , respectively. This result is well-known,¹³⁻¹⁵ but the derivation is given here to exhibit the connection between α, β -space and f_1, f_2 -space. F is a constant, depending only on $m_1:m_2:m_3$, and is given for a few cases in Figs. 14 and 16. There are a few noteworthy correspondences between Figs. 13 and 14 and between Figs. 15 and 16; Table I shows explicitly how some points and lines in Fig. 13a transform into Fig. 14a. The point

Table I. Transformation from (α, β) to (f_1, f_2) for $m_1 = m_2 = m_3$

(See Figs. 13a and 14a.)

(α, β) -space		(f_1, f_2) -space
Point $(\pi, 0)$	$\longleftrightarrow$	Curve $(0, 0.5) (0.5, 0.5)$
Point $(0, \pi)$	$\longleftrightarrow$	Curve $(0.5, 0.5) (0.5, 0)$
Point (π, π)	$\longleftrightarrow$	Curve $(0.5, 0) (0, 0.5)$
Line $\alpha = \pi$	$\longleftrightarrow$	Point $(0, 0.5)$
Line $\beta = \pi$	$\longleftrightarrow$	Point $(0.5, 0)$
Line $\alpha + \beta = \pi$	$\longleftrightarrow$	Point $(0.5, 0.5)$

at which the $f_3 = 0$ line is tangent to the ellipse corresponds to $P_3' = 0$ and $\vec{P}_1' = -\vec{P}_2'$; the points of tangency between the $f_1 = 0$ and $f_2 = 0$ lines and the ellipses have similar correspondences. When $f_1 = f_{1m}$, all three momentum vectors are colinear, $P_1' = -P_2' - P_3'$, and $v_2' = v_3'$; i.e., particles b_2 and b_3 move like a single particle of mass $(m_2 + m_3)$.

The average value, $\bar{n}$, of some quantity, $n(f_1, f_2)$, over momentum space is found from

$$\bar{n} = \int_0^{f_{1m}} \int_{f_{2-}}^{f_{2+}} n(f_1, f_2) F df_1 df_2 \quad (78)$$

where $f_{1m} = (m_2 + m_3)/M$. For example, the average value of f_1 is $\bar{f}_1 = f_{1m}/2$.

Define a quantity γ analogously to Eq. (5),

$$\gamma \equiv v_c/v'_1 \quad (79)$$

$$\gamma = v_c \sqrt{\frac{m_1}{2f_1 E}} \quad (80)$$

For a two-body disintegration, γ was fixed by v_c , E_0 , and Q , but such is not the case for a three-body disintegration, since $0 \leq f_1 \leq f_{1m}$. Let γ_0 be the minimum value of γ . Then

$$\gamma_0 = v_c \sqrt{\frac{m_1}{2f_{1m} E}} \quad (81)$$

If $m_1 \neq 0$, then

$$\gamma_0 \leq \gamma \leq \infty \quad (82)$$

Equations (79) through (82) are generally applicable, but often the three-body disintegration of interest is a reaction of the form

$$a + X \rightarrow b_1 + b_2 + b_3 + Q \quad (2)$$

where X is a stationary target particle. Then

$$\vec{v}_c = \frac{m_a \vec{v}_a}{m_a + m_X} \quad (9)$$

$$E_0 = \frac{\mu_{aX}}{m_a} E_a \quad (10)$$

$$E = E_0 + Q \quad (36)$$

and, if $m_1 + m_2 + m_3 = m_a + m_X = M$,

$$\gamma_0 = \left\{ \frac{m_1 m_a E_0}{[m_2 + m_3][(m_a + m_X)Q + m_X E_0]} \right\}^{1/2} \quad (83)$$

or

$$\gamma_0 = \left[\frac{m_1 m_a E_0}{(M - m_1)(MQ + m_X E_0)} \right]^{1/2} \quad (84)$$

Equations (12) and (84) are formally identical if m_1 is identified with m_b . Note that if $\gamma_0 = 0$, then the barycentric and laboratory coordinate systems coincide.

B. Transformations of Differential Cross Sections from BCS to LCS

1. General Discussion

It is not possible to transform solid angle from BCS to LCS for a three-body disintegration as was done for a two-body disintegration. The value of γ was fixed for a two-body disintegration, but ranges between γ_0 and ∞ for a three-body disintegration. An appropriate average must be performed. In the general case of the three-body disintegration, the differential cross section may depend upon θ^* , ϕ^* , χ^* , E , Q , f_1 , and f_2 . For any given value of E and Q , the differential cross section is $\sigma''(\theta^*, \phi^*, \chi^*, f_1, f_2)$. Transformation of σ'' from BCS to LCS is a formidable task in the most general case; in fact, we have not yet defined suitable laboratory coordinates. A general expression for the transformation of σ'' from BCS to LCS

does not seem to be suitable for the present purposes and is not given here.

Consider the case in which σ'' is a function of only θ^* , f_1 , and f_2 . Note that if particle b_1 is the observed product, then from Eq. (80)

$$\gamma = \sqrt{\frac{m_1 v_c^2}{2f_1 E}} \equiv \sqrt{\frac{\epsilon}{Ef_1}} \quad (85)$$

$$\epsilon = \frac{1}{2} m_1 v_c^2 \quad (86)$$

$$df_1 = \frac{-2\epsilon}{E\gamma^3} d\gamma$$

$$\sigma''(\theta^*, f_1, f_2) d\Omega^* df_1 df_2 = \frac{-2\epsilon}{E\gamma^3} \sigma''(\theta^*, \gamma, f_2) d\gamma df_2 d\Omega^* \quad (87)$$

Use Eqs. (25) and (26) or (30) and (31) to transform $\sigma'' d\Omega^*$,

$$\begin{aligned} \sigma''(\theta^*, \gamma, f_2) d\Omega^* &= \eta_1(\theta, \gamma, f_2) d\Omega \quad \gamma < 1 \\ &= \eta_2(\theta, \gamma, f_2) d\Omega \quad \gamma \geq 1, \end{aligned} \quad (88)$$

exactly as in the two-body case, where η_1 and η_2 are the functions obtained from the transformation.

$$\begin{aligned} \sigma''(\theta^*, f_1, f_2) d\Omega^* df_1 df_2 &= \frac{-2\epsilon}{E\gamma^3} \eta_1(\theta, \gamma, f_2) d\Omega d\gamma df_2 \quad \gamma < 1 \\ &= \frac{-2\epsilon}{E\gamma^3} \eta_2(\theta, \gamma, f_2) d\Omega d\gamma df_2 \quad \gamma \geq 1. \end{aligned} \quad (89)$$

From Eq. (73),

$$f_{2-} \leq f_2 \leq f_{2+}$$

The region of θ and γ over which Eq. (88) holds is the shaded portion of Fig. 17. We can represent this region by

$$\left. \begin{aligned} \gamma_0 \leq \gamma \leq 1 & \quad 0 \leq \theta \leq \pi \\ 1 \leq \gamma \leq \infty & \quad 0 \leq \theta \leq \arcsin(1/\gamma) \end{aligned} \right\} \quad \text{for } \gamma_0 \leq 1 \quad (90a)$$

$$\gamma_0 \leq \gamma \leq \infty \quad 0 \leq \theta \leq \arcsin(1/\gamma) \quad \text{for } \gamma_0 \geq 1 \quad (90b)$$

To obtain the angular distribution in BCS or LCS, integrate over γ and f_2 or f_1 and f_2 . Let

$$\sigma'(\theta^*) \equiv \int_0^{f_{1m}} \int_{f_{2-}}^{f_{2+}} \sigma''(\theta^*, f_1, f_2) df_2 df_1 \quad (91)$$

Then for $\gamma_0 < 1$,

$$\sigma'(\theta^*) d\Omega^* = \frac{2\epsilon}{E} d\Omega \left(\int_{\gamma_0}^1 \int_{f_{2-}}^{f_{2+}} \eta_1 df_2 \frac{d\gamma}{\gamma^3} + \int_1^{\csc \theta} \int_{f_{2-}}^{f_{2+}} \eta_2 df_2 \frac{d\gamma}{\gamma^3} \right) \quad 0 \leq \theta \leq \frac{\pi}{2} \quad (92)$$

$$\sigma'(\theta^*) d\Omega^* = \frac{2\epsilon}{E} d\Omega \int_{\gamma_0}^1 \int_{f_{2-}}^{f_{2+}} \eta_1 df_2 \frac{d\gamma}{\gamma^3} \quad \frac{\pi}{2} \leq \theta \leq \pi \quad (93)$$

Fig. 17. The region of allowed values of γ and θ for a three-body disintegration.

For $\gamma_0 \geq 1$,

$$\sigma'(\theta^*) d\Omega = \frac{2e}{E} d\Omega \int_{\gamma_0}^{\csc \theta} \int_{f_{2-}}^{f_{2+}} \eta_2 df_2 \frac{d\gamma}{\gamma^3} \quad 0 \leq \theta \leq \theta_{mo} \quad (94)$$

where

$$\theta_{mo} = \arcsin(1/\gamma_0) \quad (95)$$

One may choose to normalize $\sigma'(\theta^*) d\Omega^*$ so that $A \int \sigma'(\theta^*) d\Omega^* = 1$, but care must be exercised in attempting to normalize $\sigma''(\theta^*, f_1, f_2) d\Omega^*$. In particular, if $\int \sigma''(\theta^*, f_1, f_2) d\Omega^* = 1/A'$ and if A' is a function of f_1 and/or f_2 , Eqs. (92) to (94) give different angular dependences for σ'' and $A'\sigma''$.

Consider a further specialization. Suppose σ'' is a function only of θ^* and f_1 . Now

$$\int_{f_{2-}}^{f_{2+}} df_2 = \frac{4 m_2 m_3}{(m_2 + m_3)^2} \left[\frac{m_1}{m_3} f_1 \left(1 + \frac{m_3}{m_2} - \frac{M}{m_2} f_1 \right) \right]^{1/2} \quad (96)$$

$$f_1 = \frac{e}{E\gamma^2} \quad (97)$$

$$\int_{f_{2-}}^{f_{2+}} df_2 = 4 \left[\frac{m_1 \mu_{23} e}{(m_2 + m_3)^2 E} \right]^{1/2} \frac{\sqrt{\gamma^2 - \gamma_0^2}}{\gamma^2} \quad (97)$$

where

$$\mu_{23} = \frac{m_2 m_3}{m_2 + m_3} \quad (98)$$

Define $u(\theta)$ such that

$$\sigma(\theta^*) d\Omega^* = u(\theta) d\Omega \quad (99)$$

Substitute Eq. (97) into Eqs. (92) to (94). For $\gamma_0 < 1$,

$$u(\theta) = B \left[\int_{\gamma_0}^1 \eta_1(\theta, \gamma) \frac{\sqrt{\gamma^2 - \gamma_0^2}}{\gamma^5} d\gamma + \int_1^{\csc \theta} \eta_2(\theta, \gamma) \frac{\sqrt{\gamma^2 - \gamma_0^2}}{\gamma^5} d\gamma \right] \quad 0 \leq \theta \leq \frac{\pi}{2} \quad (100)$$

$$u(\theta) = B \int_{\gamma_0}^1 \eta_1(\theta, \gamma) \frac{\sqrt{\gamma^2 - \gamma_0^2}}{\gamma^5} d\gamma \quad \frac{\pi}{2} \leq \theta \leq \pi, \quad (101)$$

and for $\gamma_0 \geq 1$

$$u(\theta) = B \int_{\gamma_0}^{\csc \theta} \eta_2(\theta, \gamma) \frac{\sqrt{\gamma^2 - \gamma_0^2}}{\gamma^5} d\gamma \quad 0 \leq \theta \leq \theta_{mo}, \quad (102)$$

where

$$B = 8 \left[\frac{m_1 \mu_{23}}{(m_2 + m_3)^2} \right]^{1/2} \left(\frac{\epsilon}{E} \right)^{3/2}.$$

The normalized version of $u(\theta)$ is given by

$$w(\theta) = \frac{u(\theta)}{\int_0^{\theta_0} u(\theta) d\theta}, \quad (103)$$

where

$$w(\theta) = u(\theta) \sin \theta; \quad (104)$$

$$\theta_0 = \begin{cases} \pi & \text{if } \gamma_0 < 1 \\ \theta_{mo} & \text{if } \gamma_0 \geq 1 \end{cases}.$$

Equations (100) to (102) apply whenever η_1 and η_2 in Eq. (89) depend only on θ and γ , regardless of the functional form of σ'' .

A treatment similar to the above is possible if σ'' is expressed as a function of θ^* , α , and β . We do not explore this equivalent approach.

2. Two Simple Examples

Suppose that scattering is "isotropic and statistical" in BCS; i.e.,

$$\sigma''(\theta^*, \alpha, \beta) \propto H(\alpha, \beta),$$

or, equivalently,

$$\sigma''(\theta^*, f_1, f_2) \propto F(f_1, f_2) \propto \text{constant},$$

where H and F are discussed in Section III A. Equations (30), (31), and (88) yield

$$\begin{aligned} \eta_1(\theta, \gamma) &= g(\theta) & 0 \leq \theta \leq \pi \text{ for } \gamma < 1 \\ \eta_2(\theta, \gamma) &= g(\theta) & 0 \leq \theta \leq \theta_m \text{ for } \gamma \geq 1 \end{aligned}$$

Equations (100) to (104) now yield for $\gamma < 1$,

$$w(\theta) \propto \sin \theta \left[\int_{\gamma_0}^1 \frac{\left(\sqrt{1 - \gamma^2 \sin^2 \theta} + \gamma \cos \theta \right)^2 \sqrt{\gamma^2 - \gamma_0^2}}{2 \sqrt{1 - \gamma^2 \sin^2 \theta} \gamma^5} d\gamma + \int_1^{\csc \theta} \frac{(1 + \gamma^2 \cos 2\theta) \sqrt{\gamma^2 - \gamma_0^2}}{\sqrt{1 - \gamma^2 \sin^2 \theta} \gamma^5} d\gamma \right] \quad 0 \leq \theta \leq \frac{\pi}{2} \quad (105)$$

$$w(\theta) \propto \sin \theta \int_{\gamma_0}^1 \frac{\left(\sqrt{1 - \gamma^2 \sin^2 \theta} + \gamma \cos \theta \right)^2 \sqrt{\gamma^2 - \gamma_0^2}}{2 \sqrt{1 - \gamma^2 \sin^2 \theta} \gamma^5} d\gamma \quad \frac{\pi}{2} \leq \theta \leq \pi, \quad (106)$$

and for $\gamma \geq 1$

$$w(\theta) \propto \sin \theta \int_{\gamma_0}^{\csc \theta} \frac{(1 + \gamma^2 \cos 2\theta) \sqrt{\gamma^2 - \gamma_0^2}}{\sqrt{1 - \gamma^2 \sin^2 \theta} \gamma^5} d\gamma \quad 0 \leq \theta \leq \theta_{mo}. \quad (107)$$

Evaluations of the integrals in Eqs. (105) to (107) are elementary but lengthy. The results are

$$J(\theta) = \frac{j(\theta)}{\int_{\theta_0}^{\theta} j(\theta) d\theta}, \quad (108)$$

where

$$\theta_0 = \begin{cases} \pi & \text{if } \gamma_0 < 1 \\ \theta_{mo} & \text{if } \gamma_0 \geq 1 \end{cases}$$

For $\gamma_0 < 1$,

$$j(\theta) = \begin{cases} (I_1 - I_2 + I_3) \sin \theta & 0 \leq \theta \leq \frac{\pi}{2} \\ (I_1 + I_2) \sin \theta & \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \quad (109)$$

For $\gamma_0 \geq 1$,

$$j(\theta) = \begin{cases} I_3 \sin \theta & 0 \leq \theta \leq \theta_{mo} \\ 0 & \theta_{mo} \leq \theta \end{cases} \quad (110)$$

$$I_1 = \frac{2(1 - \gamma_0^2)^{3/2} \cos \theta}{3\gamma_0^2} \quad (111)$$

$$I_2 = \left\{ \frac{(1 - \gamma_0^2 - 5\gamma_0^2 \cos^2 \theta) |\cos \theta| \sqrt{1 - \gamma_0^2}}{8\gamma_0^2} + \frac{(1 - \gamma_0^2 + \gamma_0^2 \cos^2 \theta)(1 - \gamma_0^2 + 5\gamma_0^2 \cos^2 \theta)}{16\gamma_0^3} \times \left[\frac{\pi}{2} + \arctan \left(\frac{1 - \gamma_0^2 - \gamma_0^2 \cos^2 \theta}{2\gamma_0 |\cos \theta| \sqrt{1 - \gamma_0^2}} \right) \right] \right\} \quad (112)$$

$$I_3 = \frac{\pi}{8\gamma_0^3} (1 - \gamma_0^2 + \gamma_0^2 \cos^2 \theta)(1 - \gamma_0^2 + 5\gamma_0^2 \cos^2 \theta) \quad (113)$$

Some lengthy manipulations show that for all $\gamma_0 > 0$,

$$\int_{\theta_0}^{\theta} j(\theta) d\theta = \frac{\pi}{8\gamma_0^3} \quad (114)$$

$J(\theta)$ is plotted in Figs. 18 and 19 for a few values of γ_0 and is tabulated in Appendix B.

Fig. 18. Plot of $J(\theta)$ vs θ for a few values of $\gamma_0 \leq 1$ ($\gamma_0 = \gamma$). Isotropic, statistical, three-body disintegration. Curve K corresponds to the assumed, barycentric angular distribution. Compare with Fig. 3.

Fig. 19. Plot of $J(\theta)$ vs θ for a few values of $\gamma_0 \geq 1$ ($\gamma_0 = \gamma$). Isotropic, statistical, three-body disintegration. Compare with Fig. 4.

Suppose that "glory, statistical" scattering obtains in BCS; i.e.,

$$\sigma''(\theta^*, f_1, f_2) \propto 1/\sin \theta^*$$

Equations (30), (31), and (88) yield

$$\eta_1 = S(\theta) \quad 0 \leq \theta \leq \pi \text{ for } \gamma < 1$$

$$\eta_2 = S(\theta) \quad 0 \leq \theta \leq \theta_m \text{ for } \gamma \geq 1,$$

where $S(\theta)$ is given by Eq. (35). Equations (100) to (104) yield, for $\gamma_0 < 1$,

$$w(\theta) \propto \left[\int_{\gamma_0}^1 \frac{\left(\sqrt{1 - \gamma^2 \sin^2 \theta} + \gamma \cos \theta \right) \sqrt{\gamma^2 - \gamma_0^2}}{\sqrt{1 - \gamma^2 \sin^2 \theta} \gamma^5} d\gamma + \int_1^{\csc \theta} \frac{2\gamma \cos \theta \sqrt{\gamma^2 - \gamma_0^2}}{\sqrt{1 - \gamma^2 \sin^2 \theta} \gamma^5} d\gamma \right] \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad (115)$$

and

$$w(\theta) \propto \int_{\gamma_0}^1 \frac{\left(\sqrt{1 - \gamma^2 \sin^2 \theta} + \gamma \cos \theta \right) \sqrt{\gamma^2 - \gamma_0^2}}{\sqrt{1 - \gamma^2 \sin^2 \theta} \gamma^5} d\gamma \quad \frac{\pi}{2} \leq \theta \leq \pi; \quad (116)$$

for $\gamma_0 \geq 1$,

$$w(\theta) \propto \int_{\gamma_0}^{\csc \theta} \frac{2\gamma \cos \theta \sqrt{\gamma^2 - \gamma_0^2}}{\sqrt{1 - \gamma^2 \sin^2 \theta} \gamma^5} d\gamma \quad 0 \leq \theta \leq \theta_{mo} \quad (117)$$

These integrals are expressible in terms of elliptic integrals of the first and second kind. Some manipulations yield

$$\Sigma(\theta) = \frac{w(\theta)}{\int_0^{\theta_0} w(\theta) d\theta} \quad (118)$$

where

$$\theta_0 = \begin{cases} \pi & \text{if } \gamma_0 < 1 \\ \theta_{mo} & \text{if } \gamma_0 \geq 1 \end{cases}$$

For $\gamma_0 < 1$

$$w(\theta) = \begin{cases} i_1 - i_2 + i_3 & 0 \leq \theta \leq \frac{\pi}{2} \\ i_1 + i_2 & \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \quad (119)$$

For $\gamma_0 \geq 1$

$$w(\theta) = \begin{cases} i_3 & 0 \leq \theta \leq \theta_{mo} \\ 0 & \theta_{mo} \leq \theta \end{cases} \quad (120)$$

$$i_1 = -\frac{\sqrt{1 - \gamma_0^2}}{4} + \frac{\sqrt{1 - \gamma_0^2}}{8\gamma_0^2} + \frac{1}{8\gamma_0^3} \arccos \gamma_0 \quad (121)$$

$$i_2 = \cos \theta \int_{\gamma_0}^1 \sqrt{\frac{\gamma^2 - \gamma_0^2}{1 - \gamma^2 \sin^2 \theta}} \frac{d\gamma}{\gamma^4}$$

or

$$i_2 = \frac{\cos \theta}{3} \left\{ \frac{[(1 - 2\gamma_0^2 \sin^2 \theta) E(\lambda, q) + \gamma_0^2 \sin^2 \theta F(\lambda, q)]}{\gamma_0^2} - \sqrt{1 - \gamma_0^2} |\cos \theta| \right\} \quad (122)$$

$$i_3 = 2 \cos \theta \int_{\gamma_0}^{\csc \theta} \sqrt{\frac{\gamma^2 - \gamma_0^2}{1 - \gamma^2 \sin^2 \theta}} \frac{d\gamma}{\gamma^4}$$

$$i_3 = \frac{2 \cos \theta}{3\gamma_0^2} \left[(1 - 2\gamma_0^2 \sin^2 \theta) E\left(\frac{\pi}{2}, q\right) + \gamma_0^2 \sin^2 \theta F\left(\frac{\pi}{2}, q\right) \right] \quad (123)$$

$$q = \sqrt{1 - \gamma_0^2 \sin^2 \theta} \quad (124a)$$

$$\lambda = \arcsin \sqrt{\frac{1 - \gamma_0^2}{1 - \gamma_0^2 \sin^2 \theta}} \quad (124b)$$

$$E(\lambda, q) = \int_0^\lambda \sqrt{1 - q^2 \sin^2 \alpha} \, d\alpha \equiv \text{elliptic integral of the second kind.}$$

$$F(\lambda, q) = \int_0^\lambda \frac{d\alpha}{\sqrt{1 - q^2 \sin^2 \alpha}} \equiv \text{elliptic integral of the first kind.}$$

The integration over θ gives for $\gamma_0 > 0$,

$$\int_0^{\theta_0} w(\theta) d\theta = \frac{\pi^2}{16\gamma_0^3} \quad (125)$$

$\Sigma(\theta)$ is plotted in Figs. 20 and 21 for a few values of γ_0 and is tabulated in Appendix B.

Comparison of Figs. 18 and 19 with Figs. 20 and 21 reveals the differences in laboratory angular distributions caused by the differences in barycentric angular distributions for these two cases. Curve K for $\gamma_0 = 0$ gives $\zeta'(\theta^*)$ in both cases, where $\sigma''(\theta^*) d\Omega^* = \zeta'(\theta^*) d\theta^*$; i.e., whenever $\gamma_0 = 0$, the barycentric and laboratory angular distributions are identical.

C. Energy Distributions of the Products of Three-Body Disintegrations

Finding the kinetic energy distributions of three-body disintegrations is generally more complicated than the corresponding problem for two-body disintegrations. For example, in BCS, for a given Q , a two-body disintegration gives a single, calculable value for E_b' , but a three-body disintegration gives a distribution of values for E_b' that depends on σ'' . In LCS, for a given value of Q , there can be at most two calculable values of E_b for every value of θ , but a three-body reaction product may have a distribution of kinetic

Fig. 20. Plot of $\Sigma(\theta)$ vs θ for a few values of $\gamma_0 \leq 1$ ($\gamma_0 \equiv \gamma$). Glory, statistical, three-body disintegration. Curve K corresponds to the assumed barycentric angular distribution. Compare with Fig. 7.

Fig. 21. Plot of $\Sigma(\theta)$ vs θ for a few values of $\gamma_0 \geq 1$ ($\gamma_0 \equiv \gamma$). Glory, statistical, three-body disintegration. Compare with Fig. 8.

energies for every value of θ .

Throughout this discussion it is presumed that for fixed values of E_0 and Q , σ'' is a function of f_1 , f_2 , and θ^* only.

In BCS, the kinetic energy distribution of particle b_1 is given by

$$p(\theta^*, E'_1) dE'_1 = \frac{1}{E} \int_{f_{2-}}^{f_{2+}} \sigma'' \left(\theta^*, \frac{E'_1}{E}, f_2 \right) df_2 dE'_1, \quad (126)$$

where $E'_1 = f_1 E$ and $p(\theta^*, E'_1) d\theta dE'_1$ is the probability of finding a particle scattered with θ^* in $d\theta^*$ and E'_1 in dE'_1 .

The kinetic energy distribution of particle b_1 in LCS can be found as follows:

(a) Transform $\sigma''(\theta^*, f_1, f_2) d\Omega^* = \eta(\theta, \gamma, f_2) d\Omega$ as in Eq. (88).

(b) Sum over f_2 and obtain

$$N(\theta, \gamma) = \int_{f_{2-}}^{f_{2+}} \eta(\theta, \gamma, f_2) df_2. \quad (127)$$

$N(\theta, \gamma)$ may depend parametrically on γ_0 also.

(c) Find γ as a function of E_1 and θ : $\gamma = \gamma(E_1, \theta)$.

(d) Eliminate γ from $N(\theta, \gamma) \sin \theta d\gamma$ and obtain $\tilde{F}(E_1, \theta) dE_1$, which $\tilde{F}$ may depend parametrically on γ_0 .

(e) Normalize $\tilde{F}$ so that $\int \tilde{F}(E_1, \theta) d\theta dE_1 = 1$.

$\tilde{F}(E_1, \theta) dE_1 d\theta$ is the probability of finding particle b_1 scattered with E_1 in dE_1 and θ in $d\theta$.

The preceding points are illustrated in the following example. Suppose that we have "isotropic, statistical" scattering; i.e.,

$$\sigma''(\theta^*, f_1, f_2) = \text{constant}.$$

Equation (126) becomes

$$p(E'_1) dE'_1 = \frac{8}{f_{1m}^2 \pi E^2} \sqrt{E'_1(f_{1m} E - E'_1)} dE'_1 \quad 0 \leq E'_1 \leq E_{1m}. \quad (128)$$

$p(E'_1)$ is normalized to $\int_0^{E'_{1m}} p(E'_1) dE'_1 = 1$.

$$f_{1m} = \frac{m_2 + m_1}{M} \equiv \text{maximum value of } f_1. \quad (129)$$

$$E'_{1m} = f_{1m} E \equiv \text{maximum barycentric kinetic energy of particle } b_1. \quad (130)$$

Equation (128) can be rewritten

$$p(E'_1) dE'_1 = \frac{8}{\pi} \sqrt{\rho'(1 - \rho')} d\rho' \quad 0 \leq \rho' \leq 1, \quad (131)$$

where

$$\rho' \equiv E'_1 / E'_{1m}. \quad (132)$$

ρ' is the ratio of the actual kinetic energy of particle b_1 in BCS to the maximum possible kinetic energy of particle b_1 in BCS. Thus, there is a distribution of kinetic energies for particle b_1 in BCS, which is given

by Eq. (128) or (131).

Consider now the laboratory kinetic energy distribution of particle b_1 when $\sigma''(\theta^*, f_1, f_2) = \text{constant}$. From Eq. (40)

$$E_1 = E'_1 \left(\gamma \cos \theta \pm \sqrt{1 - \gamma^2 \sin^2 \theta} \right)^2, \quad (133)$$

and when $\gamma < 1$, only the plus sign is used.

$$\epsilon = \frac{1}{2} m_1 v_c^2 \quad (86)$$

$$E'_1 = \frac{\epsilon}{\gamma^2} \quad (134)$$

Define

$$r = \sqrt{\frac{E_1}{\epsilon}} \quad (135)$$

$$E_1 = \frac{\epsilon}{\gamma^2} \left(\gamma \cos \theta \pm \sqrt{1 - \gamma^2 \sin^2 \theta} \right)^2. \quad (136)$$

Solve Eq. (136) for γ

$$\gamma = (1 - 2r \cos \theta + r^2)^{-1/2} = [\sin^2 \theta + (\cos \theta - r)^2]^{-1/2}. \quad (137)$$

Perform steps (a) through (e) outlined in the fourth paragraph of this section. After some algebraic manipulations, one obtains for all γ_0

$$\tilde{F}(\theta, E_1) = F(\theta, r)/\epsilon, \quad (138a)$$

$$F(\theta, r) = \frac{4\gamma_0^3 r}{\pi} \sqrt{1 - \gamma_0^2 (1 - 2r \cos \theta + r^2)} \sin \theta. \quad (138b)$$

If $\gamma_0 \geq 1$, the allowed ranges of θ and r are given by

$$0 \leq \theta \leq \theta_{mo}, \quad \left(\gamma_0 \cos \theta - \sqrt{1 - \gamma_0^2 \sin^2 \theta} \right)^2 / \gamma_0^2 \leq r^2 \leq \left(\gamma_0 \cos \theta + \sqrt{1 - \gamma_0^2 \sin^2 \theta} \right)^2 / \gamma_0^2 \quad (139a)$$

or by

$$\left(1 - \frac{1}{\gamma_0} \right)^2 \leq r^2 \leq \left(1 + \frac{1}{\gamma_0} \right)^2, \quad 0 \leq \theta \leq \arccos \left[(1 + r^2 - \gamma_0^{-2}) / 2r \right] \quad (139b)$$

If $\gamma_0 \leq 1$, the allowed ranges of θ and r are given by

$$0 \leq \theta \leq \pi, \quad 0 \leq r^2 \leq \left(\gamma_0 \cos \theta + \sqrt{1 - \gamma_0^2 \sin^2 \theta} \right)^2 / \gamma_0^2 \quad (140a)$$

or by

$$0 \leq r^2 \leq \left(1 + \frac{1}{\gamma_0} \right)^2, \quad \begin{cases} 0 \leq \theta \leq \pi & \text{for } 0 \leq r^2 \leq \left(\frac{1}{\gamma_0} - 1 \right)^2 \\ 0 \leq \theta \leq \arccos \left[(1 + r^2 - \gamma_0^{-2}) / 2r \right] & \text{for } \left(\frac{1}{\gamma_0} - 1 \right)^2 \leq r^2 \leq \left(\frac{1}{\gamma_0} + 1 \right)^2 \end{cases} \quad (140b)$$

The regions defined by Eqs. (139) and (140) are shown in Fig. 22 for three representative values of γ_0 . Note that

Fig. 22. Allowed values of $r^2 (\equiv E_1/\epsilon)$ and θ . The allowed regions are bounded by the curve shown for each γ_0 and the coordinate axes.

$$r^2 = E_1/\epsilon \quad (141)$$

$\tilde{F}(\theta, E_1) dE_1 d\theta$ is the probability that a particle is scattered in LCS with θ in $d\theta$ and E_1 in dE_1 . $\int \int \tilde{F}(\theta, E_1) dE_1 d\theta = 1$; i.e., $\int \int F(\theta, r) d\theta d(r^2) = 1$. When the dependence on θ is integrated out, the kinetic energy distribution of particle b_1 in LCS is obtained.

When $\gamma_0 \geq 1$,

$$P_T(r) = \frac{4\gamma_0}{3\pi} \left[1 - \gamma_0^2 (1 - r)^2 \right]^{3/2} \quad \left(1 - \frac{1}{\gamma_0} \right)^2 \leq r^2 \leq \left(1 + \frac{1}{\gamma_0} \right)^2, \quad (142)$$

and when $\gamma_0 \leq 1$,

$$P_T(r) = \frac{4\gamma_0}{3\pi} \cdot \begin{cases} \left[1 - \gamma_0^2 (1 - r)^2 \right]^{3/2} - \left[1 - \gamma_0^2 (1 + r)^2 \right]^{3/2} & 0 \leq r^2 \leq \left(\frac{1}{\gamma_0} - 1 \right)^2 \\ \left[1 - \gamma_0^2 (1 - r)^2 \right]^{3/2} & \left(\frac{1}{\gamma_0} - 1 \right)^2 \leq r^2 \leq \left(1 + \frac{1}{\gamma_0} \right)^2 \end{cases} \quad (143)$$

$P_T(r) d(r^2) = P_T(r) \frac{dE_1}{\epsilon} \equiv$ probability of finding particle b_1 with kinetic energy E_1 in dE_1 , summed over all values of the scattering angle θ .

For some purposes, it is more convenient to work with distributions expressed in terms of a parameter, ρ , defined analogously to the ρ defined in Eq. (44).

$$\rho = E_1/E'_{1m}, \quad (144)$$

where $E'_{1m} \equiv$ maximum kinetic energy of particle b_1 in BCS. From Eqs. (134), (141), and (144),

$$\epsilon = \gamma_0^2 E'_{1m} \quad (145)$$

$$\rho = \gamma_0^2 r^2 \quad (146)$$

Equations (138) to (140) can be rewritten in terms of ρ :

$$\tilde{F}(\theta, E_1) = F_1(\theta, \rho) / E'_{1m} \quad (147a)$$

$$F_1(\theta, \rho) = \frac{4}{\pi} \sqrt{\rho [1 - (\gamma_0^2 - 2\gamma_0 \sqrt{\rho} \cos \theta + \rho)]} \sin \theta \quad (147b)$$

If $\gamma_0 \geq 1$, the ranges of θ and ρ are given by

$$0 \leq \theta \leq \theta_{mo}, \quad (\gamma_0 \cos \theta - \sqrt{1 - \gamma_0^2 \sin^2 \theta})^2 \leq \rho \leq (\gamma_0 \cos \theta + \sqrt{1 - \gamma_0^2 \sin^2 \theta})^2, \quad (148a)$$

or by

$$(\gamma_0 - 1)^2 \leq \rho \leq (\gamma_0 + 1)^2, \quad 0 \leq \theta \leq \arccos [(1 + \rho - \gamma_0^2) / 2 \sqrt{\rho}] \quad (148b)$$

If $\gamma_0 \leq 1$, the allowed values of θ and ρ are given by

$$0 \leq \theta \leq \pi, \quad 0 \leq \rho \leq (\gamma_0 \cos \theta + \sqrt{1 - \gamma_0^2 \sin^2 \theta})^2, \quad (149a)$$

or by

$$0 \leq \rho \leq (\gamma_0 + 1)^2, \quad \begin{cases} 0 \leq \theta \leq \pi & \text{for } 0 \leq \rho \leq (1 - \gamma_0)^2 \\ 0 \leq \theta \leq \arccos [(1 + \rho - \gamma_0^2) / 2 \sqrt{\rho}] & \text{for } (1 - \gamma_0)^2 \leq \rho \leq (1 + \gamma_0)^2 \end{cases} \quad (149b)$$

Note that $\iint F_1(\theta, \rho) d\theta d\rho = \iint \tilde{F}_1(\theta, E_1) dE_1 d\theta = 1$.

Equations (142) and (143) can be rewritten in terms of ρ . When $\gamma_0 \geq 1$,

$$P_t(\rho) = \frac{4}{3\pi\gamma_0} [1 - (\gamma_0 - \sqrt{\rho})^2]^{3/2} \quad (\gamma_0 - 1)^2 \leq \rho \leq (\gamma_0 + 1)^2, \quad (150)$$

and when $\gamma_0 \leq 1$,

$$P_t(\rho) = \frac{4}{3\pi\gamma_0} \cdot \begin{cases} [1 - (\gamma_0 - \sqrt{\rho})^2]^{3/2} - [1 - (\gamma_0 + \sqrt{\rho})^2]^{3/2} & 0 \leq \rho \leq (1 - \gamma_0)^2 \\ [1 - (\gamma_0 - \sqrt{\rho})^2]^{3/2} & (1 - \gamma_0)^2 \leq \rho \leq (1 + \gamma_0)^2 \end{cases} \quad (151)$$

$P_t(\rho) d\rho = P_t(\rho) dE_1 / E'_{1m} \equiv$ probability of finding particle b_1 with kinetic energy E_1 in dE_1 , summed over all values of θ . When $\gamma_0 = 0$, Eq. (147) becomes

$$\tilde{F}(\theta, E_1) = \frac{4}{\pi E'_{1m}} \sqrt{\rho(1 - \rho)} \sin \theta.$$

Integrate over θ to obtain

$$P_t(\rho) = \frac{8}{\pi} \sqrt{\rho(1 - \rho)} \quad 0 \leq \rho \leq 1. \quad (152)$$

If $\gamma_0 = 0$ in Eq. (151), Eq. (152) is again obtained. When $\gamma_0 = 0$, LCS and BCS coincide, and $E_1 = E'_1$. Thus $\rho = \rho'$, and Eq. (152) is identical with Eq. (131), as expected.

The forms of these functions are illustrated in Figs. 23 and 24. $\tilde{F}(\theta, r)$ is plotted in Fig. 23 for $\gamma_0 = 0.2$ and $\gamma_0 = 5.0$, and two examples of the use of this figure are given in the figure caption. $P_t(\rho)$ is plotted in Fig. 24 for a few values of γ_0 ; $P_t(\rho)$ is not symmetric about the midpoint of the range of ρ and peaks toward the lower values of ρ (lower values of E_1).

Fig. 23. Plot of $F(\theta, r)$ vs r^2 for "isotropic, statistical" scattering [$\sigma''(\theta^*, f_1, f_2) = \text{constant}$]. (a) $\gamma_0 = 0.2$; (b) $\gamma_0 = 5.0$. ($\gamma \equiv \gamma_0$ and $\theta_m \equiv \theta_{m0}$). For example, the probability for $\gamma_0 = 0.2$ that a particle is scattered into $d\theta$ at $\theta = 120^\circ$ with energy in dE_1 around $E_1 = 10 \epsilon$ is read from (a) to be $(0.0183) d\theta d(r^2) = (0.0183/\epsilon) d\theta dE_1$. As another example, note that with $\gamma_0 = 0.2$ a particle b_1 with $E_1 = 10 \epsilon$ is about 5.8 times more likely to be scattered at 120° than at 170° .

Fig. 24. Plot of $P_t(\rho)$ vs ρ for some values of $\gamma_0 (\equiv \gamma)$. Here, $\sigma''(\theta^*, f_1, f_2) = \text{constant}$. $\int P_t(\rho) d\rho = 1$. Compare with Fig. 9.

IV. DISCUSSION OF TWO-BODY AND THREE-BODY DISINTEGRATIONS

In previous sections, several formulas and results pertaining to two-body and three-body disintegrations are derived. In this section, the implications of these results are discussed further, and two- and three-body disintegrations are compared.

First, consider the barycentric energy distributions for fixed Q . For a two-body disintegration,

the value of E completely fixes the kinetic energies, E'_b and E'_y (Section II A). In a three-body disintegration the kinetic energy, E'_1 , of particle b_1 may be anywhere between 0 and $f_{1m}E$; furthermore, some values of E'_1 and normally more likely to occur than others. Generally, to know the kinetic energy distribution of b_1 , it may be necessary to specify E , θ^* , χ^* , ϕ^* , and f_2 or else to average over one or more of these quantities in $\sigma''(\theta^*, \chi^*, \phi^*, E, f_1, f_2)$.

Thus, the kinetic energy distribution of a secondary particle in BCS does not depend on the differential cross section in a two-particle disintegration, but does depend on the differential cross section in a three-particle disintegration.

Second, consider the influence of momentum space in BCS. The differential cross section, σ'' , is not commonly known for three-body disintegrations. The specific examples of σ'' treated in this paper presume that the (α, β) or (f_1, f_2) dependence of σ'' is proportional to the volume of momentum space at given (α, β) or (f_1, f_2) ; in this way the purely "kinematical" effects are exhibited. The volume of momentum space associated with (α, β) or (f_1, f_2) is proportional to $H(\alpha, \beta)$ or $F(f_1, f_2)$; these functions are plotted in Figs. 13 to 16 for a few cases. From Eqs. (65) and (77), augmented by Figs. 13 to 16, the following results may be deduced:

- (a) Whenever conservation of momentum permits, i.e., whenever a triangle can be formed with sides of length $\sqrt{m_1}$, $\sqrt{m_2}$, and $\sqrt{m_3}$, the largest volume of momentum space is associated with those values of α and β at which the energy is equipartitioned among the separating products; i.e., $f_1 = f_2 = f_3$.
- (b) The probability that particle b_1 has kinetic energy E'_1 in BCS is given by Eqs. (128) and (131) and is $\propto \sqrt{E'_1(E'_{1m} - E'_1)} dE'_1$ for fixed E , provided that $\sigma''(\theta^*, f_1, f_2)$ is an explicit function of θ^* only, i.e., provided that $\sigma''(\theta^*, \alpha, \beta) = H(\alpha, \beta)\sigma'(\theta^*)$.
- (c) When $m_3 \ll m_1 = m_2$, the volume of momentum space is small except near $\alpha + \beta = \pi$; the ellipse defined by Eq. (73) approximates the straight line $f_1 = f_2$, so $E'_1 \approx E'_2$ and $\vec{P}'_1 \approx -\vec{P}'_2$. If $m_3 \ll m_1$, $m_3 \ll m_2$, and $m_1 \neq m_2$, then $\vec{P}'_1 \approx -\vec{P}'_2$. In either case, particle b_3 may be scattered with any energy between 0 and E_{3m} , where E_{3m} is almost equal to E . The separation of b_1 and b_2 approximates a two-particle disintegration in which the total kinetic energy is not a constant.
- (d) If $m_3 \gg m_1$ and $m_3 \gg m_2$, then the ellipse defined by Eq. (73) approximates the straight line $f_3 = 0$; i.e., $E'_1 + E'_2 \approx E$. For sufficiently large m_3 , the volume of momentum space is largest for $\alpha + \beta \approx 2\pi$; thus, for large m_3 ,

$$H(\alpha, \beta) \approx \frac{8(m, m_2)^{3/2} \sin^2 \alpha \sin^2 \beta \sin^2(\alpha + \beta)}{\pi(m_1 \sin^2 \beta + m_2 \sin^2 \alpha)} \quad (153)$$

If $m_1 = m_2$, Eq. (153) becomes

$$H(\alpha, \beta) \approx \frac{\sin^2 \alpha \sin^2 \beta \sin^2(\alpha + \beta)}{\pi(\sin^2 \beta + \sin^2 \alpha)} \quad (154)$$

For a given value of β , the largest value of $H(\alpha, \beta)$ occurs when $\alpha = \beta$, i.e., when

$$H(\alpha, \alpha) = \frac{4}{\pi} \cos^2 \alpha \quad (155)$$

A plot of the contours of $H(\alpha, \beta)$ in Eq. (154) is fairly similar to Fig. 15c, except that the maximum at $(\alpha, \beta) = (\pi, \pi)$ is larger. No matter how large m_3 is, scattering is moderately probable for values of α and β considerably different from $(\alpha, \beta) = (\pi, \pi)$.

Third, compare the angular distribution in LCS of two-body and three-body disintegrations. It has been previously explained that the solid angle associated with a scattered particle cannot be usefully transformed for a three-body disintegration without knowledge of σ'' ; this contrasts with the two-body disintegration [see Eqs. (25), (26), (92), (93), and (94)]. The expressions for γ and γ_0 are formally identical for reactions (1) and (2); i.e.,

$$\gamma_0(\text{three-body}) = \left[\frac{m_0 m_a E_a}{(M - m_0)(MQ + m_X E_0)} \right]^{1/2} = \gamma(\text{two-body})$$

where m_0 = mass of the observed particle, provided that mass is conserved. Also, if $\gamma = \gamma_0 = 0$, the barycentric and laboratory coordinate systems coincide for both two-body and three-body disintegrations. Therefore, it seems sensible to compare the angular distributions in LCS of two- and three-body disintegrations for $\gamma(\text{two-body}) = \gamma_0(\text{three-body})$; in fact, the curves in the figures pertinent to three-body disintegrations are labeled with $\gamma(\equiv \gamma_0)$. Compare Figs. 3, 4, 7, and 8 with Figs. 18, 19, 20, and 21, respectively. Observe that for given γ (or γ_0), scattering is more forwardly directed for a given type of three-body disintegration than for the same type of two-body disintegration. Infinities, which occur in LCS at θ_m for two-body disintegrations in which $\gamma > 1$, do not appear in three-body disintegrations.

Fourth, the kinetic energy distributions for two-body and three-body disintegrations for "isotropic" and "isotropic, statistical" scattering are plotted in Figs. 9 and 24. Comparison of these two figures reveals several differences. Particularly

note that for $\gamma_0 < 1$, particles produced by a three-body disintegration may have no kinetic energy in LCS, whereas all particles produced by a two-body disintegration for $\gamma < 1$ must have kinetic energy in LCS. In the two-body disintegration, the kinetic energy distribution in LCS is symmetric about $E_b = E'_b(1 + \gamma^2)$, i.e., around $\rho = (1 + \gamma^2)$, and $P_i(E_b)$ approaches a δ -function as $\gamma \rightarrow 0$. In the three-body disintegration, the kinetic energy distribution in LCS is peaked at an energy $E_1 \leq E'_{1m}(1 + \gamma_0^2)$ and is therefore asymmetric; as $\gamma_0 \rightarrow 0$, the laboratory kinetic energy distribution approaches the barycentric distribution given in Eqs. (128) and (131). For a given value of θ there are, at most, two values of E_b in a two-body disintegration; these values of E_b can be calculated from Eq. (40). In a three-body disintegration, there may be a distribution of values for E_1 associated with a given value of θ ; σ'' must be known before this distribution can be calculated. Figure 23 illustrates the situation for "isotropic, statistical" scattering.

Finally, for a given value of Q , fewer laboratory measurements are necessary to determine $\sigma'(\theta^*)$ for a two-body disintegration than to determine $\sigma''(\theta^*, f_1, f_2)$ for a three-body disintegration. From Sections II B through D, it may be inferred that a laboratory measurement of either the angular distribution or the kinetic energy distribution of one particle produced in a two-body disintegration is sufficient to determine σ' . From Sections III B and C, it may be deduced that laboratory measurements of the kinetic energy and angular distributions of one type of particle produced in a three-body disintegration are sufficient to determine σ'' only if the dependence of σ'' on one of θ^* , f_1 , and f_2 is previously known. A measurement of the laboratory angular distribution of one secondary particle is sufficient to determine σ'' only if the dependence of σ'' on two of the variables θ^* , f_1 , and f_2 is known a priori.

V. APPLICATIONS

Anyone wishing to apply the formulas and results appearing in this work will have a clear idea of his application. This section, therefore, will just touch briefly on a few applications of the present results.

These formulas can be used to estimate collec-

tion efficiencies of product particles in certain sorts of experiments,^{16,17} and may therefore be useful in the reduction of some kinds of data to obtain total cross sections.

Whenever the kinetic energy of one of the product particles is of interest, the remarks in this report may be of value. For example, these results should help in ascertaining the value of Q from measured kinetic energy distributions of secondary particles in LCS.

The formulas and methods may help in the interpretation of data for angular separations of product particles in three-body disintegrations. Some data on angular separations and particle energies are available for nuclear reactions,¹⁸ but practically no theoretical or experimental results are available in atomic and molecular scattering.

It is of some interest to compare points (c) and (d) in the third paragraph of Section IV with a simple theory of β -decay.¹⁹ The fact that the neutrino is a relativistic particle of zero rest mass accounts for the difference between the usual formula for β -particle energy and Eq. (128).

The ionization of an atom or a molecule, A , by electron impact, $e + A \rightarrow A^+ + 2e$, is a three-body disintegration in which two of the particles are much lighter than the third, e.g., b_3 . In such a case $E'_1 + E'_2 \approx E$, and the ellipse defined by Eq. (73) very nearly degenerates to a straight line; therefore, the energy dependence of the volume of momentum space may be approximated by EdE . If the cross section, σ , for ionization depends primarily on the volume of momentum space, then $\sigma \propto E$ where E is the energy above threshold. The result that $\sigma \propto E$ has been previously derived by similar methods.²⁰

VI. CONCLUSIONS

Methods of transforming differential cross sections and kinetic energy distributions from BCS to LCS for two-body and three-body disintegrations have been discussed. Explicit formulas have been given and their use illustrated for $\sigma'(\theta^*) = \text{constant}$ and $\sigma'(\theta^*) \propto 1/\sin \theta^*$. Several incidental conclusions, which hold if only one value of Q is possible in a reaction, are succinctly stated below.

In the laboratory, secondary particles are scattered more forwardly in three-body disintegrations than in two-body disintegrations if the dependences

on θ^* are similar if $\gamma(\text{two-body}) = \gamma_0(\text{three-body})$.

The kinetic energy distributions of secondary particles in LCS depend on the differential cross sections in BCS. For three-body disintegrations, the kinetic energy distributions of secondary particles in BCS and LCS depend on the differential cross sections in BCS.

The laboratory kinetic energy of a particle produced in a two-body disintegration is analytically associated with one laboratory scattering angle. Particles produced in a three-body disintegration with a particular laboratory kinetic energy may be scattered at many different angles in LCS.

The differential cross section, $\sigma'(\theta^*)$, in BCS of a two-body reaction can be determined from laboratory measurements of either the kinetic energy or angular distributions of a single secondary particle. The differential cross section, $\sigma''(\theta^*, f_1 f_2)$, in BCS of a three-body reaction cannot, in general, be determined from simultaneous laboratory measure-

ments of the kinetic energy and angular distributions of a single secondary particle.

If a cross section in BCS of a three-body disintegration is proportional to the volume of momentum space available to the separating products, then the most probable values for E'_1, E'_2, E'_3 are $E'_1 = E'_2 = E'_3 = E/3$; i.e., the kinetic energy tends to be equipartitioned, provided that momentum can be simultaneously conserved.

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APPENDIX A. GLOSSARY OF SYMBOLS AND TERMS

1. Symbols

- a $\equiv$ projectile particle.
- b, b_1, Y $\equiv$ reaction products, secondary particles.
- α, β, δ $\equiv$ angles between particles b_1, b_2 , and b_3 in BCS. See Fig. 12.
- $d\tau \equiv d^3\vec{P}'_1 d^3\vec{P}'_2 d^3\vec{P}'_c$.
- $d\Omega, d\Omega^*$ $\equiv$ solid angles in LCS and BCS, respectively.
- E $\equiv$ total kinetic energy of the reaction products ($=E_0 + Q$) in BCS.
- E_i, E'_i $\equiv$ kinetic energies of the i^{th} particle in LCS and BCS, respectively.
- E_{1m} $\equiv$ maximum kinetic energy of particle b_1 in BCS.
- E_0 $\equiv$ kinetic energy in BCS before collision.
- ϵ $\equiv$ kinetic energy of particle b_1 moving with speed $v_c (= m_1 v_c^2/2)$.
- f_i $\equiv$ fraction of the total kinetic energy in BCS carried off by the i^{th} particle, b_i .
- f_{im} $\equiv$ maximum value of f_i . See Eq. (74).
- $f_{2\pm}$ $\equiv$ boundary of allowed values of f_1 and f_2 for a three-body disintegration. See Eq. (73).
- γ $\equiv$ ratio of the center-of-mass speed to the speed of the observed particle (called b or b_1 here) in BCS. ($= v_c/v_b$ or v_c/v'_1).
- γ_0 $\equiv$ ratio of the center-of-mass speed to the maximum speed of the observed particle in BCS. See Eqs. (79) to (84).
- M $\equiv$ total mass of the products of a reaction.
- m_i $\equiv$ mass of the i^{th} particle.
- m_0 $\equiv$ mass of the observed particle, either m_b or m_1 .

$\vec{P}_c$ = momentum of the center of mass ($= M\vec{v}_c$).
 $\vec{P}_i, \vec{P}'_i$ = momenta of the i^{th} particle in LCS and BCS, respectively.
 $P(E_i)$ = kinetic energy distribution of the i^{th} particle in LCS.
 Q = energy released in a reaction ($Q < 0$ for an endothermic reaction).
 $r^2 = E_1/\epsilon$.
 $\rho = E_b/E'_b$ for a two-body disintegration, and $= E_1/E'_{1m}$ for a three-body disintegration.
 $\rho' = E'_1/E'_{1m}$. See Eq. (132).
 σ = total reaction cross section.
 σ' = barycentric differential cross section that depends only on θ^* for fixed E and Q . The differential cross section in BCS of a two-body disintegration.
 σ'' = differential cross section in BCS of a three-body disintegration. May depend on θ^*, χ^*, ϕ^* , and f_1, f_2 (or α, β), for fixed E and Q .
 θ, ϕ = polar and azimuthal scattering angles of particle b or b_1 in LCS.
 θ^*, ϕ^* = polar and azimuthal scattering angles of particle b or b_1 in BCS.
 θ_m = maximum laboratory scattering angle when $\gamma \geq 1$ [$= \arcsin(1/\gamma)$].
 θ_{mo} = maximum laboratory scattering angle when $\gamma_o \geq 1$ [$= \arcsin(1/\gamma_o)$].
 $\vec{v}_i, \vec{v}'_i$ = velocities of the i^{th} particle in LCS and BCS, respectively.
 $\vec{v}_c$ = velocity of the barycentric coordinate system.
 χ^* = angle of rotation of the plane containing $\vec{P}'_1, \vec{P}'_2$, and $\vec{P}'_3$ about $\vec{P}'_1$ (see Fig. 12).

2. Terms

BCS = barycentric coordinate system = center-of-mass coordinate system.

Glory scattering $\equiv \sigma'(\theta^*) \propto 1/\sin \theta^*$, in this paper. (The term "glory scattering" normally has a broader meaning.¹⁰)

Glory, statistical scattering $\equiv \sigma''(\theta^*, f_1, f_2) = \text{constant}/\sin \theta^*$ in a three-body disintegration.

Isotropic scattering $\equiv \sigma'(\theta^*) = \text{constant}$.

Isotropic, statistical scattering $\equiv \sigma''(\theta^*, f_1, f_2) = \text{constant}$ in a three-body disintegration.

LCS = laboratory coordinate system.

Three-body disintegration = a process in which three secondary particles are produced.

Two-body disintegration = a process in which two secondary particles are produced.

APPENDIX B. TABLES OF FUNCTIONS

Values of the functions $g(\theta)$, $g_+(\theta)$, $g_-(\theta)$, $S(\theta)$, $J(\theta)$, and $\Sigma(\theta)$ are tabulated in this appendix. The values of θ are given in radians throughout. The last digit tabulated is uncertain, owing to round-off and computational errors; e.g., π is tabulated as 3.1415 and $\pi/2$ as 1.5708, but the computations were performed with $\pi = 3.141592653898$.

TABLE B I. Values of $g(\theta)$ for $\gamma < 1$.

$\theta \backslash \gamma$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.0805	0.04023	0.04865	0.05786	0.06788	0.07869	0.09030	0.10271	0.11593	0.12995	0.14477
0.1611	0.08020	0.09681	0.11495	0.13465	0.15590	0.17871	0.20309	0.22906	0.25663	0.28583
0.2416	0.11965	0.14398	0.17051	0.19923	0.23018	0.26338	0.29886	0.33668	0.37688	0.41954
0.3222	0.15833	0.18971	0.22380	0.26060	0.30018	0.34259	0.38790	0.43623	0.48771	0.54248
0.4027	0.19598	0.23355	0.27415	0.31783	0.36467	0.41477	0.46828	0.52540	0.58637	0.65151
0.4833	0.23236	0.27508	0.32097	0.37009	0.42257	0.47857	0.53833	0.60216	0.67049	0.74386
0.5638	0.26723	0.31393	0.36372	0.41669	0.47301	0.53290	0.59670	0.66489	0.73810	0.81720
0.6444	0.30037	0.34977	0.40198	0.45709	0.51531	0.57694	0.64242	0.71238	0.78771	0.86975
0.7249	0.33156	0.38232	0.43540	0.49090	0.54905	0.61020	0.67489	0.74389	0.81839	0.90028
0.8055	0.36060	0.41135	0.46374	0.51788	0.57402	0.63251	0.69394	0.75921	0.82979	0.90820
0.8860	0.38730	0.43668	0.48687	0.53800	0.59027	0.64402	0.69985	0.75869	0.82221	0.89360
0.9666	0.41149	0.45817	0.50476	0.55133	0.59805	0.64521	0.69330	0.74320	0.79663	0.85731
1.0472	0.43301	0.47577	0.51746	0.55813	0.59787	0.63684	0.67538	0.71418	0.75473	0.80095
1.1277	0.45172	0.48944	0.52512	0.55876	0.59036	0.61992	0.64754	0.67354	0.69890	0.72704
1.2083	0.46750	0.49920	0.52796	0.55371	0.57634	0.59567	0.61150	0.62365	0.63221	0.63911
1.2888	0.48025	0.50513	0.52627	0.54355	0.55673	0.56546	0.56919	0.56716	0.55830	0.54183
1.3694	0.48989	0.50733	0.52039	0.52889	0.53250	0.53069	0.52261	0.50690	0.48119	0.44108
1.4499	0.49635	0.50594	0.51069	0.51040	0.50466	0.49280	0.47372	0.44560	0.40495	0.34365
1.5305	0.49959	0.50112	0.49759	0.48876	0.47419	0.45312	0.42436	0.38577	0.33323	0.25622
1.6110	0.49959	0.49307	0.48150	0.46462	0.44200	0.41289	0.37608	0.32945	0.26886	0.18380
1.6916	0.49635	0.48200	0.46283	0.43861	0.40893	0.37314	0.33014	0.27808	0.21350	0.12826
1.7721	0.48989	0.46813	0.44200	0.41130	0.37571	0.33471	0.28743	0.23252	0.16762	0.08831
1.8527	0.48025	0.45168	0.41938	0.38321	0.34294	0.29822	0.24851	0.19304	0.13073	0.06081
1.9332	0.46750	0.43289	0.39534	0.35478	0.31109	0.26411	0.21363	0.15946	0.10171	0.04230
2.0138	0.45172	0.41198	0.37020	0.32638	0.28052	0.23262	0.18278	0.13132	0.07922	0.02990
2.0944	0.43301	0.38917	0.34426	0.29832	0.25146	0.20383	0.15577	0.10796	0.06191	0.02153
2.1749	0.41149	0.36467	0.31776	0.27083	0.22405	0.17770	0.13229	0.08869	0.04862	0.01579
2.2555	0.38730	0.33870	0.29091	0.24406	0.19835	0.15413	0.11197	0.07284	0.03838	0.01178
2.3360	0.36060	0.31143	0.26390	0.21813	0.17434	0.13292	0.09443	0.05978	0.03044	0.00893
2.4166	0.33156	0.28305	0.23685	0.19308	0.15196	0.11385	0.07926	0.04899	0.02422	0.00684
2.4971	0.30037	0.25372	0.20987	0.16893	0.13110	0.09668	0.06611	0.04001	0.01930	0.00529
2.5777	0.26723	0.22359	0.18303	0.14566	0.11163	0.08117	0.05463	0.03248	0.01534	0.00410
2.6582	0.23236	0.19278	0.15637	0.12320	0.09338	0.06708	0.04454	0.02608	0.01210	0.00317
2.7388	0.19598	0.16143	0.12991	0.10147	0.07619	0.05417	0.03556	0.02056	0.00941	0.00243
2.8193	0.15833	0.12964	0.10365	0.08038	0.05988	0.04222	0.02746	0.01571	0.00711	0.00181
2.8999	0.11965	0.09751	0.07756	0.05981	0.04429	0.03102	0.02003	0.01138	0.00511	0.00129
2.9804	0.08020	0.06514	0.05162	0.03965	0.02923	0.02037	0.01309	0.00739	0.00330	0.00082
3.0610	0.04023	0.03261	0.02578	0.01975	0.01452	0.01009	0.00646	0.00364	0.00162	0.00040
3.1415	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE B II. Values of $g(\theta)$ for $\gamma \geq 1$.

θ	$\gamma=1.0$	θ	$\gamma=1.05$	θ	$\gamma=1.1$	θ	$\gamma=1.5$	θ	$\gamma=2.0$	θ	$\gamma=3.0$	θ	$\gamma=4.0$	θ	$\gamma=5.0$
0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000
0.0402	0.08046	0.0323	0.06793	0.0292	0.06462	0.0187	0.06080	0.0134	0.06713	0.0087	0.08715	0.0064	0.11017	0.0051	0.13427
0.0805	0.16041	0.0646	0.13557	0.0585	0.12903	0.0374	0.12154	0.0268	0.13427	0.0174	0.17440	0.0129	0.22050	0.0103	0.26877
0.1208	0.23931	0.0970	0.20266	0.0877	0.19300	0.0561	0.18218	0.0402	0.20146	0.0261	0.26186	0.0194	0.33117	0.0154	0.40372
0.1611	0.31666	0.1293	0.26890	0.1170	0.25632	0.0748	0.24266	0.0537	0.26898	0.0348	0.34963	0.0259	0.44234	0.0206	0.53935
0.2013	0.39196	0.1616	0.33402	0.1462	0.31878	0.0935	0.30293	0.0671	0.33601	0.0435	0.43781	0.0323	0.55420	0.0258	0.67591
0.2416	0.46472	0.1939	0.39775	0.1755	0.38017	0.1122	0.36293	0.0805	0.40342	0.0522	0.52652	0.0388	0.66691	0.0309	0.81363
0.2819	0.53446	0.2263	0.45982	0.2048	0.44027	0.1309	0.42262	0.0939	0.47096	0.0610	0.61589	0.0453	0.78069	0.0361	0.95278
0.3222	0.60074	0.2586	0.51998	0.2340	0.49890	0.1496	0.48195	0.1074	0.53866	0.0697	0.70604	0.0518	0.89574	0.0413	1.09364
0.3624	0.66312	0.2909	0.57798	0.2633	0.55584	0.1684	0.54089	0.1208	0.60656	0.0784	0.79710	0.0583	1.01227	0.0464	1.23651
0.4027	0.72120	0.3233	0.63357	0.2925	0.61093	0.1871	0.59940	0.1342	0.67470	0.0871	0.88925	0.0647	1.13054	0.0516	1.38170
0.4430	0.77460	0.3556	0.68653	0.3218	0.66396	0.2058	0.65744	0.1476	0.74314	0.0958	0.98264	0.0712	1.25080	0.0567	1.52958
0.4833	0.82298	0.3879	0.73664	0.3511	0.71477	0.2245	0.71499	0.1611	0.81195	0.1045	1.07746	0.0777	1.37334	0.0619	1.68052
0.5236	0.86602	0.4203	0.78368	0.3803	0.76319	0.2432	0.77203	0.1745	0.88120	0.1132	1.17393	0.0842	1.49850	0.0671	1.83495
0.5638	0.90345	0.4526	0.82748	0.4096	0.80906	0.2619	0.82855	0.1879	0.95097	0.1219	1.27227	0.0907	1.62662	0.0722	1.99336
0.6041	0.93501	0.4849	0.86783	0.4388	0.85224	0.2806	0.88456	0.2013	1.02139	0.1307	1.37278	0.0971	1.75813	0.0774	2.15628
0.6444	0.96051	0.5173	0.90460	0.4681	0.89259	0.2993	0.94005	0.2148	1.09258	0.1394	1.47574	0.1036	1.89348	0.0826	2.32432
0.6847	0.97979	0.5496	0.93761	0.4974	0.92999	0.3180	0.99506	0.2282	1.16470	0.1481	1.58153	0.1101	2.03321	0.0877	2.49818
0.7249	0.99270	0.5819	0.96675	0.5266	0.96433	0.3368	1.04964	0.2416	1.23793	0.1568	1.69056	0.1166	2.17794	0.0929	2.67867
0.7652	0.99918	0.6143	0.99190	0.5559	0.99551	0.3555	1.10385	0.2550	1.31251	0.1655	1.80331	0.1231	2.32838	0.0981	2.86672
0.8055	0.99918	0.6466	1.01297	0.5851	1.02345	0.3742	1.15778	0.2685	1.38871	0.1742	1.92037	0.1295	2.48538	0.1032	3.06344
0.8458	0.99270	0.6789	1.02987	0.6144	1.04808	0.3929	1.21156	0.2819	1.46686	0.1829	2.04242	0.1360	2.64995	0.1084	3.27012
0.8860	0.97979	0.7113	1.04255	0.6437	1.06936	0.4116	1.26535	0.2953	1.54738	0.1917	2.17029	0.1425	2.82328	0.1135	3.48834
0.9263	0.96051	0.7436	1.05098	0.6729	1.08727	0.4303	1.31939	0.3087	1.63078	0.2004	2.30499	0.1490	3.00683	0.1187	3.71996
0.9666	0.93501	0.7759	1.05514	0.7022	1.10180	0.4490	1.37395	0.3222	1.71771	0.2091	2.44775	0.1555	3.20239	0.1239	3.96731
1.0069	0.90345	0.8083	1.05506	0.7314	1.11298	0.4677	1.42944	0.3356	1.80897	0.2178	2.60013	0.1619	3.41218	0.1290	4.23326
1.0472	0.86602	0.8406	1.05077	0.7607	1.12089	0.4864	1.48634	0.3490	1.90560	0.2265	2.76409	0.1684	3.63900	0.1342	4.52141
1.0874	0.82298	0.8729	1.04236	0.7899	1.12562	0.5052	1.54536	0.3624	2.00895	0.2352	2.94214	0.1749	3.88645	0.1394	4.83640
1.1277	0.77460	0.9053	1.02993	0.8192	1.12736	0.5239	1.60740	0.3759	2.12081	0.2439	3.13761	0.1814	4.15925	0.1445	5.18430
1.1680	0.72120	0.9376	1.01365	0.8485	1.12636	0.5426	1.67374	0.3893	2.24361	0.2527	3.35493	0.1878	4.46370	0.1497	5.57321
1.2083	0.66312	0.9699	0.99377	0.8777	1.12299	0.5613	1.74617	0.4027	2.38073	0.2614	3.60025	0.1943	4.80850	0.1548	6.01430
1.2485	0.60074	1.0022	0.97061	0.9070	1.11783	0.5800	1.82731	0.4161	2.53700	0.2701	3.88231	0.2008	5.20601	0.1600	6.52343
1.2888	0.53446	1.0346	0.94469	0.9362	1.11177	0.5987	1.92110	0.4296	2.71966	0.2788	4.21412	0.2073	5.67458	0.1652	7.12410
1.3291	0.46472	1.0669	0.91677	0.9655	1.10622	0.6174	2.03383	0.4430	2.94008	0.2875	4.61597	0.2138	6.24277	0.1703	7.85291
1.3694	0.39196	1.0992	0.88818	0.9948	1.10367	0.6361	2.17611	0.4564	3.21737	0.2962	5.12176	0.2202	6.95824	0.1755	8.77085
1.4096	0.31666	1.1316	0.86141	1.0240	1.10884	0.6548	2.36768	0.4699	3.58664	0.3049	5.79352	0.2267	7.90804	0.1807	9.98923
1.4499	0.23931	1.1639	0.84187	1.0533	1.13198	0.6735	2.65033	0.4833	4.12160	0.3137	6.76082	0.2332	9.27372	0.1858	11.7400
1.4902	0.16041	1.1962	0.84453	1.0825	1.20103	0.6923	3.13370	0.4967	5.01407	0.3224	8.35924	0.2397	11.5248	0.1910	14.6231
1.5305	0.08046	1.2286	0.93366	1.1118	1.43298	0.7110	4.25944	0.5101	7.02983	0.3311	11.9220	0.2462	16.5242	0.1961	21.0156
1.5708	0.00000	1.2609	∞	1.1411	∞	0.7297	∞	0.5236	∞	0.3398	∞	0.2526	∞	0.2013	∞

TABLE B III. Values of $g_+(A)$ for $\gamma \geq 1$.

θ	$\gamma=1.0$	θ	$\gamma=1.05$	θ	$\gamma=1.1$	θ	$\gamma=1.5$	θ	$\gamma=2.0$	θ	$\gamma=3.0$	θ	$\gamma=4.0$	θ	$\gamma=5.0$
0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000
0.0402	0.08046	0.0323	0.06789	0.0292	0.06447	0.0187	0.05846	0.0134	0.06041	0.0087	0.06971	0.0064	0.08100	0.0051	0.09295
0.0805	0.16041	0.0646	0.13549	0.0585	0.12873	0.0374	0.11685	0.0268	0.12081	0.0174	0.13947	0.0129	0.16207	0.0103	0.18601
0.1208	0.23931	0.0970	0.20253	0.0877	0.19256	0.0561	0.17511	0.0402	0.18119	0.0261	0.20932	0.0194	0.24331	0.0154	0.27929
0.1611	0.31666	0.1293	0.26873	0.1170	0.25573	0.0748	0.23317	0.0537	0.24154	0.0348	0.27929	0.0259	0.32479	0.0206	0.37291
0.2013	0.39196	0.1616	0.33381	0.1462	0.31803	0.0935	0.29097	0.0671	0.30185	0.0435	0.34944	0.0323	0.40658	0.0258	0.46697
0.2416	0.46472	0.1939	0.39749	0.1755	0.37925	0.1122	0.34845	0.0805	0.36212	0.0522	0.41982	0.0388	0.48879	0.0309	0.56160
0.2819	0.53446	0.2263	0.45952	0.2048	0.43918	0.1309	0.40553	0.0939	0.42233	0.0610	0.49048	0.0453	0.57151	0.0361	0.65694
0.3222	0.60074	0.2586	0.51963	0.2340	0.49762	0.1496	0.46217	0.1074	0.48249	0.0697	0.56147	0.0518	0.65482	0.0413	0.75310
0.3624	0.66312	0.2909	0.57757	0.2633	0.55438	0.1684	0.51829	0.1208	0.54259	0.0784	0.63286	0.0583	0.73885	0.0464	0.85025
0.4027	0.72120	0.3233	0.63310	0.2925	0.60925	0.1871	0.57386	0.1342	0.60265	0.0871	0.70471	0.0647	0.82370	0.0516	0.94854
0.4430	0.77460	0.3556	0.68600	0.3218	0.66206	0.2058	0.62880	0.1476	0.66266	0.0958	0.77711	0.0712	0.90951	0.0567	1.04814
0.4833	0.82298	0.3879	0.73603	0.3511	0.71263	0.2245	0.68308	0.1611	0.72264	0.1045	0.85014	0.0777	0.99641	0.0619	1.14925
0.5236	0.86602	0.4203	0.78300	0.3803	0.76079	0.2432	0.73666	0.1745	0.78262	0.1132	0.92390	0.0842	1.08456	0.0671	1.25206
0.5638	0.90345	0.4526	0.82671	0.4096	0.80638	0.2619	0.78947	0.1879	0.84261	0.1219	0.99849	0.0907	1.17414	0.0722	1.35683
0.6041	0.93501	0.4849	0.86697	0.4388	0.84924	0.2806	0.84151	0.2013	0.90266	0.1307	1.07406	0.0971	1.26536	0.0774	1.46382
0.6444	0.96051	0.5173	0.90362	0.4681	0.88924	0.2993	0.89273	0.2148	0.96281	0.1394	1.15073	0.1036	1.35843	0.0826	1.57332
0.6847	0.97979	0.5496	0.93652	0.4974	0.92625	0.3180	0.94312	0.2282	1.02313	0.1481	1.22869	0.1101	1.45362	0.0877	1.68569
0.7249	0.99270	0.5819	0.96552	0.5266	0.96015	0.3368	0.99267	0.2416	1.08369	0.1568	1.30814	0.1166	1.55124	0.0929	1.80133
0.7652	0.99918	0.6143	0.99051	0.5559	0.99083	0.3555	1.04138	0.2550	1.14458	0.1655	1.38931	0.1231	1.65163	0.0981	1.92070
0.8055	0.99918	0.6466	1.01139	0.5851	1.01821	0.3742	1.08926	0.2685	1.20593	0.1742	1.47249	0.1295	1.75522	0.1032	2.04436
0.8458	0.99270	0.6789	1.02808	0.6144	1.04220	0.3929	1.13635	0.2819	1.26789	0.1829	1.55800	0.1360	1.86251	0.1084	2.17294
0.8860	0.97979	0.7113	1.04052	0.6437	1.06274	0.4116	1.18271	0.2953	1.33065	0.1917	1.64626	0.1425	1.97410	0.1135	2.30723
0.9263	0.96051	0.7436	1.04866	0.6729	1.07978	0.4303	1.22842	0.3087	1.39445	0.2004	1.73777	0.1490	2.09069	0.1187	2.44816
0.9666	0.93501	0.7759	1.05247	0.7022	1.09330	0.4490	1.27360	0.3222	1.45959	0.2091	1.83313	0.1555	2.21320	0.1239	2.59690
1.0069	0.90345	0.8083	1.05198	0.7314	1.10329	0.4677	1.31841	0.3356	1.52647	0.2178	1.93312	0.1619	2.34271	0.1290	2.75486
1.0472	0.86602	0.8406	1.04718	0.7607	1.10977	0.4864	1.36309	0.3490	1.59558	0.2265	2.03870	0.1684	2.48064	0.1342	2.92386
1.0874	0.82298	0.8729	1.03814	0.7899	1.11279	0.5052	1.40796	0.3624	1.66759	0.2352	2.15113	0.1749	2.62877	0.1394	3.10621
1.1277	0.77460	0.9053	1.02494	0.8192	1.11242	0.5239	1.45345	0.3759	1.74338	0.2439	2.27205	0.1814	2.78945	0.1445	3.30494
1.1680	0.72120	0.9376	1.00768	0.8485	1.10880	0.5426	1.50018	0.3893	1.82415	0.2527	2.40370	0.1878	2.96584	0.1497	3.52410
1.2083	0.66312	0.9699	0.98652	0.8777	1.10213	0.5613	1.54902	0.4027	1.91156	0.2614	2.54912	0.1943	3.16229	0.1548	3.76927
1.2485	0.60074	1.0022	0.96168	0.9070	1.09272	0.5800	1.60128	0.4161	2.00804	0.2701	2.71268	0.2008	3.38497	0.1600	4.04838
1.2888	0.53446	1.0346	0.93346	0.9362	1.08103	0.5987	1.65890	0.4296	2.11717	0.2788	2.90089	0.2073	3.64303	0.1652	4.37318
1.3291	0.46472	1.0669	0.90232	0.9655	1.06780	0.6174	1.72500	0.4430	2.24464	0.2875	3.12387	0.2138	3.95078	0.1703	4.76196
1.3694	0.39196	1.0992	0.86899	0.9948	1.05430	0.6361	1.80490	0.4564	2.39999	0.2962	3.39858	0.2202	4.33203	0.1755	5.24522
1.4096	0.31666	1.1316	0.83479	1.0240	1.04294	0.6548	1.90843	0.4699	2.60075	0.3049	3.75602	0.2267	4.83030	0.1807	5.87860
1.4499	0.23931	1.1639	0.80251	1.0533	1.03890	0.6735	2.05649	0.4833	2.88378	0.3137	4.26096	0.2332	5.53636	0.1858	6.77813
1.4902	0.16041	1.1962	0.77974	1.0825	1.05619	0.6923	2.30389	0.4967	3.34497	0.3224	5.08120	0.2397	6.68499	0.1910	8.24364
1.5305	0.08046	1.2286	0.79872	1.1118	1.15338	0.7110	2.87143	0.5101	4.36720	0.3311	6.88335	0.2462	9.20758	0.1961	11.4638
1.5708	0.00000	1.2609	∞	1.1411	∞	0.7297	∞	0.5236	∞	0.3398	∞	0.2526	∞	0.2013	∞

TABLE B IV. Values of $g_{-}^{(q)}$ for $\gamma \geq 1$.

θ	$\gamma=1.0$	θ	$\gamma=1.05$	θ	$\gamma=1.1$	θ	$\gamma=1.5$	θ	$\gamma=2.0$	θ	$\gamma=3.0$	θ	$\gamma=4.0$	θ	$\gamma=5.0$
0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000
0.0402	0.00000	0.0323	0.00004	0.0292	0.00014	0.0187	0.00234	0.0134	0.00671	0.0087	0.01743	0.0064	0.02916	0.0051	0.04132
0.0805	0.00000	0.0646	0.00008	0.0585	0.00029	0.0374	0.00469	0.0268	0.01346	0.0174	0.03493	0.0129	0.05842	0.0103	0.08276
0.1208	0.00000	0.0970	0.00012	0.0877	0.00044	0.0561	0.00707	0.0402	0.02026	0.0261	0.05254	0.0194	0.08785	0.0154	0.12443
0.1611	0.00000	0.1293	0.00016	0.1170	0.00059	0.0748	0.00948	0.0537	0.02715	0.0348	0.07033	0.0259	0.11755	0.0206	0.16644
0.2013	0.00000	0.1616	0.00020	0.1462	0.00075	0.0935	0.01195	0.0671	0.03415	0.0435	0.08836	0.0323	0.14761	0.0258	0.20893
0.2416	0.00000	0.1939	0.00025	0.1755	0.00092	0.1122	0.01447	0.0805	0.04130	0.0522	0.10670	0.0388	0.17812	0.0309	0.25202
0.2819	0.00000	0.2263	0.00030	0.2048	0.00109	0.1309	0.01708	0.0939	0.04862	0.0610	0.12541	0.0453	0.20918	0.0361	0.29584
0.3222	0.00000	0.2586	0.00035	0.2340	0.00127	0.1496	0.01978	0.1074	0.05617	0.0697	0.14456	0.0518	0.24091	0.0413	0.34053
0.3624	0.00000	0.2909	0.00041	0.2633	0.00146	0.1684	0.02259	0.1208	0.06396	0.0784	0.16424	0.0583	0.27342	0.0464	0.38625
0.4027	0.00000	0.3233	0.00047	0.2925	0.00167	0.1871	0.02554	0.1342	0.07205	0.0871	0.18453	0.0647	0.30683	0.0516	0.43316
0.4430	0.00000	0.3556	0.00053	0.3218	0.00189	0.2058	0.02863	0.1476	0.08048	0.0958	0.20552	0.0712	0.34129	0.0567	0.48143
0.4833	0.00000	0.3879	0.00060	0.3511	0.00213	0.2245	0.03190	0.1611	0.08930	0.1045	0.22731	0.0777	0.37693	0.0619	0.53127
0.5236	0.00000	0.4203	0.00068	0.3803	0.00239	0.2432	0.03537	0.1745	0.09858	0.1132	0.25002	0.0842	0.41393	0.0671	0.58289
0.5638	0.00000	0.4526	0.00077	0.4096	0.00268	0.2619	0.03907	0.1879	0.10836	0.1219	0.27378	0.0907	0.45247	0.0722	0.63652
0.6041	0.00000	0.4849	0.00086	0.4388	0.00299	0.2806	0.04304	0.2013	0.11873	0.1307	0.29872	0.0971	0.49277	0.0774	0.69246
0.6444	0.00000	0.5173	0.00097	0.4681	0.00334	0.2993	0.04731	0.2148	0.12976	0.1394	0.32501	0.1036	0.53505	0.0826	0.75099
0.6847	0.00000	0.5496	0.00109	0.4974	0.00374	0.3180	0.05194	0.2282	0.14156	0.1481	0.35283	0.1101	0.57959	0.0877	0.81248
0.7249	0.00000	0.5819	0.00123	0.5266	0.00417	0.3368	0.05697	0.2416	0.15424	0.1568	0.38241	0.1166	0.62670	0.0929	0.87733
0.7652	0.00000	0.6143	0.00139	0.5559	0.00467	0.3555	0.06247	0.2550	0.16792	0.1655	0.41400	0.1231	0.67675	0.0981	0.94601
0.8055	0.00000	0.6466	0.00157	0.5851	0.00523	0.3742	0.06852	0.2685	0.18277	0.1742	0.44788	0.1295	0.73015	0.1032	1.01908
0.8458	0.00000	0.6789	0.00178	0.6144	0.00588	0.3929	0.07520	0.2819	0.19896	0.1829	0.48441	0.1360	0.78743	0.1084	1.09718
0.8860	0.00000	0.7113	0.00202	0.6437	0.00662	0.4116	0.08264	0.2953	0.21672	0.1917	0.52402	0.1425	0.84918	0.1135	1.18111
0.9263	0.00000	0.7436	0.00232	0.6729	0.00748	0.4303	0.09097	0.3087	0.23633	0.2004	0.56721	0.1490	0.91613	0.1187	1.27180
0.9666	0.00000	0.7759	0.00266	0.7022	0.00849	0.4490	0.10035	0.3222	0.25811	0.2091	0.61462	0.1555	0.98919	0.1239	1.37041
1.0069	0.00000	0.8083	0.00308	0.7314	0.00968	0.4677	0.11102	0.3356	0.28249	0.2178	0.66701	0.1619	1.06946	0.1290	1.47839
1.0472	0.00000	0.8406	0.00358	0.7607	0.01111	0.4864	0.12325	0.3490	0.31001	0.2265	0.72539	0.1684	1.15836	0.1342	1.59754
1.0874	0.00000	0.8729	0.00421	0.7899	0.01283	0.5052	0.13740	0.3624	0.34135	0.2352	0.79101	0.1749	1.25768	0.1394	1.73018
1.1277	0.00000	0.9053	0.00498	0.8192	0.01494	0.5239	0.15395	0.3759	0.37742	0.2439	0.86555	0.1814	1.36979	0.1445	1.87935
1.1680	0.00000	0.9376	0.00597	0.8485	0.01755	0.5426	0.17355	0.3893	0.41946	0.2527	0.95123	0.1878	1.49785	0.1497	2.04910
1.2083	0.00000	0.9699	0.00725	0.8777	0.02085	0.5613	0.19714	0.4027	0.46916	0.2614	1.05113	0.1943	1.64620	0.1548	2.24502
1.2485	0.00000	1.0022	0.00893	0.9070	0.02511	0.5800	0.22602	0.4161	0.52895	0.2701	1.16962	0.2008	1.82104	0.1600	2.47504
1.2888	0.00000	1.0346	0.01122	0.9362	0.03074	0.5987	0.26220	0.4296	0.60248	0.2788	1.31323	0.2073	2.03154	0.1652	2.75092
1.3291	0.00000	1.0669	0.01445	0.9655	0.03842	0.6174	0.30882	0.4430	0.69543	0.2875	1.49209	0.2138	2.29198	0.1703	3.09095
1.3694	0.00000	1.0992	0.01919	0.9948	0.04937	0.6361	0.37121	0.4564	0.81738	0.2962	1.72318	0.2202	2.62621	0.1755	3.52563
1.4096	0.00000	1.1316	0.02662	1.0240	0.06589	0.6548	0.45925	0.4699	0.98588	0.3049	2.03750	0.2267	3.07774	0.1807	4.11063
1.4499	0.00000	1.1639	0.03935	1.0533	0.09308	0.6735	0.59384	0.4833	1.23781	0.3137	2.49986	0.2332	3.73736	0.1858	4.96195
1.4902	0.00000	1.1962	0.06478	1.0825	0.14484	0.6923	0.82981	0.4967	1.66909	0.3224	3.27804	0.2397	4.83985	0.1910	6.37946
1.5305	0.00000	1.2286	0.13493	1.1118	0.27960	0.7110	1.38800	0.5101	2.66262	0.3311	5.03868	0.2462	7.3166	0.1961	9.5519
1.5708	0.00000	1.2609	∞	1.1411	∞	0.7297	∞	0.5236	∞	0.3398	∞	0.2526	∞	0.2013	∞

TABLE B V. Values of $S(\theta)$ for $\gamma < 1$.

$\theta \backslash \gamma$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0000	0.31830	0.35014	0.38197	0.41380	0.44563	0.47746	0.50929	0.54112	0.57295	0.60478
0.0805	0.31830	0.35003	0.38177	0.41352	0.44528	0.47707	0.50889	0.54075	0.57265	0.60461
0.1611	0.31830	0.34973	0.38117	0.41267	0.44424	0.47591	0.50770	0.53964	0.57175	0.60407
0.2416	0.31830	0.34922	0.38019	0.41126	0.44250	0.47395	0.50568	0.53775	0.57021	0.60314
0.3222	0.31830	0.34851	0.37881	0.40929	0.44006	0.47120	0.50282	0.53505	0.56799	0.60180
0.4027	0.31830	0.34761	0.37705	0.40677	0.43691	0.46762	0.49908	0.53147	0.56502	0.59997
0.4833	0.31830	0.34652	0.37492	0.40369	0.43304	0.46320	0.49440	0.52695	0.56119	0.59757
0.5638	0.31830	0.34525	0.37242	0.40007	0.42846	0.45790	0.48872	0.52137	0.55639	0.59448
0.6444	0.31830	0.34380	0.36957	0.39592	0.42317	0.45170	0.48199	0.51464	0.55045	0.59055
0.7249	0.31830	0.34218	0.36638	0.39124	0.41715	0.44458	0.47412	0.50660	0.54316	0.58555
0.8055	0.31830	0.34041	0.36287	0.38606	0.41042	0.43651	0.46506	0.49711	0.53428	0.57918
0.8860	0.31830	0.33850	0.35906	0.38040	0.40300	0.42748	0.45472	0.48602	0.52350	0.57102
0.9666	0.31830	0.33645	0.35497	0.37428	0.39490	0.41750	0.44307	0.47316	0.51049	0.56053
1.0472	0.31830	0.33428	0.35062	0.36775	0.38617	0.40659	0.43007	0.45839	0.49487	0.54694
1.1277	0.31830	0.33201	0.34605	0.36083	0.37684	0.39478	0.41574	0.44162	0.47626	0.52928
1.2083	0.31830	0.32964	0.34129	0.35358	0.36699	0.38215	0.40012	0.42281	0.45436	0.50635
1.2888	0.31830	0.32720	0.33635	0.34605	0.35667	0.36879	0.38332	0.40205	0.42901	0.47686
1.3694	0.31830	0.32470	0.33129	0.33829	0.34599	0.35482	0.36553	0.37955	0.40033	0.43981
1.4499	0.31830	0.32216	0.32613	0.33036	0.33503	0.34040	0.34696	0.35565	0.36881	0.39518
1.5305	0.31830	0.31959	0.32092	0.32234	0.32390	0.32570	0.32791	0.33086	0.33537	0.34468
1.6110	0.31830	0.31702	0.31569	0.31427	0.31271	0.31091	0.30870	0.30575	0.30124	0.29193
1.6916	0.31830	0.31445	0.31048	0.30625	0.30158	0.29621	0.28965	0.28096	0.26780	0.24143
1.7721	0.31830	0.31191	0.30532	0.29832	0.29062	0.28179	0.27108	0.25706	0.23628	0.19680
1.8527	0.31830	0.30941	0.30026	0.29056	0.27994	0.26782	0.25329	0.23456	0.20760	0.15975
1.9332	0.31830	0.30697	0.29532	0.28303	0.26962	0.25446	0.23649	0.21380	0.18225	0.13026
2.0138	0.31830	0.30460	0.29056	0.27578	0.25977	0.24183	0.22087	0.19499	0.16035	0.10733
2.0944	0.31830	0.30233	0.28599	0.26886	0.25044	0.23002	0.20654	0.17822	0.14174	0.08967
2.1749	0.31830	0.30016	0.28164	0.26233	0.24171	0.21911	0.19354	0.16345	0.12612	0.07608
2.2555	0.31830	0.29811	0.27755	0.25621	0.23361	0.20913	0.18189	0.15059	0.11311	0.06559
2.3360	0.31830	0.29620	0.27374	0.25055	0.22619	0.20010	0.17155	0.13950	0.10233	0.05743
2.4166	0.31830	0.29443	0.27023	0.24537	0.21946	0.19203	0.16249	0.13001	0.09345	0.05106
2.4971	0.31830	0.29281	0.26704	0.24069	0.21344	0.18491	0.15462	0.12197	0.08616	0.04606
2.5777	0.31830	0.29136	0.26419	0.23654	0.20815	0.17871	0.14789	0.11523	0.08022	0.04213
2.6582	0.31830	0.29009	0.26169	0.23292	0.20357	0.17341	0.14221	0.10966	0.07542	0.03904
2.7388	0.31830	0.28900	0.25956	0.22984	0.19970	0.16899	0.13753	0.10514	0.07159	0.03664
2.8193	0.31830	0.28810	0.25780	0.22731	0.19655	0.16541	0.13379	0.10156	0.06862	0.03481
2.8999	0.31830	0.28739	0.25642	0.22535	0.19411	0.16266	0.13093	0.09886	0.06640	0.03347
2.9804	0.31830	0.28688	0.25543	0.22394	0.19237	0.16070	0.12891	0.09697	0.06486	0.03254
3.0610	0.31830	0.28658	0.25484	0.22309	0.19133	0.15954	0.12772	0.09586	0.06396	0.03200
3.1415	0.31830	0.28647	0.25464	0.22281	0.19098	0.15915	0.12732	0.09549	0.06366	0.03183

TABLE B VI. Values of $S(q)$ for $\gamma \geq 1$.

θ	$\gamma=1.0$	θ	$\gamma=1.05$	θ	$\gamma=1.1$	θ	$\gamma=1.5$	θ	$\gamma=2.0$	θ	$\gamma=3.0$	θ	$\gamma=4.0$	θ	$\gamma=5.0$
0.0000	0.63661	0.0000	0.66845	0.0000	0.70028	0.0000	0.95492	0.0000	1.27323	0.0000	1.90985	0.0000	2.54647	0.0000	3.18309
0.0402	0.63661	0.0323	0.66848	0.0292	0.70034	0.0187	0.95513	0.0134	1.27358	0.0087	1.91043	0.0064	2.54728	0.0051	3.18411
0.0805	0.63661	0.0646	0.66859	0.0585	0.70053	0.0374	0.95576	0.0268	1.27461	0.0174	1.91218	0.0129	2.54969	0.0103	3.18717
0.1208	0.63661	0.0970	0.66877	0.0877	0.70085	0.0561	0.95681	0.0402	1.27635	0.0261	1.91510	0.0194	2.55372	0.0154	3.19230
0.1611	0.63661	0.1293	0.66903	0.1170	0.70130	0.0748	0.95830	0.0537	1.27879	0.0348	1.91921	0.0259	2.55940	0.0206	3.19952
0.2013	0.63661	0.1616	0.66936	0.1462	0.70188	0.0935	0.96022	0.0671	1.28196	0.0435	1.92454	0.0323	2.56677	0.0258	3.20887
0.2416	0.63661	0.1939	0.66977	0.1755	0.70260	0.1122	0.96260	0.0805	1.28587	0.0522	1.93112	0.0388	2.57587	0.0309	3.22042
0.2819	0.63661	0.2263	0.67027	0.2048	0.70347	0.1309	0.96545	0.0939	1.29055	0.0610	1.93900	0.0453	2.58675	0.0361	3.23424
0.3222	0.63661	0.2586	0.67086	0.2340	0.70450	0.1496	0.96880	0.1074	1.29604	0.0697	1.94822	0.0518	2.59949	0.0413	3.25041
0.3624	0.63661	0.2909	0.67154	0.2633	0.70568	0.1684	0.97266	0.1208	1.30236	0.0784	1.95886	0.0583	2.61417	0.0464	3.26905
0.4027	0.63661	0.3233	0.67233	0.2925	0.70705	0.1871	0.97706	0.1342	1.30958	0.0871	1.97097	0.0647	2.63089	0.0516	3.29028
0.4430	0.63661	0.3556	0.67322	0.3218	0.70860	0.2058	0.98205	0.1476	1.31773	0.0958	1.98465	0.0712	2.64977	0.0567	3.31425
0.4833	0.63661	0.3879	0.67424	0.3511	0.71036	0.2245	0.98766	0.1611	1.32689	0.1045	2.00000	0.0777	2.67095	0.0619	3.34113
0.5236	0.63661	0.4203	0.67540	0.3803	0.71234	0.2432	0.99395	0.1745	1.33712	0.1132	2.01714	0.0842	2.69460	0.0671	3.37113
0.5638	0.63661	0.4526	0.67670	0.4096	0.71456	0.2619	1.00095	0.1879	1.34851	0.1219	2.03620	0.0907	2.72089	0.0722	3.40449
0.6041	0.63661	0.4849	0.67817	0.4388	0.71706	0.2806	1.00875	0.2013	1.36117	0.1307	2.05735	0.0971	2.75005	0.0774	3.44148
0.6444	0.63661	0.5173	0.67982	0.4681	0.71987	0.2993	1.01742	0.2148	1.37521	0.1394	2.08078	0.1036	2.78233	0.0826	3.48244
0.6847	0.63661	0.5496	0.68169	0.4974	0.72301	0.3180	1.02705	0.2282	1.39076	0.1481	2.10671	0.1101	2.81805	0.0877	3.52774
0.7249	0.63661	0.5819	0.68379	0.5266	0.72654	0.3368	1.03774	0.2416	1.40799	0.1568	2.13539	0.1166	2.85755	0.0929	3.57783
0.7652	0.63661	0.6143	0.68617	0.5559	0.73051	0.3555	1.04963	0.2550	1.42709	0.1655	2.16715	0.1231	2.90127	0.0981	3.63324
0.8055	0.63661	0.6466	0.68887	0.5851	0.73498	0.3742	1.06286	0.2685	1.44829	0.1742	2.20235	0.1295	2.94969	0.1032	3.69462
0.8458	0.63661	0.6789	0.69194	0.6144	0.74002	0.3929	1.07762	0.2819	1.47187	0.1829	2.24143	0.1360	3.00343	0.1084	3.76273
0.8860	0.63661	0.7113	0.69544	0.6437	0.74574	0.4116	1.09412	0.2953	1.49816	0.1917	2.28493	0.1425	3.06322	0.1135	3.83848
0.9263	0.63661	0.7436	0.69946	0.6729	0.75226	0.4303	1.11264	0.3087	1.52757	0.2004	2.33350	0.1490	3.12995	0.1187	3.92300
0.9666	0.63661	0.7759	0.70410	0.7022	0.75970	0.4490	1.13352	0.3222	1.56060	0.2091	2.38796	0.1555	3.20472	0.1239	4.01769
1.0069	0.63661	0.8083	0.70948	0.7314	0.76828	0.4677	1.15716	0.3356	1.59788	0.2178	2.44932	0.1619	3.28892	0.1290	4.12429
1.0472	0.63661	0.8406	0.71579	0.7607	0.77821	0.4864	1.18412	0.3490	1.64022	0.2265	2.51886	0.1684	3.38429	0.1342	4.24499
1.0874	0.63661	0.8729	0.72323	0.7899	0.78982	0.5052	1.21507	0.3624	1.68865	0.2352	2.59824	0.1749	3.49308	0.1394	4.38265
1.1277	0.63661	0.9053	0.73211	0.8192	0.80352	0.5239	1.25093	0.3759	1.74454	0.2439	2.68964	0.1814	3.61826	0.1445	4.54101
1.1680	0.63661	0.9376	0.74284	0.8485	0.81989	0.5426	1.29292	0.3893	1.80971	0.2527	2.79599	0.1878	3.76383	0.1497	4.72510
1.2083	0.63661	0.9699	0.75600	0.8777	0.83970	0.5613	1.34276	0.4027	1.88670	0.2614	2.92135	0.1943	3.93528	0.1548	4.94186
1.2485	0.63661	1.0022	0.77244	0.9070	0.86412	0.5800	1.40286	0.4161	1.97915	0.2701	3.07150	0.2008	4.14050	0.1600	5.20121
1.2888	0.63661	1.0346	0.79345	0.9362	0.89487	0.5987	1.47687	0.4296	2.09243	0.2788	3.25504	0.2073	4.39117	0.1652	5.51792
1.3291	0.63661	1.0669	0.82109	0.9655	0.93470	0.6174	1.57047	0.4430	2.23501	0.2875	3.48546	0.2138	4.70561	0.1703	5.91505
1.3694	0.63661	1.0992	0.85888	0.9948	0.98824	0.6361	1.69321	0.4564	2.42103	0.2962	3.78527	0.2202	5.11444	0.1755	6.43121
1.4096	0.63661	1.1316	0.91342	1.0240	1.06411	0.6548	1.86263	0.4699	2.67644	0.3049	4.19580	0.2267	5.67378	0.1807	7.13714
1.4499	0.63661	1.1639	0.99879	1.0533	1.18055	0.6735	2.11553	0.4833	3.05563	0.3137	4.80355	0.2332	6.50111	0.1858	8.18091
1.4902	0.63661	1.1962	1.15224	1.0825	1.38541	0.6923	2.54767	0.4967	3.69987	0.3224	5.83311	0.2397	7.90140	0.1910	9.94686
1.5305	0.63661	1.2286	1.52518	1.1118	1.87156	0.7110	3.54146	0.5101	5.17259	0.3311	8.17935	0.2462	11.08958	0.1961	13.9659
1.5708	0.00000	1.2609	∞	1.1411	∞	0.7297	∞	0.5293	∞	0.3398	∞	0.2526	∞	0.2013	∞

TABLE B VII. Values of $J(\theta)$ for $v_0 < 1$.

θ	0.001	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.0805	0.04036	0.05550	0.07436	0.09719	0.12437	0.15627	0.19320	0.23551	0.28345	0.33727
0.1611	0.08047	0.11028	0.14725	0.19177	0.24450	0.30602	0.37688	0.45753	0.54834	0.64956
0.2416	0.12005	0.16362	0.21722	0.28124	0.35638	0.44325	0.54230	0.65383	0.77798	0.91464
0.3222	0.15884	0.21485	0.28296	0.36336	0.45653	0.56279	0.68218	0.81449	0.95919	1.11538
0.4027	0.19659	0.26334	0.34332	0.43626	0.54217	0.66073	0.79129	0.93273	1.08350	1.24148
0.4833	0.23305	0.30853	0.39734	0.49851	0.61132	0.73461	0.86671	1.00538	1.14775	1.29024
0.5638	0.26799	0.34995	0.44428	0.54914	0.66293	0.78346	0.90793	1.03288	1.15406	1.26639
0.6444	0.30118	0.38720	0.48364	0.58771	0.69683	0.80779	0.91671	1.01897	1.10917	1.18099
0.7249	0.33240	0.41999	0.51517	0.61422	0.71368	0.80941	0.89666	0.97001	1.02332	1.04960
0.8055	0.36144	0.44810	0.53883	0.62914	0.71482	0.79111	0.85278	0.89407	0.90877	0.89008
0.8860	0.38813	0.47141	0.55482	0.63328	0.70215	0.75642	0.79085	0.80000	0.77831	0.72016
0.9666	0.41228	0.48990	0.56350	0.62775	0.67791	0.70919	0.71686	0.69644	0.64381	0.55547
1.0472	0.43374	0.50361	0.56540	0.61384	0.64453	0.65330	0.63649	0.59109	0.51510	0.40789
1.1277	0.45237	0.51268	0.56114	0.59297	0.60443	0.59242	0.55468	0.49014	0.39928	0.28479
1.2083	0.46806	0.51728	0.55146	0.56657	0.55992	0.52973	0.47543	0.39802	0.30054	0.18897
1.2888	0.48070	0.51767	0.53709	0.53604	0.51306	0.46785	0.40162	0.31735	0.22036	0.11932
1.3694	0.49022	0.51411	0.51880	0.50267	0.46559	0.40880	0.33510	0.24917	0.15803	0.07203
1.4499	0.49655	0.50692	0.49736	0.46763	0.41895	0.35395	0.27676	0.19328	0.11144	0.04197
1.5305	0.49965	0.49642	0.47346	0.43190	0.37421	0.30413	0.22677	0.14862	0.07774	0.02394
1.6110	0.49952	0.48297	0.44776	0.39631	0.33211	0.25972	0.18473	0.11369	0.05399	0.01362
1.6916	0.49614	0.46689	0.42084	0.36152	0.29315	0.22073	0.14991	0.08679	0.03758	0.00786
1.7721	0.48955	0.44851	0.39311	0.32801	0.25755	0.18691	0.12142	0.06633	0.02634	0.00457
1.8527	0.47980	0.42815	0.36529	0.29612	0.22537	0.15787	0.09830	0.05086	0.01868	0.00288
1.9332	0.46694	0.40611	0.33745	0.26608	0.19652	0.13310	0.07965	0.03920	0.01344	0.00184
2.0138	0.45106	0.38266	0.30997	0.23799	0.17083	0.11208	0.06465	0.03041	0.00981	0.00123
2.0944	0.43227	0.35806	0.28304	0.21189	0.14805	0.09431	0.05259	0.02376	0.00728	0.00084
2.1749	0.41069	0.33252	0.25683	0.18773	0.12791	0.07928	0.04288	0.01869	0.00548	0.00060
2.2555	0.38646	0.30624	0.23143	0.16543	0.11011	0.06657	0.03503	0.01480	0.00418	0.00043
2.3360	0.35975	0.27938	0.20689	0.14487	0.09439	0.05579	0.02865	0.01178	0.00322	0.00032
2.4166	0.33071	0.25210	0.18322	0.12590	0.08045	0.04660	0.02342	0.00941	0.00251	0.00024
2.4971	0.29955	0.22450	0.16042	0.10836	0.06806	0.03872	0.01910	0.00752	0.00196	0.00018
2.5777	0.26646	0.19669	0.13842	0.09210	0.05697	0.03191	0.01549	0.00600	0.00154	0.00014
2.6582	0.23166	0.16872	0.11717	0.07694	0.04696	0.02596	0.01243	0.00475	0.00120	0.00011
2.7388	0.19537	0.14067	0.09658	0.06271	0.03785	0.02069	0.00980	0.00370	0.00092	0.00008
2.8193	0.15782	0.11256	0.07656	0.04925	0.02946	0.01596	0.00749	0.00280	0.00069	0.00006
2.8999	0.11926	0.08443	0.05701	0.03641	0.02163	0.01163	0.00542	0.00201	0.00049	0.00004
2.9804	0.07993	0.05628	0.03780	0.02402	0.01419	0.00760	0.00352	0.00130	0.00031	0.00002
3.0610	0.04009	0.02814	0.01884	0.01193	0.00703	0.00375	0.00173	0.00064	0.00015	0.00001
3.1415	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE B VIII. Values of $J(\theta)$ for $\gamma_0 \geq 1$.

θ	$\gamma_0=1.0$	θ	$\gamma_0=1.3$	θ	$\gamma_0=1.5$	θ	$\gamma_0=2.0$	θ	$\gamma_0=3.0$	θ	$\gamma_0=4.0$	θ	$\gamma_0=5.0$
0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000	0.0000	0.00000
0.0402	0.20067	0.0225	0.17436	0.0187	0.18687	0.0134	0.22801	0.0087	0.32215	0.0064	0.42082	0.0051	0.52109
0.0805	0.39714	0.0450	0.34717	0.0374	0.37236	0.0268	0.45471	0.0174	0.64278	0.0129	0.83980	0.0103	1.03999
0.1208	0.58529	0.0675	0.51688	0.0561	0.55508	0.0402	0.67878	0.0261	0.96037	0.0194	1.25510	0.0154	1.55448
0.1611	0.76131	0.0900	0.68198	0.0748	0.73368	0.0537	0.89892	0.0348	1.27339	0.0259	1.66487	0.0206	2.06238
0.2013	0.92169	0.1125	0.84101	0.0935	0.90684	0.0671	1.11386	0.0435	1.58036	0.0323	2.06728	0.0258	2.56149
0.2416	1.06344	0.1350	0.99260	0.1122	1.07328	0.0805	1.32234	0.0522	1.87978	0.0388	2.46053	0.0309	3.04963
0.2819	1.18406	0.1575	1.13543	0.1309	1.23179	0.0939	1.52315	0.0610	2.17017	0.0453	2.84279	0.0361	3.52463
0.3222	1.28171	0.1800	1.26830	0.1496	1.38121	0.1074	1.71509	0.0697	2.45010	0.0518	3.21228	0.0413	3.98433
0.3624	1.35516	0.2025	1.39010	0.1684	1.52047	0.1208	1.89703	0.0784	2.71813	0.0583	3.56723	0.0464	4.42657
0.4027	1.40388	0.2250	1.49986	0.1871	1.64858	0.1342	2.06789	0.0871	2.97288	0.0647	3.90587	0.0516	4.84924
0.4430	1.42801	0.2475	1.59674	0.2058	1.76462	0.1476	2.22661	0.0958	3.21296	0.0712	4.22648	0.0567	5.25021
0.4833	1.42834	0.2700	1.68002	0.2245	1.86782	0.1611	2.37224	0.1045	3.43706	0.0777	4.52735	0.0619	5.62739
0.5236	1.40625	0.2925	1.74916	0.2432	1.95747	0.1745	2.50385	0.1132	3.64388	0.0842	4.80680	0.0671	5.97871
0.5638	1.36366	0.3150	1.80374	0.2619	2.03299	0.1879	2.62062	0.1219	3.83216	0.0907	5.06318	0.0722	6.30211
0.6041	1.30296	0.3375	1.84353	0.2806	2.09392	0.2013	2.72177	0.1307	4.00070	0.0971	5.29487	0.0774	6.59558
0.6444	1.22689	0.3600	1.86843	0.2993	2.13993	0.2148	2.80660	0.1394	4.14835	0.1036	5.50028	0.0826	6.85710
0.6847	1.13845	0.3825	1.87852	0.3180	2.17078	0.2282	2.87453	0.1481	4.27397	0.1101	5.67786	0.0877	7.08472
0.7249	1.04077	0.4050	1.87403	0.3368	2.18639	0.2416	2.92501	0.1568	4.37653	0.1166	5.82611	0.0929	7.27650
0.7652	0.93704	0.4275	1.85532	0.3555	2.18678	0.2550	2.95762	0.1655	4.45501	0.1231	5.94355	0.0981	7.43051
0.8055	0.83036	0.4500	1.82293	0.3742	2.17211	0.2685	2.97201	0.1742	4.50846	0.1295	6.02876	0.1032	7.54490
0.8458	0.72367	0.4725	1.77752	0.3929	2.14267	0.2819	2.96792	0.1829	4.53601	0.1360	6.08035	0.1084	7.61783
0.8860	0.61964	0.4950	1.71988	0.4116	2.09884	0.2953	2.94519	0.1917	4.53682	0.1425	6.09700	0.1135	7.64748
0.9263	0.52060	0.5175	1.65093	0.4303	2.04116	0.3087	2.90376	0.2004	4.51014	0.1490	6.07740	0.1187	7.63211
0.9666	0.42850	0.5400	1.57169	0.4490	1.97023	0.3222	2.84364	0.2091	4.45528	0.1555	6.02032	0.1239	7.56999
1.0069	0.34482	0.5625	1.48326	0.4677	1.88680	0.3356	2.76498	0.2178	4.37160	0.1619	5.92459	0.1290	7.45944
1.0472	0.27063	0.5850	1.38686	0.4864	1.79170	0.3490	2.66797	0.2265	4.25855	0.1684	5.78906	0.1342	7.29883
1.0874	0.20649	0.6075	1.28373	0.5052	1.68585	0.3624	2.55293	0.2352	4.11566	0.1749	5.61267	0.1394	7.08656
1.1277	0.15256	0.6301	1.17520	0.5239	1.57027	0.3759	2.42027	0.2439	3.94251	0.1814	5.39439	0.1445	6.82110
1.1680	0.10857	0.6526	1.06259	0.5426	1.44605	0.3893	2.27048	0.2527	3.73877	0.1878	5.13327	0.1497	6.50094
1.2083	0.07392	0.6751	0.94728	0.5613	1.31432	0.4027	2.10416	0.2614	3.50417	0.1943	4.82842	0.1548	6.12466
1.2485	0.04769	0.6976	0.83062	0.5800	1.17629	0.4161	1.92196	0.2701	3.23854	0.2008	4.47898	0.1600	5.69084
1.2888	0.02877	0.7201	0.71396	0.5987	1.03323	0.4296	1.72465	0.2788	2.94176	0.2073	4.08420	0.1652	5.19815
1.3291	0.01592	0.7426	0.59861	0.6174	0.88641	0.4430	1.51307	0.2875	2.61381	0.2138	3.64337	0.1703	4.64531
1.3694	0.00784	0.7651	0.48584	0.6361	0.73714	0.4564	1.28813	0.2962	2.25475	0.2202	3.15583	0.1755	4.03108
1.4096	0.00326	0.7876	0.37686	0.6548	0.58673	0.4699	1.05082	0.3049	1.86470	0.2267	2.62103	0.1807	3.35428
1.4499	0.00104	0.8101	0.27280	0.6735	0.43652	0.4833	0.80220	0.3137	1.44387	0.2332	2.03845	0.1858	2.61381
1.4902	0.00020	0.8326	0.17469	0.6923	0.28781	0.4967	0.54339	0.3224	0.99255	0.2397	1.40765	0.1910	1.80859
1.5305	0.00001	0.8551	0.08348	0.7110	0.14188	0.5101	0.27558	0.3311	0.51111	0.2462	0.72826	0.1961	0.93763
1.5708	0.00000	0.8776	0.00000	0.7297	0.00000	0.5236	0.00000	0.3398	0.00000	0.2526	0.00000	0.2013	0.00000

TABLE B IX. Values of $\Sigma(\theta)$ for $v_0 < 1$.

$\theta \backslash v_0$	.001	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0000	0.31385	0.37277	0.42886	0.48817	0.55209	0.62195	0.69882	0.78343	0.87610	0.97652
0.0805	0.31884	0.37215	0.42781	0.48704	0.55137	0.62222	0.70077	0.78786	0.88386	0.98855
0.1611	0.31884	0.37171	0.42733	0.48716	0.55295	0.62627	0.70843	0.80034	0.90244	1.01449
0.2416	0.31883	0.37095	0.42627	0.48653	0.55360	0.62921	0.71464	0.81077	0.91790	1.03561
0.3222	0.31881	0.36986	0.42455	0.48481	0.55261	0.62959	0.71690	0.81520	0.92441	1.04364
0.4027	0.31880	0.36843	0.42208	0.48178	0.54941	0.62638	0.71353	0.81102	0.91819	1.03333
0.4833	0.31878	0.36667	0.41884	0.47728	0.54364	0.61894	0.70348	0.79673	0.89716	1.00200
0.5638	0.31876	0.36458	0.41479	0.47122	0.53511	0.60693	0.68628	0.77177	0.86082	0.94935
0.6444	0.31873	0.36216	0.40994	0.46357	0.52376	0.59033	0.66204	0.73653	0.81006	0.87719
0.7249	0.31871	0.35944	0.40430	0.45435	0.50969	0.56938	0.63131	0.69214	0.74701	0.78915
0.8055	0.31868	0.35641	0.39790	0.44366	0.49312	0.54457	0.59507	0.64038	0.67468	0.69011
0.8860	0.31864	0.35312	0.39081	0.43164	0.47436	0.51654	0.55454	0.58339	0.59662	0.58570
0.9666	0.31861	0.34957	0.38308	0.41846	0.45381	0.48609	0.51112	0.52354	0.51659	0.48166
1.0472	0.31857	0.34579	0.37480	0.40432	0.43189	0.45403	0.46626	0.46314	0.43813	0.38326
1.1277	0.31854	0.34182	0.36607	0.38945	0.40907	0.42120	0.42133	0.40430	0.36431	0.29476
1.2083	0.31850	0.33768	0.35697	0.37408	0.38581	0.38838	0.37754	0.34876	0.29744	0.21909
1.2888	0.31846	0.33341	0.34762	0.35843	0.36252	0.35626	0.33589	0.29782	0.23901	0.15762
1.3694	0.31841	0.32904	0.33811	0.34272	0.33959	0.32542	0.29714	0.25229	0.18962	0.11022
1.4499	0.31837	0.32460	0.32854	0.32716	0.31735	0.29630	0.26176	0.21252	0.14913	0.07548
1.5305	0.31833	0.32013	0.31900	0.31191	0.29605	0.26921	0.23001	0.17848	0.11680	0.05120
1.6110	0.31828	0.31566	0.30956	0.29711	0.27588	0.24430	0.20190	0.14980	0.09153	0.03485
1.6916	0.31824	0.31120	0.30030	0.28287	0.25695	0.22165	0.17730	0.12597	0.07209	0.02409
1.7721	0.31820	0.30681	0.29131	0.26932	0.23941	0.20129	0.15603	0.10638	0.05730	0.01708
1.8527	0.31816	0.30252	0.28266	0.25656	0.22329	0.18315	0.13779	0.09040	0.04610	0.01249
1.9332	0.31811	0.29836	0.27439	0.24462	0.20859	0.16710	0.12224	0.07743	0.03763	0.00942
2.0138	0.31807	0.29436	0.26656	0.23354	0.19528	0.15300	0.10905	0.06692	0.03119	0.00733
2.0944	0.31804	0.29053	0.25920	0.22334	0.18332	0.14068	0.09791	0.05840	0.02626	0.00587
2.1749	0.31800	0.28691	0.25234	0.21401	0.17264	0.12996	0.08852	0.05150	0.02247	0.00483
2.2555	0.31796	0.28351	0.24600	0.20555	0.16316	0.12069	0.08063	0.04590	0.01952	0.00406
2.3360	0.31793	0.28036	0.24019	0.19794	0.15481	0.11271	0.07402	0.04135	0.01721	0.00349
2.4166	0.31790	0.27746	0.23493	0.19117	0.14751	0.10588	0.06849	0.03764	0.01539	0.00306
2.4971	0.31787	0.27483	0.23022	0.18519	0.14119	0.10008	0.06390	0.03464	0.01396	0.00273
2.5777	0.31785	0.27248	0.22606	0.18000	0.13578	0.09519	0.06010	0.03220	0.01282	0.00248
2.6582	0.31783	0.27043	0.22247	0.17556	0.13121	0.09113	0.05699	0.03024	0.01192	0.00228
2.7388	0.31781	0.26868	0.21943	0.17185	0.12744	0.08783	0.05449	0.02869	0.01122	0.00213
2.8193	0.31779	0.26723	0.21695	0.16884	0.12442	0.08520	0.05253	0.02749	0.01068	0.00201
2.8999	0.31778	0.26611	0.21502	0.16653	0.12211	0.08321	0.05106	0.02659	0.01029	0.00193
2.9804	0.31777	0.26530	0.21364	0.16488	0.12048	0.08181	0.05003	0.02597	0.01001	0.00187
3.0610	0.31777	0.26481	0.21282	0.16390	0.11951	0.08099	0.04942	0.02560	0.00986	0.00184
3.1415	0.31777	0.26465	0.21254	0.16357	0.11919	0.08071	0.04922	0.02548	0.00980	0.00183

TABLE B X. Values of $\Sigma(\theta)$ for $\gamma_0 \geq 1$.

θ	$\gamma_0=1.0001$	θ	$\gamma_0=1.3$	θ	$\gamma_0=1.5$	θ	$\gamma_0=2.0$	θ	$\gamma_0=3.0$	θ	$\gamma_0=4.0$	θ	$\gamma_0=5.0$
0.0000	1.09342	0.0000	1.42353	0.0000	1.64412	0.0000	2.19697	0.0000	3.30762	0.0000	4.42395	0.0000	5.54577
0.0399	1.10026	0.0225	1.42907	0.0187	1.65015	0.0134	2.20451	0.0087	3.31845	0.0064	4.43806	0.0051	5.56314
0.0798	1.11389	0.0450	1.44093	0.0374	1.66324	0.0268	2.22114	0.0174	3.34254	0.0129	4.46969	0.0103	5.60225
0.1197	1.12999	0.0675	1.45632	0.0561	1.68043	0.0402	2.24330	0.0261	3.37499	0.0194	4.51240	0.0154	5.65515
0.1596	1.14602	0.0900	1.47355	0.0748	1.69997	0.0537	2.26889	0.0348	3.41279	0.0259	4.56231	0.0206	5.71705
0.1995	1.16010	0.1125	1.49141	0.0935	1.72059	0.0671	2.29639	0.0435	3.45382	0.0323	4.61667	0.0258	5.78454
0.2394	1.17072	0.1350	1.50893	0.1122	1.74127	0.0805	2.32457	0.0522	3.49639	0.0388	4.67324	0.0309	5.85487
0.2794	1.17667	0.1575	1.52525	0.1309	1.76113	0.0939	2.35239	0.0610	3.53902	0.0453	4.73012	0.0361	5.92568
0.3193	1.17689	0.1800	1.53965	0.1496	1.77940	0.1074	2.37893	0.0697	3.58043	0.0518	4.78562	0.0413	5.99489
0.3592	1.17085	0.2025	1.55150	0.1684	1.79540	0.1208	2.40335	0.0784	3.61942	0.0583	4.83820	0.0464	6.06060
0.3991	1.15793	0.2250	1.56022	0.1871	1.80852	0.1342	2.42490	0.0871	3.65494	0.0647	4.88647	0.0516	6.12106
0.4390	1.13784	0.2475	1.56521	0.2058	1.81819	0.1476	2.44288	0.0958	3.68597	0.0712	4.92910	0.0567	6.17463
0.4789	1.11051	0.2700	1.56625	0.2245	1.82382	0.1611	2.45665	0.1045	3.71156	0.0777	4.96483	0.0619	6.21975
0.5188	1.07606	0.2925	1.56287	0.2432	1.82519	0.1745	2.46543	0.1132	3.73083	0.0842	4.99248	0.0671	6.25496
0.5588	1.03479	0.3150	1.55475	0.2619	1.82179	0.1879	2.46902	0.1219	3.74292	0.0907	5.01092	0.0722	6.27883
0.5987	0.98720	0.3375	1.54163	0.2806	1.81327	0.2013	2.46672	0.1307	3.74680	0.0971	5.01907	0.0774	6.29000
0.6386	0.93391	0.3600	1.52332	0.2993	1.79934	0.2148	2.45806	0.1394	3.74218	0.1036	5.01559	0.0826	6.28680
0.6785	0.87572	0.3825	1.49969	0.3180	1.77974	0.2282	2.44260	0.1481	3.72807	0.1101	5.00007	0.0877	6.26869
0.7184	0.81350	0.4050	1.47065	0.3368	1.75428	0.2416	2.41995	0.1568	3.70379	0.1166	4.97123	0.0929	6.23408
0.7583	0.74824	0.4275	1.43617	0.3555	1.72280	0.2550	2.38973	0.1655	3.66865	0.1231	4.92813	0.0981	6.18174
0.7982	0.68097	0.4500	1.39630	0.3742	1.68519	0.2685	2.35163	0.1742	3.62202	0.1295	4.86988	0.1032	6.11053
0.8382	0.61275	0.4725	1.35112	0.3929	1.64139	0.2819	2.30535	0.1829	3.56331	0.1360	4.79558	0.1084	6.01932
0.8781	0.54467	0.4950	1.30076	0.4116	1.59139	0.2953	2.25064	0.1917	3.49192	0.1425	4.70440	0.1135	5.90699
0.9180	0.47776	0.5175	1.24542	0.4303	1.53522	0.3087	2.18730	0.2004	3.40733	0.1490	4.59553	0.1187	5.77250
0.9579	0.41302	0.5400	1.18533	0.4490	1.47296	0.3222	2.11514	0.2091	3.30902	0.1555	4.46817	0.1239	5.61480
0.9978	0.35135	0.5625	1.12078	0.4677	1.40472	0.3356	2.03402	0.2178	3.19651	0.1619	4.32158	0.1290	5.43289
1.0377	0.29357	0.5850	1.05210	0.4864	1.33069	0.3490	1.94385	0.2265	3.06935	0.1684	4.15503	0.1342	5.22581
1.0776	0.24036	0.6075	0.97966	0.5052	1.25106	0.3624	1.84457	0.2352	2.92713	0.1749	3.96784	0.1394	4.99263
1.1176	0.19228	0.6301	0.90387	0.5239	1.16609	0.3759	1.73615	0.2439	2.76947	0.1814	3.75937	0.1445	4.73246
1.1575	0.14972	0.6526	0.82517	0.5426	1.07607	0.3893	1.61862	0.2527	2.59605	0.1878	3.52900	0.1497	4.44447
1.1974	0.11291	0.6751	0.74403	0.5613	0.98133	0.4027	1.49204	0.2614	2.40656	0.1943	3.27618	0.1548	4.12786
1.2373	0.08192	0.6976	0.66096	0.5800	0.88224	0.4161	1.35653	0.2701	2.20076	0.2008	3.00041	0.1600	3.78192
1.2772	0.05665	0.7201	0.57648	0.5987	0.77922	0.4296	1.21223	0.2788	1.97847	0.2073	2.70124	0.1652	3.40603
1.3171	0.03683	0.7426	0.49114	0.6174	0.67271	0.4430	1.05936	0.2875	1.73957	0.2138	2.37836	0.1703	2.99966
1.3570	0.02204	0.7651	0.40552	0.6361	0.56320	0.4564	0.89819	0.2962	1.48402	0.2202	2.03152	0.1755	2.56247
1.3970	0.01173	0.7876	0.32020	0.6548	0.45125	0.4699	0.72909	0.3049	1.21196	0.2267	1.66074	0.1807	2.09440
1.4369	0.00521	0.8101	0.23583	0.6735	0.33746	0.4833	0.55256	0.3137	0.92374	0.2332	1.26637	0.1858	1.59591
1.4768	0.00167	0.8326	0.15309	0.6923	0.22259	0.4967	0.36934	0.3224	0.62023	0.2397	0.84960	0.1910	1.06860
1.5167	0.00024	0.8551	0.07292	0.7110	0.10779	0.5101	0.18097	0.3311	0.30391	0.2462	0.41411	0.1961	0.51766
1.5566	0.00000	0.8776	0.00000	0.7297	0.00000	0.5236	0.00000	0.3398	0.00000	0.2526	0.00000	0.2013	0.00000

APPENDIX C. MULTIPLE VALUES OF Q.

Suppose that a reaction may proceed in more than one inelastic reaction channel. For example, suppose that the cross section $\sigma'(\theta^*, E, Q)$ for the reaction

is nonzero for several values of Q. Then the average value, $\bar{N}$, of a quantity $N(Q)$ associated with this reaction is

$$\bar{N} = \frac{\int N(Q) g_Q dQ}{\int g_Q dQ}, \quad (C2)$$

where g_Q = statistical weight associated with a given value of Q. For a discrete value of Q, say Q_1 , $g_Q = g_{Q_1} \delta(Q - Q_1)$.

Many of the previous results depend on γ or γ_0 , both of which depend on Q; furthermore, σ' depends on θ^* , E_0 , and Q. If several values of Q are permitted in a reaction, then the total or average reaction cross section must be computed by an appropriate formula, such as Eq. (C2).

Often it is of interest to deduce $\sigma'(\theta^*, E_0, Q)$ from laboratory scattering data. The following example serves to illustrate the problem. Suppose that in reaction (C1) two reaction channels, 1 and 2, are permitted and have Q-values Q_1 and Q_2 , respectively. Let the scattering be isotropic in BCS for both channels and all values of E_0 . Let γ_1 and γ_2 be computed from Q_1 and Q_2 , respectively. Suppose that $Q_2 > Q_1 > 0$, $m_a m_b / (m_Y m_X) = 1$, and $m_a + m_X = m_b + m_Y = M$. Then from Eqs. (10) and (12),

$$\gamma = \left[\frac{m_a m_b E_0}{m_Y m_X (E_0 + Q)} \right]^{1/2},$$

and

$$0 \leq \gamma_i \leq \left(\frac{m_a m_b}{m_Y m_X} \right)^{1/2} = 1 \quad i = 1, 2. \quad (C3)$$

Now refer to Fig. 3. Since $\gamma_1 \neq \gamma_2$, the laboratory angular distribution of particle b produced in reaction channel 1 is different from the laboratory distribution of particle b produced in channel 2, and as E_0 is varied these distributions change differently. Since the total cross sections, σ_1 and σ_2 , for channels 1 and 2 depend on E_0 arbitrarily, the angular distribution of particle b in the laboratory may exhibit complicated behavior as a func-

tion of E_0 . In this example Q_1 , Q_2 , and the angular distributions in BCS are known, so it is possible to find σ_1/σ_2 from the laboratory angular distribution of b as a function of E_0 . If $\sigma_1(E_0) + \sigma_2(E_0)$ is then measured, σ_1 and σ_2 are known. But if, for instance, the barycentric angular distribution of b produced in channel 1 were not known, it might be very hard to deduce σ_1/σ_2 from the angular distribution of particle b in LCS.

In an atomic or molecular reaction, the barycentric angular distributions are not normally known for any reaction channel. Very often it is not obvious a priori which values of Q may yield significant reaction cross sections; it is possible that even when exothermic channels ($Q > 0$) are available, endothermic channels ($Q < 0$) yield larger reaction cross sections.^{21,22} We consider below what laboratory measurements are necessary to determine the cross section $\sigma'(\theta^*, Q, E_0)$ for reaction (C1).

The laboratory angular distribution $d(E_0, \theta)$ for a two-particle disintegration is given by

$$d(E_0, \theta) = \int \sigma'(\theta^*(\gamma, \theta), E_0, Q) f_Y(\theta) g_Q dQ, \quad (C4)$$

where $\theta^*(\gamma, \theta)$, $f_Y(\theta)$, and γ can be deduced from Sections III A, B, and C. If $d(E_0, \theta)$ is measured, it is generally impossible to find a unique σ' which satisfies Eq. (C4). A measurement of the laboratory angular distribution $d(E_0, \theta)$ is insufficient to determine σ' . Suppose that the distribution $D(E_0, \theta, E_b)$ for producing particle b with laboratory kinetic energy E_b and scattering angle θ were measured. Then one could find $\sigma'(\theta^*, E_0, Q)$ from

$$D(E_0, \theta, E_b) d\theta dE_b = D(E_0, \theta, E_b) \frac{\partial(\theta, E_b)}{\partial(\theta^*, Q)} d\theta^* dQ \quad (C5)$$

$$= g_Q \sigma'(\theta^*, E_0, Q) d\theta^* dQ.$$

$$\sigma'(\theta^*, E_0, Q) = \frac{D(E_0, \theta, E_b)}{2\pi g_Q \sin \theta^*} \frac{\partial(\theta, E_b)}{\partial(\theta^*, Q)}, \quad (C6)$$

where $\theta = \theta(\theta^*, Q)$ and $E_b = E_b(\theta^*, Q)$ on the right-hand side of Eq. (C6). From Eq. (66)

$$\theta = \arccot \left(\frac{\cos \theta^* + \gamma}{\sin \theta^*} \right). \quad (C7)$$

$$E_b = E_b' (1 + \gamma^2 + 2\gamma \cos \theta^*) \quad (C8)$$

where

$$E'_b = \frac{m_Y}{m_Y + m_b} (E_o + Q) \quad (37)$$

γ is given by Eq. (C7) or, when mass is conserved, by

$$\gamma = \left[\frac{m_a m_b E_o}{m_X m_Y (E_o + Q)} \right]^{1/2}, \quad (C8)$$

so

$$E'_b = \frac{m_Y}{M} (E_o + Q) = \frac{\epsilon}{\gamma^2} \quad (C9)$$

with

$$\epsilon = \frac{1}{2} m_b v_c^2 = \frac{m_a m_b E_o}{m_X M}, \quad (C10)$$

where

$$M \equiv m_Y + m_b = m_X + m_a.$$

These formulas can be combined to give

$$\frac{\partial(\theta, E_b)}{\partial(\theta^*, Q)} = \frac{m_Y}{M}. \quad (C11)$$

Thus the transformation $\theta = \theta(\theta^*, Q)$, $E_b = E_b(\theta^*, Q)$ given by Eqs. (C7) and (38) is one-to-one. The inverse transformation is also one-to-one.¹² Figure C1 shows two examples in which regions of (θ^*, Q) -space are mapped into regions of (θ, E_b) -space. Substitution of Eq. (C9) into Eq. (40) yields the boundaries in (θ, E_b) -space shown in Fig. C1,

$$E_b = \frac{\epsilon}{\gamma^2} \left(\gamma \cos \theta \pm \sqrt{1 - \gamma^2 \sin^2 \theta} \right)^2, \quad (C12)$$

where the plus sign only is used if $\gamma < 1$, corresponding to the lines $Q = \text{constant}$; $0 \leq \theta^* \leq \pi$, in (θ^*, Q) -space. Note that γ is a function of Q alone for fixed E_o . Simultaneous measurements in LCS of the kinetic energy and angular distributions are sufficient to find the reaction cross section in BCS for a two-particle disintegration when more than one value of Q is possible, provided that all the masses are known.

The transformation $(\theta, E_b) \rightarrow (\theta^*, Q)$ may also be useful occasionally. $\theta^* = \theta^*(\theta, E_b)$ is given by Eqs. (13) to (16) and Eq. (137) with $r \equiv \sqrt{E_b/\epsilon}$. $Q = Q(\theta, E_b)$ is given by Halliday.⁷

$$Q = E_b \left(1 + \frac{m_a}{m_Y} \right) - E_a \left(1 - \frac{m_a}{m_Y} \right) - \frac{2}{m_Y} (E_a E_b m_a m_b)^{1/2} \cos \theta. \quad (C13)$$

Fig. C1. Transformations between (θ^*, Q) -space and (θ, E_b) -space. Examples for $\gamma < 1$ and $\gamma > 1$ are shown. The arrows show how the boundaries transform.

Sometimes the laboratory kinetic energy distribution, $P(E_b)$, of a secondary particle, b , is measured to determine the probability with which a given two-body disintegration proceeds as a function of Q . The results of such an investigation do not have an unequivocal interpretation, unless $D(E_o, \theta, E_b)$ is measured and $P(E_b)$ determined by integrating D over θ , $P(E_b) = \int D(E_o, \theta, E_b) d\theta$ for given E_o . In most cases, even the values of Q for which σ' is nonzero cannot be obtained unambiguously from $P(E_b)$ alone. These results can be deduced from Eq. (C6) or from the results of Section II D, where it is shown that $P(E_b)$ depends not only on the value of Q but also on the angular dependence of $\sigma'(\theta^*, E_o, Q)$.

Suppose that the laboratory distribution, $D(E_o, \theta, E_1)$, for producing particle b_1 with laboratory kinetic energy E_1 and scattering angle θ in a

three-body disintegration were measured. From Section III it can be deduced that

$$D(E_0, \theta, E_1) = \int G(E_1, \theta, \gamma_0, E_0, Q) g_Q dQ, \quad (C14)$$

where $G(E_1, \theta, \gamma_0, E_0, Q)$ is an unnormalized form of the $\tilde{F}$ that was discussed in the fourth paragraph of Section III C. The explicit dependence of G on E_0 and Q comes from the explicit dependence of $\sigma''(\theta^*, f_1, f_2, E_0, Q)$ on E_0 and Q ; however, γ_0 is also a function of E_0 and Q . Generally, it is impossible to find a unique function G which satisfies Eq. (C14). When several values of Q are possible, si-

multaneous laboratory measurements of the kinetic energy and angular distributions of particle b_1 are insufficient to determine even G uniquely and, hence, do not determine σ'' .

It has been assumed throughout that Q is a continuous variable. The formulas are easily generalized to the case in which Q is discrete, and the conclusions of this appendix are then unaltered. The interpretation of scattering data for which several values of Q are permitted is much more complicated than when the reaction proceeds with a single value of Q .

APPENDIX D. COMPUTATIONAL DETAILS

$$1. \text{ Derivation of } d\vec{P}'_1 d\vec{P}'_2 = \frac{\partial(P'_{1x}, P'_{1y}, P'_{1z}, P'_{2x}, P'_{2y}, P'_{2z})}{\partial(\alpha, \beta, E, \chi^*, \theta^*, \varphi^*)} d\alpha d\beta dE d\chi^* d\theta^* d\varphi^*$$

Direct evaluation of the 6×6 determinant $\frac{\partial(P'_{1x}, \dots, P'_{2z})}{\partial(\alpha, \dots, \varphi^*)}$ is lengthy. An outline of a less arduous method is given here. Consider a coordinate system in which

$$\vec{P}'_1 = (0, 0, P'_{1z}) \quad (D1)$$

and $\vec{P}'_2$ and $\vec{P}'_3$ lie in the $y''z''$ plane; in this coordinate system P'_{ij} is independent of $\theta^*, \varphi^*, \chi^*$ and depends only on α, β , and E . The transformation of $\vec{P}'_i = (P'_{ix}, P'_{iy}, P'_{iz})$ to this coordinate system depends only upon θ^*, χ^* , and φ^* .

$$P'_{ij} = \sum_k a_{jk} P''_{ik} \quad i = 1, 2, 3 \quad (D2)$$

$$k, j = x, y, z$$

Let

$$d\tau \equiv d\vec{P}'_1 d\vec{P}'_2 = K d\theta^* d\chi^* d\varphi^* dP''_{1z} dP''_{2y} dP''_{2z} \quad (D3)$$

Now

$$\frac{\partial P'_{ij}}{\partial P''_{sk}} = a_{jk} \delta_{si} \quad (D4)$$

and

$$\frac{\partial P'_{ij}}{\partial \theta^*} = \sum_k \frac{\partial a_{jk}}{\partial \theta^*} P''_{ik} = \sum_k \theta^{a_{jk}} P''_{ik} \quad (D5)$$

etc; the asterisk on θ^* is dropped for simplicity of notation in $\theta^{a_{jk}}$, etc.

$$K = \frac{\partial(P'_{1x}, \dots, P'_{2z})}{\partial(P''_{1z}, P''_{2y}, \dots, \chi^*)} \quad (D6)$$

Some algebra yields

$$K = \begin{vmatrix} a_{1z} & a_{2z} & a_{3z} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{1y} & a_{2y} & a_{3y} \\ 0 & 0 & 0 & a_{1z} & a_{2z} & a_{3z} \end{vmatrix} \quad (D7)$$

$$= P'_{1z} \begin{pmatrix} \theta^{a_{1z}} & \theta^{a_{2z}} & \theta^{a_{3z}} \\ \phi^{a_{1z}} & \phi^{a_{2z}} & \phi^{a_{3z}} \\ \chi^{a_{1z}} & \chi^{a_{2z}} & \chi^{a_{3z}} \end{pmatrix} + P'_{2z} \begin{pmatrix} \theta^{a_{1z}} & \theta^{a_{2z}} & \theta^{a_{3z}} \\ \phi^{a_{1z}} & \phi^{a_{2z}} & \phi^{a_{3z}} \\ \chi^{a_{1z}} & \chi^{a_{2z}} & \chi^{a_{3z}} \end{pmatrix} + P'_{2y} \begin{pmatrix} \theta^{a_{1y}} & \theta^{a_{2y}} & \theta^{a_{3y}} \\ \phi^{a_{1y}} & \phi^{a_{2y}} & \phi^{a_{3y}} \\ \chi^{a_{1y}} & \chi^{a_{2y}} & \chi^{a_{3y}} \end{pmatrix}$$

From Eqs. (52) to (54) and Fig. 12,

$$P''_{2z} = P'_1 \frac{\sin \beta \cos(\alpha + \beta)}{\sin \alpha}, \quad (D8)$$

$$P''_{2y} = -P'_1 \frac{\sin \beta \sin(\alpha + \beta)}{\sin \alpha}, \quad (D9)$$

$$P'_{1z} = P'_1 = (2 M_0 E)^{1/2} \sin \alpha \quad (D10)$$

The matrix $\tilde{A}$, where $\underline{A} = (a_{ij})$, is given on p. 10 of Whittaker's book²³ where his ψ is the present χ^* . More manipulations yield

$$K = (2 M_0 E)^{3/2} \sin^2 \alpha \sin \beta \sin(\alpha + \beta) \sin^2 \theta^* \quad (D11)$$

Now write

$$d\tau = K R d\theta^* d\chi^* d\phi^* d\alpha d\beta dE \quad (D12)$$

$$R = \frac{\partial(P'_{1z}, P'_{2y}, P'_{2z})}{\partial(\alpha, \beta, E)} \quad (D13)$$

$$R = \frac{\partial P'_1}{\partial E} \begin{vmatrix} 0 & 0 & 1 \\ (-P'_1 \csc^2 \alpha \sin 2\beta)/2 & P'_1(\cot \alpha \cos 2\beta - \sin 2\beta) & 0 \\ P'_1 \sin^2 \beta \csc^2 \alpha & -P'_1(\cos 2\beta + \cot \alpha \sin 2\beta) & 0 \end{vmatrix} \quad (D14)$$

Since

$$\frac{\partial P'_1}{\partial E} = \left(\frac{M_0}{2E} \right)^{1/2} \sin \alpha,$$

$$R = (2E)^{1/2} M_0^{3/2} \sin \beta \sin(\alpha + \beta)$$

$$KR = 4 M_0^3 E^2 \sin^2 \alpha \sin^2 \beta \sin^2(\alpha + \beta) \sin \theta \quad (D15)$$

Substitution of Eq. (D15) into Eq. (D12) yields Eq. (59).

2. Evaluation of $\partial(\alpha, \beta)/\partial(f_1, f_2)$

$$\frac{\partial(\alpha, \beta)}{\partial(f_1, f_2)} = \begin{vmatrix} \frac{\partial \alpha}{\partial f_1} & \frac{\partial \beta}{\partial f_1} \\ \frac{\partial \alpha}{\partial f_2} & \frac{\partial \beta}{\partial f_2} \end{vmatrix} \quad (D16)$$

From Eqs. (52) to (54) and Fig. 12, it may be deduced that

$$\sqrt{m_1 f_1} \sin \beta = \sqrt{m_2 f_2} \sin \alpha \quad (D17)$$

Equation (D16) readily reduces to

$$\frac{\partial(\alpha, \beta)}{\partial(f_1, f_2)} = \frac{1}{2\sqrt{m_1 f_1} \cos \beta} \begin{vmatrix} \frac{\partial \alpha}{\partial f_1} & -\sqrt{\frac{m_1}{f_1}} \sin \beta \\ \frac{\partial \alpha}{\partial f_2} & \sqrt{\frac{m_2}{f_2}} \sin \alpha \end{vmatrix}$$

Use Eq. (D17) to obtain

$$\frac{\partial(\alpha, \beta)}{\partial(f_1, f_2)} = \frac{1}{2} \sqrt{\frac{m_2 f_2}{m_1 f_1}} \frac{\sin \alpha}{\cos \beta} \left(\frac{1}{f_2} \frac{\partial \alpha}{\partial f_1} + \frac{1}{f_1} \frac{\partial \alpha}{\partial f_2} \right) \quad (D18)$$

From Fig. 12 and $P'_1 = \sqrt{2 m_1 f_1 E}$,

$$\cos \alpha = \frac{m_1 f_1 - m_2 f_2 - m_3(1 - f_1 - f_2)}{2\sqrt{m_2 m_3 f_2(1 - f_1 - f_2)}} \quad (D19)$$

$$\cos \beta = \frac{m_2 f_2 - m_1 f_1 - m_3(1 - f_1 - f_2)}{2\sqrt{m_1 m_3 f_1(1 - f_1 - f_2)}} \quad (D20)$$

Differentiate Eq. (D19) by f_1 and by f_2 and substitute these results and the expression for $\cos \beta$, Eq. (D20), into Eq. (D18) to obtain Eq. (71),

$$\frac{\partial(\alpha, \beta)}{\partial(f_1, f_2)} = \frac{1}{4f_1 f_2(1 - f_1 - f_2)} \quad (71)$$

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