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**OPTIMAL L^∞ ESTIMATES
FOR SOME GALERKIN METHODS
FOR THE DIRICHLET PROBLEM**

CHARLES I. GOLDSTEIN AND RIDGWAY SCOTT

AUGUST 1977

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ABSTRACT

The goal of this paper is to establish optimal L^∞ error estimates for a few different finite element type methods for the Dirichlet problem in a bounded domain. The methods are selected so as to avoid the necessity of imposing boundary conditions on the trial functions, usually difficult in practice.

Three specific methods are treated. These are the method of interpolated boundary conditions and two methods of Nitsche. The trial spaces, S^h , consist of continuous piecewise polynomials of degree $\leq K-1$ with $K > 2$.

The model problem treated is $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$, $\Omega \subset \mathbb{R}^2$. For each method, an approximate solution, u^h , is constructed for h sufficiently small. The main result of this report states that the following error estimate holds for each method:

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)}.$$

It is the purpose of this paper to establish optimal L^∞ estimates for a few finite element type methods for the Dirichlet problem in a bounded domain. The problem of proving optimal maximum norm estimates for finite element methods has been solved recently by a number of people. We shall mention some of these papers and refer the reader to these references for a more complete list. For the case of linear, elliptic problems on irregular meshes, see [1] - [4]. L^∞ estimates for nonlinear problems are treated in [5] - [7]. Interior L^∞ estimates for linear elliptic problems are treated in [8]. In addition, optimal uniform estimates were proved earlier for one-dimensional problems and higher dimensional problems on a regular mesh. For references to these papers, see the publications cited above.

A common feature of these papers is the requirement that the trial functions must satisfy the essential boundary condition. In practice it is often difficult to construct finite element spaces of trial functions that satisfy the boundary conditions, particularly when the boundaries are curved. For this reason, several methods have been devised and analyzed in order to avoid this requirement. The error estimates for these methods have been obtained in the L^2 norm. For a brief survey of many of these techniques, see [9].

It is our intention to establish optimal L^∞ estimates for three specific methods of this type. The methods we shall treat are the Nitsche method, [10], the Nitsche method with "nearly zero boundary conditions", [11], and the method of interpolated boundary conditions, [12]. In this report, for simplicity of exposition, we shall treat the model problem, $-\Delta u = f$, in a two-dimensional domain, with $u=0$ on the boundary. In a subsequent publication, we shall treat the case of variable coefficients and an N -dimensional domain ($N \geq 2$). (For the interpolated boundary condition method, we are limited at this time to $N=2$, as were the formulation and L^2 estimates in [12]).

We now outline briefly the remainder of this paper. In Section 1, we shall define our notation and prove some preliminary results to be employed later. In Section 2, we shall establish the L^∞ estimates for the two Nitsche methods. In Section 3, we shall prove this estimate for the method of interpolated boundary conditions. The arguments employed in Sections 2 and 3 are extensions of those developed in [1] to treat the Neumann problem in which no boundary terms had to be considered. These arguments are based on obtaining certain Sobolev norm estimates for the difference between the Green's function and its projection with respect to the given Galerkin method. Such estimates are of interest in their own right.

Section 1 - Preliminaries

Let Ω denote a bounded domain in R^N , $N \geq 2$, with smooth boundary, $\partial\Omega$.* We consider the boundary value problem

$$(1.1) \quad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The smoothness of f will be specified implicitly later in terms of u .

We shall employ standard notation for Sobolev spaces. In particular

$$|u|_{W_p^M(\Omega)} = \left(\sum_{|\alpha|=M} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

and

$$\|u\|_{W_p^M(\Omega)} = \left(\sum_{s=0}^M |u|_{W_p^s(\Omega)}^p \right)^{\frac{1}{p}},$$

where M is a nonnegative integer and $1 \leq p < \infty$. Note that $D^\alpha u$ represents the weak derivative of u of order α . For $p = \infty$, set

$$|u|_{W_\infty^M(\Omega)} = \sum_{|\alpha|=M} \|D^\alpha u\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u\|_{W_\infty^M(\Omega)} = \sum_{s=0}^M |u|_{W_\infty^s(\Omega)}. \quad \text{Define}$$

$$W_p^M(\Omega) = \{u \mid \|u\|_{W_p^M(\Omega)} < \infty\} \quad \text{and} \quad H^M(\Omega) = W_2^M(\Omega).$$

$H^M(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the H^M norm. The dual of $H^M(\Omega)$ will be denoted by $H^{-M}(\Omega)$.

*The results of this section will be stated and proved in R^N with $N \geq 2$ unless otherwise specified.

We now describe some general properties of the finite element methods to be discussed in detail in Sections 2 and 3. These methods consist of a finite element subspace, S_K^h (with K an integer greater than 1 and h in $(0, h_0]$, $h_0 > 0$) and a bounded bilinear form, $a(\cdot, \cdot)$, defined on suitably chosen Hilbert spaces. The spaces S_K^h are finite dimensional. In the case of a Neumann boundary condition associated with the operator $-\Delta + 1$, the bilinear form, $a(\cdot, \cdot)$, is usually given by

$$a(u, v) = \int_{\Omega} uv + \int_{\Omega} \nabla u \cdot \nabla v \quad \text{for } u, v \text{ in } H^1(\Omega).$$

The spaces, S_K^h , in this case are usually piecewise polynomials not subject to any boundary conditions. In our case of the Dirichlet boundary condition, either the spaces S_K^h , or the form $a(\cdot, \cdot)$ (or both) are required to be more complicated in order to take account of boundary terms. (From now on, we shall omit the subscript K when referring to S_K^h). We first describe what is meant by a quasi-uniform triangulation, T^h , of Ω . This means that T^h consists of simplices, t^h , such that each t^h contains a sphere of radius $c_1 h$ and is contained in a sphere of radius $c_2 h$, with $0 < c_1 < c_2$ independent of h and t^h . The boundary "simplices" have (possibly) one curved face, whereas all interior faces are flat.

Next, let $a(,)$ denote a bounded bilinear form defined on $H^1(\Omega) \times H^1(\Omega)$ and S^h a finite dimensional subspace of $H^1(\Omega)$ (for each h in $(0, h_0]$). In order for the results of sections 2 and 3 to hold, the following conditions on $a(,)$ and S^h will be required.

(A.1) If $v_1 \in S^h$ and $a(v_1, v) = 0$ for each v in S^h , then $v_1 = 0$.

Using assumption (A.1) and the finite dimensionality of S^h , we now define our finite element approximation, u^h , to the solution of (1.1) by means of the equation

$$(1.2) \quad a(u^h, v) = (f, v) \text{ for each } v \text{ in } S^h,$$

where $(,)$ denotes the L^2 inner product.

We shall also require certain approximation properties of the spaces, S^h . The validity of the following assumptions, as well as other approximation results geared to specific methods will be indicated in the next two sections.

(A.2) Suppose $1 < p \leq \infty$ and $\frac{N}{p} + 1 < s \leq K$ or $p = 1$ and

$N + 1 \leq s \leq K$. Let $u \in H^s$.

Then there exists a u^A in S^h and a constant C such that, for all $t^h \in T^h$,

$$\sum_{j=0}^1 h^{\frac{N+j}{p}} \|u - u^A\|_{w_{\infty}^j(t^h)} + \sum_{j=0}^1 h^j \|u - u^A\|_{w_p^j(t^h)} \leq Ch^s |u|_{w_p^s(t^h)}.*$$

*Throughout this paper, we shall use the same letter C to denote different constants.

In practice, the function v^A is often constructed as the interpolate of v . We shall see examples of this in sections 2 and 3.

(A.3) The triangulation, T^h , is quasi-uniform. Furthermore, the spaces, S^h , consist of continuous functions, v , such that v restricted to t^h is a polynomial of degree $\leq K'$ for some $K' \geq K-1$ and for each $t^h \in T^h$.

The following inverse inequalities are consequences of condition (A.3).

$$(1.3) \quad \|v\|_{W_q^s(t^h)} \leq Ch^{t-s-N(\frac{1}{p}-\frac{1}{q})} \|v\|_{W_p^t(t^h)}, \text{ for all } t^h \in T^h, v \in S^h,$$

where $1 \leq p, q \leq \infty$, and $0 \leq t \leq s \leq K'$. For a proof of (1.3), see, e.g., [1].

Finally, we shall be dealing with finite element methods that yield optimal mean square error estimates in the sense of

(A.4) Suppose that $-1 \leq s \leq K-2$ and $u \in H^2(\Omega)$. Then

$$\|u-u^h\|_{H^{-s}(\Omega)} \leq Ch^{s+2} \|u\|_{H^2(\Omega)}.$$

We shall see that each of the finite element methods dealt with below satisfy (A.1) - (A.4) in addition to certain other conditions. The main result we shall prove for each of these methods is that

$$(1.4) \quad \|u-u^h\|_{L^\infty(\Omega)} \leq Ch^K \text{ for } K \geq 3.$$

Remark: Although the arguments of sections 2-4 may also be applied to the case of piecewise linear elements ($K=2$) to obtain results analogous to those in [1], this is unnecessary. For instance, suppose Ω is replaced by an inscribed polygon, Ω^h , and the trial functions are made to vanish on $\partial\Omega^h$ and in $\Omega - \Omega^h$. Since $\text{dist}(\partial\Omega^h, \partial\Omega) \leq ch^2$, we have

$$\|u\|_{L^\infty(\Omega - \Omega^h)} \leq ch^2 \|\nabla u\|_{L^\infty(\Omega)}.$$

Defining u^h to be the Ritz projection with respect to the trial space, we note that $\|u - u^h\|_{L^\infty(\Omega^h)}$ has been estimated, e.g., in [4].

We thus obtain an estimate for $\|u - u^h\|_{L^\infty(\Omega)}$.

Let G_{x_0} denote the Green's function for (1.1) with singularity at x_0 in Ω , let $G_{x_0}^h$ denote its finite element approximation, and let δ_{x_0} denote the Dirac delta function. We shall show that the proof of (1.4) may be reduced to estimates of the following type:

$$\|G_{x_0} - G_{x_0}^h\|_{W_1^1(\Omega)} \leq Ch,$$

with analogous estimates for boundary terms.

To lay the foundation for the proof of this estimate, we shall state a few basic results concerning G_{x_0} and $G_{x_0}^h$ that will be applied later. Our first result is an extension of Proposition 1 of [1] that is applicable to nonstandard finite element approximations.

Note: None of the results of this section requires condition (A.2) except Lemma 1.3.

Lemma 1.1: Suppose that $0 \leq s \leq K - 2$, $s > \frac{N}{2} - 2$, and conditions (A.1), (A.3) and (A.4) hold. Then there exist functions $\tilde{G}_{x_0}$ in $C^\infty(\Omega) \cap \dot{H}^1(\Omega)$, $G_{x_0}^h$ in S^h , and $\tilde{\delta}_{x_0}$ in $C_0^\infty(\Omega)$, satisfying the following conditions:

- (i) $-\Delta \tilde{G}_{x_0} = \tilde{\delta}_{x_0}$,
- (ii) $a(G_{x_0}^h, v) = (\tilde{\delta}_{x_0}, v) = v(x_0)$ for each v in S^h ,
- (iii) $\max_{x \in \Omega} |D_x^\alpha \tilde{\delta}_{x_0}(x)| \leq C_\alpha h^{-N-|\alpha|}$, where $|\alpha| \geq 0$,

and

- (iv) $\|G_{x_0} - G_{x_0}^h\|_{H^{-s}(\Omega)} \leq Ch^{s+2-\frac{N}{2}}$.

Furthermore the support of $\tilde{\delta}_{x_0}$ is contained in the sphere of radius $C^1 h$ centered at x_0 , and the constants C_α , C^1 and C are independent of h and the point x_0 in Ω .

Note: We shall delete the subscript x_0 from $\tilde{\delta}_{x_0}$, etc. from now on.

Proof: Let t^h be a triangle in T^h containing x_0 . We begin by constructing a function $\tilde{\delta}(x) \in C_0^\infty(t^h)$ and satisfying condition (iii).

Denote by B_ρ the ball of radius ρ centered at the origin. Define a function $\phi(x) \in C_0^\infty(B_1)$ such that $\phi(x) \geq 0$ on B_1 , $\phi(x) > 0$ on $B_{\frac{1}{4}}$ and $\phi(x) = 0$ on $B_1 - B_{\frac{1}{2}}$.

By the quasi-uniformity of T^h , the simplex t^h is contained in a ball of radius $c_2 h$ and contains a ball of radius $c_1 h$, centered at a point x_1 in t^h . Set $\gamma = c_2/c_1 > 1$ and consider the space, $\bar{S}$, of polynomials of degree $\leq K'$ defined on B_γ . Define

$$((P, Q)) \equiv \int_{B_\gamma} P(x) Q(x) \phi(x) dx$$

for all P and Q in $\bar{S}$. Since ϕ is strictly positive on $B_{\frac{1}{4}}$ and non-negative everywhere, this defines an inner-product on $\bar{S}$. Thus there is a $Q_{x_0} \in \bar{S}$ satisfying

$$((Q_{x_0}, P)) = P((x_0 - x_1)/c_1 h) \quad \text{for all } P \in \bar{S}.$$

Thus defining $\tilde{\delta} \in C_0^\infty(t^h)$ by

$$\tilde{\delta}(x) = (c_1 h)^{-N} Q_{x_0}((x - x_1)/c_1 h) \phi((x - x_1)/c_1 h)$$

guarantees the second equality of (ii) will hold.

To verify condition (iii), observe that by the equivalence of norms on $\bar{S}$, for any nonnegative integer m there is a constant c_m such that

$$\|P\|_{W_\infty^m(B_\gamma)} \leq c_m ((P, P))^{\frac{1}{2}} \quad \text{for all } P \in \bar{S}.$$

Thus for any x_0 such that $|x_0 - x_1| \leq c_2 h$,

$$\begin{aligned}
\|Q_{x_0}\|_{W_\infty^m(B_Y)} &\leq c_m ((Q_{x_0}, Q_{x_0}))^{\frac{1}{2}} \\
&= c_m \sup_{0 \neq P \in \bar{S}} ((Q_{x_0}, P)) / ((P, P))^{\frac{1}{2}} \\
&\leq c_0 c_m \sup_{0 \neq P \in \bar{S}} ((Q_{x_0}, P)) / \|P\|_{L^\infty(B_Y)} \\
&= c_0 c_m \sup_{0 \neq P \in \bar{S}} P((x_0 - x_1)/c_1 h) / \|P\|_{L^\infty(B_Y)} \\
&= c_0 c_m.
\end{aligned}$$

Condition (iii) now follows from the definition of $\tilde{\delta}$.

Next, let $\tilde{G}(x)$ denote the unique solution of (1.1) with $f = \tilde{\delta}$. Set $G^h = \tilde{G}^h \epsilon S^h$. Hence, we have $a(G^h, v) = (\tilde{\delta}, v) = v(x_0)$ for each v in S^h . We have thus proved (i) and (ii).

The lemma will now follow once we prove (iv).

Note that

$$\|G - G^h\|_{H^{-s}(\Omega)} \leq \|G - \tilde{G}\|_{H^{-s}(\Omega)} + \|\tilde{G} - G^h\|_{H^{-s}(\Omega)}.$$

It now follows from (A.4), (iii), and elliptic regularity theory that $\|\tilde{G} - G^h\|_{H^{-s}(\Omega)} \leq ch^{s+2} \|\tilde{G}\|_{H^2(\Omega)} \leq ch^{s+2} \|\tilde{\delta}\|_{L^2(\Omega)} \leq ch^{s+2-\frac{N}{2}}$.

Finally, consider $\|G - \tilde{G}\|_{H^{-s}(\Omega)}$. Suppose $\psi \in H^s(\Omega)$ and Ψ satisfies

(1.1) with $f = \psi$. Then it follows from integration by parts, the

boundary conditions on $G, \tilde{G}$, and Ψ and the facts that $G \in H^{2-\frac{N}{2}-\epsilon}(\Omega)$ for each $\epsilon > 0$ and $\tilde{G} \in H^{s+2}(\Omega)$ with $s > \frac{N}{2} - 2$ that

$$(G - \tilde{G}, \Psi) = (G - \tilde{G}, -\Delta \Psi) = (\delta - \tilde{\delta}, \Psi) \equiv E_h(\Psi),$$

where the linear functional E_h is defined by the last equality.

Changing coordinates, we have

$$E_h(\Psi) = E_1(\bar{\Psi})$$

where $\Psi(x) = \bar{\Psi}((x - x_1)/c_1 h)$ and

$$E_1(\bar{\Psi}) = \bar{\Psi}((x_0 - x_1)/c_1 h) - \int_{B_Y} \bar{\Psi}(x) Q_{x_0}(x) \bar{\Phi}(x) dx.$$

From the choice of Q_{x_0} , we have $E_1(\bar{\Psi} + P) = E_1(\bar{\Psi})$ for all $P \in \bar{S}$.

Also, Sobolev's inequality implies that

$$|E_1(\bar{\Psi})| \leq C \|\bar{\Psi}\|_{H^{s+2}(B_Y)}.$$

Thus the Bramble-Hilbert lemma [14] implies that for some constant c ,

$$|E_1(\bar{\Psi})| \leq C |\bar{\Psi}|_{H^{s+2}(B_Y)}.$$

Scaling back, we find

$$|E_h(\Psi)| \leq C h^{s+2-N/2} |\Psi|_{H^{s+2}(\tilde{t}^h)},$$

where $\tilde{t}^h = \{x \in \mathbb{R}^N : |x - x_1| \leq c_2 h\}$.

Since $\tilde{t}^h$ may not lie completely in Ω , we must employ an extension operator $\mathcal{E} : H^{s+2}(\Omega) \rightarrow H^{s+2}(\mathbb{R}^N)$, and thus we obtain, for $\psi \in H^{s+2}(\Omega)$,

$$\begin{aligned} |E_h(\psi)| &= |E_h(\mathcal{E}\psi)| \\ &\leq Ch^{s+2-N/2} \|\mathcal{E}\psi\|_{H^{s+2}(\mathbb{R}^N)} \\ &\leq Ch^{s+2-N/2} \|\psi\|_{H^{s+2}(\Omega)}. \end{aligned}$$

Using the equation above and elliptic regularity, we thus have

$$\begin{aligned} |(G-\tilde{G}, \psi)| &= |E_h(\psi)| \\ &\leq Ch^{s+2-N/2} \|\psi\|_{H^{s+2}(\Omega)} \\ &\leq Ch^{s+2-N/2} \|\psi\|_{H^s(\Omega)} \end{aligned}$$

Therefore $\|G-\tilde{G}\|_{H^{-s}(\Omega)} \leq Ch^{s+2-N/2}$ and the proof of the lemma is complete.

Remark: Analogous results may be proved for general (nonself-adjoint) elliptic operators of order $2m$ and for more general boundary conditions using the same arguments. Note that the only assumption implying a relationship between $a(\cdot)$ and (1.1) is (A.4).

We next state an important estimate concerning the function,

$G_{x_0}(x)$.

$$(1.5) \quad |D_x^\alpha G_{x_0}(x)| \leq \begin{cases} C |\ln|x-x_0||, & \text{for } |\alpha| = 0, N = 2 \\ C |x-x_0|^{2-N-\alpha}, & \text{for } N \geq 3 \text{ or } N = 2, |\alpha| > 0. \end{cases}$$

The constant, C , is independent of x_0 and x in Ω (up to the boundary). For a proof of this result, see [13].

We shall also require the following local estimates for $G - G^h = G_{x_0} - G_{x_0}^h$. For each boundary "simplex" t^h , denote $\partial t^h \cap \partial \Omega$ by $\partial' t^h$.

Lemma 1.2: Suppose $N \geq 2$, t^h is an arbitrary simplex in Ω , and condition (A.3) holds. Then for each h there exists a constant, $\bar{C}$, such that

$$(a) \quad \|G - \bar{C}\|_{L^p(t^h)} \leq Ch^{\frac{N}{p} + 2 - N}, \text{ where } p < N/(N-2) \text{ when } N > 2 \text{ and } p < \infty$$

when $N = 2$, and

$$(b) \quad \oint_{\partial' t^h} |G - \bar{C}|^2 + h^{-1} \int_{t^h} |G - \bar{C}|^2 \leq Ch \text{ for } N = 2$$

for each boundary "simplex", t^h . The constant, C , is independent of x_0, h , and t^h in both (a) and (b).

Proof: (a) Let B_h denote the ball of radius h centered at x_0 . Now consider B_1 and apply the usual arguments used in proving the Bramble-Hilbert lemma, [14], to obtain a constant, $\bar{C}$, such that

$$\|w - \bar{C}\|_{L^p(B_1)} \leq C \|\nabla w\|_{L^{p'}(B_1)}, \text{ where } w \in L^p(B_1), |\nabla w| \in L^{p'}(B_1), \text{ and}$$

$p' < p$ is chosen so that $N/p - N/p' + 1 > 0$. Scaling by h , we find

$$\|w - \bar{C}\|_{L^p(B_h)} \leq Ch^{N/p - N/p' + 1} \|\nabla w\|_{L^{p'}(B_h)}$$

Employing (1.5) and the hypothesis on p , we may choose a p' such that the previous inequality holds, $G_\epsilon L^p(B_h)$ and $\nabla G_\epsilon L^{p'}(B_h)$. Thus since t^h is contained in a ball of radius $c_2 h$, (1.5) yields

$$\begin{aligned} \|G - \bar{C}\|_{L^p(t^h)} &\leq Ch^{N/p - N/p' + 1} \|\nabla G\|_{L^{p'}(B_{c_2 h})} \\ &\leq Ch^{N/p + 2 - N}. \end{aligned}$$

(b) Let t^h denote an arbitrary boundary "simplex". Suppose that the origin of our coordinate system is at the intersection of the straight edges of t^h . Let n denote the outward directed normal to ∂t^h . We now apply Green's theorem to the vector $w^2 \frac{\underline{x}}{h} = (w^2 \frac{x_1}{h}, w^2 \frac{x_2}{h})$

and consequently obtain

$$\begin{aligned} \oint_{\partial t^h} w^2 \frac{\underline{x}}{h} \cdot n &= \oint_{\partial t^h} w^2 \frac{\underline{x}}{h} \cdot n = \int_{t^h} \sum_{i=1}^2 \frac{1}{h} \frac{\partial}{\partial x_i} (x_i w^2) \\ &= \frac{2}{h} \int_{t^h} w^2 + \int_{t^h} \sum_{i=1}^2 \frac{2x_i}{h} w \frac{\partial w}{\partial x_i}. \end{aligned}$$

We see from the quasi-uniformity assumption that the angles of t^h are bounded away from 0 and π . We thus conclude from the smoothness of $\partial \Omega$ that

$$(1.6) \quad \int_{\partial, t^h} w^2 \leq c \left| \oint_{\partial, t^h} w^2 \frac{x}{h} \cdot n \right| \leq c |h^{-1} \int_{t^h} w^2 + \int_{t^h} \sum_{i=1}^2 \frac{x_i}{h} w \frac{\partial w}{\partial x_i} |,$$

where C is independent of t^h .

We now complete the proof of (b) by combining Holder's inequality, part (a), and (1.6) with $w = G - \bar{C}$, $p > 2$, and $\frac{1}{p'} + \frac{1}{p} = 1$ to obtain:

$$\begin{aligned} \int_{\partial, t^h} |G - \bar{C}|^2 &\leq c(h^{-1} \int_{t^h} |G - \bar{C}|^2 + \left| \int_{t^h} \sum_{i=1}^2 \frac{x_i}{h} (G - \bar{C}) \frac{\partial G}{\partial x_i} \right|) \\ &\leq c(h+h)^{\frac{4}{p'}-3} \left(\int_{t^h} |G - \bar{C}|^p \right)^{\frac{2}{p}} + h^{\frac{3-4}{p'}} \left(\int_{t^h} |\nabla G|^{p'} \right)^{\frac{2}{p'}} \\ &\leq c(h+h)^{\frac{4}{p'}-3} \left(h^{\frac{4}{p}} + h^{\frac{3-4}{p'}} \right)^{\frac{2}{p}} \left(h^{\frac{4}{p'}} \right)^{\frac{2}{p'}} \\ &\leq Ch. \quad \text{Q.E.D.} \end{aligned}$$

Corollary 1.1: Suppose $N \geq 2$, t^h is an arbitrary simplex in Ω , and conditions (A.1), (A.3), and (A.4) hold. Then there exists a constant, C , independent of x_0 , h , and t^h such that

$$(a) \quad \int_{t^h} |\nabla(G - G^h)| \leq Ch,$$

$$(b) \oint_{\partial' t^h} |G - G^h| \leq Ch \text{ for each boundary "simplex", } t^h,$$

$$(c) \int_{t^h} r^3 |\nabla(G - G^h)|^2 \leq Ch^3 \text{ for } N = 2, \text{ and}$$

$$(d) \oint_{\partial' t^h} |G - G^h|^2 \leq Ch \text{ for } N = 2 \text{ and each boundary "simplex", } t^h.$$

Proof: (a) First note that by (1.5), we have $\int_{t^h} |\nabla G| \leq Ch$. Now

let $\bar{C}$ be the constant obtained from Lemma 1.2 (a). Using (1.3), Lemma 1.1 and Lemma 1.2 (a) (with $p = 1$), we thus obtain

$$\begin{aligned} \int_{t^h} |\nabla G^h| &= \int_{t^h} |\nabla(G^h - \bar{C})| \leq Ch^{-1} \int_{t^h} |G^h - \bar{C}| \leq Ch^{-1} \cdot (h \|G - G^h\|_{L^2(t^h)} \\ &\quad + \|G - \bar{C}\|_{L^1(t^h)}) \leq Ch. \end{aligned}$$

This proves (a).

(b) We may apply (1.3) and the same arguments as in Lemma 1.2 (b) to obtain the following estimate for a suitable constant, $\bar{C}$.

$$\begin{aligned} \oint_{\partial' t^h} |G - G^h| &\leq \oint_{\partial' t^h} (|G - \bar{C}| + |G^h - \bar{C}|) \\ &\leq C(h^{-1} \int_{t^h} |G - \bar{C}| + \int_{t^h} |\nabla G| \\ &\quad + h^{-1} \int_{t^h} |G^h - \bar{C}|). \end{aligned}$$

Each of these terms was shown in (a) to be bounded by $O(h)$. We have thus proved (b).

(c) We apply (1.5), (1.3), Lemma 1.1, and Lemma 1.2 (a) to see that

$$\begin{aligned}
 \int_{t^h} r^3 |\nabla(G-G^h)|^2 &\leq \int_{t^h} r^3 |\nabla G|^2 + \int_{t^h} r^3 |\nabla G^h|^2 \\
 &\leq ch^3 + ch^3 \int_{t^h} |\nabla(G^h - \bar{C})|^2 \\
 &\leq ch^3 + ch \int_{t^h} |G^h - \bar{C}|^2 \\
 &\leq ch^3 + ch \left(\int_{t^h} |G-G^h|^2 + \int_{t^h} |G - \bar{C}|^2 \right) \\
 &\leq ch^3.
 \end{aligned}$$

(d) Apply Lemma 1.2, (1.3), and Lemma 1.1 to obtain

$$\begin{aligned}
 \oint_{\partial' t^h} |G-G^h|^2 &\leq \oint_{\partial' t^h} |G-\bar{C}|^2 + \oint_{\partial' t^h} |G^h - \bar{C}|^2 \\
 &\leq ch + ch^{-1} \int_{t^h} |G^h - \bar{C}|^2 \\
 &\leq ch + ch^{-1} \left(\int_{t^h} |G-G^h|^2 + \int_{t^h} |G-\bar{C}|^2 \right) \\
 &\leq ch. \quad \text{Q.E.D.}
 \end{aligned}$$

Finally, we show that it suffices to prove estimate (1.4) for the subdomain $\Omega_1 \subset \Omega$, defined by $\Omega_1 = \{x \text{ in } \Omega \mid \text{dist}(x, \partial\Omega) \geq c_0 h\}$ for some fixed, sufficiently small constant c_0 . Thus, Ω_1 is obtained from Ω by removing a "skin" of width $c_0 h$. The following result will simplify somewhat the arguments in the following sections.

Lemma 1.3: Suppose that c_0 is sufficiently small, conditions (A.1)-(A.3) hold, and (1.4) holds for Ω_1 .

Then (1.4) holds for Ω .

Proof: Suppose that x_0 is in a boundary "simplex", t^h , and $\text{dist}(x_0, \partial\Omega) < c_0 h$. It follows from (A.2) that

$$(1.7) \quad |u(x_0) - u^h(x_0)| \leq |u(x_0) - u^A(x_0)| + |u^A(x_0) - u^h(x)| \\ \leq Ch^K + |u^A(x_0) - u^h(x_0)|.$$

Set $t' = t^h \cap \Omega_1$. Since $u^A - u^h \in S^h$, it follows from (A.2), (A.3), and the hypothesis of the lemma that

$$|u^A(x_0) - u^h(x_0)| \leq \max_{x \text{ in } t^h} |u^A(x) - u^h(x)| \\ \leq C \max_{x \text{ in } t'} |u^A(x) - u^h(x)| \leq C \|u^A - u^h\|_{L^\infty(\Omega_1)} \\ \leq C (\|u - u^A\|_{L^\infty(\Omega_1)} + \|u - u^h\|_{L^\infty(\Omega_1)}) \leq Ch^K.$$

Combining this with (1.7), we have thus proved the lemma. Q.E.D.

Section 2 - Two Methods of Nitsche

A. First Method

In this section we shall prove estimate (1.3) for the methods developed in [10] and [11] with $\Omega \subset \mathbb{R}^2$. We begin by describing the first method. The trial space, S^h , will consist of continuous piecewise polynomials of degree $< K$, defined on a quasi-uniform mesh and not subject to any boundary conditions, where $K > 2$. For reasons of technical convenience, we shall assume in sections 2 and 3 that S^h is generated by a nodal basis of Lagrange type which, since the mesh is quasi-uniform, is uniform in the sense of [15], Sect. 3.1. The bilinear form, $a_Y(.,.)$, is defined on $H^1(\Omega) \times H^1(\Omega)$ by the formula

$$(2.1) \quad a_Y(u, v) = \int_{\Omega} \nabla u \cdot \nabla v - \oint_{\partial\Omega} (u_n v + uv_n) + \gamma h^{-1} \oint_{\partial\Omega} uv,$$

where $u, v \in H^1(\Omega)$, $\oint_{\partial\Omega}$ denotes the integral over the boundary of Ω , u_n denotes the normal derivative of u in the direction of the outward normal to Ω , and $\gamma > 0$ is sufficiently large (to be determined later).

Conditions (A.1) and (A.4) of Section 1 follow from the results of [10] and (A.3) is immediate. In addition to (A.2), we have the following approximation result.

$$(2.2) \quad \int_{\Omega} |\nabla(u - u^A)|^2 + \oint_{\partial\Omega} (h |(u - u^A)_n|^2 + h^{-1} |u - u^A|^2) \\ \leq Ch^{2K-2} \|u\|_{H^K(\Omega)}^2 \quad \text{for each } u \text{ in } H^K(\Omega),$$

where the function u^A in S^h is the same as that in (A.2). The

proof of both (A.2) and (2.2) is based on the Bramble-Hilbert Lemma and follows along the same lines as the argument in [15]. The function, u^A , is simply the interpolate, u^I , of u .

Now, let u denote the unique solution of (1.1). Using (A.1), we obtain a unique function u^h in S^h such that

$$(2.3) \quad a_Y(u^h, v) = (f, v) = a_Y(u, v) \text{ for each } v \text{ in } S^h,$$

employing Green's theorem and the definition, (2.1), of a_Y to obtain the last equality. We now state our main result.

Theorem 2.1: Suppose $K \geq 3$ and $u \in W_\infty^K(\Omega)$. Then for Y sufficiently large we have

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)}.$$

The remainder of this subsection will be devoted to proving Theorem 2.1. We first observe that by Lemma 1.3 we may replace Ω by $\Omega_1 = \{x \text{ in } \Omega \mid \text{dist}(x, \partial\Omega) \geq c_0 h\}$. Now suppose x_0 is a fixed point in Ω_1 and $x_0 \in t_0^h$. Define Ω_h^0 to consist of the closure of the union of t_0^h with each of its neighboring triangles in Ω . Let Ω_h denote the complement of Ω_h^0 in Ω . Finally, denote $\partial\Omega_h^0 \cap \partial\Omega(\partial\Omega_h \cap \partial\Omega)$ by $\partial'\Omega_h^0(\partial'\Omega_h)$.

Note: The results of Corollary 1.1 hold with $t^h(\partial't^h)$ replaced by $\Omega_h^0(\partial'\Omega_h^0)$ using the same arguments.

Using (2.1), (2.3), and properties of the Green's function $G_{x_0}(x)$, and its finite element approximation, $G_{x_0}^h(x)$, obtained from Lemma 1.1, we see that

$$(2.4) \quad \begin{cases} u(x_0) - u^h(x_0) = \int_{\Omega} \nabla(u-u^h) \cdot \nabla G_{x_0} - \oint_{\partial\Omega} (u-u^h) \frac{\partial G_{x_0}}{\partial n} \\ = a_Y(u-u^h, G_{x_0}) = a_Y(u-u^h, G_{x_0} - G_{x_0}^h) \\ = a_Y(u-u^A, G_{x_0} - G_{x_0}^h). \end{cases}$$

Again, we shall omit the subscript, x_0 , from G_{x_0} , etc. The following estimate now follows from (2.1), (A.2), and (2.4):

$$(2.5) \quad |u(x_0) - u^h(x_0)| \leq Ch^{K-1} \|u\|_{W_{\infty}^K(\Omega)} \left(\|G-G^h\|_{W_1^1(\Omega)} + \|G-G^h\|_{L^1(\partial\Omega)} + h\|(G-G^h)_n\|_{L^1(\partial\Omega)} \right)$$

In order to simplify the right hand side of (2.5), we shall need the following Lemma.

Lemma 2.1: Suppose $v \in S^h$ and t^h is an arbitrary boundary triangle.

Then

$$\oint_{\partial t^h} |v_n| \leq C_0 h^{-1} \int_{t^h} |\nabla v|,$$

where C_0 is independent of v , h , and t^h .

Proof: $v_n = n \cdot \nabla v$. By (1.3), we have

$$\oint_{\partial t^h} |v_n| \leq Ch \sup_{t^h} |\nabla v| \leq C_0 h^{-1} \int_{t^h} |\nabla v|. \quad \text{Q.E.D.}$$

We now consider the last term on the right side of (2.5).

$$(2.6) \quad \|(G-G^h)_n\|_{L^1(\partial\Omega)} \leq \|(G-G^h)_n\|_{L^1(\partial'\Omega_h^0)} \\ + \|(G-G^h)_n\|_{L^1(\partial'\Omega_h)}$$

Using Lemma 2.1, (1.5), and the fact that $x_0 \in \Omega_1$, we conclude that

$$\int_{\partial'\Omega_h^0} |(G-G^h)_n| \leq \oint_{\partial'\Omega_h^0} |G_n| + \oint_{\partial'\Omega_h^0} |G_n^h| \\ \leq c + c_0 h^{-1} \int_{\Omega_h^0} |\nabla G^h|.$$

Using the argument of Corollary 1.1, we readily see that $\int_{\Omega_h^0} |\nabla G^h| \leq ch$.

Hence, we have

$$(2.7) \quad \oint_{\partial'\Omega_h^0} |(G-G^h)_n| \leq c.$$

We next consider $\oint_{\partial'\Omega_h} |(G-G^h)_n|$. Since the approximation con-

dition, (A.2), holds with $G^A = G^I$ in Ω_h , we deduce the following estimate for each boundary "triangle", $t^h \subset \Omega_h$, using condition (A.2) and (1.5):

$$\begin{aligned} \oint_{\partial, t^h} |(G-G^I)_n| &\leq Ch |G-G^I|_{W_\infty^1(t^h)} \leq Ch \cdot h^{-2} \cdot h^{K-1} |G|_{W_1^K(t^h)} \\ &\leq Ch^{K-2} |G|_{W_1^K(t^h)}. \end{aligned}$$

Summing over all such t^h , we deduce

$$\oint_{\partial, \Omega_h} |(G-G^I)_n| \leq Ch^{K-2} |G|_{W_1^K(\Omega_h)} \leq c.$$

We now combine this inequality with Lemma 2.1 to conclude that

$$\begin{aligned} (2.8) \quad \oint_{\partial, \Omega_h} |(G-G^h)_n| &\leq \oint_{\partial, \Omega_h} |(G-G^I)_n| + \oint_{\partial, \Omega_h} |(G^I-G^h)_n| \\ &\leq c + c_0 h^{-1} \int_{\Omega_h} |\nabla(G^I-G^h)| \\ &\leq c + c_0 h^{-1} \left(\int_{\Omega_h} |\nabla(G-G^h)| + \int_{\Omega_h} |\nabla(G-G^I)| \right). \end{aligned}$$

We now apply Lemma 2.2(d) below to the last term on the right side of (2.8). Employing polar coordinates, (r, θ) (with $r = |x-x_0|$), we readily obtain

$$\begin{aligned} (2.9) \quad h^{-1} \int_{\Omega_h} |\nabla(G-G^I)| &\leq Ch^{-1} \left(\int_{\Omega_h} r^{-3} \right)^{\frac{1}{2}} \cdot \left(\int_{\Omega_h} r^3 |\nabla(G-G^I)|^2 \right)^{\frac{1}{2}} \\ &\leq Ch^{-\frac{3}{2}} \cdot h^{\frac{3}{2}} = c. \end{aligned}$$

Combining (2.5) - (2.9), we conclude that

$$(2.10) \quad |u(x_0) - u^h(x_0)| \leq Ch^{K-1} \|u\|_{W_\infty^K(\Omega)} \left(h + \|G - G^h\|_{W_1^1(\Omega)} + \|G - G^h\|_{L^1(\partial\Omega)} \right)$$

To simplify the right side of (2.10), we first note that $\int_\Omega |G - G^h| \leq C \|G - G^h\|_{L^2(\Omega)} \leq Ch$ by Lemma 1.1(iv). Hence, we are left with estimating $\|\nabla(G - G^h)\|_{L^1(\Omega)} + \|G - G^h\|_{L^1(\partial\Omega)}$.

We now apply Corollary 1.1 to (2.10) and conclude that

$$(2.11) \quad |u(x_0) - u^h(x_0)| \leq Ch^{K-1} \|u\|_{W_\infty^K(\Omega)} \left(h + \|\nabla(G - G^h)\|_{L^1(\Omega_h)} + \|G - G^h\|_{L^1(\partial'\Omega_h)} \right)$$

Set $E = G - G^h$. In view of (2.11), it now suffices to prove

$$(2.12) \quad \int_{\Omega_h} |\nabla E| + \oint_{\partial'\Omega_h} |E| \leq Ch.$$

We first reduce (2.12) to weighted norm estimates as follows.

Note that

$$\int_{\Omega_h} |\nabla E| \leq \left(\int_{\Omega_h} r^{-3} \right)^{\frac{1}{2}} \left(\int_{\Omega_h} r^3 |\nabla E|^2 \right)^{\frac{1}{2}} \leq Ch^{-\frac{1}{2}} \left(\int_{\Omega_h} r^3 |\nabla E|^2 \right)^{\frac{1}{2}}$$

and

$$\oint_{\partial'\Omega_h} |E| \leq \left(\oint_{\partial'\Omega_h} r^{-3} \right)^{\frac{1}{2}} \left(\oint_{\partial'\Omega_h} r^3 |E|^2 \right)^{\frac{1}{2}} \leq Ch^{-1} \left(\oint_{\partial'\Omega_h} r^3 |E|^2 \right)^{\frac{1}{2}}.$$

Estimate (2.12) will thus follow immediately from estimate

$$(2.13) \quad \int_{\Omega} r^3 |\nabla E|^2 + h^{-1} \oint_{\partial\Omega} r^3 |E|^2 \leq Ch^3.$$

We begin our proof of (2.13) by noting the following identity, which is an immediate consequence of the definition, (2.1), of $a_Y(\cdot)$.

$$(2.14) \quad \int_{\Omega} r^3 |\nabla E|^2 + \gamma h^{-1} \oint_{\partial\Omega} r^3 E^2 = a_Y(r^3 E, E) - \int_{\Omega} E \nabla r^3 \cdot \nabla E \\ + \oint_{\partial\Omega} r^3 E E_n + \int_{\partial\Omega} E (r^3 E)_n.$$

The next two estimates are readily obtained, using the fact that $r \geq 3\frac{h}{\kappa}$ on $\partial\Omega$ for some fixed $\kappa > 0$.

$$\left| \int_{\Omega} E \nabla r^3 \cdot \nabla E \right| \leq 3 \int_{\Omega} r^2 |E| |\nabla E| \\ \leq \frac{3}{4} \int_{\Omega} r^3 |\nabla E|^2 + 3 \int_{\Omega} r E^2$$

and

$$\left| \oint_{\partial\Omega} E (r^3 E)_n \right| \leq 3 \oint_{\partial\Omega} r^2 E^2 + \oint_{\partial\Omega} r^3 |E E_n| \\ \leq \kappa h^{-1} \oint_{\partial\Omega} r^3 E^2 + \int_{\partial\Omega} r^3 |E E_n|.$$

Substituting these estimates in (2.14), we obtain

$$\frac{1}{4} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - \kappa) h^{-1} \oint_{\partial\Omega} r^3 E^2 \\ \leq a_Y(r^3 E, E) + 3 \int_{\Omega} r E^2 + 2 \oint_{\partial\Omega} r^3 |E E_n|.$$

Also note that

$$\oint_{\partial\Omega} r^3 |EE_n| \leq \frac{\epsilon}{2} h^{-1} \oint_{\partial\Omega} r^3 E^2 + \frac{h}{2\epsilon} \oint_{\partial\Omega} r^3 E_n^2 \text{ for } \epsilon > 0.$$

Combining the last two estimates, we deduce

$$(2.15) \quad \frac{1}{4} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - \kappa - \epsilon) h^{-1} \oint_{\partial\Omega} r^3 E^2 \leq a_{\gamma}(r^3 E, E) \\ + 3 \int_{\Omega} r E^2 + \frac{h}{\epsilon} \oint_{\partial\Omega} r^3 E_n^2.$$

Before proceeding further, we prove the following lemma:

Lemma 2.2: (a) $h^{-1} \oint_{\partial, \Omega_h} r^3 (G - G^I)^2 \leq Ch^3,$

(b) $h \oint_{\partial, \Omega_h} r^3 (G - G^I)_n^2 \leq Ch^3,$

(c) $\oint_{\partial, \Omega_h} r^2 (G - G^I)^2 \leq Ch^3,$

and

(d) $\int_{\Omega_h} r^3 |\nabla (G - G^I)|^2 \leq Ch^3$

Proof: We shall only give the proof of (a), since the proofs of (b) - (d) are entirely analogous. Consider an arbitrary boundary triangle, $t^h \subset \Omega_h$. Let $r_{\max}$ and $r_{\min}$ denote the maximum and minimum values of r in t^h . It is easily seen that

$$(2.16) \quad r_{\max} \leq C_1 r_{\min},$$

where the constant C_1 is independent of x_0^h , and $t^h \subset \Omega_h$. We now apply (A.2) (with $G^I = G^A$), (1.5), and (2.16) to obtain

$$\begin{aligned}
h^{-1} \oint_{\partial' t^h} r^3 (G-G^I)^2 &\leq Ch^{-1} r^3 \cdot h \|G-G^I\|_{L^\infty(t^h)}^2 \\
&\leq Cr_{\max}^3 h^4 \|G\|_{H^3(t^h)}^2 \leq Cr_{\min}^3 h^4 \|G\|_{H^3(t^h)}^2 \\
&\leq Ch^4 \int_{t^h} r^3 \cdot r^{-6} \cdot r dr d\theta \leq Ch^4 \int_{t^h} r^{-2} dr d\theta.
\end{aligned}$$

Summing over all of these boundary triangles, we deduce

$$h^{-1} \oint_{\partial' \Omega_h} r^3 (G-G^I)^2 \leq Ch^4 \int_{\Omega_h} r^{-2} dr d\theta \leq Ch^3. \quad \text{Q.E.D.}$$

We now return to (2.15) and estimate the last term as follows:

$$(2.17) \quad h \int_{\partial \Omega} r^3 E_n^2 \leq h \oint_{\partial' \Omega_h} r^3 E_n^2 + h \oint_{\partial' \Omega_h} r^3 E_n^2.$$

Combining (1.5), Lemma 2.1, and the argument of Corollary 1.1(c), we readily see that

$$(2.18) \quad h \oint_{\partial' \Omega_h} r^3 E_n^2 \leq Ch^3.$$

To estimate the last term in (2.17), we first note that

$$(2.19) \quad h \oint_{\partial' \Omega_h} r^3 E_n^2 \leq 2h \oint_{\partial' \Omega_h} r^3 |(G-G^I)_n|^2 + 2h \oint_{\partial' \Omega_h} r^3 |(G^I-G^h)_n|^2.$$

Consider the last term and let t^h denote an arbitrary boundary triangle in Ω_h . Apply (2.16) and Lemma 2.1 to obtain

$$\begin{aligned} \oint_{\partial' t^h} r^3 |(G^I - G^h)_n|^2 &\leq r_{\max}^3 \oint_{\partial' t^h} |(G^I - G^h)_n|^2 \\ &\leq c_1^3 r_{\min}^3 c_0 h^{-1} \int_{t^h} |\nabla (G^I - G^h)|^2 \\ &\leq 2c_1^3 c_0 h^{-1} \left(\int_{t^h} r^3 |\nabla (G - G^I)|^2 + \int_{t^h} r^3 |\nabla (G - G^h)|^2 \right). \end{aligned}$$

Summing over all boundary triangles in Ω_h , we see that

$$\begin{aligned} h \oint_{\partial' \Omega_h} r^3 |(G^I - G^h)_n|^2 \\ \leq 2c_1^3 c_0 \left(\int_{\Omega_h} r^3 |\nabla (G - G^I)|^2 + \int_{\Omega_h} r^3 |\nabla E|^2 \right). \end{aligned}$$

Combining the last estimate with Lemma 2.2(b) and (d), (2.15), and (2.17) - (2.19), we deduce

$$\begin{aligned} \left(\frac{1}{4} - \frac{4c_1^3 c_0}{\epsilon} \right) \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - \kappa - \epsilon) h^{-1} \oint_{\partial \Omega} r^3 E^2 \\ \leq Ch^3 + a_Y(r^3 E, E) + 3 \int_{\Omega} r E^2. \end{aligned}$$

Choose $\epsilon = 32C_1^3C_0$, so that $\frac{4C_1^3C_0}{\epsilon} = \frac{1}{8}$. Hence, we have

$$(2.20) \quad \frac{1}{8} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - \kappa - 32C_1^3C_0)h^{-1} \oint_{\partial\Omega} r^3 E^2 \leq Ch^3 + a_Y(r^3 E, E) + 3 \int_{\Omega} r E^2.$$

In order to estimate the last term, we need the following

lemma:

Lemma 2.3: Suppose $\epsilon > 0$. Then

$$\int_{\Omega} r E^2 \leq \epsilon \int_{\Omega} r^3 |\nabla E|^2 + C(\epsilon)h^3.$$

The proof of Lemma 2.3 is the same as that in [1] (Lemma 4)

and is based on Lemma 1.1 and Corollary 1.1. We now combine

Lemma 2.3 (with $\epsilon = \frac{1}{48}$) and estimate (2.20). This yields

$$(2.21) \quad \begin{aligned} & \frac{1}{16} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - \kappa - 32C_1^3C_0)h^{-1} \oint_{\partial\Omega} r^3 E^2 \\ & \leq a_Y(E, r^3 E) + Ch^3 \end{aligned}$$

In order to estimate $a_Y(E, r^3 E)$, we note that by (2.3), we have

$$a_Y(E, r^3 E) = a_Y(E, r^3 E - v) \quad \text{for each } v \text{ in } S^h.$$

We define a suitable function, v in S^h , as follows. Set $v = 0$ at all vertices of T^h contained in Ω_h^0 and set $v = (r^3(G^I - G^h))^I$ at all vertices contained in Ω_h .

By (2.1), we have

$$\begin{aligned}
 (2.22) \quad a_Y(E, r^3 E - v) &= \int_{\Omega_h^0} \nabla E \cdot \nabla(r^3 E - v) - \oint_{\partial \Omega_h^0} E(r^3 E - v)_n \\
 &\quad - \int_{\partial \Omega_h^0} E_n(r^3 E - v) + \gamma h^{-1} \oint_{\partial \Omega_h^0} E(r^3 E - v) + \int_{\Omega_h} \nabla E \cdot \nabla(r^3 E - v) \\
 &\quad - \oint_{\partial \Omega_h} E(r^3 E - v)_n - \oint_{\partial \Omega_h} E_n(r^3 E - v) + \gamma h^{-1} \oint_{\partial \Omega_h} E(r^3 E - v).
 \end{aligned}$$

Consider the first four terms on the right side of (2.22) and note that $v=0$ and $r \leq Ch$ in Ω_h^0 . Employing Corollary 1.1(c) and Lemma 1.1, we see that

$$\begin{aligned}
 (2.23) \quad \left| \int_{\Omega_h^0} \nabla E \cdot \nabla(r^3 E) \right| &\leq \int_{\Omega_h^0} r^3 |\nabla E|^2 + 3 \int_{\Omega_h^0} r^2 |E| |\nabla E| \\
 &\leq Ch^3 + \frac{3}{2} \int_{\Omega_h^0} r^3 |\nabla E|^2 + \frac{3}{2} \int_{\Omega_h^0} r |E|^2 \leq Ch^3 + Ch \int_{\Omega_h^0} |E|^2 \\
 &\leq Ch^3.
 \end{aligned}$$

We now consider the three boundary terms on $\partial \Omega_h^0$ and let t^h denote one of the boundary triangles in Ω_h^0 . It is readily seen that

$$\left| \oint_{\partial t^h} E_n r^3 E \right| \leq C \oint_{\partial t^h} (r^4 E_n^2 + r^2 E^2).$$

It now follows easily from (2.18) and Corollary 1.1(d) that

$$\left| \oint_{\partial t^h} E_n r^3 E \right| \leq Ch^3.$$

Summing over the small group of triangles comprising Ω_h^0 , we conclude that

$$(2.24) \quad |\phi_{\partial, \Omega_h^0} E r^3 E| \leq Ch^3.$$

Note that
$$|\phi_{\partial, \Omega_h^0} E(r^3 E)_n| \leq C(\phi_{\partial, \Omega_h^0} r^2 E^2 + r^4 E_n^2)$$

and
$$|h^{-1} \phi_{\partial, \Omega_h^0} E r^3 E| \leq C \phi_{\partial, \Omega_h^0} r^2 E^2 \quad \text{since } r \leq Ch \text{ in } \Omega_h^0.$$

Hence, we obtain the following estimate in the same way as (2.24).

$$(2.25) \quad |\phi_{\partial, \Omega_h^0} E(r^3 E)_n| + |\gamma h^{-1} \phi_{\partial, \Omega_h^0} E(r^3 E)| \leq Ch^3.$$

Now, we combine (2.22), (2.23), (2.24), and (2.25) to obtain

$$(2.26) \quad |a_Y(E, r^3 E - v)| \leq Ch^3 + \left| \int_{\Omega_h} \nabla E \cdot \nabla (r^3 E - v) \right| \\ + |\phi_{\partial, \Omega_h^0} E(r^3 E - v)_n| + |\phi_{\partial, \Omega_h^0} E_n(r^3 E - v)| + |\gamma h^{-1} \phi_{\partial, \Omega_h^0} E(r^3 E - v)|$$

Substituting (2.26) into (2.21), we are left with the following estimate:

*The number of such triangles is bounded by a constant independent of h since the triangulations are quasi-uniform.

$$\begin{aligned}
(2.27) \quad & \frac{1}{16} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - \kappa - 32C_1^3 C_0) h^{-1} \oint_{\partial\Omega} r^3 E^2 \\
& \leq Ch^3 + \left| \int_{\Omega_h} \nabla E \cdot \nabla (r^3 E - v) \right| + \left| \oint_{\partial', \Omega_h} E (r^3 E - v)_n \right| \\
& \quad + \left| \oint_{\partial', \Omega_h} E (r^3 E - v) \right| + \left| \gamma h^{-1} \oint_{\partial', \Omega_h} E (r^3 E - v) \right|.
\end{aligned}$$

We next proceed to estimate the last four terms in (2.27).

First, let Ω_h^1 denote that subset of Ω_h consisting of all those triangles, t^h , such that each vertex of t^h is contained in Ω_h . Set $\Omega_h^2 = \Omega_h - \Omega_h^1$, $\partial' \Omega_h^1 = \partial\Omega \cap \Omega_h^1$, and $\partial' \Omega_h^2 = \partial\Omega \cap \Omega_h^2$. Note that

$$(2.28) \quad r^3 E - v = r^3 (G - G^I) + (r^3 (G^I - G^h) - (r^3 (G^I - G^h))^I) \quad \text{in } \Omega_h^1.$$

We therefore begin by estimating the following expression.

$$\begin{aligned}
(2.29) \quad & \left\{ \begin{aligned}
& \left| \int_{\Omega_h^1} \nabla (r^3 (G - G^I)) \cdot \nabla E \right| + \left| \gamma h^{-1} \oint_{\partial', \Omega_h^1} r^3 (G - G^I) E \right| \\
& + \left| \oint_{\partial', \Omega_h^1} r^3 (G - G^I) E_n \right| + \left| \oint_{\partial', \Omega_h^1} (r^3 (G - G^I))_n E \right| \\
& \leq C_\epsilon \int_{\Omega_h^1} r^{-3} |\nabla (r^3 (G - G^I))|^2 + \epsilon \int_{\Omega_h} r^3 |\nabla E|^2 + \gamma^2 h^{-1} C_\epsilon \oint_{\partial', \Omega_h^1} r^3 (G - G^I)^2 \\
& + \epsilon h^{-1} \oint_{\partial', \Omega_h} r^3 E^2 + C_\epsilon h^{-1} \oint_{\partial', \Omega_h^1} r^3 (G - G^I)^2 + \epsilon h \oint_{\partial', \Omega_h} r^3 E_n^2 \\
& + C_\epsilon h \oint_{\partial', \Omega_h^1} r^3 (G - G^I)_n^2 + \epsilon h^{-1} \oint_{\partial', \Omega_h} r^3 E^2 \\
& + C_\epsilon \oint_{\partial', \Omega_h^1} r^2 (G - G^I)^2 + \epsilon \oint_{\partial', \Omega_h} r^2 E^2,
\end{aligned} \right.
\end{aligned}$$

where $\epsilon > 0$ is arbitrary.

It may be seen from the same arguments as those employed in the proof of Lemma 2.2 that the 1st, 3rd, 5th, 7th, and 9th terms on the right side are estimated by Ch^3 . The 2nd, 4th, and 8th terms appear on the left side of (2.27) except for the factor of ϵ . The 6th and 10th terms were already treated in the argument leading to (2.20). We may thus make ϵ sufficiently small and combine (2.28) and (2.29) with (2.27) to obtain

$$\begin{aligned}
(2.30) \quad & \frac{1}{K_1} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - K_2 C_0) h^{-1} \oint_{\partial \Omega} r^3 E^2 \\
& \leq Ch^3 + \left| \int_{\Omega_h} \nabla(r^3 \phi - v) \cdot \nabla E \right| + |\gamma h^{-1} \oint_{\partial \Omega_h} (r^3 \phi - v) E| \\
& + |\oint_{\partial \Omega_h} (r^3 \phi - v) E_n| + |\oint_{\partial \Omega_h} (r^3 \phi - v)_n E| + \left| \int_{\Omega_h} \nabla E \cdot \nabla(r^3 E - v) \right| \\
& + |\oint_{\partial \Omega_h} E(r^3 E - v)_n| + |\oint_{\partial \Omega_h} E_n(r^3 E - v)| + |\gamma h^{-1} \oint_{\partial \Omega_h} E(r^3 E - v)|,
\end{aligned}$$

where $\phi = G^I - G^h$, $v = (r^3 \phi)^I$, and K_1, K_2 denote fixed constants independent of γ and C_0 .

Again we choose an arbitrary $\epsilon > 0$ and estimate the first four integrals on the right side of (2.30) by

$$\begin{aligned}
(2.31) \quad & C_\epsilon \int_{\Omega_h} r^{-3} |\nabla(r^3 \phi - v)|^2 + \epsilon \int_{\Omega_h} r^3 |\nabla E|^2 \\
& + C_\epsilon \gamma^2 h^{-1} \oint_{\partial \Omega_h} r^{-3} (r^3 \phi - v)^2 + \epsilon h^{-1} \oint_{\partial \Omega_h} r^3 E^2 + C_\epsilon h^{-1} \int_{\partial \Omega_h} r^{-3} (r^3 \phi - v)^2 \\
& + \epsilon h \oint_{\partial \Omega_h} r^3 E_n^2 + C_\epsilon h \oint_{\partial \Omega_h} r^{-3} (r^3 \phi - v)_n^2 + \epsilon h^{-1} \oint_{\partial \Omega_h} r^3 E^2.
\end{aligned}$$

We note that the 2nd, 4th and 8th terms appear on the left side of (2.30), except for the factor of ϵ . Furthermore, the 6th term has already been dealt with when estimate (2.20) was proved.

We may thus reduce (2.30) to the following estimate by choosing ϵ sufficiently small in (2.31).

$$(2.32) \quad \frac{1}{K_3} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - K_4 C_0) h^{-1} \oint_{\partial\Omega} r^3 E^2$$

$$\leq C \left(h^3 + \int_{\Omega_h^1} r^{-3} |\nabla(r^3 \phi - v)|^2 + h^{-1} \oint_{\partial\Omega_h^1} r^{-3} (r^3 \phi - v)^2 \right.$$

$$\left. + h \oint_{\partial\Omega_h^1} r^{-3} (r^3 \phi - v)_n^2 + \left| \int_{\Omega_h^2} \nabla E \cdot \nabla(r^3 E - v) \right| \right.$$

$$\left. + \left| \oint_{\partial\Omega_h^2} E(r^3 E - v)_n \right| + \left| \oint_{\partial\Omega_h^2} E_n(r^3 E - v) \right| + \left| \gamma h^{-1} \oint_{\partial\Omega_h^2} E(r^3 E - v) \right| \right)$$

where K_3 and K_4 denote fixed constants independent of γ and C_0 .

To estimate the first three integrals on the right, we first prove

Lemma 2.4: (a) $h^{-1} \oint_{\partial\Omega_h^1} r^{-3} (r^3 \phi - v)^2 \leq C \int_{\Omega_h^1} r \phi^2,$

(b) $h \oint_{\partial\Omega_h^1} r^{-3} (r^3 \phi - v)_n^2 \leq C \int_{\Omega_h^1} r \phi^2,$

and

(c) $\int_{\Omega_h^1} r^{-3} |\nabla(r^3 \phi - v)|^2 \leq C \int_{\Omega_h^1} r \phi^2.$

Proof: We shall only prove (a) since the proofs of (b) and (c) are very similar. We consider an arbitrary boundary triangle, $t^h \subset \Omega_h^1$, and employ condition (A.2). Hence, we obtain

$$h^{-1} \oint_{\partial t^h} r^{-3} (r^3 \phi - v)^2 \leq Cr_{\min}^{-3} |r^3 \phi - v|_{L^\infty(t^h)}^2$$

$$\leq Cr_{\min}^{-3} h^{2K} |r^3 \phi|_{W_\infty^K(t^h)}^2.$$

$$\text{But } h^K |r^3 \phi|_{W_\infty^K(t^h)} \leq h^K \sum_{j=1}^K |r^3|_{W_\infty^j(t^h)} |\phi|_{W_\infty^{K-j}(t^h)}$$

since $D^\beta \phi = 0$ when $|\beta| = K$.

Next we observe using (2.16) and (1.3) that

$$|r^3|_{W_\infty^j(t^h)} \leq Cr_{\max}^{3-j} \leq Cr_{\min}^{3-j} \leq Cr_{\min}^2 h^{1-j}$$

and

$$|\phi|_{W_\infty^{K-j}(t^h)} \leq Ch^{j-K-1} \|\phi\|_{L^2(t^h)} \leq Cr_{\min}^{-\frac{1}{2}} h^{j-K-1} \|r^{\frac{1}{2}} \phi\|_{L^2(t^h)}.$$

We thus obtain the estimate:

$$h^K |r^3 \phi|_{W_\infty^K(t^h)} \leq Cr_{\min}^{\frac{3}{2}} \|r^{\frac{1}{2}} \phi\|_{L^2(t^h)}.$$

Hence we conclude that

$$h^{-1} \oint_{\partial t^h} r^{-3} (r^3 \phi - v)^2 \leq C \int_{t^h} r \phi^2.$$

Finally, we sum over all boundary triangles, $t^h \subset \Omega_h^1$, to deduce

$$h^{-1} \oint_{\partial \Omega_h^1} r^{-3} (r^3 \phi - v)^2 \leq C \int_{\Omega_h^1} r \phi^2. \quad \text{Q.E.D.}$$

We now return to (2.32). It follows from Lemma 2.4 and (2.32)

that

$$\begin{aligned}
 (2.33) \quad & \frac{1}{K_3} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - K_4 C_0) h^{-1} \oint_{\partial \Omega} r^3 E^2 \\
 & \leq C(h^3 + \int_{\Omega_h} r \phi^2 + |\int_{\Omega_h} \nabla E \cdot \nabla(r^3 E - v)| \\
 & + |\oint_{\partial, \Omega_h} E(r^3 E - v)_n| + |\oint_{\partial, \Omega_h} E_n(r^3 E - v)| \\
 & + |\gamma h^{-1} \oint_{\partial, \Omega_h} E(r^3 E - v)|
 \end{aligned}$$

$$\text{But } \int_{\Omega_h} r \phi^2 = \int_{\Omega_h} r (G^I - G^h)^2 \leq 2 \left(\int_{\Omega_h} r (G - G^I)^2 + \int_{\Omega_h} r (G - G^h)^2 \right)$$

Using the arguments of Lemma 2.2, we readily obtain

$$(2.34) \quad \int_{\Omega_h} r (G - G^I)^2 \leq Ch^3.$$

Using Lemma 2.3, we see that

$$\int_{\Omega_h} r (G - G^h)^2 \leq \epsilon \int_{\Omega} r^3 |\nabla E|^2 + C_{\epsilon} h^3 \text{ for } \epsilon > 0.$$

We thus choose ϵ sufficiently small and combine the last three estimates with (2.33) to conclude that

$$\begin{aligned}
 (2.35) \quad & \frac{1}{K_5} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - K_6 C_0) h^{-1} \oint_{\partial \Omega} r^3 E^2 \leq Ch^3 + |\int_{\Omega_h} \nabla E \cdot \nabla(r^3 E - v)| \\
 & + |\oint_{\partial, \Omega_h} E(r^3 E - v)_n| + |\oint_{\partial, \Omega_h} E_n(r^3 E - v)| + |\gamma h^{-1} \oint_{\partial, \Omega_h} E(r^3 E - v)|,
 \end{aligned}$$

where K_5 and K_6 are fixed constants, independent of γ and C_0 .

We are thus left with estimating the last four terms in (2.35). We first note that there exist constants C_1 and $C_2 > 0$ such that $C_1 h \leq |x - x_0| \leq C_2 h$ for each x in Ω_h^2 . We shall also require the following two lemmas.

Lemma 2.5: Suppose that $v \in S^h$, $t^h \in T^h$, and P_i , $i = 1, \dots, L$, denotes the nodal points associated with t^h . Then

$$\|v\|_{L^\infty(t^h)} \leq C \max_{i=1, \dots, L} |v(P_i)|,$$

where j denotes a nonnegative integer and C is independent of v , h , and t^h .

Lemma 2.5 follows from the fact that the nodal basis for S^h is uniform. (See [15], Section 3.1). The following lemma was proved in [1] (Lemma 6) using (1.3), the triangle inequality, Lemma 1.1, and (2.34).

Lemma 2.6: $\|G^I - G^h\|_{L^\infty(\Omega_h)} \leq C.$

We now consider

$$\left| \int_{\Omega_h^2} \nabla E \cdot \nabla (r^3 E - v) \right| \leq \left| \int_{\Omega_h^2} \nabla E \cdot \nabla (r^3 E) \right| + \left| \int_{\Omega_h^2} \nabla E \cdot \nabla v \right|.$$

Employing the same argument as in the derivation of (2.23), we obtain

$$\left| \int_{\Omega_h^2} \nabla E \cdot \nabla (r^3 E) \right| \leq C h^3.$$

Hence, we see that

$$(2.36) \quad \left| \int_{\Omega_h^2} \nabla E \cdot \nabla (r^3 E - v) \right| \leq C h^3 + \left| \int_{\Omega_h^2} \nabla E \cdot \nabla v \right|.$$

In a similar manner, we may employ the same arguments as those used in the derivation of (3.24) and (3.25) to deduce

$$(2.37) \quad \begin{aligned} & \left| \oint_{\partial \Omega_h^2} E(r^3 E - v)_n \right| + \left| \oint_{\partial \Omega_h^2} E_n(r^3 E - v) \right| + \left| \gamma h^{-1} \oint_{\partial \Omega_h^2} E(r^3 E - v) \right| \\ & \leq C h^3 + \left| \oint_{\partial \Omega_h^2} E v_n \right| + \left| \oint_{\partial \Omega_h^2} E_n v \right| + \left| \gamma h^{-1} \oint_{\partial \Omega_h^2} E v \right|. \end{aligned}$$

We next prove the following estimate:

$$(2.38) \quad \left| \int_{\Omega_h^2} \nabla E \cdot \nabla v \right| + \left| \oint_{\partial \Omega_h^2} E v_n \right| + \left| \oint_{\partial \Omega_h^2} E_n v \right| + \left| \gamma h^{-1} \oint_{\partial \Omega_h^2} E v \right| \leq C h^3.$$

Let t^h denote an arbitrary triangle contained in Ω_h^2 and consider the first term in (2.38). It follows easily from (1.3), Corollary 1.1(a), Lemma 2.5, Lemma 2.6, and the definition of v that

$$\begin{aligned} \left| \int_{t^h} \nabla E \cdot \nabla v \right| & \leq \|v\|_{W_\infty^1(t^h)} \int_{t^h} |\nabla E| \leq C h \cdot h^{-1} \|r^3 (G^I - G^h)\|_{L^\infty(t^h)} \\ & \leq C h^3 \|G^I - G^h\|_{L^\infty(t^h)} \leq C h^3. \end{aligned}$$

Summing over the small group of triangles comprising Ω_h^2 , we conclude that

$$(2.39) \quad \left| \int_{\Omega_h^2} \nabla E \cdot \nabla v \right| \leq Ch^3.$$

We next observe that

$$\begin{aligned} & \left| \oint_{\partial \Omega_h^2} E_n v \right| + \left| \oint_{\partial \Omega_h^2} E v_n \right| + \left| \gamma h^{-1} \oint_{\partial \Omega_h^2} E v \right| \\ & \leq \frac{1}{2} \left(\oint_{\partial \Omega_h^2} r^4 E_n^2 + \oint_{\partial \Omega_h^2} r^{-4} v^2 + \oint_{\partial \Omega_h^2} r^2 E^2 + \oint_{\partial \Omega_h^2} r^{-2} v_n^2 \right. \\ & \quad \left. + \gamma^2 \oint_{\partial \Omega_h^2} r^2 E^2 + h^{-2} \oint_{\partial \Omega_h^2} r^{-2} v^2 \right) \end{aligned}$$

Applying the same argument as in the proof of (2.18) to the 1st term and Corollary 1.1(d) to the 3rd and 5th terms on the right, we conclude that

$$\begin{aligned} (2.40) \quad & \left| \oint_{\partial \Omega_h^2} E_n v \right| + \left| \oint_{\partial \Omega_h^2} E v_n \right| + \left| \gamma h^{-1} \oint_{\partial \Omega_h^2} E v \right| \leq Ch^3 \\ & + \frac{1}{2} \left(\oint_{\partial \Omega_h^2} r^{-4} v^2 + \oint_{\partial \Omega_h^2} r^{-2} v_n^2 + h^{-2} \oint_{\partial \Omega_h^2} r^{-2} v^2 \right). \end{aligned}$$

Again suppose that $t^h \subset \Omega_h^2$ and apply Lemma 2.5, Lemma 2.6, and the definition of v . We thus obtain

$$\begin{aligned} \oint_{\partial t^h} r^{-4} v^2 &\leq \int_{\partial t^h} r^{-4} \|v\|_{L^\infty}^2 \leq \int_{\partial t^h} r^{-4} \|r^3 (G^I - G^h)\|_{L^\infty(t^h)}^2 \\ &\leq Ch^6 \cdot h^{-3} \|G^I - G^h\|_{L^\infty(t^h)}^2 \leq Ch^3. \end{aligned}$$

Summing over each t^h in Ω_h^2 , we deduce $\int_{\partial \Omega_h^2} r^{-4} v^2 \leq Ch^3$. Applying

an analogous argument to the last two terms in (2.40), we see that

$$(2.41) \quad \left| \oint_{\partial \Omega_h^2} E_n v \right| + \left| \oint_{\partial \Omega_h^2} E v_n \right| + \left| \gamma h^{-1} \oint_{\partial \Omega_h^2} E v \right| \leq Ch^3.$$

Combining (2.39) and (2.41), we thus obtain (2.38). We now combine (2.38) with (2.35) - (2.37) to finally conclude that

$$\frac{1}{K_7} \int_{\Omega} r^3 |\nabla E|^2 + (\gamma - K_8 C_0) h^{-1} \oint r^3 E^2 \leq Ch^3,$$

where K_7 and K_8 are fixed constants independent of C_0 and γ .

Choosing $\gamma \geq K_8 C_0 + \frac{1}{2}$, we have thus proved Theorem 2.1.

B. Second Nitsche Method - Nearly Zero Boundary Conditions

We have just proved that when no boundary conditions are imposed on functions in S^h , then the constant γ appearing in (2.1) must be sufficiently large. An alternative approach, studied in [11], consists of setting $\gamma=0$ in (2.1) while imposing certain "nearly zero" boundary conditions on the functions in S^h . Hence the bilinear form we consider is defined by

$$(2.42) \quad a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v - \oint (u_n v + u v_n) \text{ for each } u, v \text{ in } H^1(\Omega).$$

As before, S^h will consist of piecewise continuous polynomials of degree $\leq K-1$ ($K \geq 2$) defined on a quasi-uniform mesh. However, we shall now require the following additional condition.

$$(A.5) \quad \left(\oint_{\partial t^h} |v|^2 \right)^{\frac{1}{2}} \leq C h^{\alpha} \left(\int_{t^h} |\nabla v|^2 \right)^{\frac{1}{2}}$$

for each boundary triangle, t^h , and each v in S^h , where $\alpha > \frac{1}{2}^*$ and C is independent of h and v .

S^h is again assumed to be generated by a uniform nodal basis of Lagrange type. Specific examples may be found in [11], [12], [16], and [17], where the functions of S^h satisfy the boundary conditions at certain nodal points on $\partial\Omega$. Our main result of this

*Actually, our arguments allow the case $\alpha = \frac{1}{2}$ provided C is sufficiently small.

subsection will be an analogue of Theorem 2.1 with respect to the form $a(\cdot, \cdot)$ given by (2.42) and these trial spaces, S^h . As before, we shall require conditions (A.1) - (A.4).

Condition (A.3) again follows immediately. Condition (A.1) holds provided h is sufficiently small. The proof is analogous to the corresponding proof for the first Nitsche method and may be found in [11]. This enables us to construct a unique solution, u^h in S^h , of the equation

$$(2.43) \quad a(u^h, v) = a(u, v) \quad \text{for each } v \text{ in } S^h,$$

provided h is sufficiently small.

If we now assume the approximation property (A.2) we may also conclude that the optimal mean square estimate (A.4) holds for h sufficiently small by arguments analogous to those employed for the first Nitsche method. The proof may be found in [11]. Finally, we note that the required approximation properties have been proved for the concrete applications of this method that we have cited above.

We are now ready to state and prove the following analogue of Theorem 2.1 for the present situation.

Theorem 2.2: Suppose $K \geq 3$, $u \in W_\infty^K(\Omega)$, and u^h is given by (2.43). Then there exists a positive h_0 sufficiently small such that

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)} \quad \text{for all } h \leq h_0.$$

Since the proof is entirely analogous to that of Theorem 2.1, we shall merely make the following observations. Using conditions (A.1) - (A.4) and the same arguments as before, we may reduce the proof of Theorem 2.2 to proving

$$(2.44) \quad \|G-G^h\|_{W_1^1(\Omega)} + \|G-G^h\|_{L^1(\partial\Omega)} \leq Ch.$$

To treat the second term, note that the following estimate follows from Sobolev's inequality [18] (pp. 128-130):

$$(2.45) \quad |u|_{L^1(\partial\Omega)} \leq C |u|_{W_1^1(\Omega)}.$$

Hence we are left with proving

$$(2.46) \quad \|G-G^h\|_{W_1^1(\Omega)} \leq Ch.$$

In order to prove (2.46), we employ the same reasoning as in subsection A to reduce this to proving

$$(2.47) \quad \int_{\Omega} r^3 |\nabla E|^2 \leq Ch^3.$$

To prove (2.47), we begin with the following equality:

$$(2.48) \quad \int_{\Omega} r^3 |\nabla E|^2 = a(r^3 E, E) - \int_{\Omega} E \nabla r^3 \cdot \nabla E \\ + \oint_{\partial\Omega} r^3 E E_n + \oint_{\Omega} E (r^3 E)_n.$$

The right side of (2.48) may now be estimated in the same way as before, with the following exception. The expression, $h^{-1} \oint_{\partial\Omega} r^3 E^2$, must now be estimated explicitly, since it does not appear on the left side of (2.48) with a coefficient, γ , that may be made sufficiently large (as in the analogous equality (2.14)). To this end, we employ condition (A.5) and proceed as follows.

Note that

$$(2.49) \quad h^{-1} \oint_{\partial\Omega} r^3 E^2 \leq h^{-1} \oint_{\partial' \Omega_h^0} r^3 E^2 + h^{-1} \oint_{\partial' \Omega_h} r^3 E^2.$$

Since $r \leq Ch$ in Ω_h^0 , it follows readily from Corollary 1.1(d) that

$$(2.50) \quad h^{-1} \oint_{\partial' \Omega_h^0} r^3 E^2 \leq Ch^3.$$

We next apply the triangle inequality and Lemma 2.2(a) to obtain

$$(2.51) \quad \begin{aligned} h^{-1} \oint_{\partial' \Omega_h} r^3 E^2 &\leq C(h^{-1} \oint_{\partial' \Omega_h} r^3 (G - G^I)^2 + h^{-1} \oint_{\partial' \Omega_h} r^3 (G^I - G^h)^2) \\ &\leq C(h^3 + h^{-1} \oint_{\partial' \Omega_h} r^3 (G^I - G^h)^2). \end{aligned}$$

Now consider the second term on the right and apply (2.16) and condition (A.5) to an arbitrary boundary triangle $\tau^h \subset \Omega_h$. Hence,

$$\begin{aligned}
h^{-1} \oint_{\partial, t^h} r^3 (G^I - G^h)^2 &\leq Ch^{-1} r_{\max}^3 \oint_{\partial, t^h} (G^I - G^h)^2 \\
&\leq Ch^{-1} r_{\min}^3 h^{2\alpha} \int_{t^h} |\nabla (G^I - G^h)|^2 \\
&\leq Ch^{2\alpha-1} \int_{t^h} r^3 |\nabla (G^I - G^h)|^2 \\
&\leq Ch^{2\alpha-1} \left(\int_{t^h} r^3 |\nabla (G^I - G)|^2 + \int_{t^h} r^3 |\nabla (G - G^h)|^2 \right)
\end{aligned}$$

Summing over all boundary triangles, $t^h \subset \Omega_h$, and applying

Lemma 2.2(d), we deduce

$$\begin{aligned}
h^{-1} \oint_{\partial, \Omega_h} r^3 (G^I - G^h)^2 &\leq Ch^{2\alpha-1} \left(\int_{\Omega_h} r^3 |\nabla (G^I - G)|^2 + \int_{\Omega_h} r^3 |\nabla (G - G^h)|^2 \right) \\
&\leq Ch^{2\alpha-1} (h^3 + \int_{\Omega_h} r^3 |\nabla (G - G^h)|^2).
\end{aligned}$$

Since $2\alpha-1 > 0$, we see that for arbitrary $\epsilon > 0$, we may choose h sufficiently small that we obtain the estimate

$$(2.52) \quad h^{-1} \oint_{\partial, \Omega} r^3 (G^I - G^h)^2 \leq Ch^3 + \epsilon \int_{\Omega} r^3 |\nabla (G - G^h)|^2.$$

Combining (2.49) - (2.52), we conclude that

$$h^{-1} \oint_{\partial \Omega} r^3 E^2 \leq Ch^3 + \epsilon \int_{\Omega} r^3 |\nabla (G - G^h)|^2.$$

The remainder of the proof of Theorem 2.2 now proceeds in the same manner as that of Theorem 2.1.

Section 3 - Interpolated Boundary Conditions

In Section 2A, we showed that no boundary conditions need be imposed on the trial functions provided that the bilinear form is perturbed sufficiently. In this section we shall show that optimal L^∞ error estimates also hold when the bilinear form is left unchanged, provided that a suitable set of interpolation points are chosen on $\partial\Omega$ for the trial functions.

It may be seen from the example in [16] that arbitrarily chosen interpolation points do not suffice to yield optimal L^2 error estimates. However, it was proved in [12] that a proper choice of boundary interpolation nodes do yield optimal L^2 error estimates in R^2 . We now proceed to apply these results to obtain optimal L^∞ error estimates in R^2 .

We begin by describing the trial space, S^h , and the bilinear form, $a(\cdot, \cdot)$. S^h will consist of those continuous functions, v , such that v is a polynomial of degree $\leq K-1$ in each triangle $t \in T^h$, where T^h again denotes a quasi-uniform triangulation of Ω . Furthermore, if t^h is an arbitrary boundary triangle, then v vanishes at the two boundary nodes of t^h as well as the $K-2$ Lobatto quadrature points for $\partial t^h = \partial\Omega \cap t^h$. For the definition of these quadrature points, see [12] as well as the references cited there. The properties of these points are crucial to the proof of Lemmas 3.2 and 3.3 below. The bilinear form we now employ is simply given by

$$(3.1) \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \quad \text{for each } u, v \text{ in } H^1(\Omega).$$

We next consider properties (A.1) - (A.4) of Section 1. Condition (A.1) follows from the estimate, $|a(v, v)| \geq \delta \|v\|_{H^1(\Omega)}^2$, for h sufficiently small and each v in S^h , where $\delta > 0$ is independent of v and h . This was proved in [12]. Condition (A.3) is immediate. The optimal mean square estimate, (A.4), was proved in [12].

Finally, we note that the approximation condition, (A.2), may be proved using the arguments of [12] (the Appendix) with u^A given by the interpolate of u in S^h . Hence $u^A = u$ at each of the interior nodes and boundary vertices as well as at the boundary edge nodes. In addition, we have the following result.

Lemma 3.1: (a) Suppose that $t^h \in T^h$ and $u \in W_{\infty}^1(t^h)$. Then

$$\|u^A\|_{W_{\infty}^1(t^h)} \leq C \|u\|_{W_{\infty}^1(t^h)},$$

(b) Suppose that $u \in W_p^s(\Omega)$, $p > 2$, and $2 \leq s \leq K$. Then

$$\|u^A\|_{W_p^{s-1}(t^h)} \leq C \|u\|_{W_p^{s-1}(t^h)} \quad \text{for each triangle, } t^h \in T^h,$$

$$t^h \cap \partial\Omega \neq \emptyset \quad \sum \|u^A\|_{W_p^{s-1}(t^h)} \leq Ch^{\frac{1}{p}} \|u\|_{W^{s-1+\frac{1}{p}}(\Omega)},$$

and

$$\|u^A\|_{W_p^j(t^h)} \leq Ch^{\frac{2-j}{p}} \|u\|_{L^{\infty}(t^h)}$$

for each triangle t^h , and $j = 0, 1$.

The proof of Lemma 3.1 follows from the same arguments as in [12] and hence will be omitted. Now, let u denote the solution of (1.1). Applying condition (A.1), we immediately obtain a unique solution, u^h in S^h , of the equation

$$(3.2) \quad a(u^h, v) = (f, v) \quad \text{for each } v \text{ in } S^h.$$

The main result of this section is the following theorem.

Theorem 3.1: Suppose $K \geq 3$ and $u \in W_\infty^K(\Omega)$. Then

$$\|u - u^h\| \leq Ch^K \|u\|_{W_\infty^K(\Omega)}.$$

As in Section 2, we may replace Ω by Ω_1 . The proof of Theorem 3.1 will be based on the following two lemmas. First, suppose that Γ is an arbitrary measurable subset of $\partial\Omega$. We define $(W_p^s(\Gamma))^*$ to consist of those functions v such that

$$\|v\|_{(W_p^s(\Gamma))^*} = \sup_{\substack{\phi \in W_p^s(\Gamma) \\ \phi \neq 0}} \frac{|\int_\Gamma v \phi|}{\|\phi\|_{W_p^s(\Gamma)}} < \infty.$$

Lemma 3.2: Suppose that $-1 \leq s \leq K - 2$, $p \geq 1$, $\frac{1}{p'} = 1 - \frac{1}{p}$, and $v \in S^h$. Then

$$\|v\|_{(W_p^s(\partial', t^h))^*} \leq Ch^{s+2-\frac{1}{p'}} \|v\|_{W_{p'}^1(t^h)} \quad \text{for each boundary triangle, } t^h.$$

Lemma 3.3: Suppose $0 \leq s \leq r \leq K - 2$, $p \geq 1$, $\frac{1}{p'} = 1 - \frac{1}{p}$, and

$$u \in W_{p'}^{K-1+\frac{1}{p'}}(\Omega) \cap W_{p'}^{2K-2}(\partial\Omega).$$

Then

$$(a) \quad \|u-u^A\|_{(W_p^r(\partial', t^h))^*} \leq Ch^{K+s} \left(h^{-\frac{1}{p'}} \|u\|_{W_{p'}^{K+s-r-1}(t^h)} + \|u\|_{W_{p'}^{K+s}(\partial', t^h)} \right)$$

for each boundary triangle, t^h , and

$$(b) \quad \|u-u^A\|_{(W_p^r(\partial\Omega))^*} \leq Ch^{K+s} \left(\|u\|_{W_{p'}^{K+s-r-1+\frac{1}{p'}}(\Omega)} + \|u\|_{W_{p'}^{K+s}(\partial\Omega)} \right).$$

The proofs of Lemmas 3.2 and 3.3 follow from the same arguments as those of Lemmas 1 and 6 of [12], respectively (where it was assumed that $p = p' = 2$) and hence need not be repeated here. We now proceed to prove Theorem 3.1. Combining (3.1), (3.2), Lemma 1.1, integration by parts, and properties of the Green's function and the solution, u , of (1.1), we obtain for each fixed point x_0 in Ω_1 :

$$\begin{aligned}
u(x_0) - u^h(x_0) &= a(u - u^h, G) + \oint_{\partial\Omega} u^h G_n \\
&= a(u - u^h, G - G^h) + \oint_{\partial\Omega} (G^h u_n + u^h G_n) \\
&= a(u - u^A, G - G^h) + \oint_{\partial\Omega} (u^A G_n - u^h G_n + G^h u_n + u^h G_n).
\end{aligned}$$

We thus have

$$\begin{aligned}
(3.3) \quad |u(x_0) - u^h(x_0)| &= |a(u - u^A, G - G^h) + \oint_{\partial\Omega} (u^A G_n + G^h u_n)| \\
&\leq |a(u - u^A, G - G^h)| + |\oint_{\partial\Omega} u^A G_n| + |\oint_{\partial\Omega} G^h u_n|.
\end{aligned}$$

In order to estimate the right side of (3.3), recall from Section 2 that Ω_h^0 denotes the closure of the union of the triangle containing x_0 with each of its neighboring triangles and Ω_h denotes the complement of Ω_h^0 in Ω . We now estimate the second term on the right side of (3.3). Since $u=0$ on $\partial\Omega$, $x_0 \in \Omega_1$, and $K > 2$, we apply (1.5), Lemma 3.3 (b) (with $s = r = K - 2$, $p' = \infty$, and $p = 1$) to obtain

$$\begin{aligned}
(3.4) \quad |\oint_{\partial\Omega} u^A G_n| &= |\oint_{\partial\Omega} (u^A - u) G_n| \leq \|u - u^A\|_{(W_1^{K-2}(\partial\Omega))^*} \|G_n\|_{W_1^{K-2}(\partial\Omega)} \\
&\leq Ch^{2-K} \|u - u^A\|_{(W_1^{K-2}(\partial\Omega))^*} \\
&\leq Ch^{2-K} (h^{2K-2} \|u\|_{W_\infty^{K-1}(\Omega)} + h^{2K-2} \|u\|_{W_\infty^{2K-2}(\partial\Omega)}) \\
&\leq Ch^K \|u\|_{W_\infty^{K-1}(\Omega)}.
\end{aligned}$$

We have thus estimated correctly the second term on the right side of (3.3).

We next estimate the last term in (3.3) and note that

$$|\oint_{\partial\Omega} G^h u_n| \leq |\oint_{\partial'\Omega_h^0} G^h u_n| + |\oint_{\partial'\Omega_h} G^h u_n|. \quad \text{Using the triangle in-}$$

equality and the boundary condition for G , we have

$$(3.5) \quad |\oint_{\partial'\Omega_h} G^h u_n| \leq |\oint_{\partial'\Omega_h} u_n (G^h - G^A)| + |\oint_{\partial'\Omega_h} u_n (G^A - G)|.$$

Employing Lemma 3.2 (with $p = \infty$, $p' = 1$, and $s = K - 2$) and summing over all boundary triangles in Ω_h , we conclude that

$$(3.6) \quad |\oint_{\partial'\Omega_h} u_n (G^h - G^A)| \leq Ch^{K-1} \|u_n\|_{W_\infty^{K-2}(\partial\Omega) \cap \partial t^h \cap \partial\Omega \neq \emptyset} \sum_{\partial t^h \cap \partial\Omega \neq \emptyset} \|G^h - G^A\|_{W_1^1(t^h)} \\ \leq Ch^{K-1} \|u\|_{W_\infty^{K-1}(\Omega)} \sum_{\partial t^h \cap \partial\Omega \neq \emptyset} (\|G - G^h\|_{W_1^1(t^h)} + \|G - G^A\|_{W_1^1(t^h)}).$$

Employing Lemma 3.3 (a) (with $p = \infty$, $p' = 1$, and $s = r = K - 2$) and summing over all boundary triangles in Ω_h , we obtain

$$(3.7) \quad |\oint_{\partial'\Omega_h} u_n (G^A - G)| \leq Ch^{2K-2} \|u_n\|_{W_\infty^{K-2}(\partial\Omega) \cap \partial t^h \cap \partial\Omega \neq \emptyset} \left(h^{-1} \|G\|_{W_1^{K-1}(t^h)} \right. \\ \left. + \|G\|_{W_1^{2K-2}(\partial't^h)} \right) \\ \leq Ch^{2K-3} \|u\|_{W_\infty^{K-1}(\Omega)} \sum_{\partial t^h \cap \partial\Omega \neq \emptyset} \|G\|_{W_1^{K-1}(t^h)} \\ \leq Ch^K \|u\|_{W_\infty^{K-1}(\Omega)},$$

where we used (1.5) in the last step. Hence we may combine (3.5) - (3.7) to deduce

$$(3.8) \quad \left| \oint_{\partial' \Omega_h} G^h u_n \right| \leq Ch^K \|u\|_{W_\infty^{K-1}(\Omega)} + Ch^{K-1} \|u\|_{W_\infty^{K-1}(\Omega)} \cdot \left(\begin{array}{c} \|G-G^h\|_{W_1^1(\Omega_h)} \\ + \|G-G^A\|_{W_1^1(\Omega_h)} \end{array} \right)$$

It now follows from (A.2) and (1.5) that $\|G-G^A\|_{W_1^1(\Omega_h)} \leq Ch$,

employing the same arguments as in Section 2. Combining this estimate with (3.8), we see that

$$(3.9) \quad \left| \int_{\partial' \Omega_h} G^h u_n \right| \leq Ch^K \|u\|_{W_\infty^{K-1}(\Omega)} + Ch^{K-1} \|G-G^h\|_{W_1^1(\Omega_h)}$$

We next consider $\left| \oint_{\partial' \Omega_h^0} G^h u_n \right|$ and apply Lemma 3.2 (with $p=\infty$, $p'=1$, and $s=K-2$) to each boundary triangle, $t \in \Omega_h^0$. Hence

$$\begin{aligned} \left| \oint_{\partial' \Omega_h} G^h u_n \right| &\leq Ch^{K-1} \|u_n\|_{W_\infty^{K-2}(\partial\Omega)} \|G^h\|_{W_1^1(t^h)} \\ &\leq Ch^{K-1} \|u\|_{W_\infty^{K-1}(\Omega)} (\|G-G^h\|_{W_1^1(t^h)} + \|G\|_{W_1^1(t^h)}) \end{aligned}$$

Summing over the small group of boundary triangles contained in Ω_h^0 , we see that

$$(3.10) \quad |\phi_{\partial, \Omega_h^0} G^h u_n| \leq Ch^{K-1} \|u\|_{W_\infty^{K-1}(\Omega)} \left(\|G-G^h\|_{W_1^1(\Omega_h^0)} + \|G\|_{W_1^1(\Omega_h^0)} \right)$$

It follows from Corollary 1.1 (a) and Lemma 1.1 that $\|G-G^h\|_{W_1^1(\Omega_h^0)} \leq Ch$.

Furthermore, it follows readily from (1.5) that $\|G\|_{W_1^1(\Omega_h^0)} \leq Ch$.

Combining the last two estimates with (3.10), we conclude that

$$(3.11) \quad |\phi_{\partial, \Omega_h^0} G^h u_n| \leq Ch^K \|u\|_{W_\infty^{K-1}(\Omega)}.$$

Finally, we combine (3.9) and (3.11) to obtain

$$(3.12) \quad |\phi_{\partial \Omega} G^h u_n| \leq Ch^K \|u\|_{W_\infty^{K-1}(\Omega)} + Ch^{K-1} \|G-G^h\|_{W_1^1(\Omega_h^0)}.$$

Before estimating the last term in (3.12), we consider the first term on the right side of (3.3). Employing (A.2) and the definition of $a(\cdot, \cdot)$, we see that

$$|a(u-u^A, G-G^h)| \leq C \|u-u^A\|_{W_\infty^1(\Omega)} \|G-G^h\|_{W_1^1(\Omega)}$$

$$\begin{aligned} &\leq Ch^{K-1} \|u\|_{W_{\infty}^K(\Omega)} \|G-G^h\|_{W_1^1(\Omega)} \\ &\leq Ch^{K-1} \|u\|_{W_{\infty}^K(\Omega)} (\|G-G^h\|_{W_1^1(\Omega_h)} + \|G-G^h\|_{W_1^1(\Omega_h^0)}) \end{aligned}$$

Again combining (1.5), Lemma 1.1 and Corollary 1.1 (a), we see that

$$\|G-G^h\|_{W_1^1(\Omega_h^0)} \leq Ch. \text{ We thus see that both the last term in (3.12)}$$

and the first term in (3.3) will be correctly estimated once we prove:

$$(3.13) \quad \|G-G^h\|_{W_1^1(\Omega_h)} \leq Ch.$$

We shall prove (3.13) in the same manner as in Section 2.

Using Lemma 1.1, it suffices to prove $\|\nabla(G-G^h)\|_{L^1(\Omega_h)} \leq Ch$. Using

the Schwarz inequality, we obtain the estimate:

$$\begin{aligned} \|\nabla(G-G^h)\|_{L^1(\Omega_h)} &\leq \left(\int_{\Omega_h} r^{-3} \right)^{\frac{1}{2}} \left(\int_{\Omega_h} r^3 |\nabla(G-G^h)|^2 \right)^{\frac{1}{2}} \\ &\leq Ch^{-\frac{1}{2}} \left(\int_{\Omega} r^3 |\nabla(G-G^h)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, it suffices to prove

$$(3.14) \quad \int r^3 |\nabla(G-G^h)|^2 \leq Ch^3$$

Set $E = G - G^h$ and note that

$$\int_{\Omega} r^3 |\nabla E|^2 = a(r^3 E, E) - \int_{\Omega} E \nabla r^3 \cdot \nabla E.$$

But $|\int_{\Omega} E \nabla r^3 \cdot \nabla E| \leq 3(\int_{\Omega} r^3 |\nabla E|^2)^{\frac{1}{2}} (\int_{\Omega} r E^2)^{\frac{1}{2}} \leq \frac{3}{4} \int_{\Omega} r^3 |\nabla E|^2 + 3 \int_{\Omega} r E^2.$

Thus

$$\int_{\Omega} r^3 |\nabla E|^2 \leq C(|a(r^3 E, E)| + \int_{\Omega} r E^2).$$

Combining the last estimate with Lemma 2.3, we conclude that

$$(3.15) \quad \int_{\Omega} r^3 |\nabla E|^2 \leq C(|a(r^3 E, E)| + h^3).$$

We next employ (3.2), integration by parts, and the properties of G to obtain

$$(3.16) \quad a(E, r^3 E) = a(E, r^3 E - v) + \oint_{\partial \Omega} \frac{\partial G}{\partial n} v \quad \text{for each } v \text{ in } S^h.$$

We now define v as in Section 2 by setting $v=0$ at all nodes of T^h contained in Ω_h^0 and setting $v = (r^3(G^A - G^h))^A$ at all nodes contained in Ω_h . Consider the last term in (3.16) and define Ω_h^1 , $\partial \Omega_h^1$, Ω_h^2 , and $\partial \Omega_h^2$ in the same way as in Section 2.

Let t^h denote an arbitrary boundary triangle contained in Ω_h^1 . Employing (1.3), (2.16), Lemma 3.1 (a), and Lemma 3.2 (with $p=p'=2$ and $s=0$), we obtain

$$\begin{aligned}
|\oint_{\partial, t^h} \frac{\partial G}{\partial n} v| &\leq (\oint_{\partial, t^h} r^2 |\frac{\partial G}{\partial n}|^2)^{\frac{1}{2}} (\oint_{\partial, t^h} r^{-2} |v|^2)^{\frac{1}{2}} \\
&\leq (\oint_{\partial, t^h} r^2 |\frac{\partial G}{\partial n}|^2)^{\frac{1}{2}} r_{\min}^{-1} h^{\frac{3}{2}} \|r^3 (G^A - G^h)^A\|_{H^1(t^h)} \\
&\leq (\oint_{\partial, t^h} r^2 |\frac{\partial G}{\partial n}|^2)^{\frac{1}{2}} r_{\min}^{-1} h^{\frac{5}{2}} \|r^3 (G^A - G^h)^A\|_{W_{\infty}^1(t^h)} \\
&\leq (\oint_{\partial, t^h} r^2 |\frac{\partial G}{\partial n}|^2)^{\frac{1}{2}} r_{\min}^{-1} h^{\frac{5}{2}} \|r^3 (G^A - G^h)\|_{W_{\infty}^1(t^h)} \\
&\leq (\oint_{\partial, t^h} r^2 |\frac{\partial G}{\partial n}|^2)^{\frac{1}{2}} r_{\min}^{-1} r_{\max}^3 h^{\frac{3}{2}} \|G^A - G^h\|_{H^1(t^h)} \\
&\leq C_{\epsilon} h^3 \oint_{\partial, t^h} r^2 |\frac{\partial G}{\partial n}|^2 + \epsilon r_{\max}^{-2} \|r^3 (G^A - G^h)\|_{H^1(t^h)}^2.
\end{aligned}$$

Summing over all boundary triangles, $t^h \subset \Omega_h^1$, and applying (1.5), we see that

$$\begin{aligned}
|\oint_{\partial, \Omega_h^1} \frac{\partial G}{\partial n} v| &\leq C_{\epsilon} h^3 + \epsilon \int_{\Omega_h^1} r^{-2} \left(\begin{aligned} &r^6 (G^A - G^h)^2 \\ &+ 3r^4 (G^A - G^h)^2 \\ &+ r^6 |\nabla (G^A - G^h)|^2 \end{aligned} \right) \\
&\leq C_{\epsilon} h^3 + \epsilon \left(\int_{\Omega_h^1} Cr^2 (G^A - G^h)^2 + r^4 |\nabla (G^A - G^h)|^2 \right)
\end{aligned}$$

We may now apply Lemma 2.3 and the argument of Lemma 2.2 to conclude as in Section 2 that

$$(3.17) \quad \left| \int_{\partial, \Omega_h^1} \frac{\partial G}{\partial n} v \right| \leq C_{\epsilon} h^3 + \epsilon \int_{\Omega} r^3 |\nabla G|^2$$

for $\epsilon > 0$ arbitrary.

Note that $v=0$ in Ω_h^0 . Hence we are left with considering Ω_h^2 . Let t^h denote an arbitrary boundary triangle in Ω_h^2 . We observe that Lemma 2.5 was proved in [12] (the Appendix) for the space S^h considered here. Applying Lemma 2.5 and the definition of v , we readily obtain

$$(3.18) \quad \|v\|_{L^\infty(\Omega_h^2)} \leq C \|r^3 (G^h - G^A)\|_{L^\infty(\Omega_h^2)}.$$

Combining (3.18), (1.5), and the fact that $r \leq Ch$ in Ω_h^2 , we conclude that

$$(3.19) \quad \left| \oint_{\partial\Omega_h^2} \frac{\partial G}{\partial n} v \right| \leq \|v\|_{L^\infty(\Omega_h^2)} \|G_n\|_{L^1(\partial\Omega_h^2)} \leq C \|r^3 (G^h - G^A)\|_{L^\infty(\Omega_h^2)} \\ \leq Ch^3 \|G^h - G^A\|_{L^\infty(\Omega_h^2)}.$$

We also observe that Lemma 2.6 again holds for the same reasons as in Section 2. Estimate (3.19) and Lemma 2.6 together yield

$$(3.20) \quad \left| \oint_{\partial\Omega_h^2} \frac{\partial G}{\partial n} v \right| \leq Ch^3.$$

Finally, we combine (3.17) and (3.20) to obtain

$$(3.21) \quad \left| \oint_{\partial\Omega} \frac{\partial G}{\partial n} v \right| \leq C_\epsilon h^3 + \epsilon \int_{\Omega} r^3 |\nabla E|^2.$$

To complete the proof of Theorem 3.1, we combine (3.21) with (3.16) and make ϵ sufficiently small. We are thus left with proving $|a(E, r^3 E - v)| \leq Ch^3 + \epsilon \int_{\Omega} r^3 |\nabla E|^2$, again with ϵ sufficiently small. This estimate follows in exactly the same way as in Section 2. The proof of Theorem 3.1 is thus complete.

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