1855

176
12/14/81
(15)

I-339⁽¹⁾

LD. 132

NOVEMBER 1981

PPPL-1855
uc-20g

MASTER

ISLETS OF STABILITY
BEYOND PERIOD DOUBLING

BY

R.S. MACKAY

PLASMA PHYSICS LABORATORY

**PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY**

This work was supported by the U.S. Department of Energy
Contract No. DE-AC02-76-CHO 3073. Reproduction, transla-
tion, publication, use and disposal, in whole or in part,
by or for the United States government is permitted.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

Printed in the United States of America.

Available from:

National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151

Price: Printed Copy \$ * ; Microfiche \$3.50

<u>*PAGES</u>	<u>NTIS Selling Price</u>
1-25	\$5.00
26-50	\$6.50
51-75	\$8.00
76-100	\$9.50
101-125	\$11.00
126-150	\$12.50
151-175	\$14.00
176-200	\$15.50
201-225	\$17.00
226-250	\$18.50
251-275	\$20.00
276-300	\$21.50
301-325	\$23.00
326-350	\$24.50
351-375	\$26.00
376-400	\$27.50
401-425	\$29.00
426-450	\$30.50
451-475	\$32.00
476-500	\$33.50
500-525	\$35.00
526-550	\$36.50
551-575	\$38.00
576-600	\$39.50

For documents over 600
pages, add \$1.50 for each
additional 25 page increment.

ISLETS OF STABILITY BEYOND PERIOD DOUBLING

R.S. MacKay,

Plasma Physics Lab, Princeton University,

P.O.Box 451, Princeton NJ 08544

Period doubling of a periodic orbit of an area preserving map appears to lead to the elimination of all closed invariant curves in the vicinity. We show, however, that there are always islets of stability again beyond the accumulation point of the period doubling sequence.

--:--

A fixed point of an area preserving map T is said to be Moser stable if there are closed invariant curves, topologically equivalent to concentric circles around it, and arbitrarily close to it. These invariant circles trap all points inside them. Thus a Moser stable point is stable. Hyperbolic points are always unstable, but an elliptic point is typically Moser stable, except at multipliers $e^{\pm 2\pi i/3}$ and possibly $\pm i$ [1]. By the multipliers of a periodic orbit we mean the eigenvalues of the linearization of the map around it. When it loses stability by transition to inversion hyperbolic at multipliers -1 , as a parameter is varied, it acquires a neighbourhood in which there are no such circles around it. Closed curves may remain further

out, and if the 2-cycle to which it generically gives birth [2] is elliptic, the period 2 points will possess their own closed curves around them, invariant under T^2 (Fig. 1).

Many periodic orbits of 1-parameter families of area preserving maps possess infinite period doubling sequences, with universal self-similarity properties (see [3,4] and references therein). For example, the intervals in parameter between successive period doublings shorten by a factor converging to a universal number $\delta = 8.721097200...$. In fact there exists a universal 1-parameter map T_p^* such that given a one parameter family, and an n -cycle with a period doubling sequence, $T_p^{2^{2^n}}$ (suitably rescaled) is well approximated by T_p^* , for large n , sufficiently near the accumulation point p^*, x^*, y^* . The trapped area associated with the period doubling sequence shrinks to zero as p approaches p^* , although not monotonically (Fig. 2). In fact it goes down to zero every time the multipliers of the appropriate 2^n -cycle pass through $e^{\pm 2\pi i/3}$ (3rd order resonance), and presumably it has a dense set of jump discontinuities, as outermost curves are destroyed or created. The area decreases by a factor of about 33 each time p comes closer to p^* by a factor δ . This is consistent with the value one should expect of $31/24 = 32.87569642...$, where β, α are scalings in two directions [3], and the factor of 2 comes from doubling the periods.

We conjectured [3,5], as did others [6,8], that beyond p^* , T_p^* has no closed curves invariant under some power (in particular, no stable periodic orbits), and so no trapped area.

As T_p^* asserts itself only locally, a map with a period doubling sequence can in general possess closed invariant curves not associated with the period doubling sequence even beyond the accumulation point (e.g. see [5]), or a periodic orbit may regain stability preceded by a period doubling sequence in reverse. The conjecture would imply, however, that locally there are no closed invariant curves beyond period doubling. It was supported by a search in the quadratic map (in DeVogelaere coordinates):

$$(: x' = f(x) - y \qquad f(x) = px - (1-p)x^2$$

$$y' = x - f(x')$$

for points whose orbits do not escape for some parameter value p . The quadratic map has a simple escape criterion for $p < 1$, viz an orbit is unbounded (in positive or negative time or both) if it has a point (x,y) with $x < -1$. By iterating this criterion one step forward and backward, one sees that it is sufficient to search only a finite area for bounded orbits, viz :

$$|y| \leq 1 + f(x) = (1+x)(1 - (1-p)x) \quad .$$

By further iteration one can reduce the area more. For a grid spacing of .005 in x , y , and p , for a substantial interval of parameter beyond the accumulation point of the period doubling sequence of the fixed point at 0, we found that all points escaped.

The conjecture, however, turns out to be false. Just as

for maps of the interval[7], there are tangent bifurcations beyond period doubling which produce stable periodic orbits. The only difference is that in 1-D maps the only generic bifurcations of periodic orbits are tangent and period doubling, and under the hypothesis of negative Schwarzian derivative, there can be at most one stable periodic orbit at a time. Thus it is clear, for example, that many of the 2^n fixed points of f^n , $f(x) = ax(1-x)$, present at $a = 4$, will be born by tangent bifurcations after the period doubling sequence of the fixed point accumulates (at $a = 3.5700\dots$). In area preserving maps, periodic orbits give birth to others densely while elliptic, so one might have expected to obtain them all without many tangent bifurcations.

The period doubling sequence, however, of the fixed point of the quadratic map accumulates from above at $p^* = -1.26631127692\dots$, and there is a tangent bifurcation at $p_0 = -1.5596292\dots$, producing two 5-cycles (see Appendix). The 5-cycle of positive index is stable up to $p'_0 = -1.5596327\dots$, that is, for an interval of only $\Delta p = 3.5 \times 10^{-6}$, so it is not surprising that it did not show up on the grid. It is reproduced as a stable 5×2^n -cycle (this was checked as far as $n = 9$) from parameters p_n to p'_n , with p_n and p'_n converging to p^* at rate δ , as before, and similar scaling in x (and presumably y). Thus, I claim that they belong to $\mathcal{T}_p^*$, too.

Similarly, there are two 6-cycles (see Appendix) which are not attached to the bifurcation tree of the fixed point. They pop up by tangent bifurcation at $p = -1.584447622\dots$ and the 6-cycle of positive index is stable for an interval of 5.2×10^{-9} .

Fig.4 in [3] indicates that there will be a similar 8-cycle popping up beyond p^* . The least period for which this occurs in the quadratic map is 5, but the universal map even possesses stable fixed points beyond p^* , for example corresponding to the period 3 points above, three factors of δ , α , β away from p^* , x^* , y^* .

Following Karney et al.[8], the appropriate power of a map in the neighbourhood of a tangent bifurcation is well approximated by the quadratic map (though not with our choice of coordinates/parametrization). Thus the whole picture of period doubling followed by islets of stability will be repeated in miniature.

The trapped area beyond p^* presumably also scales like $|3\alpha/2|$ with each factor of δ in parameter, but giving a different universal function from that of Fig. 2. Although it is possible that the set of parameter values beyond p^* for which there is a stable periodic orbit is dense, I expect its measure is small.

Incidentally, Newhouse [10] shows that for the quadratic map, there is a parameter value p_0 such that for $p < p_0$, the only bounded invariant set is a hyperbolic set (of zero area) on which the map is equivalent to a shift of doubly infinite sequences of two symbols (horseshoe). In particular, all 2^n fixed points of $\mathcal{Q}^n$ in $\mathcal{Q}^2$ are real and hyperbolic. The best p_0 I have found, using Newhouse's argument, is $-\sqrt{5} = -2.236$.

ACKNOWLEDGEMENTS

I wish to thank Charles Karney, John Greene, and Sheldon Newhouse for their help, and H.Greenwood for the use of computing facilities at Keele University, England. This work was supported by the U.S. Department of Energy Contract No. DE-AC02-75-CHO-3073 and the U.K. Science Research Council grant B/80/3001.

APPENDIX

The tangent bifurcation producing the stable 5-cycle is symmetric with respect to the symmetry S [3] of the quadratic map:

$$\begin{aligned} S: x' &= x, \\ y' &= -y. \end{aligned}$$

Thus it was located as a tangency of $\mathcal{Q}^2(\text{Fix}(\mathcal{Q}S))$ and $\text{Fix}(S)$ (Fig. 3), where $\text{Fix}(K)$ is the fixed line of symmetry K [3]. The possibility of such a tangent bifurcation was suggested by counting symmetric periodic orbits in $\mathcal{Q}^2$. A point (x_0, y_0) on $\text{Fix}(\mathcal{Q}S)$ is fixed under $\mathcal{Q}^{2n+1}$ if its n th iterate (x_n, y_n) lies on $\text{Fix}(S)$. Thus we require:

$$\begin{aligned} x_0 &= x_n - f(x_n), \\ y_n &= 0. \end{aligned}$$

Using $x_1 = f(x_0) - y_0$,

$$x_k = 2f(x_{k-1}) - x_{k-2}, \text{ and}$$

$$y_n = x_{n-1} - f(x_n),$$

we see that y_n is a polynomial of degree 2^{n+1} in x_0 . So there are 2^{n+1} fixed points in $\mathcal{Q}^2$ or $\mathcal{Q}^{2n+1}$ on $\text{Fix}(\mathcal{Q}S)$. Not all of these have period $2n+1$, however. Removing points with period corresponding to a proper factor of $2n+1$, we obtain table 1 for

the numbers of symmetric periodic orbits of odd period (odd symmetric periodic orbits have precisely one point on each symmetry line of the family $\{Q^k(S)\}$). Two of the symmetric cycles appear at 1/5-bifurcation of the fixed point at the origin, and two more at 2/5-bifurcation. This leaves two more, corresponding to the tangent bifurcation above.

One can do the same analysis for symmetric even periodic orbits, distinguishing between those belonging to S and those of $\bar{S}$. Even symmetric periodic orbits have precisely two points on each symmetry line of one of the half-families $\{L^n: n \text{ odd}\}$, $\{L^n: n \text{ even}\}$, and none on the other). One finds 2^{n+1} fixed points in L^2 of L^{2n} on $\text{Fix}(S)$ and 2^n on $\text{Fix}(\bar{S})$. Hence table 2.

REFERENCES

- [1] J.K.Moser, "Stable and Random Motions," (Princeton University Press, 1973).
- [2] K.R.Meyer, Trans.AMS 149 (1970) 95.
- [3] J.M.Greene, R.S.MacKay, F.Vivaldi, and P.J.Feigenbaum, Physica 3D (1981) 469.
- [4] P.Collet, J-P.Eckmann, and M.Roch, Physica 3D (1981) 487.
- [5] R.S.MacKay, "Period Doubling as a Universal Route to Locally Complete Stochasticity," to appear in "Nonequilibrium Problems in Statistical Mechanics," vol.2, Ann. N.Y.Acad. Sci., L.Reichl, and V.Szechenyi (Wiley, New York, 1981).
- [6] J.B.McLaughlin, J.Stat.Phys. 24 (1981) 375.
- [7] P.Collet and J.P.Eckmann, "Iterated Maps on the Interval as Dynamical Systems" (Birkhäuser, 1980).
- [8] C.F.F.Karney, A.B.Rechester, and R.B.White, "Effect of Noise on the Standard Mapping," Plasma Physics Report PPPL-1782, Princeton (1981), to appear in Physica 4D.

- [9] R.H.G.Helleman, "Self Generated Chaotic behaviour in Nonlinear Mechanics," in "Fundamental Problems in Statistical Mechanics," vol.5, ed. E.G.D.Cohen (N.Holland, 1980) p.165.
- [10] S.E.Newhouse, in "Dynamical Systems," ed. J.Guckenheimer, (Birkhäuser, Boston, 1980) especially p.18.

FIGURE CAPTIONS

Figure 1: Some orbits of the quadratic map for three parameter values, exhibiting period doubling.

Figure 2: Trapped area in the quadratic map against $p - p^*$. Parameter values a_i , $i=1-3$, correspond to the first three period doublings, and b_i , c_i , $i=1,2$, to 4th and 3rd order resonances, respectively.

Figure 3: Tangency of $Q^2(\text{Fix}(QS))$ and $\text{Fix}(S)$ (i.e., $y = 0$).

TABLES

n	F_n	S_n
1	2	2
3	4	2
5	8	6
7	16	14
9	32	28
11	64	62

Table 1: Numbers F_n of fixed points of Q^n lying on $\text{Fix}(QS)$, and S_n of symmetric n -cycles of Q , for n odd.

n	F_n^0	F_n^1	S_n^0	S_n^1
2	4	2	1	0
4	8	4	2	1
6	16	8	5	2
8	32	16	12	6
10	64	32	27	12
12	128	64	54	27

Table 2: Numbers F_n^i of fixed points of Q^n lying on $\text{Fix}(Q^iS)$, and S_n^i of n -cycles of Q belonging to Q^iS , for $i=0,1$ and n even.

* 3170116

Fig. 1

#81T0189

Fig. 2

#81T0188

Fig. 3