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A Study of Coupling Between Neutral- Air Motion and the Ionosphere ✓

Paul A. Bernhardt

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Los Alamos Los Alamos National Laboratory
Los Alamos, New Mexico 87545

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A STUDY OF COUPLING BETWEEN NEUTRAL AIR MOTION AND THE IONOSPHERE

by

Paul A. Bernhardt

ABSTRACT

The coupling between (1) an acoustic wave originating at or below the earth's surface and (2) the ionosphere is described by equations of continuity and motion. The plasma concentration is influenced by collisional and electrostatic forces. Above 130 km altitude, ion-neutral collisions are rare and the plasma tends to be tied to the magnetic field lines. In this region only the magnetic field aligned components of the acoustic disturbance influences the plasma concentration. Below 120 km altitude, ion-neutral collisions dominate over the magnetic field and the plasma responds isotropically to the disturbance. In this lower region, motion of plasma across magnetic field lines produces electric fields and currents. The acoustic wave in the ionosphere may be detected by observations of changes in electron concentration and magnetic field intensity.

I. INTRODUCTION

Pressure waves in the upper atmosphere can produce detectable changes in the plasma concentration of the ionosphere. Sources of such disturbances include polar substorms,¹ large ground-level detonations of high explosives,² volcanic eruptions,³ earthquakes, and large meteoroids.⁴

The following sequence of events results from the sudden release of energy at a point below the earth's surface. The surface rises and falls producing first, a compression wave followed by a rarefaction wave, which propagates radially from the point of origin. As the wave propagates to high altitudes, it is refracted by the atmospheric density gradients. The

velocity perturbation of the wave increases as the atmosphere becomes more rarefied.

At approximately 100-km altitude, the wave enters the lower (E-region) ionosphere. Interaction with the plasma in the region produces electric fields and currents. This interaction can be via conductivity modulation in the E-region or via direct "dynamo" action of the wave. The electric fields can induce plasma drifts in the overlying F-region plasma. This occurs before the acoustic wave enters the F-region. The currents produce fluctuations in the magnetic field, which can be observed on the ground. The disturbance of the neutral velocity caused by the wave couples to the plasma via ion-neutral collisions. This coupling perturbs the concentration of the plasma in the vicinity of the wave. The purpose of this work is to calculate the ionospheric disturbances produced by a propagating pressure wave.

In the first part of this report, perturbations in plasma concentration are derived for acoustic waves propagating in the mid-latitude, upper atmosphere. The coupled equations of continuity and motion are solved numerically, simulating the daytime ionosphere. The magnitude of the perturbation of the electron concentration by an acoustic wave is shown to be dependent on (1) the strength of the wave, (2) its propagation vector relative to the magnetic field direction, and (3) the ratio of the ion-neutral collision frequency to the ion-gyro frequency. An approximate, analytic expression for the coupling of the acoustic wave to the plasma concentration is derived. Good agreement is obtained between the analytical and numerical models. Numerical calculations illustrate the plasma perturbations as a function of time and position.

The second part deals with perturbation of the electric fields in E-region by the dynamo action of the acoustic wave. The neutral air motion couples to the plasma via ion-neutral collisions. Ion motion across magnetic field lines sets up electrostatic fields in the E-region. These fields map up magnetic field lines, perturbing the F-region. Thus, E-region disturbances can produce a precursor in the overlying ionosphere. The effects of the E-region dynamo cause fluctuations in the magnetic field

on the ground. The electric-current density is modified during the passage of the acoustic wave. The integrated effects of fluctuations in current become manifested as magnetic field changes. The magnitude of these changes depend on (1) the strength of the acoustic wave, (2) the orientation of the wave relative to the magnetic field, and (3) the plasma concentration of the E-region. The present modeling only considers the generation of current patterns by the acoustic wave. The modulation of existing current systems via conductivity modification is not included in this study. Numerical calculations of temporal variations in the magnetic flux density on the earth's surface are presented.

The results show that the largest perturbation in the ionosphere is produced by direct action of the wave. The electron concentration changes on the order of 10% at 120-km altitude for a ± 40 m/s neutral velocity perturbation. Above 130-km altitude, the magnitude of underdense electron perturbation is larger than the magnitude of the overdense perturbation. The results also show that smaller disturbances associated with acoustic wave propagation in the ionosphere include electric and magnetic field fluctuations.

II. THEORY

This section presents the theory for construction of models of the ionospheric perturbations. Assuming a flat earth over the localized disturbed region, the coordinate system is $(x,y,z) = (\text{south, east, up})$. A description of the unperturbed atmosphere involves the quantities in Table I.

TABLE I
UNDISTURBED ATMOSPHERE PARAMETER LIST

<u>Parameter</u>	<u>Description</u>
$N_o(z)$	Electron concentration
$\nu(z)$	Ion-neutral collision frequency
$T_e(z)$	Electron temperature
$T_i(z)$	Ion temperature
$\overline{B}(z,y,z)$	Magnetic flux density
$q^o(z)$	Electron production rate
$\ell_o(z)$	Electron loss rate
$\overline{g}(z)$	Gravitational acceleration
$\overline{E}_o(x,y,z)$	Electric potential
$\overline{U}_n(z)$	Neutral wind velocity
$N_{no}(z)$	Neutral concentration
$\overline{J}(x,y,z)$	Current density
$\overline{v}_o(x,y,z)$	Plasma velocity
$\sigma_o(x,y,z)$	Direct conductivity
$\sigma_1(x,y,z)$	Pedersen conductivity
$\sigma_2(x,y,z)$	Hall conductivity

The input to the models is a functional description of the perturbed air motion (Table II).

TABLE II
NEUTRAL AIR DISTURBANCE

<u>Parameter</u>	<u>Description</u>
$N_n'(x,y,z,t)$	Perturbation of neutral concentration
$\overline{U}_n'(x,y,z,t)$	Perturbation of neutral velocity

The output of the models include the quantities given in Table III.

TABLE III

ATMOSPHERIC PERTURBATIONS

<u>Parameter</u>	<u>Description</u>
$N'(x,y,z,t)$	Changes in electron concentration
$\vec{v}'(x,y,z,t)$	Changes in plasma velocity
$\Psi'(x,y,z,t)$	Changes in electric potential
$\vec{E}'(x,y,z,t)$	Changes in electric field
$\vec{J}'(x,y,z,t)$	Changes in current density
$q'(x,y,z,t)$	Changes in electron production
$\ell'(x,y,z,t)$	Changes in electron loss

A. Plasma Concentration

In this section the equation describing the changes in electron concentration produced by a pressure wave is derived. The results of the derivation are summarized by Eq. (19), which is suitable for numerical solution. Further simplification for a spherical propagating disturbance yield an analytic expression (Eq. (24)).

Modeling of the disturbances in the ionosphere involves solving the continuity equation

$$\frac{\partial N_0}{\partial t} + \nabla \cdot (N_0 \vec{v}_0) = q_0 - \ell_0 \quad (1)$$

with the plasma velocities given by the equation of motion

$$m \frac{d\bar{v}}{dt} = mg - \frac{\nabla P}{N_0} + e(\bar{E} + \bar{v} \times \bar{B}) - m\bar{v}(\bar{v} - \bar{U}_n) , \quad (2)$$

where m is particle mass and

$$\frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \nabla \bar{v} .$$

A separate pair of Eqs. (1) and (2) applies to the electrons and to each ion species in the medium.

The coordinate system for the plasma calculation is related to the magnetic field direction (see Fig. 1). The three coordinate vectors are

$$\bar{p} = \cos I \bar{z} - \sin I \bar{x} \quad (3)$$

$$\bar{q} = \sin I \bar{z} + \cos I \bar{x} \quad (4)$$

$$\bar{y} = \bar{y} \quad (5)$$

where $\bar{x}, \bar{y}, \bar{z}$ are the unit vectors in the (x, y, z) system, $\bar{p}$ and $\bar{q}$ are unit vectors in the magnetic meridional plane, $\bar{p}$ is perpendicular to $\bar{B}$, $\bar{q}$ is tangent to $\bar{B}$, $\bar{y}$ is the unit vector perpendicular to the magnetic meridian, and I is the magnetic dip angle. In this study, the magnetic field in the disturbed region is assumed to be composed of straight field lines with constant dip angle. The dip angle is found from

$$\sin I = \frac{2 \cos \theta}{\Delta} \quad (6)$$

$$\cos I = \frac{\sin \theta}{\Delta} ,$$

where $\Delta = (1 + 3 \cos^2 \theta)^{1/2}$ and θ is the geomagnetic co-latitude. The

Fig. 1. Coordinate geometry for ionospheric calculations.

magnetic field coordinates (p, q, y) are found by rotation of the (x, y, z) coordinates

$$p = z \cos I - x \sin I , \quad (7)$$

$$q = z \sin I + x \cos I , \quad \text{and} \quad (8)$$

$$y = y . \quad (9)$$

In this coordinate system, the electron and ion equations of motion are solved assuming that

1. because of collisional dominance, inertial (dv/dt) terms can be neglected;

2. the plasma is quasi-neutral (i.e., $N_e \approx N_i$);
3. the perfect gas law applies (i.e., $P = NkT$);
4. gravitational acceleration of electrons is unimportant;
5. the plasma velocity is equal to the ion velocity ($\bar{v}_i$); and
6. pressure gradients perpendicular to $\mathbf{B}$ can be neglected.

With these assumptions, manipulation of Eq. (2) yields

$$v_{op} = \frac{1}{1 + \rho^2} \left[\rho^2 U_{np} + \rho \left(U_{ny} + \frac{E_{op}}{B} + \frac{g_p}{\Omega} \right) + \frac{E_{oy}}{B} \right] \quad (10)$$

$$v_{oq} = U_{nq} + D_a \left[\frac{1}{H_a} - \frac{1}{N_o} \frac{\partial N_o}{\partial q} - \frac{1}{T_p} \frac{\partial T_p}{\partial q} \right], \text{ and} \quad (11)$$

$$v_{oy} = \frac{1}{1 + \rho^2} \left[\rho^2 U_{ny} + \rho \left(-U_{np} + \frac{E_{oy}}{B} \right) - \frac{g_p}{\Omega} - \frac{E_{op}}{B} \right] \quad (12)$$

where $\rho = \nu/\Omega$, $\Omega = eB/m_i$ is the ion-gyro frequency, ν is the ion-neutral collision frequency, $g_p = -g \cos I$, $g_q = g \sin I$, $D_a = kT_p/m_i \nu$ is the ambipolar diffusion coefficient, $H_a = kT_p/m_i g_q$ is the scale height, and $T_p = T_e + T_i$ is the plasma temperature. Above 120 km, $\rho < 0.1$. The perturbed plasma velocity is, to the first order,

$$v'_p = \frac{1}{1 + \rho^2} \left[\rho^2 U'_{np} + \rho \left(U'_{ny} + \frac{E'_p}{B} \right) + \frac{E'_y}{B} \right], \quad (13)$$

$$v'_q = U'_{nq} + D_a \left[-\frac{1}{N_o} \frac{\partial N'}{\partial q} + \frac{N'}{N_o^2} \frac{\partial N_o}{\partial q} \right] - \frac{N'_n}{N_{no}} \left[v_{oq} - U_{nq} \right], \quad (14)$$

$$\mathbf{v}'_y = \frac{1}{1 + \rho^2} \left[\rho^2 U'_{ny} + \rho \left(-U'_{np} + \frac{E'_y}{B} \right) - \frac{E'_p}{B} \right] . \quad (15)$$

The second-order term ρ' has been dropped, but could be added if necessary in future studies. These equations are identical to those used by Davis,⁵ except that the electric field $(\vec{E} \times \vec{B})$ drifts and the perpendicular velocity components (U'_{np} and U'_{ny}) have been added.

The ionization production (q_0) is assumed to be unaffected by the air disturbance. The ionization loss in the unperturbed atmosphere is approximately⁵

$$\ell_0 = \frac{\alpha_0 \beta_0 N_0^2}{\alpha_0 N_0 + \beta_0} \quad (16)$$

and the perturbed loss is given by

$$\ell' = \ell_0 \left[\frac{N'}{N_0} \left(\frac{\gamma_0 N_0 + 2\beta_0}{\alpha_0 N_0 + \beta_0} \right) + \frac{N'_n}{N_{no}} \left(\frac{\alpha_0 N_0}{\alpha_0 N_0 + \beta_0} \right) \right] \quad (17)$$

where $\alpha_0 \approx 1.1 \times 10^{-13} \text{ m}^3/\text{s}$ is the effective recombination coefficient below 140 km, $\beta_0 = \gamma[N_2]$ is the recombination rate above 180 km, and $\gamma \approx 1.4 \times 10^{-18} \text{ m}^3/\text{s}$. $N'_n/N_{no} = U'_n/C_{no}$ where C_{no} is the atmospheric speed of sound.

The continuity equation for the ionospheric perturbation is

$$\frac{\partial N'}{\partial t} + \nabla \cdot (N_0 \vec{v}' + N' \vec{v}) = q' - \ell' . \quad (18)$$

Substituting Eqs. (10) through (17) into (18) yields

$$\frac{\partial N'}{\partial t} + \frac{\partial N'}{\partial p} v_{op} + \frac{\partial N'}{\partial y} v_{oy} = D_a \frac{\partial^2 N'}{\partial q^2} + A \frac{\partial N'}{\partial q} + B N' + C, \quad (19)$$

where

$$A = \frac{\partial D_a}{\partial q} - \left[U_{nq} + D_a \left(\frac{1}{H_a} - \frac{1}{T_p} \frac{\partial T_p}{\partial q} \right) \right]$$

$$B = \frac{-\alpha_o \beta_o N_o (\alpha_o N_o + 2\beta_o)}{(\alpha_o N_o + \beta_o)^2} - \frac{\partial}{\partial q} \left[U_{nq} + D_a \left(\frac{1}{H_a} - \frac{1}{T_p} \frac{\partial T_p}{\partial q} \right) \right]$$

and

$$C = \frac{\partial}{\partial q} \left[-N_o (U'_{nq} - \frac{N'_n}{N_{nc}} \{ v_{oq} - U_{nq} \}) \right] - \frac{N'_n}{N_{no}} \frac{g_o^2}{\beta_o N_o} - \frac{\partial [N_o v'_p]}{\partial p} - N_o \frac{\partial v'_y}{\partial y}.$$

A transformation is used to remove the terms $\partial N'/\partial p$ and $\partial N'/\partial \phi$. Above 150 km (where $\rho \ll 1$), Eqs. (10) and (12) simplify to

$$v_{op} = E_{oy}/B \quad (20)$$

and

$$v_{oy} = \frac{-g_p}{\Omega} - \frac{E_{op}}{B}. \quad (21)$$

In the F-region, the electric potential is assumed constant along magnetic-

field lines because of the essentially infinite parallel conductivity. Consequently, the $\overline{ExB}$ plasma velocity is independent of coordinate q . In the local region of the disturbance, $-g_p/\Omega$ is nearly constant. Thus, the perpendicular velocities, v_{op} and v_{oy} , are functions of p and y only.

Consider the transformation

$$t = t^* , \quad \frac{dp}{dt} = v_{op}(p,y), \text{ and } \frac{dy}{dt} = v_{oy}(p,y) \quad (22)$$

with initial conditions $p = p^*$ and $y = y^*$ when $t = t_0$. These equations describe the motion of magnetic tubes of plasma under the influence of electric fields. In the initial modeling studies, E_p and E_y will be taken to be constant, yielding from Eq. (22)

$$\begin{aligned} t &= t^* \\ p &= p^* + v_{op}(t - t_0) \\ y &= y^* + v_{oy}(t - t_0) . \end{aligned} \quad (23)$$

Equation (23) is the characteristic of the motion of a drifting plasma tube. With this transformation, the left side of Eq. (19) reduces to $\partial N'/\partial t^*$.

An analytic approximation to the solution of Eq. (19) is derived in Appendix A for a spherical disturbance centered at $(x,y,z) = [0, 0, h_c(x)]$, here $h_c(x)$ is a time dependent altitude. This solution is given as

$$N'(P,y,q,t) = N_0 \left\{ \left[\left(\frac{\partial r}{\partial q} \right)^2 + \rho^2 \right] / (1 + \rho^2) \right\} \frac{U'_n}{-\partial r_0 / \partial t} , \quad (24)$$

where $r = \sqrt{x^2 + y^2 + [z - h_c(t)]^2}$ is radius, r_0 is the radius to the location of the disturbance and U'_n is the perturbed neutral velocity.

Equation (24) has the following physical interpretation. The term $\partial r / \partial q$ is the projection of the unit vector in the direction of the velocity perturbation along the magnetic field lines. The term $-\partial r_0 / \partial t$ is the propagation velocity of the disturbance. Thus, the coupling factor between neutral velocity and electron density perturbations is $[(\partial r / \partial q)^2 + \rho^2] / (1 + \rho^2)$. In regions where ion-neutral collisions dominate (i.e., $\rho \gg 1$), plasma responds isotropically to the disturbance. In regions where the ions are tied to the magnetic field (i.e., $\rho \ll 1$), only the field-aligned component of the velocity perturbation affects the plasma concentrations.

B. E-Region Electric Fields

In Section B, a number of equations representing the electrostatic potential produced by the pressure wave are derived. The form of the equations differs according to the assumptions made in the derivation. Equation (37) describes the two-dimensional potential in a horizontally stratified E-region where the scale length of the disturbance is much larger than the E-layer thickness. The three-dimensional potential in the rectangular (x, y, z) coordinate system of the ionosphere is given by Eq. (40). Equations (B10) and (B11) describe the potential in a spherical coordinate (r, θ, ϕ) system centered at the pressure wave (see Appendix B). Equations (63) and (64) present the potential in the rectangular coordinate system of the magnetic field. Numerical methods are required to solve all of these equations.

The electric field perturbation ($\bar{E}$) is produced by dynamo action in the E-region. If the neutral wind disturbance is nearly constant throughout the region, the solution for the electric fields is two-dimensional and the E-region can be treated as a layer with thickness Δz_E , terminating at lower boundary z_E . The currents in the E-region obey the charge conservation law

$$\nabla \cdot \vec{J} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0 . \quad (25)$$

Integrated quantities will be used in the E-region:

$$\bar{J} = \int_{z_E}^{z_E + \Delta z_E} \vec{J} dz . \quad (26)$$

From Eq. (25),

$$-j_z = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} . \quad (27)$$

The E-region horizontal currents are related to the total electric fields by the equations

$$J_x = \Sigma_{xx} E_{tx} + \Sigma_{xy} E_{ty} \quad (28)$$

$$J_y = -\Sigma_{xy} E_{tx} + \Sigma_{yy} E_{ty}$$

where $\Sigma_{xx} = (\Sigma_0 \Sigma_1)/A$, $\Sigma_{xy} = (\Sigma_0 \Sigma_2 \sin I)/A$, $\Sigma_{yy} = (\Sigma_0 \Sigma_1 \sin^2 I + \Sigma_3 \Sigma_1 \cos^2 I)/A$, $A = \Sigma_0 \sin^2 I + \Sigma_1 \cos^2 I$, $\Sigma_3 = (\Sigma_1 + \Sigma_2)/\Sigma_1$ and Σ_0 , Σ_1 , and Σ_2 are the height-integrated direct, Pedersen, and Hall conductivities, respectively.

The total electric field is the sum of the electrostatic and induced fields

$$\vec{E}_t = \vec{E}_E + \vec{U}_E \times \vec{B} , \quad (29)$$

where $\vec{E}_E$ and $\vec{U}_E$ are the electric field and neutral wind vectors in the E-region. The individual electric field components are

$$E_{tx} = E_{Ex} + U_{Ey} (-\sin I) B , \quad (30)$$

and

$$E_{ty} = E_{Ey} + [U_{Ez}(-\cos I) - U_{Ex}(-\sin I)] B . \quad (31)$$

The electrostatic fields map up the magnetic field lines into the F-region.

$\vec{E}_E$ may be written as the gradient of a potential

$$\vec{E}_E = -\nabla \psi_0 . \quad (32)$$

The fields in the F-region are

$$E_{Op} = - \frac{\partial \psi_0}{\partial p} . \quad (33)$$

and

$$E_{O\phi} = - \frac{\partial \psi_0}{\partial \phi} . \quad (34)$$

Substituting Eqs. (28), (30), (31), and (32) into (27) and assuming that Σ_{xx} , Σ_{xy} , Σ_{yy} , $B \sin I$, and $B \cos I$ are constant in x and y yields the equation for the electric potential

$$\begin{aligned} \Sigma_{xx} \frac{\partial^2 \psi_0}{\partial x^2} + \Sigma_{yy} \frac{\partial^2 \psi_0}{\partial y^2} = j_z + B \sin I \left[(\Sigma_{xy} + \Sigma_{yy}) \frac{\partial U_{Ex}}{\partial x} + (\Sigma_{xy} - \Sigma_{xx}) \frac{\partial U_{Ey}}{\partial y} \right] \\ - B \cos I (\Sigma_{xy} + \Sigma_{yy}) \frac{\partial U_{Ex}}{\partial z} . \end{aligned} \quad (35)$$

The perturbed electric potential can be found from

$$\begin{aligned} \Sigma_{xx} \frac{\partial^2 \Psi'}{\partial x^2} + \Sigma_{yy} \frac{\partial^2 \Psi'}{\partial y^2} + E_{xx}' \frac{\partial^2 \Psi_0}{\partial x^2} + \Sigma_{yy}' \frac{\partial^2 \Psi_0}{\partial y^2} = j_z' \\ + B \sin I \left[(\Sigma_{xy} + \Sigma_{yy}) \frac{\partial U_{Ex}'}{\partial x} + (\Sigma_{xy}' + \Sigma_{yy}') \frac{\partial U_{Ex}}{\partial x} + (\Sigma_{xy} - \Sigma_{xx}) \frac{\partial U_{Ey}'}{\partial y} \right. \\ \left. + (\Sigma_{xy}' - \Sigma_{xx}') \frac{\partial U_{Ey}}{\partial y} \right] + B \cos I \left[(\Sigma_{xy} + \Sigma_{yy}) \frac{\partial U_{Ez}'}{\partial z} + (\Sigma_{xy}' + \Sigma_{yy}') \frac{\partial U_{Ez}}{\partial z} \right] \quad (36) \end{aligned}$$

If we assume that the unperturbed electric fields are zero (i.e., neutral winds are uniform), this equation reduces to

$$\begin{aligned} \Sigma_{xx} \frac{\partial^2 \Psi'}{\partial x^2} + \Sigma_{yy} \frac{\partial^2 \Psi'}{\partial y^2} = j_z' + B \sin I \left[(\Sigma_{xy} + \Sigma_{yy}) \frac{\partial U_{Ex}'}{\partial x} + (\Sigma_{xy} - \Sigma_{xx}) \frac{\partial U_{Ey}'}{\partial y} \right] \\ - B \cos I (\Sigma_{xy} + \Sigma_{yy}) \frac{\partial U_{Ez}'}{\partial z} . \quad (37) \end{aligned}$$

The perturbed electric fields are $E_p' = -\partial \Psi' / \partial p$ and $E_\phi' = -\partial \Psi' / \partial \phi$.

If the neutral wind perturbation has significant altitude variation within the E-region, a three-dimensional solution may be employed. This is probably the case for pressure waves that are produced by ground-based detonations. The scale size of these waves is ~10 km, whereas the E-region thickness is ~50 km. The derivation of the three-dimensional electric field equations is similar to the derivation for the two-dimensional case.

The currents are related to the electrostatic fields and neutral winds by a generalized ohms law

$$\vec{j} = \vec{\sigma} \cdot (\vec{E}_s + \vec{u}_n \times \vec{B}) , \quad (38)$$

where $\vec{j} = (j_x, j_y, j_z)$, $\vec{E}_s = (-\partial\psi_0/\partial x, \partial\psi_c/\partial y, \partial\psi_0/\partial z)$, and

$$\bar{\sigma} = \begin{bmatrix} \sigma_{aa} & \sigma_{ab} & \sigma_{ac} \\ \sigma_{ba} & \sigma_{bb} & \sigma_{bc} \\ \sigma_{ca} & \sigma_{cb} & \sigma_{cc} \end{bmatrix}$$

is the conductivity tensor. The components of $\bar{\sigma}$ are

$$\sigma_{aa} = \sigma_1 \sin^2 I + \sigma_0 \cos^2 I ,$$

$$\sigma_{ab} = -\sigma_{ba} = \sigma_2 \sin I ,$$

$$\sigma_{ac} = +\sigma_{ca} = (\sigma_0 - \sigma_1) \sin I \cos I ,$$

$$\sigma_{bb} = \sigma_1 ,$$

$$\sigma_{bc} = -\sigma_{cb} = \sigma_2 \cos I , \text{ and}$$

$$\sigma_{cc} = \sigma_1 \cos^2 I + \sigma_0 \sin^2 I ,$$

where

$$\sigma_0 = \frac{N_0 e}{B} \left(\frac{1}{\rho_e} + \frac{1}{\rho_I} \right) \text{ is the direct conductivity,}$$

$$\sigma_1 = \frac{N_0 e}{B} \left(\frac{\rho_e}{1 + \rho_e^2} + \frac{\rho_I}{1 + \rho_I^2} \right) \text{ is the Pedersen conductivity,}$$

$$\sigma_2 = \frac{N_0 e}{B} \left(\frac{1}{1 + \rho_e^2} - \frac{1}{1 + \rho_I^2} \right) \text{ is the Hall conductivity,}$$

$$\rho_I = \rho = \frac{v}{\Omega} , \rho_e = \frac{v_e}{\Omega_e} , \Omega_e = \frac{eB}{m_e} \text{ is the electron gyrofrequency,}$$

and ν_e is the electron-neutron collision frequency. Using $\nabla \cdot \vec{j} = 0$, the equation for the potential is found to be

$$\begin{aligned} & \sigma_{aa} \frac{\partial^2 \psi_o}{\partial x^2} + \sigma_{bb} \frac{\partial^2 \psi_o}{\partial y^2} + \sigma_{cc} \frac{\partial^2 \psi_o}{\partial z^2} + 2\sigma_{ac} \frac{\partial^2 \psi_o}{\partial x \partial z} - \frac{\partial \sigma_{ac}}{\partial z} \frac{\partial \psi_o}{\partial x} + \frac{\partial \sigma_{bc}}{\partial z} \frac{\partial \psi_o}{\partial y} - \frac{\partial \sigma_{cc}}{\partial z} \frac{\partial \psi_o}{\partial z} \\ & = B_z \left(-\sigma_{ab} \frac{\partial U_{nx}}{\partial x} - \sigma_{bb} \frac{\partial U_{nx}}{\partial y} + \sigma_{bc} \frac{\partial U_{nx}}{\partial z} + \sigma_{aa} \frac{\partial U_{ny}}{\partial x} - \sigma_{ab} \frac{\partial U_{ny}}{\partial y} + \sigma_{ac} \frac{\partial U_{ny}}{\partial z} \right. \\ & \left. + B_x \left(-\sigma_{ac} \frac{\partial U_{ny}}{\partial x} - \sigma_{bc} \frac{\partial U_{ny}}{\partial y} - \sigma_{cc} \frac{\partial U_{ny}}{\partial z} + \sigma_{ab} \frac{\partial U_{nz}}{\partial x} + \sigma_{bb} \frac{\partial U_{nz}}{\partial y} - \sigma_{bc} \frac{\partial U_{nz}}{\partial z} \right) \right). \quad (39) \end{aligned}$$

The equation for the perturbed potential can be written in terms of perturbed conductivities (σ'_{aa} , σ'_{ab} , etc.) and perturbed neutral winds ($\partial U'_x/\partial x$, $\partial U'_x/\partial y$, etc.), as was Eq. (36). If we assume either (1) that the perturbation of the conductivity (σ') is negligible, or (2) that the unperturbed winds ($\bar{U}_n$) are uniform, the equation for the perturbed potential becomes

$$\begin{aligned} & \sigma_{aa} \frac{\partial^2 \psi'}{\partial x^2} + \sigma_{bb} \frac{\partial^2 \psi'}{\partial y^2} + \sigma_{cc} \frac{\partial^2 \psi'}{\partial z^2} + 2\sigma_{ac} \frac{\partial^2 \psi'}{\partial x \partial z} - \frac{\partial \sigma_{ac}}{\partial z} \frac{\partial \psi'}{\partial x} + \frac{\partial \sigma_{bc}}{\partial z} \frac{\partial \psi'}{\partial y} - \frac{\partial \sigma_{cc}}{\partial z} \frac{\partial \psi'}{\partial z} \\ & = B_z \left(-\sigma_{ab} \frac{\partial U'_{nx}}{\partial x} - \sigma_{bb} \frac{\partial U'_{nx}}{\partial y} + \sigma_{bc} \frac{\partial U'_{nx}}{\partial z} + \sigma_{aa} \frac{\partial U'_{ny}}{\partial x} - \sigma_{ab} \frac{\partial U'_{ny}}{\partial y} + \sigma_{ac} \frac{\partial U'_{ny}}{\partial z} \right) \\ & + B_x \left(-\sigma_{ac} \frac{\partial U'_{ny}}{\partial x} - \sigma_{bc} \frac{\partial U'_{ny}}{\partial y} - \sigma_{cc} \frac{\partial U'_{ny}}{\partial z} + \sigma_{ab} \frac{\partial U'_{nz}}{\partial x} + \sigma_{bb} \frac{\partial U'_{nz}}{\partial y} - \sigma_{bc} \frac{\partial U'_{nz}}{\partial z} \right). \quad (40) \end{aligned}$$

The neutral air disturbance from a point source is usually localized in a thin shell whose center was originally located at the source. Numerical calculations of the electric fields are most efficiently computed in a reference frame which places most grid locations on this shell. The

calculation of the electric potential in a spherical coordinate system with a moving center is described in Appendix B.

Even if the scale size of the disturbance is smaller than the E-region thickness, a two-dimensional set of equations may be formulated, assuming that the electric potential is constant along magnetic field lines. This is justified as follows. The currents relative to the magnetic field lines are

$$\begin{aligned} j_p &= \sigma_1 \left(-\frac{\partial \Psi}{\partial p} + BU_{ny} \right) - \sigma_2 \left(-\frac{\partial \Psi}{\partial y} - BU_{np} \right) , \\ j_y &= \sigma_2 \left(-\frac{\partial \Psi}{\partial p} + BU_{ny} \right) + \sigma_1 \left(-\frac{\partial \Psi}{\partial y} - BU_{np} \right) , \text{ and} \\ j_q &= \sigma_0 \left(-\frac{\partial \Psi}{\partial q} \right) . \end{aligned} \quad (41)$$

Current conservation requires that the perpendicular currents j_p and j_y are of the same magnitude as j_q . The parallel conductivity (σ_0) is much greater than perpendicular conductivities (σ_1 and σ_2) above 100-km altitude.⁶ Consequently, $\partial \Psi / \partial q$ is much smaller than $\partial \Psi / \partial p$ and $\partial \Psi / \partial y$. By neglecting $\partial \Psi / \partial q$ in the formulation, j_q becomes indeterminate and magnetic field changes due to current fluctuations cannot be accurately computed (see Section D).

Following the procedures used in this section, the perturbed electric potential written in (p,q,y) coordinates is found from

$$\begin{aligned} \sigma_1 \frac{\partial^2 \Psi'}{\partial p^2} + \sigma_1 \frac{\partial^2 \Psi'}{\partial y^2} + \left(\frac{\partial \sigma_1}{\partial p} + \frac{\partial \sigma_2}{\partial y} \right) \frac{\partial \Psi'}{\partial p} + \left(-\frac{\partial \sigma_2}{\partial p} + \frac{\partial \sigma_1}{\partial y} \right) \frac{\partial \Psi'}{\partial y} - \frac{\partial j_q'}{\partial q} \\ = \frac{\partial}{\partial p} (\sigma_1 BU'_{ny}) + \frac{\partial}{\partial p} (\sigma_2 BU'_{np}) + \frac{\partial}{\partial y} (-\sigma_1 BU'_{np}) + \frac{\partial}{\partial y} (\sigma_2 BU'_{ny}) . \end{aligned} \quad (42)$$

Assuming that Ψ' is independent of q and integrating over q yields

$$\begin{aligned} \Sigma_1 \frac{\partial^2 \psi'}{\partial p^2} + \Sigma_1 \frac{\partial^2 \psi'}{\partial y^2} &= \frac{\partial}{\partial p} \left[\int_{q_0}^{q_1} (\sigma_1 B U'_{ny} + \sigma_2 B U'_{np}) dq \right] \\ &+ \frac{\partial}{\partial y} \left[\int_{q_0}^{q_1} (-\sigma_1 B U'_{np} + \sigma_2 B U'_{ny}) dq \right] + j'_q(q_1) - j'_q(q_0) \quad , \end{aligned} \quad (43)$$

where

$$\Sigma_1 = \int_{q_0}^{q_1} \sigma_1 dq \quad ,$$

and q_0 and q_1 are the values of q at the bottom and top of the E-region, respectively. The field-aligned currents at the top, $j_q(q_1)$, and bottom, $j_q(q_0)$, of the E-region are described in the next section.

Using Eq. (42), a potential $\psi'(p, y)$ is obtained. This potential may then be substituted back into Eq. (41) to give the potential variation along the magnetic field line represented by $\psi''(p, y, q)$

$$\begin{aligned} \sigma_0 \frac{\partial^2 \psi''}{\partial q^2} + \frac{\partial \sigma_0}{\partial q} \frac{\partial \psi''}{\partial q} &= -\sigma_1 \left[\frac{\partial^2 \psi'}{\partial p^2} + \frac{\partial^2 \psi'}{\partial y^2} \right] - \frac{\partial \sigma_1}{\partial p} \frac{\partial \psi'}{\partial p} + \frac{\partial \sigma_2}{\partial p} \frac{\partial \psi'}{\partial y} \\ &+ \frac{\partial}{\partial p} [B(\sigma_1 U_{ny} + \sigma_2 U_{np})] + \frac{\partial}{\partial y} [B(\sigma_2 U_{ny} - \sigma_1 U_{np})] \quad . \end{aligned} \quad (44)$$

The total electric potential is $\psi'(p, y) + \psi''(p, y, q)$. This procedure may be continued to derive higher order estimates of the potential.

In summary, four sets of equations have been presented to describe the perturbed electrostatic potential in the E-region. Equation (37) can be used if the scale size of the neutral air disturbance is much larger than the E-region thickness. Equation (40) provides a general three-dimensional

description of the potential in the flat-earth coordinate system. Equations (B10) and (B11) provide a three-dimensional description in the coordinate system of a spherical pressure wave. Finally, Eqs. (43) and (44) describe the perturbed potential in coordinates relative to the geomagnetic field.

The E-region electric potential can be changed by (1) modulation of the conductivity and (2) dynamo action of the perturbed neutral velocity in the E-region. The former effect requires the presence of nonzero currents and electric fields in the E-region. The latter effect is independent of an existing electric field or current. Equation (36) contains both effects. The other equations only provide the electric potential produced by dynamo action of the neutral air wave. These equations can be extended to include both effects by analogy to Eq. (36).

C. F-Region Currents

The E- and F-regions are electrically coupled by field aligned currents. This section describes this coupling.

The boundary conditions at the top and bottom of the E-region are given in terms of the current. For example, from Eq. (38),

$$j'_z(x, y, z) = \sigma_{ac} \left(-\frac{\partial \Psi'}{\partial x} + U'_{ny} B_z \right) - \sigma_{bc} \left(-\frac{\partial \Psi'}{\partial y} + U'_{nz} B_x - U'_{nx} B_z \right) \\ + \sigma_{cc} \left(-\frac{\partial \Psi'}{\partial z} - U'_{ny} B_x \right) . \quad (45)$$

At the bottom of the E-region ($z = z_E$), the vertical current is zero [i.e., $j'_z(x, y, z_E) = 0$]. At the top of the E-region ($z = z_E + \Delta z_E$), the vertical current perturbation is found by matching the field-aligned current (j'_{qE}) from the upper ionosphere

$$j'_z(x, y, z_E + \Delta z_E) = j'_{qE} \cos I . \quad (46)$$

In the F-region, charge conservation requires that $\nabla \cdot \vec{j}' = 0$. In the (p, q, ϕ) coordinate system, this becomes

$$j'_{qE} = \int_{q_E}^{q_F} \left(\frac{\partial j'_p}{\partial p} + \frac{\partial j'_y}{\partial y} \right) dq + j'_{qF} , \quad (47)$$

where q_E is the upper E-region boundary and q_F is the boundary in the F-region above the perturbation. j'_{qF} is the current entering at the upper boundary. This current can be set to zero if the two hemispheres are assumed to be symmetric. For calculations involving two conjugate regions, the currents entering the topside ionospheres on the same field line must have identical magnitudes and opposite signs.

The perpendicular current perturbations (j'_p and j'_y) are found from the plasma velocity perturbations. The unperturbed currents are

$$j_{op} = [-N_0 e \rho (v_{oy} - U_{ny})] = \frac{N_0 e \rho}{1 + \rho^2} \left[-U_{ny} + \frac{E_p}{B} + \frac{g_p}{\Omega} - \rho \left(U_{np} + \frac{E_\phi}{B} \right) \right] \quad (48)$$

and

$$j_{o\phi} = N_0 e \left[\rho (v_{op} - U_{np}) - \frac{g_p}{\Omega} \right] = \frac{N_0 e \rho}{1 + \rho^2} \left[-U_{np} + \frac{E_y}{B} + \rho \left(U_{ny} + \frac{E_p}{B} \right) - \frac{g_p}{\rho \Omega} \right] . \quad (49)$$

The perturbed currents are found to be

$$\begin{aligned} j'_p &= -N' \rho e (v_{oy} - U_{ny}) - N_0 e \rho (v'_y - U'_{ny}) \\ &= \frac{N' e \rho}{1 + \rho^2} \left[U_{n\phi} + \frac{E_p}{B} + \frac{g_p}{\Omega} - \rho \left(-U_{np} + \frac{E_y}{B} \right) \right] + \frac{N_0 e \rho}{1 + \rho^2} \left[U'_{ny} + \frac{E'_p}{B} - \rho \left(-U'_{ny} + \frac{E'_y}{B} \right) \right] \end{aligned}$$

and

$$\begin{aligned}
j'_y &= N' \rho e (v_{op} - U_{np}) + N_o e \rho (v'_p - U'_{np}) - N' e \frac{g_p}{\Omega} \\
&= \frac{N' e \rho}{1 + \rho^2} \left[(-U_{np} + \frac{E_y}{B}) + \rho (U_{ny} + \frac{E_p}{B}) - \frac{g_p}{\Omega \rho} \right] \\
&+ \frac{N_o e \rho}{1 + \rho^2} \left[(-U'_{np} + \frac{E'_y}{B}) + \rho (U'_{ny} + \frac{E'_p}{B}) \right] .
\end{aligned} \tag{50}$$

Again, terms containing ρ' have been dropped.

Electric fields and currents are transmitted up magnetic field lines from the E-region to the F-region. The perturbation of the F-region by the electric fields is discussed in Section IV. A. Coupling via currents is beyond the scope of this study.

D. Magnetic Field Fluctuations

Previous treatments of geomagnetic disturbances have not considered a three-dimension model of E-region currents⁷ with inclined magnetic fields. The effect of currents self consistently closing upon themselves reduces their amplitude and, consequently, the amplitude of magnetic field fluctuations.

The perturbed currents in the E-region may produce detectable changes in the magnetic flux density on the ground. The magnetic-field perturbations are given by

$$\bar{B}(x, y, 0) = \frac{\mu}{4\pi} \iiint \frac{\bar{J} \times \bar{a}_r}{r^2} d^3\bar{x} , \tag{51}$$

where the volume integration is taken over the E-region and $\vec{a}_r$ is the vector from the location of the current to the point $(x,y,0)$ on the ground (see Fig. 2). Thus,

$$\vec{a}_r = \frac{(x - x_0)\vec{x} + (y - y_0)\vec{y} + (-z_0)\vec{z}}{r} \quad (52)$$

where $r = (x - x_0)^2 + (y - y_0)^2 + z_0^2$. The individual magnetic field components are

Fig. 2. Geometry for calculating magnetic field fluctuations from E-region currents.

$$B'(x,y,0) = \frac{\mu}{4\pi} \int_{z_0} \int_{y_0} \int_{x_0} \frac{j'_y z_0 - j'_z (y - y_0)}{r^3} dx dy dz , \quad (53)$$

$$B'_y(x,y,0) = \frac{\mu}{4\pi} \int_{z_0} \int_{y_0} \int_{x_0} \frac{j'_x z_0 + j'_z (x - x_0)}{r^3} dx_0 dy_0 dz_0 , \quad (54)$$

and

$$B'_z(x,y,0) = \frac{\mu}{4\pi} \int_{z_0} \int_{y_0} \int_{x_0} \frac{j'_x (x - x_0) - j'_y (y - y_0)}{r^3} dx_0 dy_0 dz_0 , \quad (55)$$

where $\mu = 4\pi \times 10^{-7}$ H/m. Depending on the E-region model, the current densities for these equations can be given by Eqs. (28), (B2), or (41).

III. NUMERICAL CALCULATIONS

The theory presented in the previous section and the Appendixes provide the basis for computation of disturbances produced by pressure waves in the ionosphere. In this section, numerical calculations are used to obtain estimates of perturbations in electric fields, currents, magnetic fields, and electron concentration.

All calculations will be one in the background atmosphere given in Fig. 3. The neutral atmospheric composition and temperature (Figs. 3a and b) above 120 km are based on the MSIS model atmosphere.^{8,9} Below 120 km, the CIRA reference atmosphere¹⁰ is used. The electron temperature (Fig. 3b) is a function of neutral concentration, electron concentration, and neutral temperature

$$T_e = T_n + \frac{1}{A_0 N_{no} + A_1 N_o + A_2} ,$$

where $A_0 = 5 \times 10^{-18}$, $A_1 = 2 \times 10^{-15}$, and $A_2 = 6.67 \times 10^{-4}$. This empirical expression takes into account: (1) cooling of the electrons by neutrals, (2) heat conduction by electrons, and (3) heating of ionospheric electrons by heat transport from the overlying protonosphere. The ion temperature (Fig. 3b) is given by

$$T_i = \frac{T_n + (\alpha N_o / N_{no}) T_e^{-1/2}}{1 + (\alpha N_o / N_{no}) T_e^{-3/2}} ,$$

where $\alpha = 6 \times 10^6$. This equation assumes steady-state heat balance between the neutral, ion, and electron gases. The background electron concentration is illustrated in Fig. 3c. This electron profile is taken from the analytic model described by Booker.¹¹ All the atmospheric parameters are representative of daytime, solar-maximum conditions.

Fig. 3. Profiles of neutral concentration, temperatures, and electron concentration in the unperturbed atmosphere.

A. Perturbation of Electron Concentration - Direct Effects

The perturbation in electron concentration is calculated using Eq. (19). This second-order, partial differential equation is solved numerically. Indirect effects due to external electric fields (E_{op} , E_{oy} , E'_p , and E'_y) and the associated plasma drifts are neglected. A spherical acoustic wave is used in the simulations

$$U'_n(r, \phi, \theta) = \frac{a(t)}{2} r_0(t) \left[\tanh\left(\frac{r_0}{g\lambda_1} + \frac{\Delta r_1}{\lambda_1}\right) - \tanh\left(\frac{r_0}{g\lambda_2} - \frac{\Delta r_2}{\lambda_2}\right) \right] \quad (56)$$

where $r_0 = r - r'(t)$, r , ϕ , and θ are spherical coordinates [see Appendix A, Eq. (A1)], $r'(t) = 90 \exp[(t - 300)/304.5] - h_c(t)$ is the radius (km) of the spherical wave, $h_c(t) = 18 + 55.5 \tan^{-1}[(T - 400)/108]$ is the altitude center of the wave, $a(t)$ is the wave amplitude, Δr_1 and Δr_2 describe the thickness of the disturbance, λ_1 and λ_2 govern the steepness at the lower and upper edges of the wave and $g(\phi)$ gives the angular dependence of the wave. The velocity has no azimuthal (θ) dependence. The direction of the velocity is radially away from the point $(x, y, z) = [0, 0, h_c(t)]$. This function was derived by S. I. Warshaw and P. Dubois (Lawrence Livermore National Laboratory, personal communication, 1981).

The numerical results from Eq. (19) are compared with the analytical results of Eq. (24). All computations are made along a magnetic field line that intersects $(x, y) = (0, 0)$ at 80-km altitude. Figure 4 illustrates the disturbed region 300 s after the launch of the acoustic wave. The slanted line in the figure denotes the magnetic field line on which the calculations are made. Figure 5 shows profiles of the disturbance along the magnetic field line, 294.5 s after the start of the perturbing wave. Figure 5a presents a graph of the N-wave which has a peak velocity of ± 13.5 m/s. The resulting electron density perturbation (Figs. 5c and 5d) near 88.4-km altitude is $\pm 1.8 \times 10^9 \text{ cm}^{-3}$ or about 4.5% of the unperturbed electron concentration (Fig. 5b). The numerical (Fig. 5c) and analytical (Fig. 5d) results are identical.

Fig. 4. Velocity and electron concentration perturbations 300 s after the start of an earth-based disturbance. The slanted line is the magnetic-field line on which detailed calculations are made.

The maximum perturbation of the electron concentration occurs near 120-km altitude. Contours of the velocity perturbations and change in electron concentration are illustrated in Fig. 6. The electron perturbations in Fig. 6b are calculated using the analytic formulation. Details of the results along the magnetic field line are shown in Fig. 7. At $t = 397.9$ s, the velocity perturbation extends from 118 to 124.4 km altitude. The amplitude of the N-wave is ~ 40 m/s (Fig. 7a). The computed electron concentrations differ slightly, depending on whether the numerical or analytical formulations are used. The numerical results (Fig. 7c) are about $1 \times 10^9 \text{ cm}^{-3}$ less than the analytical results (Fig. 7d). The difference in the results is due, in part, to diffusion which was neglected in the analytic formula. Also, the graph of the numerical calculation shows a slight ripple (due to numerical errors). The magnitude of the electron perturbation is 10% of the unperturbed electron concentration (Fig. 7b). Above 130-km altitude, the ratio of ion-neutral collision frequency becomes much less than zero. Consequently, the velocity

Fig. 5. Detailed profiles at $t = 294.5$ s, showing (a) velocity perturbation, (b) perturbed total electron concentration, and the electron concentration calculated by (c) numerical and (d) analytical methods.

VELOCITY (m/s)

ELECTRON CONCENTRATION (10^7 m^{-3})

Fig. 6. Velocity and electron concentration perturbations at $t = 400 \text{ s}$ after the start of an earth-based disturbance. The slanted line is the magnetic-field line on which detailed calculations are made.

components perpendicular to the magnetic field have negligible effect on the plasma concentration. At $t = 461.8 \text{ s}$, the N-wave is at 146-km altitude with minimum and maximum velocities of -30 and $+24 \text{ m/s}$, respectively (Figs. 8 and 9a). The absolute value of the underdense portion of the electron concentration is 1.5 times the value of the overdense region of the disturbance (Fig. 9c). The analytic results (which neglect diffusion) give a ratio of the underdense to overdense portions of 1.2 (Fig. 9d). Consequently, the approximate analysis that neglects diffusion should be used cautiously above 130-km altitude. The disturbance at $t = 461.8 \text{ s}$ is 7% of the unperturbed electron concentration (Fig. 9b).

Fig. 7. Detailed profiles at $t = 397.9$ s, showing (a) velocity perturbation, (b) perturbed total electron concentration, and the electron concentration calculated by (c) numerical and (d) analytical methods.

Fig. 8. Velocity and electron concentration perturbations at $t = 460 \text{ s}$ after the start of an earth-based disturbance. The slanted line is the magnetic-field line on which detailed calculations are made.

The calculations are extended to 600 s after initiation of the wave. The field line is at the edge of the main portion of the disturbance (Figs. 10a and b). The wave is centered at 218-km altitude, having peak amplitude of -9 and $+7 \text{ m/s}$ (Fig. 11a). The analytic (Fig. 11c) and numerical (Fig. 11d) results have the largest difference of any of the

Fig. 9. Detailed profiles at $t = 461.8$ s, showing (a) velocity perturbation, (b) perturbed total electron concentration, and the electron concentration calculated by (c) numerical and (d) analytical methods.

Fig. 10 Velocity and electron concentration perturbations at $t = 600 \text{ s}$ after the start of an earth-based disturbance. The slanted line is the magnetic-field line on which detailed calculations are made.

examples presented. The 11% perturbation in the electron concentration is visually masked by the layer profile (Fig. 11b).

Figure 12a is a plot of the maximum electron disturbance as a function of altitude. The values for this figure are based on the numerical solution of Eq. (19). The absolute value of the electron perturbation is

Fig. 11. Detailed profiles at $t = 599$ s, showing (a) velocity perturbation, (b) perturbed total electron concentration, and the electron concentration calculated by (c) numerical and (d) analytical methods.

Fig. 12. Absolute (a) and relative (b) maximum perturbation in electron concentration. The numerical values are the maximum disturbance produced by the wave at each altitude. The rapid fluctuations are numerical in origin.

stored as the perturbing wave propagates upward through the ionosphere. A broad peak in amplitude of the disturbance occurs near 185 km. At higher altitudes, the amplitude increases because of the increase in the background plasma concentration. The relative perturbation in electron concentration shows a peak of 10.3% near 120-km altitude (Fig. 12b).

The primary factors controlling the magnitude of the electron disturbance are (1) the background electron concentration (N_0), (2) the perturbation of the neutral velocity (U'_n) and (3) the velocity of the disturbance ($\partial r_0 / \partial t$). Secondary effects include coupling of wave energy parallel and (in the lower atmosphere) perpendicular to the magnetic field, and diffusion in the upper atmosphere. Chemical loss rates have an insignificant effect on the ionospheric disturbance.

B. Currents and Electric Fields in the E-Region - Two-Dimensional Calculations

The previous section neglected electrostatic fields that may be produced by dynamo action in the E-region. In this section, these fields are calculated for cases in which the problem can be cast into two-dimensional rather than three-dimensional form. These two cases occur when (1) the scale of the velocity field is much larger than the E-region thickness, or (2) the electric potential is constant along magnetic field lines. These calculations produce such quantities as E-region electric fields and currents, F-region plasma drifts, and changes in the magnetic field intensity on the ground.

The first set of calculations deals with a "thick" disturbance in a "thin" E-region. The electric fields and currents are found by solving Eq. 37. At the boundary, the potential is set to zero. The velocity perturbation is described by the following function:

$$U_r(r) = 5.0 \frac{(r - V_r t)}{\sqrt{2\pi} \sigma_r} \exp \left[-\frac{(r - V_r t)^2}{2\sigma_r^2} \right] \text{ m/s} , \quad (57)$$

where $V_r = 880$ m/s is the velocity of the disturbance, $\sigma_r = 20$ km is the distance from the center of the disturbance to peaks of the "N" wave, and t is time. The peak amplitude of the velocity disturbance is 1.2 m/s. This function provides an approximate description of the radial velocity disturbance originating on the ground at $r = 0$.

The E-region is taken to be an infinitely thin layer at 105-km altitude. Numerical values for the height-integrated conductivities are $\Sigma_{xx} = 6.5$ mho, $\Sigma_{xy} = 20.0$ mho, and $\Sigma_{yy} = 5.3$ mho. These typical daytime values are at 40-degrees latitude. The dip angle is $I = 59.2$ degrees. In all graphs of the computations, the center is the location of the disturbance source. The side of the graphs is 400 km on a side.

Figure 13 illustrates the E-region disturbance 100 seconds after the start of the pressure wave. The horizontal velocity field is away from the axis of the disturbance (Fig. 13a). The electrostatic field points to a region that is 40-km north of the center of the disturbance. The maximum electric field is 2.55×10^{-4} V/m (Fig. 13b). The electric field and wind disturbances produce the current circulation illustrated in Fig. 13c. To the north of the disturbance, the current is westward and to the south of the disturbance the current flows toward the east. The noncircularity of the disturbance is due to the slanted magnetic lines.

Figure 14 illustrates the F-region plasma drifts associated with the E-region electrostatic fields at $t = 100$ s. The drifts are found from $\vec{v}_d = \vec{E} \times \vec{B}/B^2$. Figure 14a shows the plasma drift velocity in a plane perpendicular to the magnetic field lines, entered at 300-km altitude. Since this plane is tilted, part of the plasma drift is in the vertical direction. At $t = 100$ s, the plasma drift on the east side of the disturbance is up and the drift on the west side is down (Fig. 14b). The magnitude of the drift is of the same order as the neutral velocity in the E-region. The upward drift will raise the F-layer, changing the local

Fig. 13. Perturbations in neutral velocities, electric fields, and currents produced by an N-wave entering the E-region.

Fig. 14. F-region plasma drifts produced by the electric fields illustrated in Fig. 13b.

plasma concentration in regions of nonzero vertical gradients. Figure 15 illustrates the changes in magnetic field strength on the ground (see Sec. II.D). The horizontal component of $\vec{B}$ is increased by almost 6×10^{-10} Wb/m². Its direction is nearly north-south, pointing toward the disturbance center (Fig. 15b). The solenoidal action of the currents causes a 5.6×10^{-10} Wb/m² enhancement of the vertical magnetic field to the north of the source of the disturbance.

Figures 16, 17, and 18 illustrate the E-region perturbations, F-region drifts, and ground magnetic fluctuations at $t = 140$ seconds. The positive (upward) component of the neutral wind is affecting the E-region ~ 100 km from the axis of the disturbance. The negative (downward) component of the disturbance influences the center region (Fig. 16). In the 40 seconds following the results presented in Figs. 13, 14, and 15, the directions of the electric fields, currents, plasma drifts, and magnetic disturbances have reversed directions.

At $t = 180$ s, the pressure wave has penetrated the on-axis portion of the E-region. The negative velocity component encircles the axis at a radius of ~ 80 km. The positive component is 160 km from the axis of the source (Fig. 19a). The largest disturbances in electric fields and currents are to the north of the wave axis (Figs. 19b and 19c). The general structure of the drifts and magnetic fields shown in Figs. 20 and 21 expand in time following the expansion of the neutral wind source.

C. E-Region Currents and Electric Fields - Three-Dimension Calculations

In this section, the relatively "thin" atmospheric wave given by Eq. (56) is used to drive the E-region dynamo. The electric potential is calculated from Eqs. (43) and (44). The field aligned currents below and above the E-region are assumed to be zero. On the side boundaries (i.e., constant p and constant y) the potential is set equal to zero. Both constant and derivative boundary conditions have been used, yielding nearly identical results inside the region. The potential Ψ' is calculated in the (p, y) coordinate system, centered on the source of the disturbance. The

Fig. 15. Magnetic-field fluctuations produced by the currents illustrated in Fig. 13c.

Fig. 16. Perturbations in neutral velocities, electric fields, and currents produced by an N-wave having both positive and negative velocity components in the E-region.

Fig. 17. F-region plasma drifts produced in the electric fields illustrated in Fig. 16b.

Fig. 19. Perturbations in neutral velocity, electric fields, and currents produced by an N-wave having completely penetrated the E-region.

Fig. 21. Magnetic-field fluctuations produced by the currents illustrated in Fig. 19c.

space is 200 km by 200 km perpendicular to the magnetic field lines. The E-region is taken to be between 80 and 130 km altitude.

The results of the computations are illustrated for the times indicated below:

<u>Time after start of wave (s)</u>	<u>Height of wave apex (km)</u>
330	85.4
380	124.9
430	171.0

At 330 s the wave has just penetrated the bottom of the E-region giving the potential distribution shown in Figure 22a. At this time the maximum electric field is 0.71×10^{-5} V/m. As the wave propagates upward, the potential distribution changes both in shape and in amplitude. At $t = 380$ s (Fig. 22b) the maximum electric field is 0.613×10^{-4} V/m. The pattern of the electric potential at 430 s is illustrated in Fig. 22c. The potential maintains a similar shape after this time. The maximum electric field is 1.0×10^{-4} V/m at $t = 430$ s.

The plasma drift vectors in the F-region produced by the E-region electric fields is illustrated in Fig. 23. The maximum drift perpendicular to the magnetic fields is 2.07 m/s. The changes in direction of the drift pattern at $t = 430$ s shows the effects of the overdense and underdense portions of the wave (Fig. 23c).

The vertical component of the plasma drift is illustrated in Fig. 24. The maximum perturbation of the drift is to the west of the disturbance at 0.87 m/s at $t = 430$ s, the absence of drifts in the center of the region is because the center portion of the wave is above the E-region.

The next three illustrations (Figs. 25 - 27) show the patterns of magnetic field on the ground due to dynamo action of the wave. The E-region current densities are integrated using the formulas of Section II.D. The patterns at $t = 330$ s are representative of the structure as the wave enters the bottom of the E-region. The transition is shown by the

Fig. 22. Electric potential at the top of the E-region produced by a thin wave in a relatively thick E-layer.

Fig. 23. Plasma drifts in the F-region associated with E-region electric fields.

Fig. 24. Vertical component of F-region plasma drifts shown in Fig.23.

Fig. 25. Fluctuations in the horizontal component in magnetic field.

Fig. 26. Fluctuations in the vertical component of the magnetic field.

VERTICAL MAGNETIC FLUX

Fig. 27. Contributions to the vertical component of the magnetic field perturbations. B_z' is produced by the electric potential perpendicular to the ambient magnetic field. B_z'' is produced by the perturbation in neutral velocity and also the field aligned component of the electric potential.

patterns at $t = 380$ s. The magnetic field patterns after the wave has penetrated the top of the E-region are illustrated at $t = 430$ s.

The maximum horizontal magnetic flux density is 0.7×10^{-12} Wb/m² at $t = 430$ seconds (Fig. 24c). The maximum vertical flux is 0.8×10^{-12} Wb/m² downward (Fig. 26c). As the wave propagates upward, the magnetic perturbation changes direction from predominantly upward (Fig. 26a,b) to predominantly downward (Fig. 26c). The magnetic density is composed of contributions due to (1) the neutral wind, (2) the first order potential Ψ' , and (3) the potential variation along the field line Ψ'' . Figure 27 illustrates a decomposition of the flux density. The contribution to the flux density by currents due to the potential Ψ' is shown in Fig. 27a. The magnetic flux density due to currents driven by the neutral wind and the potential Ψ'' is illustrated in Fig. 27b. The two components are comparable and, in certain regions, tend to cancel. The sum of the two components is shown in Fig. 26c.

IV. CONCLUSIONS AND RECOMMENDATIONS

A. Results and Conclusions

A propagating acoustic wave produces localized perturbations of the ionospheric electron concentrations. For the specific acoustic wave used in this study, a 10% change in electron concentration was computed.

The E-region modification described here deals with the dynamo action of the wave moving ions across magnetic field lines. A two-dimensional model of the E-region electrostatic fields produced field strengths on the order of 0.5×10^{-3} V/m and ground magnetic flux densities of 0.590×10^{-9} Wb/m². A more realistic three-dimensional model produced electric fields of 0.1×10^{-3} V/m and magnetic flux densities of 0.8×10^{-12} Wb/m². The reduction in the fields in the three-dimensional model is a result of the localized nature of the disturbance. In the E-region, the thickness of the wave is on the order of 1 km. The overdense and underdense portions of the disturbance tend to cancel out because of integration over the wave. Also, the electric field produced by the wave tends to produce currents that cancel the currents driven by the neutral velocity of the wave.

The perturbation of the F-region plasma concentration by the E-region electric fields is estimated as follows. The maximum electric field is on the order of 10^{-3} V/m. The field direction can be reversed in 20 s. The plasma drifts associated with the electric field can be as large as 30 m/s. Sustained plasma drifts produced by localized E-region electric fields last less than 20 s at one location. The relative electron perturbation produced by this drift is found to be [see Eq. (19)]:

$$\frac{\partial N'}{\partial t} \frac{1}{N_0} = -v' \frac{\partial \ln N_0}{\partial p} = v'_p \cos I \frac{\partial \ln N_0}{\partial z} \quad (58)$$

Using $v'_p = 30$ m/s, $\Delta t = 20$ s, and $(\partial \ln N_0 / \partial z) = 10^{-6} \text{m}^{-1}$, one obtains a 0.06% change in electron concentration. This is much smaller than that produced by the direct effects of the wave.

We conclude that remote sensing of an acoustic wave in the ionosphere may be accomplished by looking at:

(1) the magnetic field fluctuations on the ground near the source of the acoustic wave,

(2) the precursor in the F-region associated with electric fields in the E-region, and

(3) the direct effect of the wave as it passes through the E- and F-regions. The magnitude of the underdense perturbation tends to be larger than the magnitude of the overdense perturbation.

Items(1) and(2) probably produce perturbations, which are much smaller than those that occur naturally. Only item 3) produces effects large enough to provide a unique, measurable signature.

B. Recommendations

Theoretical work needs to be done on conductivity modification produced by the acoustic wave. Modulation of ambient electric currents by conductivity modification could produce larger magnetic field amplitudes than were computed using the dynamo theory described here.

A measurement program to verify the theoretical work is recommended. Measurements of ground-based magnetic fields following the generation of an acoustic wave is essential for validation of the dynamo calculations. Radio wave techniques should be employed to observe the precursor and direct effects of the acoustic wave on the ionosphere. Assymetry in the ionospheric disturbance is predicted. Future modeling should be attempted to relate the strength of the ionospheric disturbance to the strength of the source. Also, unique properties of the ionospheric disturbance, such as ratio of underdense to overdense perturbations, should be examined.

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APPENDIX A

IONOSPHERIC PERTURBATION BY A SPHERICAL DISTURBANCE

An approximate analytic solution is derived for a spherical air disturbance centered at $(x,y,z) = [0, 0, h_c(t)]$ where $h_c(t)$ is a time-dependent altitude. The spherical coordinate system is

$$\begin{aligned} r^2 &= x^2 + y^2 + [z - h_c(t)]^2 & x &= r \cos\theta \sin\phi \\ \phi &= \tan^{-1} \{ \sqrt{x^2 + y^2} / [z - h_c(t)] \} & y &= r \sin\theta \sin\phi \\ \theta &= \tan^{-1}(y/x) & z &= r \cos\theta \end{aligned} \quad (A1)$$

where r is radius, ϕ is the polar angle with respect to the z axis, and θ is the azimuth with respect to the x axis. The velocity disturbance is assumed to be radial

$$U'_n(r, \phi, \theta) = U'_n \bar{r}$$

where $\bar{r}$ is the unit vector in the r -direction. The velocity components in the magnetic coordinates are

$$\begin{aligned} U'_{nq} &= \frac{\partial r}{\partial q} U'_n = [\sin\phi \cos\theta \cos I + \cos\phi \sin I] U'_n \equiv G_q U'_n \\ U'_n &= \frac{\partial r}{\partial y} U'_n = [\sin\phi \sin\theta] U'_n \\ U'_{np} &= \frac{\partial r}{\partial p} U'_n = [\sin\phi \cos\theta \sin I + \cos\phi \cos I] U'_n \end{aligned} \quad (A2)$$

Neglecting electric fields, the perpendicular plasma velocities become

$$v'_p = G_p U'_n \quad v'_y = G_y U'_n \quad (A3)$$

where

$$G_p = \frac{\rho^2 [-\sin\phi \cos\theta \sin I + \cos\phi \cos I] + \rho \sin\phi \sin\theta}{1 + \rho^2} = \frac{\rho^2 \frac{\partial r}{\partial p} + \rho \frac{\partial r}{\partial y}}{1 + \rho^2}$$

$$G_y = \frac{\rho^2 \sin\phi \sin\theta - \rho [-\sin\phi \cos\theta \sin I + \cos\phi \cos I]}{1 + \rho^2} = \frac{\rho^2 \frac{\partial r}{\partial p} + \rho \frac{\partial r}{\partial y}}{1 + \rho^2} .$$

The electron perturbation can be written as

$$N'(q,t) = E(q,t) U'_n + F(q,t) . \quad (A4)$$

In a non-dissipative medium, E is the ratio of the unperturbed concentration to the wave velocity, and F is zero. In the ionosphere, diffusion and chemical processes affect E and F . For approximate solutions, it is assumed that

$$\frac{1}{E} \frac{\partial E}{\partial q} \ll \frac{1}{U_n} \frac{\partial U'_n}{\partial q} , \text{ and } EU'_n \ll F.$$

The background atmosphere determines the spatial gradient of $E(q,t)$. Thus, E is expected to be much smoother than U'_n .

Substitution of Eq. (A4) into (19) and retention of first order terms yields

$$E \frac{\partial U'_n}{\partial t} = D_a \left[E \frac{\partial^2 U'_n}{\partial q^2} \right] + A \left[E \frac{\partial U'_n}{\partial q} \right] - \left[C_1 \frac{\partial U'_n}{\partial q} + G_p \frac{\partial U'_n}{\partial p} + G_y \frac{\partial U'_n}{\partial y} \right] N_o \quad (A5)$$

where $C_1 = G_q - v_{0q}/C_{no}$, and the unperturbed plasma drifts v_{0p} and v_{0y} are zero. Chemical terms can be dropped because the time constant of the wave is much shorter than the recombination time in the ionosphere. Again, the background atmosphere is assumed to be smoother than the velocity disturbance U'_n . Consequently, terms for $[\partial(\cdot)/\partial q]U'_n$ have been dropped in favor of terms $[\partial U'_n/\partial q](\cdot)$ where $(\cdot)$ is a function of the background atmosphere.

Solving Eq. (A5) for E yields

$$E(q, t) = \frac{(C_1 \frac{\partial U'_n}{\partial q} + G_p \frac{\partial U'_n}{\partial p} + G_y \frac{\partial U'_n}{\partial y}) N_0}{D_a \frac{\partial^2 U'_n}{\partial^2} + A \frac{\partial U'_n}{\partial q} - \frac{\partial U'_n}{\partial t}} \quad (A6)$$

Since U'_n is the solution of a wave equation, it has the approximate form

$$U'_n [r - r'(t)] = U'_n [r_0(t)] \quad (A7)$$

The q , p , y , and t derivatives become

$$\frac{\partial U'_n}{\partial q} = \frac{\partial U'_n}{\partial r_0} \frac{\partial r_0}{\partial q}, \quad \frac{\partial U'_n}{\partial p} = \frac{\partial U'_n}{\partial r_0} \frac{\partial r_0}{\partial p}, \quad \frac{\partial r_0}{\partial y} = \frac{\partial U'_n}{\partial r_0} \frac{\partial r_0}{\partial y} \quad (A8)$$

and

$$\frac{\partial U'_n}{\partial t} = (\partial U'_n / \partial r_0) \frac{\partial r_0}{\partial t}$$

If the diffusion can be neglected, Eq. (A6) reduces to

$$E(q,t) = [G_q \frac{\partial r_o}{\partial q} + G_p \frac{\partial r_o}{\partial p} + G_y \frac{\partial r_o}{\partial y}] / (-\frac{\partial r_o}{\partial t}) \quad (A9)$$

Since r' is only a function of time, the subscript o may be dropped from the spatial derivatives (i.e., $\partial r_o / \partial p = \partial r / \partial p$, etc.). Substituting for G_q , G_p , and G_y from Eqs. (A2) and (A3) and using the expression for E from Eq. (A4) yields

$$\begin{aligned} \frac{N'(p,y,q,t)}{N_o} &= \left\{ \left(\frac{\partial r}{\partial q} \right)^2 + \left[\left(\frac{\partial r}{\partial p} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right] \frac{\rho^2}{1 + \rho^2} \right\} \frac{U'_n}{-\partial r_o / \partial t} \\ &= \left\{ \left[\left(\frac{\partial r}{\partial q} \right)^2 + \rho^2 \right] / (1 + \rho^2) \right\} \frac{U'_n}{-\partial r_o / \partial t} \end{aligned} \quad (A10)$$

where the relation $\left(\frac{\partial r}{\partial q} \right)^2 + \left(\frac{\partial r}{\partial p} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 = 1$ is used.

APPENDIX B

ELECTRIC POTENTIAL IN A SPHERICAL REFERENCE FRAME

For practical reasons, it may be necessary to calculate the electric potential in the reference frame of the neutral air disturbance. Assume that the disturbance is spherical, centered at altitude $z = h_c(t)$. This center can move vertically with time but remains horizontally fixed at $x = 0$ and $y = 0$. A spherical coordinate system centered at $z = h_c$, $x = 0$, and $y = 0$ is described by

$$\begin{aligned} r^2 &= x^2 + y^2 + (z - h_c)^2 \\ \phi &= \tan^{-1}[\sqrt{x^2 + y^2}/(z - h_c)] \\ \theta &= \tan^{-1}(y/x). \end{aligned} \tag{B1}$$

In this system, the currents and electric fields are related by

$$\begin{bmatrix} j'_r \\ j'_\phi \\ j'_\theta \end{bmatrix} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\phi} & \sigma_{r\theta} \\ \sigma_{\phi r} & \sigma_{\phi\phi} & \sigma_{\phi\theta} \\ \sigma_{\theta r} & \sigma_{\theta\phi} & \sigma_{\theta\theta} \end{bmatrix} \begin{bmatrix} E'_{tr} \\ E'_{t\phi} \\ E'_{t\theta} \end{bmatrix} \tag{B2}$$

where

$$\begin{aligned} \sigma_{rr} &= \sigma_o C_r^2 + \sigma_1 [1 - C_r^2] \\ \sigma_{\phi r} &= (\sigma_o - \sigma_1) C_r C_\phi + \sigma_2 C_\theta \\ \sigma_{\theta r} &= (\sigma_1 - \sigma_o) C_\theta C_r - \sigma_2 C_\phi \\ \sigma_{r\phi} &= (\sigma_o - \sigma_1) C_r C_\phi - \sigma_2 C_\theta \\ \sigma_{\phi\phi} &= \sigma_o C_\phi^2 + \sigma_1 [1 - C_\phi^2] \\ \sigma_{\theta\phi} &= (\sigma_1 - \sigma_o) C_\phi C_\theta - \sigma_2 C_r \end{aligned}$$

$$\sigma_{\theta\phi} = (\sigma_1 - \sigma_0)C_\phi C_\theta - \sigma_2 C_r$$

$$\sigma_{r\theta} = (\sigma_1 - \sigma_0)C_r C_\theta + \sigma_2 C_\phi$$

$$\sigma_{\phi\theta} = (\sigma_1 - \sigma_0)\phi\theta + \sigma_2 C_r$$

$$\sigma_{\theta\theta} = \sigma_0 C_\theta^2 + \sigma_1 [1 - C_\theta^2]$$

and

$$C_r = \cos\theta \cos\phi \sin\theta + \sin\theta \cos\phi$$

$$C_\phi = \cos\theta \cos\phi \cos\phi - \sin\theta \sin\phi$$

$$C_\theta = \cos\theta \sin\theta .$$

The total electric field components are

$$E_{tr} = - \frac{\partial \Psi'}{\partial r}$$

$$E_{t\phi} = - \frac{1}{r} \frac{\partial \Psi'}{\partial \phi} - U_r' B C_\theta \quad (B3)$$

$$E = - \frac{1}{r \sin\phi} \frac{\partial \Psi'}{\partial \theta} - U_r' B C_\theta ,$$

where U_r' is the perturbation in the neutral air motion. In spherical coordinates conservation of current becomes

$$\nabla \cdot \vec{J} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 j_r) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi j_\phi) + \frac{1}{r \sin \phi} \frac{\partial j_\theta}{\partial \theta} \equiv 0. \quad (B4)$$

Substituting the expression for $\vec{J}$ yields the partial differential equation for the potential

$$\begin{aligned} & \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \sigma_{rr} \frac{\partial \Psi'}{\partial r} + r \sigma_{r\phi} \frac{\partial \Psi'}{\partial \phi} + \sigma_{r\theta} \frac{r}{\sin \phi} \frac{\partial \Psi'}{\partial \theta} \right] \\ & + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left[\sin \phi \sigma_{\phi r} \frac{\partial \Psi'}{\partial r} + \frac{\sin \phi}{r} \sigma_{\phi\phi} \frac{\partial \Psi'}{\partial \phi} + \frac{\sigma_{\phi\theta}}{r} \frac{\partial \Psi'}{\partial \theta} \right] \\ & + \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} \left[\sigma_{\theta r} \frac{\partial \Psi'}{\partial r} + \frac{\sigma_{\theta\phi}}{r} \frac{\partial \Psi'}{\partial \phi} + \frac{\sigma_{\theta\theta}}{r \sin \phi} \frac{\partial \Psi'}{\partial \theta} \right] \\ & + \frac{B}{r^2} \frac{\partial}{\partial r} (r^2 U'_r \sigma_{3r}) + \frac{B}{r \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi U'_r \sigma_{3\phi}) \\ & + \frac{B U'_r}{r \sin \phi} \frac{\partial}{\partial \theta} (\sigma_{3\theta}) = 0, \end{aligned} \quad (B5)$$

where

$$\sigma_{3r} = -\sigma_2 (C_\theta^2 + C_\phi^2)$$

$$\sigma_{3\phi} = \sigma_1 C_\theta + \sigma_2 C_r C_\phi$$

$$\sigma_{3\theta} = \sigma_1 C_\phi - \sigma_2 C_r C_\theta.$$

The radial velocity perturbation U'_r is assumed to have no azimuthal variation (i.e., $\partial U'_r / \partial \theta = 0$).

The Fourier transform is used to remove the θ derivatives. Define

$$\Psi'(\theta) = \sum_{n=0}^{N-1} \hat{\Psi}_n \exp(jn\theta)$$

$$\hat{\Psi}_n = \sum_{m=0}^{N-1} \frac{\Psi'(\theta_m) \exp(-jn\theta_m)}{N}, \quad (B6)$$

where $\theta_m = 2\pi m/N$. $\hat{\Psi}_n$ is the Fourier transform of $\Psi(\theta)$. This is written as $F[\Psi'(\theta)] = \hat{\Psi}_n$ and $F^{-1}(\hat{\Psi}_n) = \Psi'(\theta)$. The derivative with respect to θ transforms as follows

$$F\left[\frac{\partial \Psi'(\theta)}{\partial \theta}\right] = jn\hat{\Psi}_n. \quad (B7)$$

The conductivity elements (σ_{rr} , $\sigma_{r\phi}$, etc.) can be written in the form

$$\sigma(\theta) = \sigma_0 + \sigma_1 e^{j\theta} + \sigma_2 e^{j2\theta} + \sigma_1^* e^{-j\theta} + \sigma_2^* e^{j2\theta}, \quad (B8)$$

where $*$ denotes complex conjugate. The product $\sigma\Psi$ transforms to

$$F[\sigma(\theta)\Psi'(\theta)] = \sigma_0 \hat{\Psi}_n + \sigma_1 \hat{\Psi}_{n-1} + \sigma_2 \hat{\Psi}_{n-2} + \sigma_1^* \hat{\Psi}_{n+1} + \sigma_2^* \hat{\Psi}_{n+2}. \quad (B9)$$

With these identifies, Eq. (B5) becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \sum_{i=-2}^2 \sigma_{rri} \frac{\partial \hat{\Psi}_{n-i}}{\partial r} + r \sum_{i=-2}^2 \sigma_{r\phi i} \frac{\partial \hat{\Psi}_{n-i}}{\partial \phi} + \frac{r}{\sin \phi} \sum_{i=-2}^2 \sigma_{r\theta i} j(n-i) \hat{\Psi}_{n-i} \right]$$

$$\begin{aligned}
& + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left[\sin \phi \sum_{i=-2}^2 \sigma_{\phi r i} \frac{\partial \Psi_{n-1}}{\partial r} + \frac{\sin \phi}{r} \sum_{i=-2}^2 \sigma_{\phi \phi i} \frac{\partial \hat{\Psi}_{n-1}}{\partial \phi} \right. \\
& + \frac{1}{r} \sum_{i=-2}^2 \sigma_{\phi \theta i} j(n-i) \hat{\Psi}_{n-1} \left. \right] + \frac{1}{r \sin \phi} (jn) \left[\sum_{i=-2}^2 \sigma_{\theta r i} \frac{\partial \hat{\Psi}_{n-1}}{\partial r} + \frac{1}{r} \sum_{i=2}^2 \sigma_{\theta \phi i} \frac{\partial \hat{\Psi}_{n-1}}{\partial \phi} \right. \\
& + \frac{1}{r \sin \phi} \sum_{i=-2}^2 \sigma_{\theta \theta i} j(n-i) \hat{\Psi}_{n-1} \left. \right] + \frac{B}{2} \frac{\partial}{\partial r} (r^2 U'_r \sigma_{3rn}) + \frac{B}{r \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi U'_r \sigma_{d\phi n}) \\
& + \frac{B U'_r}{r \sin \phi} (jn) \sigma_{3\theta n} = 0 \quad , \tag{B10}
\end{aligned}$$

where $\sigma_{xx-i} = \sigma_{xxi}^*$. Since $\Psi(\theta)$ is real, $\hat{\Psi}_n = \hat{\Psi}_{N-n}^*$. Consequently, the range of n for Eq. (B10) is $0 \leq n \leq N/2$.

A singular solution exists for $\phi = 0$. At this location, $\hat{\Psi}_{n \neq 0} = 0$,

$$\frac{\partial \hat{\Psi}_0}{\partial \phi} = \frac{\partial \hat{\Psi}_2}{\partial \phi} = 0 \quad , \quad \text{and} \quad \frac{\partial \hat{\Psi}_1}{\partial \phi} \neq 0 \quad .$$

L'Hospital's rule is used to remove the $1/\sin \phi$ terms. In the limit, as $\phi \rightarrow 0$, Eq. (B10) becomes

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial r} \left\{ r^2 \sigma_{rro} \frac{\partial \hat{\Psi}_0}{\partial r} + r [(\sigma_{r\phi 1} - j \sigma_{r\theta 1}) \frac{\partial \hat{\Psi}_{-1}}{\partial \phi} + (\sigma_{r\phi 1}^* + j \sigma_{r\theta 1}^*) \frac{\partial \hat{\Psi}_1}{\partial \phi}] \right\} \\
& + \frac{2}{r} \left[\frac{\partial \sigma_{\phi ro}}{\partial \phi} \frac{\partial \hat{\Psi}_0}{\partial r} + \sigma_{\phi r 1} \frac{\partial^2 \hat{\Psi}_{-1}}{\partial \phi \partial r} + \sigma_{\phi r 1}^* \frac{\partial^2 \hat{\Psi}_1}{\partial \phi \partial r} + \sigma_{\phi \phi 0} \frac{\partial^2 \hat{\Psi}_0}{\partial \phi^2} + \frac{\partial \sigma_{\phi \phi 1}}{\partial \phi} \frac{\partial \hat{\Psi}_{-1}}{\partial \phi} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{\partial \sigma_{\phi\phi 1}^*}{\partial \phi} \frac{\partial \hat{\psi}_1}{\partial \phi} + \sigma_{\phi\phi 2} \frac{\partial^2 \phi_2}{\partial \psi^2} + \sigma_{\phi\phi 2}^* \frac{\partial \hat{\psi}_{-2}}{2}] - B \cos^2 I \frac{\partial (\sigma_2 U_r')}{\partial r} \\
& - \frac{B U_r'}{r} (1 + \sin^2 I) = 0.
\end{aligned}
\tag{B11}$$