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**LATTICE GAUGEFIXING AND OTHER TOPICS IN
LATTICE GAUGE THEORY***

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In memory of Grandma, who read me poetry,
and of Grandpa, who showed me baseball.

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ABSTRACT OF THE DISSERTATION

Lattice Gaugefixing and Other Topics in Lattice Gauge Theory

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We present results from four projects. In the first, quark and gluon propagators and effective masses and $\Delta I = 1/2$ Rule operator matching coefficients are computed numerically in gaugefixed lattice QCD. In the second, the same quantities are evaluated analytically in the strong coupling, $N \rightarrow \infty$ limit. In the third project, the Schwinger model is studied in covariant gauges, where we show that the effective electron mass varies with the gauge parameter and that longitudinal gaugefixing ambiguities affect operator product expansion coefficients (analogous to $\Delta I = 1/2$ Rule matching coefficients) determined by matching gauge variant matrix elements. However, we find that matching coefficients even if shifted by the unphysical modes are ξ invariant. In the fourth project, we show that the strong coupling parallelogram lattice Schwinger model has a different thermodynamic limit than the weak coupling continuum limit. As a function of lattice skewness angle these models span the $\Delta = -1$ critical line of 6-vertex models which, in turn, have been identified as $c = 1$ conformal field theories.

Chapter 1

Introduction and Summary

There is no evidence that the $SU(3)$ color gauge symmetry of QCD is broken in nature either explicitly or spontaneously. In lattice QCD, since by Elitzzer's theorem local gauge symmetry cannot break spontaneously the only way to see nonvanishing gauge variant correlation functions is if one breaks gauge symmetry by hand. Accordingly, lattice gaugefixing has drawn considerable attention in recent years. As described in the references of Chapt. 2 gluon, quark and photon propagators, effective masses, and wavefunctions have been studied in various " λ -gauges"

$$\lambda \partial_0 A^0 + \nabla \cdot \vec{A} = 0. \quad (1.1)$$

As summarized in Chaps. 3 and 4, the numerical work has spawned analytical and computational studies of longitudinal and topological gaugefixing ambiguities and their effects on gluon, quark and photon correlation functions and operator product expansion coefficients determined from gauge covariant matching conditions.

As a practical matter, lattice gaugefixing enables one to use nonlocal gauge variant operators, tantamount to "improved" interpolation operators, for numer-

ically evaluating gauge *invariant* correlation functions. In these projects, one is not interested in gauge variant quantities. Lattice gaugefixing is merely a technical intermediate step for evaluating form factors of gauge invariant processes.

In part of this Thesis, we will adopt the opposite philosophy. We will study gauge *variant* quark and gluon correlation functions. Assuming confinement, since quark and gluons are unphysical nothing guarantees their masses are gauge invariant. Indeed, we will find that their masses vary with λ .

Call the weighted sum of (naive) lattice operators whose matrix elements reproduce matrix elements of a continuum QCD operator the “lattice representation” of said continuum operator and the weights “lattice matching coefficients.” Verifying the $\Delta I = 1/2$ Rule in lattice gauge theory is a longstanding unsolved problem because it has not been possible to determine all the lattice matching coefficients of the continuum electroweak Hamiltonian responsible for $K \rightarrow \pi$ matrix elements. ($K \rightarrow \pi$ matrix elements are related to $K_s^0 \rightarrow \pi^+\pi^-$ matrix elements by chiral perturbation theory.) Specifically, the unknown coefficients correspond to a lower dimensional lattice operator which occurs in the lattice representations of several continuum operators. Since the coefficients of lower dimensional operators are proportional to inverse lattice spacing, they are inaccessible to weak coupling perturbation theory(WCPTh). In parts of this Thesis, we will use lattice gaugefixing as a technical device to determine values of these matching coefficients. The idea is to nonperturbatively replicate what is done in WCPTh, that is, to make the lattice representations of the continuum operators reproduce continuum quark correlation functions. This gauge covariant matching condition imposes constraints on the lattice matching coefficients which is sufficient to determine them. We find that matching coefficients evaluated in this way are numerically λ -independent—as they must be if they ultimately contribute to the matrix elements of gauge invariant

continuum operators.

Nonperturbative gaugefixing is a complex subject. To get a handle on what is going on, let us focus momentarily on the exactly solvable Schwinger model in Coulomb gauge and also in covariant gauges parametrized by gauge parameter ξ . (We were not able to solve the model in λ gauges.) While the actual situation in four dimensional QCD and lattice QCD may be quite different, it is helpful to have a litmus test for discarding broad arguments (“Mass is gauge invariant in perturbation theory; hence quark mass is gauge invariant.”) which do not distinguish between two dimensional $U(1)$ gauge theories and higher dimensional $SU(N)$ ones. Details are given in Chapt. 4; here we will just summarize some results.

In the Schwinger model, quarks are confined but the photon is physical and has a mass from the $U_A(1)$ anomaly. We define “effective mass” as the inverse correlation length of the zero momentum propagator. While the photon has a gauge invariant effective mass (equal to its physical mass), the effective quark mass varies with gauge parameter ξ . Coulomb gauge—the unitary gauge for the Schwinger model—has no unphysical modes and the quark propagator is the *physical* amplitude for quark propagation. Dynamically, dielectric breakdown of the vacuum is responsible for confinement—in two dimensional empty space an electric field $\vec{E}$ is constant and cannot die away. A quark’s electric field produces a, hence, constant stream of quark-antiquark pairs which move at the speed of light away from the original quark with a constant flux. As a result the quark propagator—the amplitude to have just one quark at $t \neq 0$ if there is a $t = 0$ quark—vanishes and the effective Coulomb gauge quark mass diverges. In covariant gauges, the presence of unphysical modes (or, alternatively, Gupta-Bleuler physical-state conditions on the Hilbert space) ruins the identification of the quark propagator with this amplitude. The covariant gauge quark propagator is not the physical amplitude for quark propagation and it

does not vanish despite confinement.

In the Schwinger model, not all matching conditions can be satisfied. In the satisfiable ones, different gauge variant matching conditions (relations between different gauge variant correlation functions) may lead to different (or similar) values for matching coefficients. The differences stem from unphysical gauge variant modes due to gaugefixing ambiguities—gauge variant operators transform differently under residual gauge symmetries. Nonetheless matching coefficients are ξ independent and, thus, plausibly gauge invariant. (This point is discussed in Chapt. 4, and differs from a conclusion of Ref. [1].) Thus, matching with different gauge variant correlation functions correspond to adopting *physically* inequivalent definitions of the matching coefficients. Hence the use of quark matrix elements (as opposed to, for example, diquark matrix elements) should be justified in a physical way before proceeding. This will be done in Section 2.2.8.

As an analytical study of effective quark mass and $\Delta I = 1/2$ Rule matching coefficients, a λ -gauge, $N \rightarrow \infty$, quenched strong coupling expansion in an infinite volume lattice has been developed in Chapt. 3. As proven in Chapt. 5, the thermodynamical limit of $\beta = 0$ lattice gauge theories may be completely different field theories than the continuum limit (the $\beta \rightarrow \infty$ thermodynamical limit). Hence the $\beta = 0$, $N \rightarrow \infty$ solution should be viewed as a toy model and not something necessarily related to QCD.

This Thesis is comprised of four parts. Part I was completed between 9 – 1988 and 1 – 1991; Part II between 1 – 1991 and 3 – 1991 and also 1 – 1992 and 3 – 1992; Part III between 4 – 1991 and 10 – 1991; Part IV between 3 – 1992 and 5 – 1992.

Part I (Chapt. 2) is a numerical study of quark and gluon propagators and $\Delta I = 1/2$ Rule quark matching coefficients in quenched lattice QCD gaugefixed to $\lambda\partial_0 A_0 + \nabla \vec{A} = 0$ gauges. We put forth a method for evaluating lattice operator

matching coefficients by matching lattice quark correlation functions. $\Delta I = 1/2$ Rule matching coefficients so determined are λ -independent modulo jackknife errors. However, better statistics are needed to resolve $\Delta I = 1/2$ enhancement. We define effective quark and gluon masses to be the inverse correlation lengths of the corresponding zero-momentum propagators. In the chiral limit, $\lambda = 1$ effective quark mass $M_q^{(1)} \sim 350(40) \text{ MeV}$ at $\beta \sim 6.0$. In contrast to the apparently λ -independent $\Delta I = 1/2$ Rule matching coefficients, $M_q^{(\lambda)}$ decreases $\sim 10\%$ as λ increases from 1 to 2 and another $\sim 5\%$ from $\lambda = 2$ to 25. Gluon and heavy quark masses have similar λ dependence and all diverge when $\lambda \rightarrow 0$. Furthermore, the quark dispersion relation seems to be λ -dependent.

Part II (Chapt. 3) is an analytical study of effective quark mass and $\Delta I = 1/2$ Rule matching coefficients in the color $N \rightarrow \infty$ and strong coupling limit of quenched lattice QCD. We construct an $N \rightarrow \infty$ Wilson loop picture of λ gaugefixing in which (i) the gaugefixed expectation value of a link chain C is the weighted sum over Wilson loops made by joining to C all selfavoiding chains $\tilde{C}$ which close C . (ii) Weights $A_{\tilde{C}}$, containing all the λ dependence, are proportional to the $\beta = 0$ gaugefixed expectation value of $\tilde{C}$. (iii) $A_{\tilde{C}}$ equals path-products of coefficients from the trace expansion of the gaugefixing Boltzmann weight. Using (ii) and (iii) we deduce formulae for $\beta = 0$, $N \rightarrow \infty$ effective quark masses and matching coefficients. In qualitative agreement with Part I numerical results, our expression for the effective quark mass decreases with increasing λ and the quark dispersion relation is not covariant when $\lambda \neq 1$. Quark matching coefficients are λ -independent in this limit.

Part III (Chapt. 4) is a study of gaugefixing in the exactly solvable Schwinger model. Upon constructing the covariant gauge quantization of this model, we derive expressions for the full quark and photon propagators and effective masses. We find

that, while the physical photon has a gauge invariant effective mass, the effective quark mass varies with gauge parameter ξ . Additionally, longitudinal gaugefixing ambiguities affect values of operator product expansion coefficients obtained from matching gauge variant matrix elements. Nonetheless we find that even if matching coefficients are shifted by gaugefixing modes they remain ξ independent and, thus, possibly gauge invariant. Chapt. 4 differs from Ref. [1] in this last point.

Part IV (Chapt. 5) is a study of the lattice Schwinger model, which has not been solved except in the strong coupling limit. We construct a new Fierz formulation of the hopping expansion for strongly coupled Wilson fermions. In this framework it is straightforward to interpret such systems as interacting dimer, monomer, and loop gasses. As an application, we show that the strong coupling Schwinger model on parallelogram lattices with nonbacktracking Wilson fermions span, as a function of the lattice skewness angle, the $\Delta = -1$, central charge $c = 1$ critical line of 6-vertex models. The Fierz hopping expansion also applies to backtracking Wilson fermions. While backtracking systems seem to have richer structure, we have not been able to identify them with exactly solved models. The point, however, is clear: assuming that the weak coupling continuum limit of the lattice Schwinger model is the continuum Schwinger model, $\beta = 0$ lattice gauge theories have thermodynamical continuum limits distinct from the weak coupling continuum limits.

Chapter 2

Gaugefixing Technique for Lattice Matching Coefficients

2.1 Motivation and Results

2.1.1 Matching Coefficients

Computation of $\Delta I = 1/2$ Rule pseudoscalar matrix elements in lattice QCD requires knowing electroweak operators of the form (Operators $O_{\pm}$ are defined in Section 2.2.8.)

$$O_{\pm}^{\text{cont}}[LL] = z_{\pm}^J J^{\text{latt}} + \sum_{\Gamma_1 \Gamma_2} z_{\pm}^{\Gamma_1 \Gamma_2} O_{\pm}^{\text{latt}}[\Gamma_1 \Gamma_2] , \quad (2.1)$$

$$\Gamma_1 \Gamma_2 \in \{LL, SS, PP, TT, LR\} , \quad J(0) \equiv \bar{s}(0)d(0) . \quad (2.2)$$

Proportional to $(m_s + m_d)a^{-2}$, the $z_{\pm}^J$ coefficients are beyond weak coupling perturbation theory or the usual lattice methods.[2] Matching condition[3, 4]

$$\langle s | O_{\pm}^{\text{cont}}[LL] | d \rangle = 0 , \quad (2.3)$$

the parity even quark-equivalent of the parity odd hadronic renormalization condition that $K \not\mapsto$ vacuum, implies that the lattice operators satisfy

$$z_{\pm}^J = - \sum_{\Gamma_1 \Gamma_2} \alpha_{O_{\pm}[\Gamma_1 \Gamma_2]} z_{\pm}^{\Gamma_1 \Gamma_2} , \quad (2.4)$$

$$\langle s | O_{\pm}^{\text{latt}}[\Gamma_1 \Gamma_2] | d \rangle \equiv \alpha_{O_{\pm}[\Gamma_1 \Gamma_2]} \langle s | J^{\text{latt}}(0) | d \rangle. \quad (2.5)$$

Such quark matrix elements vanish in the absence of gaugefixing. Since by confinement they are not physical, Eq. (2.5) may give a value of $\alpha_{O_{\pm}[\Gamma_1 \Gamma_2]}$ different from its physical sector value. Gauge variance of $\alpha_{O_{\pm}[\Gamma_1 \Gamma_2]}$ would be one signal of such a discrepancy.

2.1.2 Free Particle Ansatz

On $\beta = 5.7$ and 6.0 quenched Wilson lattices in “ λ gauges[4]”

$$\lambda \partial_0 A^0 + \nabla \cdot \vec{A} = 0 , \quad (2.6)$$

effective quark and gluon masses M_q and M_g are evaluated by matching gaugefixed $\vec{p} = 0$ propagators to free particle forms at large t_E . In particular, if $|q\rangle$ is the lightest state coupling to q , inserting

$$1_{\text{one particle}} = \sum_{r=1}^2 \int \frac{d\vec{p}}{(2\pi)^3 2E_q} |q_p^{(r)}\rangle \langle q_p^{(r)}| \quad (2.7)$$

between ψ_x and $\bar{\psi}_0$ in the quark propagator yields (V_x is the gauge transformation which brings ψ_x into λ gauge.)

$$\lim_{t_E \rightarrow \infty} \sum_x e^{i\vec{p} \cdot \vec{x}} \langle V_x \psi_x \bar{\psi}_0 V_0^\dagger \rangle = Z_q \left(\frac{M_q + i\vec{p}}{2E_q^{(\lambda)}} \right) e^{-E_q^{(\lambda)} |t_E|} \quad (2.8)$$

where the “ (λ) ” superscript allows for λ dependence, $E_q^{(\lambda)}(\vec{p}=0) \equiv M_q^{(\lambda)}$, and $Z_q^{\text{free}} = 1$. The Dirac matrices and free particle wavefunctions are given in Appendix A.1. Motivated by CPT relation $M_\pi^2 \propto m_q$, fitting to

$$M_q^{(\lambda)} = b^{(\lambda)} M_\pi^2 + M_c^{(\lambda)} \quad (2.9)$$

in the chiral regime yields $b^{(1)} \sim 2.7(.3) \times 10^{-4}/MeV$ and $M_c^{(1)} \sim 350(40)MeV$ at $\beta \sim 6.0$.

We offer no justification for using the free particle ansatz. Rather, we view it as a metaphorical device for describing our results. Going beyond this would require overcoming several interpretational pitfalls. First, even in the absence of confinement, as in QED_{3+1} , the meaning of gauge variant propagators is subtle. In axial gauge, $\langle \Psi_y \bar{\Psi}_x \rangle$ has a physical interpretation.[5] The gauge *invariant* propagator of a $\vec{p}=0$ hydrogen atom in the approximation that the heavy proton is stationary satisfies

$$\lim_{t_B \rightarrow \infty} \langle \bar{\Psi}_{(t,\vec{x})} e^{-i \int_0^t dt' A_0(t',\vec{x})} \Psi_{(0,\vec{x})} \rangle e^{-M_P t_B} = Z_H e^{-M_H t_B} \quad (2.10)$$

where $M_H(M_P)$ is the atomic(proton) mass. If $\{V_x\}$ are the $U(1)$ gauge transformations taking each background electromagnetic field into axial gauge, then

$$V_{(t,\vec{x})} e^{-i \int_0^t dt' A_0(t',\vec{x})} V_{(0,\vec{x})}^\dagger \equiv 1. \quad (2.11)$$

In this gauge Eq. (2.10) becomes

$$\lim_{t_B \rightarrow \infty} \langle [\bar{\Psi}_{(t,\vec{x})} V_{(t,\vec{x})}^\dagger] 1 [V_{(0,\vec{x})} \Psi_{(0,\vec{x})}] \rangle = Z_H e^{-(M_H - M_P) t_B}. \quad (2.12)$$

Since the LHS of (2.12) is proportional to the trace of the axial gauge electron

propagator, the effective axial gauge electron mass is $M_e^{\text{axial}} = M_H - M_P$. (While $\Psi_{(t,\vec{x})}$ is not summed over $\vec{x}$ as is customary, the projection to $\vec{p} = 0$ is nonetheless enforced as $t_E \rightarrow \infty$.) This argument extends to QCD. Note that the identification of $M_H - M_P$ with the electron propagator pole is not gauge covariant. In gauges where $A_0 \neq 0$, the RHS of Eq. (2.11) is not unity and the LHS of Eq. (2.12) is not the electron propagator. Although $M_H - M_P$ is gauge invariant M_e^{axial} is not.

Second is the issue of gauge dependence. In the Schwinger model, where electrons are confined, the effective electron mass varies with the covariant gauge parameter as described in Chapt. 4. Similarly, as described in Section 2.2, effective quark and gluon masses vary with gauge parameter λ . Furthermore, the dependence of quark energy E_q on momentum $\vec{p}$ depends on λ so that free particle relation $E_q^2 = M_q^2 + \vec{p}^2$ cannot be simultaneously obeyed in all gauges.

Third, we compute only correlation functions on the lattice, not quark matrix elements. As exemplified by Eq. (A.12), Z_q is necessary to convert correlation functions to matrix elements. Z_q is ambiguous because, in addition to its statistical noise being exponentially magnified relative to the M_q noise, the quark propagator is not a true exponential. As discussed in Section (2.2.5), Z_q can only be computed within a nonexponential (hence, not “free particle”) model.

2.1.3 Lattice Gaugefixing

Gaugefixed lattice QCD is given in terms of[6]

$$[\Theta]_\theta \equiv z_\theta^{-1} \int [d\theta] e^{-S^\theta} \Theta, \quad z_\theta \equiv \int [d\theta'] e^{-S^{\theta'}}, \quad \theta \in \{f, v, u\}, \quad (2.13)$$

$$S^f \equiv \sum_{x,y} \bar{\psi}_x [\delta_{x,y} + K_B \sum_n (r - \hbar) U_{x,n} \delta_{y,x+\hat{n}}] \psi_y, \quad (2.14)$$

$$S^v[\Omega] \equiv - \sum_{\mathbf{x}, n} \frac{J_n}{2} \text{Tr} \Omega_{\mathbf{x}, n}, \quad n \in \pm\{0, \dots, D-1\}, \quad (2.15)$$

$$J_{-n} \equiv J_n \equiv \lambda_n / \xi, \quad \lambda_\mu \equiv (\lambda, \bar{\lambda}, \dots, \bar{\lambda}), \quad \Omega_{\mathbf{x}, n} \equiv V_{\mathbf{x}} U_{\mathbf{x}, n} V_{\mathbf{x}+\hat{n}}^\dagger, \quad (2.16)$$

$$S^u[U] = S^u[\Omega] = \beta \text{Re} \sum_{\square} \text{Tr} \square, \quad \langle \Theta \rangle \equiv \left[\left[\langle \Theta \rangle_f \right]_v \right]_u, \quad (2.17)$$

where the sum ranges over all lattice plaquettes “ $\square$.” D is the spacetime dimensionality. Quarks are quenched by choice. Consistency with the Fadeev-Popov method requires $\{V_{\mathbf{x}}\}$ to be quenched. Appendix A.2 summarizes the numerical implementation of this system when $\xi \rightarrow 0$.

As an independent study of numerical results, we develop a λ gauge strong coupling expansion on an infinite volume lattice. Details are provided in Chapt. 3. We briefly summarize results here for comparison to numerical data. S^v is minimized site-by-site in numerical simulations to achieve λ gauge. Similarly, the $\xi \rightarrow 0$ limit corresponds to λ gauge. Integrating out $\{V_{\mathbf{x}}\}$ when $\xi \equiv 1/(2N)$ and $N \rightarrow \infty$ yields

$$\langle V_y(y \blacktriangleleft_w) V_w^\dagger \rangle_{\alpha j} = \delta_{\alpha j} \sum_{w \tilde{\mathbf{A}}_y} A_{w \tilde{\mathbf{A}}_y}(J) \text{Tr}[y \blacktriangleleft_w w \tilde{\mathbf{A}}_y]_u \quad (N \mapsto \infty), \quad (2.18)$$

$$A_{w \tilde{\mathbf{A}}_y} = \frac{1}{N^2} \text{Tr} \langle V_y y \tilde{\mathbf{A}}_w V_w^\dagger \rangle_{|\beta=0}. \quad (2.19)$$

α and j are color indices ranging from 1 to N . Eq. (2.18) says the gaugefixed expectation value of a link chain $y \blacktriangleleft_w$ is the weighted sum over all Wilson loops made by joining to $y \blacktriangleleft_w$ a selfavoiding link segment $w \tilde{\mathbf{A}}_y$. (2.19) identifies weights $A_{w \tilde{\mathbf{A}}_y}$ with the $\beta = 0$ gaugefixed expectation value of $w \tilde{\mathbf{A}}_y$. All of the gauge dependence is in these weights.

The gluon does not propagate at $\beta = 0$, where links oscillate randomly. At $D = 4, \beta = 0, r = 1, N \rightarrow \infty$, and in an orthogonal trace approximation (related to

details of the $N \rightarrow \infty$ limit) described in Chapt. 3, the quark propagator dispersion relation is

$$E_q^{(\lambda)^2} = M_q^{(\lambda)^2} + \sum_i \bar{g}_i^2 p_i^2 + \mathcal{O}(a^3), \quad E_q^{(\lambda)} \equiv \pm i p_0, \quad (2.20)$$

$$\bar{g}_i(\lambda) = \begin{cases} \frac{3}{4\lambda} & \lambda \leq \frac{1}{2}; \\ \frac{3\lambda}{4\lambda-1} & \lambda \geq \frac{1}{2}. \end{cases} \quad (2.21)$$

Since $\bar{g}_i(\lambda \neq 1) \neq 1$, the dispersion relation is not free-particle form except in Landau gauge. At $\vec{p} = 0$ $\bar{g}_i$ dependence drops out, but $M_q^{(\lambda)}$ remains λ dependent. $M_q^{(\lambda)}$ varies with λ , diverges in Coulomb gauge, and is analytic and linear in m_q near $m_q \rightarrow 0$. Following to (2.9), Taylor expansion

$$M_q^{(\lambda)} = M_c^{(\lambda)} + B^{(\lambda)} m_q + \mathcal{O}(m_q^2) \quad (2.22)$$

yields at $\beta = 0$ and $r = 0$ that

$$M_c^{(\lambda)} = \frac{\sqrt{7}}{242\lambda} \begin{cases} 67 - 25\lambda^2 & \lambda \leq \frac{1}{2}; \\ \frac{-25+200\lambda+872\lambda^2}{4(4\lambda-1)} & \lambda \geq \frac{1}{2}; \end{cases} \quad (2.23)$$

$$B^{(\lambda)} = \begin{cases} \frac{20203+4753\lambda^2}{29282\lambda} & \lambda \leq \frac{1}{2}; \\ \frac{4753-38024\lambda+399296\lambda^2}{117128\lambda(4\lambda-1)} & \lambda \geq \frac{1}{2}. \end{cases} \quad (2.24)$$

The qualitative decrease of $M_c^{(\lambda)}$ with increasing λ in strong coupling gives us confidence that the numerical λ dependence of M_q is not a finite volume artifact or due to details of how M_q is extracted in the numerical simulations. Figure A.1 compares the strong coupling $M_c^{(\lambda)}$ function to the numerical effective quark masses rescaled so that they all agree at $\lambda = 1$.

In the same limit, matching coefficients for $\bar{s}d$ subtraction of $\Delta I = 1/2$ Rule

operators satisfy

$$\alpha_{O_{\pm}[\Gamma_1\Gamma_2]} = (\delta_{\Gamma_1\Gamma_2,SS} \mp \frac{1}{2^{D/2}N} f_{\Gamma_1\Gamma_2}^{SS}) \langle \bar{\psi}_0 \psi_0 \rangle \quad (2.25)$$

where Fierz coefficients $f_{\Gamma_1\Gamma_2}^{SS}$ are given in Eq. (A.5). Since $\langle \bar{\psi}_0 \psi_0 \rangle$ is gauge invariant these $\beta = 0$ matching coefficients, which are directly related to physical continuum decay rates, are gauge invariant. At $\beta = 6.0$, $\alpha_{O_{\pm}[\Gamma_1\Gamma_2]}$ modulo statistics seems to be λ independent.

Of course, λ independence at $\beta = 0$ and $N \rightarrow \infty$ does not imply gauge independence outside of λ gauges or the $\beta = 0$, $N \rightarrow \infty$ limit.

2.2 Numerical Results

Figures are given in Appendix A.7. Table 2.1 lists the lattices we have used. Ideally one wants to have β as large as possible to be near the scaling limit. However, since lattice spacing decreases the closer to the continuum, more and more lattice sites are required to simulate a lattice of a given spatial size. Since the lattice must be larger than the Compton wavelengths of the particles (usually hadrons) being studied, values of β are restricted by computer resources.

Both the plaquettes and gauge transformations have periodic boundary conditions in time and space. Gaugefixing does not affect Z_N symmetry associated with finite temperature deconfinement. Quark propagators are periodic in space and have Von Neumann boundary conditions in time. On the $\beta = 5.7$ lattice, the hopping constant values studied are $K = .094, .155, .160, .164, .166$. On the $\beta = 6.0$ lattice, $K = .118, .152, .154, .155$. Unless otherwise specified, dimensionless values such as aM are quoted.

<i>gauge</i>	lattice size fermion	β (K_c)	$a^{-1}(\text{GeV})$	spatial size(fm)	#confs
$16^3 \times 24$	$16^3 \times 25$	5.7 (.170)	~ 1	~ 3.2	16
$24^3 \times 40(8)$	$24^3 \times 39$	6.0 (.157)	~ 1.7	~ 2.8	8
$24^3 \times 40(12)$	$24^3 \times 39$	6.0 (.157)	~ 1.7	~ 2.8	12

Table 2.1: Lattices used

Unless otherwise indicated, effective quark and gluon masses $M_q^{(\lambda)}$ and $M_g^{(\lambda)}$ are computed according to

$$M \equiv \sum_{t_E \in \text{plateau}} \log(P(t_E)/P(t_E + 1)) , \quad (2.26)$$

$$P(t_E) \equiv \frac{1}{2} \text{Tr} \sum_{\vec{x}} \langle (V\psi)_x (\bar{\psi} V^\dagger)_0 + (t_E \rightarrow -t_E) \rangle , \quad x \equiv (t_E, \vec{x}). \quad (2.27)$$

If the plateau is not specified, it is $t_E = \{5, 6, 7\}$ at $\beta = 5.7$ and $t_E = \{8, 9, 10\}$ at $\beta = 6.0$.

For the $24^3 \times 40$ lattice, there are four “exceptional” configurations, where the critical hopping constant K_c is shifted from that of the other 8 configurations. One would not expect to resolve a slight K_c shift using the effective quark mass M_q since, between $K = .155$ and K_c , M_q changes by 20%—less than the jackknife error of a single-configuration- M_q . No anomalous effective quark or gluon mass shifts are seen in the exceptional configurations up to statistics.

2.2.1 Gluon Propagator

The λ gauge gluon field is defined on the lattice as

$$A_\mu(x) \equiv \left(\frac{\Omega_{x,\mu} - \Omega_{x,\mu}^\dagger}{2i} \right) - \frac{1}{N} (\text{trace}). \quad (2.28)$$

The lattice gluon propagator was first studied by Mandula and Ogilvie in Landau gauge.[7] Upon matching the nonperturbative propagator to the free, $\vec{p} = 0$ gluon propagator, they found a nonvanishing effective gluon mass $M_g^{(1)} \sim 500 \text{ MeV}$. In an analogous calculation, effective photon mass in compact QED vanishes.[8]

λ	24by40(8) t=7,8,9	24by40(12) t=7,8,9	16by24 t=4	16by24 t=5
0	—	—	.32(.66)	—
.5	—	—	.66(.099)	.61(.16)
1	.53(.030)	.50(.045)	.52(.044)	.49(.071)
2	.50(.034)	.45(.045)	.57(.038)	.62(.092)
2p	—	—	.51(.028)	.62(.053)
10	.40(.050)	.36(.044)	.35(.027)	.36(.044)
25	—	—	.34(.023)	.36(.034)

Table 2.2: $M_g(\text{GeV})$ Scaling and λ Dependence

We have reproduced their results on our larger lattices. In a periodic box of length T , the free gluon propagator is

$$\sum_{i=1}^3 \int d\vec{x} \text{Tr} \langle A_i(x) A_i(0) \rangle = Z_g [e^{-M_g t_E} + e^{-M_g (T-t_E)}] \quad (2.29)$$

$$\approx Z_g e^{-M_g t_E} \quad (T \gg t_E). \quad (2.30)$$

Table 2.2 lists effective gluon masses defined by matching to (2.30). $M_g^{(\lambda)}$ decreases with increasing λ . Table 2.3 depicts the jackknifed ratios

$$R_g^{(\lambda)} = M_g^{(\lambda)} / M_g^{(1)}. \quad (2.31)$$

Figure A.2 and Table 2.4 show the single configuration 24by40 M_g time-averaged over $t_E = 7, 8, 9$. (It was not possible to extract useful single configuration results on 16by24.) The marked tendency for single configuration effective masses to decrease with increasing λ lends credence to the configuration-averaged λ behavior. Because

different configurations are in randomly different Gribov copies, the small variation of the single configuration masses relative to what might be expected from the jackknife variance suggests Gribov effects are unimportant.

Ratio	24by40(8) $t = 7, 8, 9$	24by40(12) $t = 7, 8, 9$	16by24 $t = 4$	16by24 $t = 5$
$R_g^{(2)}$	.94(.041)	.92(.042)	1.09(.14)	1.26(.27)
$R_g^{(10)}$	.75(.079)	.73(.051)	0.67(.065)	.73(.12)
$R_g^{(25)}$	—	—	0.65(.073)	.73(.14)

Table 2.3: Jackknifed ratio of gluon masses

configuration	$\lambda = 1$	$\lambda = 2$	$\lambda = 10$
02	.49	.39	.29
03	.49	.54	.30
04	.65	.60	.50
05	.61	.50	.54
06	.56	.59	.52
07	.51	.57	.47
08	.50	.44	.47
09	.48	.42	.28
51	.80	.71	.41
52	.50	.36	.33
53	.31	.29	.24
54	.34	.37	.28

Table 2.4: Single Configuration Gluon Masses

2.2.2 Heavy Quark Propagator

In the static approximation, the quark propagator in a background gauge field is

$$[\psi_x \bar{\psi}_0]_f^{\text{heavy}} = \frac{1}{2} \delta(\vec{x}) \hat{U}_{0,t_B \hat{0}} (1 \pm \gamma_0) e^{-m_q |t_B|} \quad (2.32)$$

where $\hat{U}_{0,t_E\hat{0}}$ is a straight Wilson line between $(0, \vec{0})$ and $(t_E, \vec{0})$. The expectation value of the Wilson line can be parametrized as

$$\frac{1}{N_s N} \lim_{t_E \rightarrow \infty} \sum_{\vec{y}} \text{Tr} \langle V_{(t_E, \vec{y})} \hat{U}_{\vec{y}, \vec{y}+t_E\hat{0}} V_{(0, \vec{y})}^\dagger \rangle = Z_H e^{-V_H^{(\lambda)} |t_E|} \quad (2.33)$$

where N_s is the number of sites per lattice timeslice and the sum over $\vec{y}$ is for statistical enhancement.

Expression (2.32) has two free fermion propagator properties: (a) equality of scalar and γ^0 components and (b) free particle electromagnetic form factors. We cannot predict, however, that $\text{Tr} \langle V_x \hat{U}_{0,t_E\hat{0}} V_0^\dagger \rangle$ has exponential time dependence (as would be required of a free quark). Z_H and $V_H^{(\lambda)}$ are real due to charge conjugation invariance. Within the free particle ansatz, $V_H^{(\lambda)}$ can be interpreted as

$$V_H^{(\lambda)} = \lim_{m_q \rightarrow \infty} (M_q^{(\lambda)} - m_q). \quad (2.34)$$

Following (2.12), in axial gauge V_H is the binding energy between a heavy quark and another heavy quark (the “proton”). This suggests that $V_H^{(A_0=0)}$ vanishes and, indeed, it does.

λ_t	Z_H	$V_H^{(\lambda)}$
1.0	2.13(.145)	.454(.0095)
2.0	2.24(.089)	.408(.006)
10.0	4.17(.415)	.336(.007)
axial	1.0	0.0

Table 2.5: 24by40 heavy quark

Table 2.5 contains results of matching the λ gauge heavy quark propagator at large $|t_E|$ to the RHS Eq. (2.33). $V_H^{(\lambda)}$ decreases with increasing λ .

2.2.3 Quark Propagator

Clifford expanding the quark propagator yields

$$\sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \langle V_x \psi_x \bar{\psi}_0 V_0^\dagger \rangle = [\mathcal{A}(E_q, t_E) + i\mathcal{B}(E_q, t_E)\not{p}] e^{-E_q|t_E|}. \quad (2.35)$$

The pseudoscalar and axial vector terms are suppressed by parity. Global color symmetry requires $\mathcal{A}$ and $\mathcal{B}$ to be color singlets. $\mathcal{A}$ and $\mathcal{B}$ are real by hermiticity symmetry. Comparing (2.8) to (2.35) yields

$$\mathcal{A} = \frac{1}{M_q} \mathcal{B} = Z_q \frac{M_q}{2E_q}. \quad (2.36)$$

The relation between $\mathcal{A}$ and $\mathcal{B}$ is a consequence of the Dirac equation, which free fermion propagators satisfy.

time	sc vs vec	.002 vs .0002
1	1.59(.057)	1.01(.061)
2	1.20(.038)	1.00(.056)
3	1.10(.030)	0.98(.053)
4	1.05(.027)	0.98(.056)
5	1.02(.038)	0.96(.066)
6	1.01(.043)	0.97(.077)
7	1.00(.045)	0.98(.090)
8	1.00(.048)	0.98(.109)
9	0.98(.040)	0.98(.126)
10	1.02(.054)	0.98(.137)
11	1.06(.069)	1.02(.148)
12	1.00(.053)	1.01(.164)

Table 2.6: $\mathcal{B}/(M_q^{(1)}\mathcal{A})$; and $M_q^{(1)}$ at $\epsilon(1, \mathcal{U}_1) = .002$ divided by at .0002

As Table 2.6 shows, the effective propagator satisfies the first equality of Eq. (2.36)—“Scalar Vs. Vector” in this Table means $\mathcal{B}/(M_q\mathcal{A})$. Table 2.7 shows the chiral behavior of the effective quark mass. Figure A.4 shows the ratio $M_q^{(\lambda)}/M_q^{(1)}$ as a

K	$M_q^{(1)}$	$M_q^{(2)}$	$M_q^{(10)}$
.118	1.74(.04)	1.67(.03)	1.59(.015)
.152	.54(.02)	.49(.02)	.46(.03)
.154	.46(.03)	.41(.03)	.39(.03)
.155	.43(.03)	.38(.03)	.35(.04)

Table 2.7: $M_q^{(\lambda)}$ (GeV), 24by40, $a^{-1} = 1.7\text{GeV}$

function of $M_q^{(1)}$. Table 2.8 has the single configuration quark masses for $K = .152$.

config #	one config quark mass
02	.575
03	.539
04	.530
05	.508
06	.606
07	.448
08	.576
09	.598
51	.579
52	.619
53	.521
54	.482

Table 2.8: Single Configuration $M_q^{(1)}$, 24by40

λ	$\vec{p}$	E_q	E_q^2	$s(\lambda, \vec{p})$
1	(0, 0, 0)	.31(.012)	.098(.0078)	.85(.28)
1	Perm(1, 0, 0)	.40(.027)	.157(.023)	1.1(.41)*
1	Perm(2, 0, 0)	.61(.078)*	.38(.100)*	—
2	(0, 0, 0)	.29(.012)	.084(.0072)	.61(.13)
2	Perm(1, 0, 0)	.35(.017)	.125(.012)	.70(.24)*
2	Perm(2, 0, 0)	.48(.15)	.235(.194)	—
10	(0, 0, 0)	.27(.017)	.072(.0089)	.85(.28)
10	Perm(1, 0, 0)	.32(.017)	.100(.011)	1.1(.41)*
10	Perm(2, 0, 0)	.39(.028)	.154(.022)	—

Table 2.9: Dispersion Relation and Slope $s(\lambda, \vec{p})$, $K = .152$, 24by40, * $t_E = 7, 8$

We have checked numerically that, if the quark is given nonzero 3-momenta $\vec{p}$, it retains the free particle form in the vector components. Table 2.9 shows the momentum dependence of effective quark energy defined by matching the scalar component of the quark propagator to $\sim e^{-E_q t_B}$. "Perm" means the propagator has been averaged over the three degenerate directions for a given $\vec{p}^2$. Since in the continuum limit

$$E_q^2 = M_q^{(\lambda)^2} + s(\lambda, \vec{p}) \vec{p}^2, \quad (2.37)$$

define

$$s(\lambda, \vec{p}) \equiv \frac{E_q^2(\vec{p} + \delta_p) - E_q^2(\vec{p})}{(\vec{p} + \delta_p)^2 - \vec{p}^2} \quad (2.38)$$

where δ_p is the minimum momentum increase on the lattice. For example, if $\vec{p} = (0, 0, 0)$ then $\delta_p = \text{Perm}(1, 0, 0)$. In Table 2.9 $s(\lambda, \vec{p})$ decreases with increasing λ as predicted by Eq. (2.20).

2.2.4 M_ρ and M_q

For $K = .155$ on the $\beta = 6.0$ lattice, the ratio

$$R(\lambda) \equiv M_\rho / M_q^{(\lambda)} \quad (2.39)$$

of the vector meson mass M_ρ over quark mass increases from $R(1) = 1.94(.15)$ to $R(10) = 2.38(.29)$. At other hopping constants, $R(10)$ is also greater than 2. At $K = .154$ $R(10) = 2.33(.22)$ and, at $K = .152$, $R(10) = 2.27(.15)$. At $(r = 0, \beta = 0)$, $R(1) = 2.6$ and $R(10) = 4.2$ and $R(\lambda)$ is always greater than 2. This suggests that

$$M_\rho > 2M_q^{\min} \quad (2.40)$$

where $M_q^{\min}$ is the minimum value of effective quark mass at the hopping constant value of M_ρ in all possible gauges (including λ gauges). Figure A.3 shows that $R(1)$ is independent of hopping constant.

2.2.5 Correlated Fits

In an effort to uncover an underlying gauge invariant mass, correlated fits[9] of the quark propagator were done to[4]

- $P_1 \equiv Ze^{-m_A t}$ (“one exponential”),
- $P_2 \equiv Cte^{-m_C t}$,
- $P_3 \equiv At^B e^{-m_B t}$, and
- $P_4 \equiv Ae^{-m_A t} + Be^{-m_B t}$ (“two exponential”).

(A and B here are not related to the same symbols used previously.) P_2 and P_3 are motivated by the form of the electron propagator in covariant gauges, where it has branch cuts corresponding to virtual longitudinal photon modes. In the Block-Nordsieck approximation the $\vec{p} = 0$ effective electron propagator in the $m_e t_E \rightarrow \infty$ limit is[10]

$$\int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \frac{e^{ip_0 t_E}}{(p_0^2 + m_e^2)^{1+b}} \longrightarrow Z t_E^{-b} e^{-m_e t_E} \equiv e^{-M_e^{(\xi)} t} \quad (2.41)$$

where $b \equiv \hbar\alpha_{em}(3 - \xi)/2\pi$ and ξ is the covariant gauge parameter. At finite t_E , effective mass $M_e^{(\xi)} = m_e + \frac{b}{t_E} \log t_E$ is ξ dependent. Eq. (2.41) implies that in covariant gauges infrared photon corrections to the free quark propagator are *wholly* gauge artifacts. In $\xi = 3$ (Yennie) gauge, the effective propagator equals the (suitably renormalized) free electron propagator. Since the difference between $M_e^{(\xi)}$ and $M_e^{(3)}$ is due to gauge transformations, it is not computable. The physical electron mass is retrieved in the limit $\lim_{t \rightarrow \infty} M_e^{(\xi)} = m_e$.

The numerical quark propagator is not a good fit to P_1 , but is to P_2, P_3 with $B \sim 1$, and also P_4 with $A \approx -B$. Moreover, P_3 is a good fit for all available λ when $m_B^{(\lambda)}$ is fixed at its $\lambda = 2$ value and only A and B are varied with λ . However, when A, B and m_B are simultaneously varied, the best fit m_B decreases with increasing λ . χ^2 fitting meson propagators to P_4 gives smaller mass m_a consistent with meson masses obtained in the usual way, a strong signal for A and B , and widely split excitation masses.

t_i	t_f	A	B	m_a	m_b	χ^2/dof
1	7	1.2(.48)	-1.0(.44)	.56(.045)	.99(.230)	2.90/3
2	7	8.7(.86)	-8.4(.95)	.66(.023)	.70(.026)	1.39/2
3	7	7.9(3.03)	-7.7(4.10)	.67(.032)	.72(.040)	0.67/1
3	8	8.8(1.10)	-8.7(1.31)	.68(.029)	.73(.035)	5.71/2

Table 2.10: Sample determination of fit timeslices

K	$M_q^{(1)}$	$m_A^{(1)}$
.094	2.91(.005)	2.90(.008)
.155	.90(.008)	.89(.018)
.160	.70(.009)	.70(.010)
.164	.53(.009)	.53(.008)
.166	.43(.010)	.42(.009)
.168	.32(.012)	.27(.010)

Table 2.11: Compare effective to fit mass, 16by24

Timeslices used are different for each fit unless otherwise specified, and determined as follows:

- Step 1: Compute the best fit for a set of trial timeslices. For the K=.160, $\lambda = 0.5$ quark fit to P_4 , the trial fits are given in Table 2.10.
- Step 2: Timeslices 2 – 7 and 3 – 7 yield the same small χ^2/dof (χ^2 per degree of freedom.) while the χ^2 costs of including $t_E = 1$ is 1.5 and $t_E = 8$ is greater

than 5. Therefore, timeslices 2 – 7 is used for this propagator. The results for the two good fits, 2 – 7 and 3 – 7, are consistent within errors. Table 2.11 compares $M_q^{(\lambda)}$ to $m_A^{(\lambda)}$.

Errors are computed from the parameter space covariance matrix $\mathcal{C}$, determined from the numerical Hessian and derivatives of the trial function. For good fits, errors calculated in this way agree within a factor of two to standard deviations obtained by jackknifing the fit.

K	t_i	t_f	$\mathcal{A}$	$\mathcal{B}$	m_a	m_b	χ^2/dof
.094	4	9	3.1(.37)	6.4(1.93)	2.90(.008)	3.47(.148)	0.48/2
.155	3	8	1.5(.24)	4.4(1.12)	.89(.018)	1.63(.190)	1.01/2
.160	3	8	1.5(.12)	5.0(1.14)	.70(.010)	1.57(.128)	1.27/2
.164	2	9	1.7(.07)	10.2(1.19)	.53(.008)	1.80(.079)	1.18/4
.166	2	8	1.8(.08)	10.3(1.48)	.42(.009)	1.74(.097)	1.84/3
.168	2	7	2.7(.30)	10.3(3.74)	.27(.010)	1.76(.258)	1.38/2

Table 2.12: Pseudoscalar meson fits to P_4

λ	Z	m_A	χ^2/dof
0.0	1.20(1.00)	.54(.160)	.0048/1
0.5	.76(.210)	.54(.041)	3.41/1
1.0	.69(.153)	.47(.019)	7.37/1
2.0	.60(.083)	.42(.026)	2.12/1
10.0	.24(.054)	.36(.026)	9.44/1
25.0	.17(.034)	.39(.034)	13.43/1

Table 2.13: 16by24 $K = .160$ fits to P_1

Table 2.12 shows results of correlated pseudoscalar meson fits to P_4 . Table 2.14 shows results of a correlated fit to P_1 using timeslices $t_E = [8, 9, 10]$. Errors are from jackknifing in this case. Table 2.13 shows results of a correlated fit to P_1 using timeslices $t_E = \{5, 6, 7\}$. Errors are from jackknifing. Tables 2.15 and 2.17 show results of a correlated fit to P_3 using respectively timeslices $t_E = [8, 9, 10, 11]$ and $t_E = [5, 6, 7, 8]$.

λ	Z	m_A	χ^2/dof
1.0	.80(.100)	.54(.036)	2.10/1
2.0	.70(.043)	.49(.016)	1.70/1
10.0	.52(.067)	.46(.033)	3.73/1

Table 2.14: $24by40(12)$ $K = .152$ fits to P_1

λ	A	B	m_B	χ^2/dof
1	.30(.110)	.89(.320)	.71(.065)	.21/1
2	.32(.059)	.70(.152)	.63(.032)	.21/1
10	.20(.243)	.82(.902)	.61(.136)	.04/1

Table 2.15: $K = .152$, $24by40(12)$ fits to P_3

λ	t_i	t_f	C	m_C	χ^2/dof
0.0	4	8	.1(.04)	.67(.058)	0.92/3
0.5	3	7	.5(.03)	.72(.016)	1.64/3
1.0	3	8	.5(.03)	.69(.018)	3.54/4
2.0	5	9	.4(.03)	.63(.015)	2.18/3
10.0	4	9	.2(.02)	.60(.011)	2.72/4

Table 2.16: P_2 fits

λ	A	B	m_B	χ^2/dof
0.5	.28(.166)	1.74(.742)	.84(.128)	4.59/1
1.0	.46(.148)	.97(.433)	.67(.078)	.39/1
2.0	.63(.235)	.36(.565)	.52(.104)	.03/1
10.0	.24(.054)	.89(.348)	.59(.068)	.31/1
25.0	.15(.058)	.95(.472)	.58(.070)	.25/1

Table 2.17: $K = .160$, $16by24$ fits to P_3 .

λ	$M_q^{(\lambda)}$	m_A	m_a
0.0	.50(.120)	.54(.054)	.55(.056)
0.5	.52(.022)	.51(.018)	.66(.023)
1.0	.49(.017)	.50(.016)	.66(.018)
2.0	.44(.014)	.46(.012)	.50(.017)
10.0	.42(.013)	.47(.030)	.58(.015)

Table 2.18: $K = .160$ λ dependence

λ	P_1	P_4	P_2
0	BEST	same as P_1	plausible
.5	poor	BEST	plausible
1	poor	BEST	plausible
2	plausible	BEST	plausible
10	poor	BEST	plausible

Table 2.19: Summary of quark mass fits

As shown in Figure A.5 and Tables 2.13, 2.14, 2.16, 2.15, 2.17, 2.18, and 2.19 quark masses decrease with increasing λ regardless of which form one fits to. Define

- $P_R(\lambda_2, \lambda_1)$ is the ratio of the numerical λ_2 propagator over the λ_1 propagator;
- $M_D(\lambda_2, \lambda_1) = M_q^{(\lambda_2)} - M_q^{(\lambda_1)}$;
- $M_R(\lambda_2, \lambda_1) = M_q^{(\lambda_2)} / M_q^{(\lambda_1)}$;

where $M_q^{(\lambda)}$ refers to the log-ratio definition of effective quark mass. Table 2.20 gives representative values for these quantities.

K	$M_q^{(\lambda_1=1)}$	$M_q^{(\lambda_2=2)}$	P_R	M_D	M_R
.118	1.74(.039)	1.67(.033)	1.30(.075)	-.0719(.013)	.959(.0072)
.152	.539(.021)	.491(.021)	1.12(.023)	-.0478(.0093)	.911(.017)
.154	.463(.028)	.414(.026)	1.11(.018)	-.0484(.012)	.895(.024)
.155	.426(.033)	.377(.030)	1.10(.020)	-.0493(.014)	.884(.030)

Table 2.20: 24by40 quark

K	$M_q^{(1)}$	$S^{(2)}$	$S^{(10)}$	$S^{(10)}/S^{(2)}$
.118(8)	1.74(.040)	.96(.005)	.91(.018)	.95(.016)
.152(12)	.54(.017)	.92(.016)	.84(.029)	.92(.019)
.154(8)	.46(.027)	.91(.022)	.84(.052)	.93(.035)
.155(8)	.42(.032)	.90(.028)	.83(.065)	.92(.045)

Table 2.21: 24by40 M_q ratios

t	$t(\text{GeV}^{-1})$	$P_R(10, 1)$	$M_D(10, 1)$
1	0.6	.684(.0258)	.228(.029)
2	1.2	.723(.0229)	-.095(.024)
3	1.8	.757(.0190)	-.076(.019)
4	2.4	.791(.0186)	-.075(.015)
5	2.9	.822(.0196)	-.067(.011)
6	3.5	.855(.0201)	-.066(.013)
7	4.1	.895(.0223)	-.077(.011)
8	4.7	.939(.0280)	-.083(.013)
9	5.3	.990(.0356)	-.089(.017)
10	5.9	1.042(.0494)	-.088(.024)
11	6.5	1.102(.0710)	-.094(.033)
12	7.1	1.167(.0854)	-.098(.028)
13	7.6	1.241(.1073)	-.105(.033)
14	8.2	1.307(.1328)	-.089(.037)
15	8.8	1.376(.1685)	-.088(.041)
16	9.4	1.464(.2130)	-.105(.042)
17	10.0	1.548(.2389)	-.094(.048)
18	10.6	1.652(.2998)	-.111(.063)
19	11.2	1.744(.4180)	-.092(.093)

Table 2.22: 24by40(12) quark comparisons

Table 2.21, with time-average plateau $t_E = [8, 9, 10]$, depicts

$$S^{(\lambda)} \equiv M_q^{(\lambda)} / M_q^{(1)} . \quad (2.42)$$

Tables 2.21 through 2.22 give values for these quantities on the 24by40(12) lattice. Since $1 > S^{(2)} > S^{(10)}$, mass strictly decreases with increasing λ with the rate of decrease slowing down with increasing λ . The configuration-by-configuration decrease of this pattern suggests that Gribov ambiguities are *not* responsible. Both $S^{(2)}$ and $S^{(10)}$ increase with decreasing K , but $S^{(10)}$ increases faster. In Table 2.23 the time-average plateau is $t_E = [5, 6, 7]$.

K	$M_q^{(1)}$	$S^{(0.5)}$	$S^{(2)}$	$S^{(10)}$	$S^{(25)}$
.094	1.80(.080)	.98(.051)	.89(.044)	.82(.038)	.80(.035)
.155	.58(.017)	1.06(.055)	.89(.036)	.84(.036)	.83(.045)
.160	.49(.017)	1.07(.063)	.90(.043)	.87(.045)	.84(.061)
.164	.41(.017)	1.10(.077)	.93(.056)	.93(.062)	.86(.089)
.166	.36(.019)	1.12(.091)	.98(.071)	1.00(.080)	.87(.119)
.168	.33(.024)	1.12(.120)	1.08(.150)	1.09(.140)	.81(.173)

Table 2.23: 16by24 M_q ratios

Table 2.24 gives some of these ratios on 24by40 configuration-by-configuration. Timeaveraging is over $t_E = [8, 9, 10]$. This table shows that the mass ratio $S^{(10)}$ is reproduced on nearly all configurations and the λ variance is not just an effect of Gibbs averaging. Table 2.25 shows that two propagators with the same log-ratio mass, even if they have different K and λ , have the same shape over many timeslices.

config	$S^{(10)}(K = .118)$	$S^{(10)}(K = .152)$	$S^{(10)}(K = .155)$
02	0.88	0.99	1.17
03	0.95	0.90	0.84
04	0.79	0.79	0.58
05	0.91	0.82	0.81
06	0.94	0.88	0.84
07	0.91	0.83	0.85
08	1.00	1.02	1.09
09	0.89	0.65	0.54
51	—	0.80	—
52	—	0.75	—
53	—	0.85	—
54	—	0.85	—

Table 2.24: 24by40 single configuration M_q ratios

t	$t(\text{GeV}^{-1})$	$P_R(10, 1)$	$M_D(10, 1)$
1	0.6	.686(.0370)	.217(.046)
2	1.2	.672(.0310)	.037(.034)
3	1.8	.667(.0245)	.011(.020)
4	2.4	.663(.0190)	.011(.023)
5	2.9	.658(.0193)	.013(.022)
6	3.5	.655(.0203)	.009(.026)
7	4.1	.653(.0235)	.005(.019)
8	4.7	.652(.0267)	.001(.014)
9	5.3	.654(.0291)	-.004(.024)
10	5.9	.652(.0323)	.004(.031)
11	6.5	.650(.0423)	.006(.041)
12	7.1	.653(.0533)	-.009(.047)
13	7.6	.653(.0633)	.002(.052)
14	8.2	.644(.0696)	.024(.049)
15	8.8	.635(.0953)	.022(.062)
16	9.4	.638(.1300)	-.006(.074)
17	10.0	.647(.1656)	-.025(.075)
18	10.6	.660(.2250)	-.034(.097)
19	11.2	.679(.3425)	-.049(.159)

Table 2.25: 24by40; $\lambda = 10$ is $K = .152$; $\lambda = 1$ is $K = .154$

2.2.6 Scalar Operator Correlation Functions

If $X(0)$ is a Lorentz scalar, then

$$\langle q_p^{(r)} | X(0) | q_{p'}^{(r)} \rangle = f_X(M_q, M_q', p \cdot p') \quad (2.43)$$

Therefore, if $\vec{p} = \vec{p}' = 0$ and $M_q = M_q'$, Eq. (A.12) implies that

$$\sum_{\vec{x}, \vec{y}} \langle V_{\vec{x}} s_{\vec{x}} X(0) \bar{d}_{\vec{y}} V_{\vec{y}}^\dagger \rangle = A_X(t) + B_X(t) \gamma^0 \quad (2.44)$$

where $t_x \equiv -t_y \equiv t$. A_X and B_X are color singlets and, while they depend on gauge, we suppress their λ dependence. Hermiticity implies

$$A_X(t) = A_X(-t), \quad B_X(t) = -B_X(-t). \quad (2.45)$$

As depicted in Table 2.26, in the example $X = J \equiv \bar{s}d$, they seem to satisfy at large t_E the Dirac equation property

$$A_X(t) = B_X(t) \propto e^{-M_X t_E}, \quad M_X = 2M_q. \quad (2.46)$$

Note that because the signal for the heavier mass M_J vanishes faster than lighter mass M_q , one needs to compare M_J at earlier ($t_E \sim 4$) against M_q at later ($t_E \sim 5, 6, 7$) timeslices.

2.2.7 Other Correlation Functions

In this section we point out obstacles to computing more complicated correlation functions. Consider correlation functions corresponding to electromagnetic current

t	M_q	A_J	B_J/A_J	M_J	M_J/M_J
1	1.14(.024)	33.7(3.78)	1.02(.034)	1.13(.059)	-2.90(0.51)
2	.292(.029)	14.5(2.72)	1.03(.015)	0.84(.095)	0.99(.028)
3	.439(.026)	5.53(1.36)	1.04(.021)	0.96(.075)	0.99(.019)
4	.496(.023)	1.92(0.54)	1.04(.023)	1.05(.069)	1.00(.016)
5	.545(.023)	.580(0.19)	1.00(.026)	1.19(.086)	1.03(.020)
6	.590(.019)	.130(.067)	0.91(.106)	1.45(.187)	1.06(.071)
7	.617(.027)	.0150(.024)	0.84(.281)	1.77(.658)	1.04(0.13)
8	.649(.031)	-.0017(.0092)	0.86(1.92)	1.40(1.55)	1.08(0.32)
9	.697(.037)	-.0006(.0037)	1.19(1.91)	1.61(2.26)	0.80(1.58)

Table 2.26: $K_u = .155$, $16by24$, $\lambda = 1.0$, $J(0) = \bar{s}_0 d_0$

matrix element

$$\langle q_p^{(r)} | j^\mu(0) | q_{p'}^{(s)} \rangle = \sqrt{\frac{M_q^2}{E_q E_{q'}}} \bar{U}^{(r)} [F_1(q^2) \gamma^\mu - F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2M_q}] U^{(s)} \quad (2.47)$$

where $q \equiv p - p'$. We will not discuss problems associated with the $1/N$ suppressed bubble graph in the Wick contraction of $L^\mu(p, p')$ and, also, $\mathcal{O}(a)$ differences between the conserved lattice $U(1)$ current and $j^\mu(0)$. The effective quark electric charge radii and magnetic moments are given by the Sach combinations. The magnetic moment $\hat{\mu}$ is

$$\lim_{q^2 \rightarrow 0} [F_1(q^2) + F_2(q^2)] = \hat{\mu} / \left(\frac{e}{2M_q} \right). \quad (2.48)$$

By Eq. (A.12) the vector correlation function can be parametrized as

$$L^\mu(p, p') \equiv \lim_{t \rightarrow \infty} \sum_{\vec{x}, \vec{y}} e^{i(\vec{p} \cdot \vec{x} - \vec{p}' \cdot \vec{y})} \langle V_x s_x J^\mu(0) \bar{d}_y V_y^\dagger \rangle \quad (2.49)$$

$$= \frac{M_q \sqrt{Z_q Z_{q'}}}{4E_q^{3/2} E_{q'}^{3/2}} (i\not{p} + M_q) \left[F_1 \gamma^\mu - F_2 \sigma^{\mu\nu} \frac{q_\nu}{2M_q} \right] (i\not{p}' + M_q) e^{-(E_q + E_{q'})t} \quad (2.50)$$

$$= \frac{M_q \sqrt{Z_q Z_{q'}}}{4E_q^{3/2} E_{q'}^{3/2}} e^{-(E_q + E_{q'})t} \left\{ [M_q^2 - p \cdot p' + iM_q \not{r}] r^\mu F_1 - i(M_q^2 + p \cdot p') r^\mu F_2 \right. \\ \left. + 2i[(p \cdot p' + M_q^2) \gamma^\mu + \frac{1}{2} \not{q} \gamma^\mu - \epsilon^{\alpha\mu\nu\beta} p_\alpha p'_\nu \gamma_5 \gamma_\beta] F_2 + i r^\mu p_\alpha p'_\beta \sigma^{\alpha\beta} F_1 \right\} \quad (2.51)$$

$$+[M_q^2 q_\nu \sigma^{\mu\nu} - p^\mu \sigma^{\nu\beta} p_\nu p'_\beta - (M_q^2 + p \cdot p') \sigma^{\mu\beta} p'_\beta + \frac{1}{2} \epsilon^{\alpha\mu\nu\gamma} \epsilon_{\beta\delta\lambda\gamma} p_\alpha p'^{\beta'} p_{\nu'} \sigma^{\delta\lambda} F_1] F_1 \}$$

where $r \equiv q + q'$, $F_i \equiv F_i(q^2)$ and Eq. (2.51) is the Clifford expansion of (2.50).

Since F_i are both proportional to powers of Z_q which has no plateau, the magnetic moment is not computable. Since this problem stems from deviations of the effective quark propagator from exponential form, it is insurmountable within the free-particle ansatz.

But suppose we boldly ignore $\mathcal{O}(a)$ corrections and assume $F_1(0) = 1$. Then the ratio $\lim_{q^2 \rightarrow 0} F_2(q^2)/F_1(q^2)$ gives the magnetic moment. At least one of $\{\vec{p}, \vec{p}'\}$ must be nonzero to get information on $F_2(q^2)$. Suppose $\vec{p}' = 0$ and only one direction of $\vec{p} \neq 0$. Then according to (2.51) $F_2(q^2)$ is proportional to $\text{Tr}(\gamma_5 \gamma_i L^j)$ for several combinations of i, j . The only way to pick out F_1 without F_2 mixing in is to compute $\text{ReTr} L^i$ with i such that $p_i \neq 0$. But since $\text{ReTr} L^i \propto M_q(M_q - E_q)$ there is an additional systematic error coming from $(M_q - E_q)$ due to finite lattice spacing dispersion relation effects.

2.2.8 $\Delta I = 1/2$ Rule in Lattice QCD

The ratio of the $\Delta I = 1/2$ part of $K_s^0 \rightarrow \pi^+ \pi^-$ divided by the $\Delta I = 3/2$ part is given by [2, 11]

$$\frac{3}{2} \times \left(\frac{1}{3} + \mathbf{R}_2^+ + \frac{c_-}{c_+} (\mathbf{R}_1 + \mathbf{R}_2^-) \right) \quad (2.52)$$

where $c_\pm$ are QCD renormalization group coefficients,

$$\mathbf{R}_1 \equiv \frac{\langle\langle O_-^{uu}[LL] \rangle\rangle_{\text{fig8}}}{\langle\langle O_+^{uu}[LL] \rangle\rangle_{\text{fig8}}}, \quad \mathbf{R}_2^\pm \equiv \frac{\langle\langle O_\pm^{uu}[LL] - O_\pm^{cc}[LL] \rangle\rangle_{\text{eye}}}{\langle\langle O_+^{uu}[LL] \rangle\rangle_{\text{fig8}}}, \quad (2.53)$$

$$O_\pm^{qq}[\Gamma_1 \Gamma_2] \equiv F_{\Gamma_1 \Gamma_2}^{qq} \pm H_{\Gamma_1 \Gamma_2}^{qq}, \quad \langle\langle \Theta \rangle\rangle \equiv \langle K^+ | \Theta | \pi^+ \rangle, \quad (2.54)$$

$$F_{\Gamma_1\Gamma_2}^{qq} = \bar{s}_0\Gamma_1 d_0 \cdot \bar{q}_0\Gamma_2 q_0, \quad H_{\Gamma_1\Gamma_2}^{qq} = \bar{s}_0\Gamma_1 q_0 \cdot \bar{q}_0\Gamma_2 d_0. \quad (2.55)$$

$O_{\pm}^{qq}[LL]$ are renormalized to give vanishing $K \rightarrow 0$ transition amplitude.[12] Martinelli and Bernard *et al* have computed the coefficients $z_{\pm}^{\Gamma_1\Gamma_2}$, defined in Eq. (2.1), in weak coupling perturbation theory.[13] The diagonally renormalized $O_{\pm}^{qq}$ operators are part of the effective electroweak Hamiltonian at hadronic scales. Their coupling constants $c_{\pm}$ are run down by the renormalization group from $\sim M_w$ where the $c_{\pm}$ are determined from tree level electroweak interactions.

K_q	$\alpha_{H_{SS}^{qq}}$	$\alpha_{H_{PP}^{qq}}$	$\alpha_{H_{VV}^{qq}}$	$\alpha_{H_{AA}^{qq}}$	$\alpha_{H_{TT}^{qq}}$	$\alpha_{H_{LL}^{qq}}$
.155	.29(.10e-2)	.28(.24e-2)	1.1(.63e-2)	-1.2(.30e-2)	-3.4(.011)	-.018(.68e-2)
.160	.30(.15e-2)	.29(.27e-2)	1.2(.73e-2)	-1.2(.30e-2)	-3.5(.012)	-.021(.90e-2)
.164	.30(.21e-2)	.29(.24e-2)	1.2(.67e-2)	-1.2(.46e-2)	-3.5(.012)	-.022(.011)
.166	.30(.30e-2)	.29(.25e-2)	1.2(.60e-2)	-1.2(.60e-2)	-3.6(.012)	-.023(.015)
.168	.31(.60e-2)	.29(.35e-2)	1.2(.11e-1)	-1.2(.99e-2)	-3.6(.016)	-.035(.021)

Table 2.27: 16by24, $\lambda = 1$, $t = 2 - 4$

K_q	$\alpha_{H_{SS}^{qq}}$	$\alpha_{H_{PP}^{qq}}$	$\alpha_{H_{VV}^{qq}}$	$\alpha_{H_{AA}^{qq}}$	$\alpha_{H_{TT}^{qq}}$	$\alpha_{H_{LL}^{qq}}$
.152	1.4(.011)	1.4(.78e-2)	5.5(.021)	-5.5(.030)	-17.(.073)	-.084(.020)
.154	1.4(.013)	1.4(.013)	5.5(.032)	-5.6(.034)	-17.(.093)	-.081(.020)
.155	1.4(.013)	1.4(.016)	5.5(.040)	-5.6(.037)	-17.(.11)	-.078(.025)

Table 2.28: 24by40, $\lambda = 1$, $t = 3 - 5$

Computing $\mathbf{R}_{1,2}^{\pm}$ on the lattice requires knowing “matching” coefficients $z_{\pm}^{\Gamma_1\Gamma_2}$ in $O_{\pm}^{\text{cont}}[LL]$, the lattice representation of the continuum 4-fermi operators. Odd parity operators do not contribute to the the parity conserving matrix elements of $\mathbf{R}_{1,2}^{\pm}$ and can be ignored. Mixing with mass dimension > 6 , suppressed by positive powers of lattice spacing a , is forbidden by our renormalization scheme. At weak coupling, figure 8 graphs induce mixing with the offdiagonal operators. Eye graphs give rise to a penguin operator O_{pen} , which is suppressed by GIM cancellation and has a very small coefficient at one loop. Although color magnetic moment $\mathcal{F} \equiv \bar{s}\sigma_{\mu\nu}F^{\mu\nu}d$

has mixing coefficient proportional to m_q/a , it is absent at one loop. In two loops, its coefficient is GIM suppressed. We will neglect O^{pen} and $\mathcal{F}$, the two operators which disrupt the $O_{\pm}^{\text{qq}}[\Gamma_1\Gamma_2]$ pattern of $O_{\pm}^{\text{cont}}[LL]$.

$\alpha_{O_+[\Gamma\Gamma]}$	$\alpha_{O_-[\Gamma\Gamma]}$	$\alpha_{F_{\Gamma\Gamma}^{\text{qq}}}$	$\alpha_{H_{\Gamma\Gamma}^{\text{qq}}}$	$\alpha_{F_{\Gamma\Gamma}^{\text{cc}}}$	$\alpha_{H_{\Gamma\Gamma}^{\text{cc}}}$
-1.1(.010)	-1.3(.93e-2)	-3.5(.010)	.29(.10e-2)	-2.2(.64e-3)	.19(.45e-4)
.096(.002)	-.096(.0023)	-.29(.37)e-4	.28(.003)	-.40(.42)e-6	.19(.10e-3)
.39(.006)	-.39(.006)	-.37(.41)e-3	1.1(.63e-2)	-.25(.24)e-4	.75(.34e-3)
-.41(.003)	.41(.003)	.11(.38)e-3	-1.2(.003)	-.65(1.3)e-6	-.75(.15e-3)
1.2(.01)	1.2(.01)	-.002(.001)	-3.4(.011)	-.42(.34)e-4	-2.2(.74e-3)
-.016(.007)	.018(.006)	.67(1.1)e-3	-.018(.007)	-.52(.30)e-4	-.55(.31)e-3

Table 2 29: 16by24, $\lambda = 1$, $K_q = .155$, $t = 2 - 4$; $\Gamma = S, P, V, A, T, L$ rows.

Due to confinement there is no guarantee that α_{Θ} is related to $z_{\pm}^J$ in Eq. (2.1). All we can do is check that α_{Θ} has all the properties expected of $z_{\pm}^J$: (i) uniqueness—the scalar and vector components of Eq. (2.5) give the same α_{Θ} ; and (ii) gauge invariance— α_{Θ} must be gauge invariant. Pursuant to (2.44) and (2.5), define

$$\alpha_X^A = \lim_{t \rightarrow \infty} A_X(t)/A_J(t), \quad \alpha_X^B = \lim_{t \rightarrow \infty} B_X(t)/B_J(t). \quad (2.56)$$

Table 2.31 shows that, following identity (2.46), $\alpha_X^A = \alpha_X^B$. We define

$$\alpha_X \equiv \frac{1}{2}(\alpha_X^A + \alpha_X^B), \quad (2.57)$$

$$\alpha_{O_{\pm}[\Gamma_1\Gamma_2]} = (\alpha_{F_{\Gamma_1\Gamma_2}^{\text{qq}}} \pm \alpha_{H_{\Gamma_1\Gamma_2}^{\text{qq}}}) - (\alpha_{F_{\Gamma_1\Gamma_2}^{\text{cc}}} \pm \alpha_{H_{\Gamma_1\Gamma_2}^{\text{cc}}}). \quad (2.58)$$

At $\beta = 6.0$ and $K = .154$

$$\frac{\alpha_{O_+[\text{SS}]}^{\lambda=2}}{\alpha_{O_+[\text{SS}]}^{\lambda=1}} = 1.00(.002), \quad \frac{\alpha_{O_-[\text{LL}]}^{\lambda=2}}{\alpha_{O_-[\text{LL}]}^{\lambda=1}} = 1.00(.092) \quad (2.59)$$

and the gauge invariance test is passed.

$\alpha_{O_+[\Gamma\Gamma]}$	$\alpha_{O_-[\Gamma\Gamma]}$	$\alpha_{F_{\Gamma\Gamma}^{qq}}$	$\alpha_{H_{\Gamma\Gamma}^{qq}}$	$\alpha_{F_{\Gamma\Gamma}^{cc}}$	$\alpha_{H_{\Gamma\Gamma}^{cc}}$
-2.6(.05)	-3.1(.070)	-17.(.07)	1.4(.01)	-14.(.01)	1.1(.002)
.22(.01)	-.22(.008)	.69(.96)e-3	1.4(.008)	-.10(.19)e-4	1.1(.80e-3)
.93(.02)	-.93(.017)	.45e-4(.002)	5.5(.02)	.68(5.5)e-4	4.5(.003)
-.99(.02)	.99(.023)	-.80e-3(.003)	-5.5(.03)	-.17(.11)e-3	-4.6(.006)
-2.9(.057)	2.9(.061)	-.002(.003)	-17.(.07)	.20(4.4)e-4	-14.(.014)
-.078(.019)	.062(.014)	-.009(.004)	-.084(.02)	-.43(.31)e-3	-.014(.004)

Table 2.30: 24by40, $\lambda = 1$, $K_q = .152$, $t = 3 \dots 5$; $\Gamma = S, P, V, A, T, L$ rows.

Since $\bar{s}d$ does not contribute to figure 8 graphs, $\langle\langle O_{\pm}^{uu'}[LL] \rangle\rangle'$ is not modified by subtraction. (u' is the same mass as u , but cannot contract with u to make eye-graphs.) $\langle\langle O_{+}^{uu'}[LL] \rangle\rangle'$ transforms as an $SU_L(4) \otimes SU_R(4)$ (84, 1) and $\langle\langle O_{-}^{uu'}[LL] \rangle\rangle'$ as a (20, 1) so that both have (momentum)² chiral behavior. Following the technique[14] used to improve $\langle\bar{K}^0 | (\bar{s}d\bar{s}d)_{LL}^{\text{cont}} | K^0 \rangle$, the coefficients $z_{\pm}^{\Gamma_1\Gamma_2} / z_{\pm}^{LL}$ are determined by solving

$$\sigma_{\pm} p \cdot p = \sum_{\Gamma_1\Gamma_2} z_{\pm}^{\Gamma_1\Gamma_2} \langle\langle (O_{\pm}^{uu'}[\Gamma_1\Gamma_2])^{\text{latt}} \rangle\rangle'. \quad (2.60)$$

Eq. (2.60), a matching condition for determining the lattice coefficients $z_{\pm}^{\Gamma_1\Gamma_2}$ in Eq. (2.1), arises in the following way. By the converse $\Theta^{\text{latt}} = \sum_i B_i(\mu) \Theta_i^{\text{cont}}$ of Eq. (2.1), the chiral behavior of all naive lattice operators is

$$\langle\langle O_{\pm}^{\text{latt}}[\Gamma_1\Gamma_2] \rangle\rangle = \sigma_{\pm}^{(0),\Gamma_1\Gamma_2} + \sigma_{\pm}^{(1),\Gamma_1\Gamma_2} M_{\pi}^2 + \sigma_{\pm}^{(2),\Gamma_1\Gamma_2} p \cdot p. \quad (2.61)$$

One can correct this naive behavior either with Eq. (2.1), the $\mathcal{O}(g^2)$ weak coupling perturbation theory(WCPT) $z_{\pm}^{\Gamma_1\Gamma_2}$ values for $O_{\pm}^{\text{cont}}[LL] \equiv (O_{\pm}[LL])^{\text{pert}}$, or the “no-mixing” approximation

$$(O_{\pm}[LL])^{\text{pert}} \equiv z_{\pm}^{LL} \widetilde{O}_{\pm}^{\text{latt}}[LL]. \quad (2.62)$$

Numerically (at current precision levels) the final answer is insensitive to $\mathcal{O}(g^2)$ WCPTh corrections which are tiny compared to statistical errors of the matrix element, the chiral corrections obtained, and *their* statistical errors. Therefore, we improve $(O_{\pm}[LL])^{\text{pert}}$ by forcing its matrix elements to reproduce the continuum chiral behavior, Eq. (2.60). To order p^0 , this provides one condition on $\sigma_{\pm}^{(0)}[L]$ and one on $\sigma_{\pm}^{(1)}[L]$ of the $O_{\pm}^{\text{cont}}[LL]$ matrix element. The coefficients of two trial operators inserted to improved $(O_{\pm}[LL])^{\text{pert}}$ can be determined by these conditions. Hence we consider replacing $(O_{\pm}[LL])^{\text{pert}}$ with

$$(O_{\pm}[LL])^{\text{chiral}} \equiv (O_{\pm}[LL])^{\text{pert}} + z_{\pm}^{LL} \sum_{i=1}^2 r_{\pm}^{\Gamma_1^i \Gamma_2^i} O_{\pm}^{uu'}[\Gamma_1^i \Gamma_2^i]. \quad (2.63)$$

Appendix A.3 gives the algorithm which takes into account all $\Gamma_1^i \Gamma_2^i$ permutations.

On the other hand (the “ $\sim$ ” means $\bar{s}d$ has been subtracted)

$$\langle\langle \tilde{O}_{\pm}^{u'u'}[LL] - \tilde{O}_{\pm}^{\text{cc}}[LL] \rangle\rangle \quad (2.64)$$

contain $SU_L(3) \otimes SU_R(3)$ irreps $(\mathbf{8}, \mathbf{1})$, $(\mathbf{3}, \bar{\mathbf{3}})$, and $(\bar{\mathbf{3}}, \mathbf{3})$. These irreps have more chiral terms so that the $\langle \bar{K}^0 | (\bar{s}d\bar{s}d)_{LL}^{\text{cont}} | K^0 \rangle$ recipe is inapplicable. The alternatives are either using the WCPTh values or using the $z_{\pm}^{\Gamma_1 \Gamma_2} / z_{\pm}^{LL}$ values occurring in the figure 8 calculation.

Figure A.6 depicts the results from both WCPTh and chiral improvement of $(\text{eye graphs})/z_{\pm}^{LL}$ at various stages. Step #1 is the raw lattice matrix element. Step #2 is with $\bar{s}d$ subtraction only. Step #3 is if WCPTh values of $z_{\pm}^{\Gamma_1 \Gamma_2} / z_{\pm}^{LL}$ are used to complete Eq. (2.1). Step #4 is if CPTH behavior is enforced nonperturbatively using the figure-8 graphs. The pair of horizontal lines are the $\langle O_{+}^{uu} \rangle_{\text{fig8}} \pm \sigma$ values. Since the eye-graph errors are much larger than $\langle O_{+}^{uu} \rangle_{\text{fig8}} \pm \sigma$, the $\mathbf{R}_{1,2}^{\pm}$

t	$\left(\frac{\alpha_{O+[SS]}^B}{\alpha_{O+[SS]}^A}\right)_{\lambda=1}$	$\left(\frac{\alpha_{O+[LL]}^B}{\alpha_{O+[LL]}^A}\right)_{\lambda=1}$	$\left(\frac{\alpha_{O+[SS]}^B}{\alpha_{O+[SS]}^A}\right)_{\lambda=2}$	$\left(\frac{\alpha_{O+[LL]}^B}{\alpha_{O+[LL]}^A}\right)_{\lambda=2}$
1	1.02(.033)	1.05(.079)	1.00(.001)	.971(.099)
2	1.03(.015)	.995(.036)	1.00(.001)	.980(.041)
3	1.04(.021)	.948(.045)	1.00(.001)	.864(.051)
4	1.04(.023)	.868(.105)	1.00(.001)	.843(.126)
5	.998(.026)	.737(.351)	1.00(.001)	.947(.157)
6	.915(.106)	1.29(1.02)	1.00(.003)	1.06(.264)
7	.846(.275)	1.19(1.03)	.996(.008)	1.55(1.02)
8	.849(1.32)	1.22(.730)	.990(.036)	1.07(3.08)
9	1.15(1.19)	.910(2.21)	1.00(.142)	.431(2.96)

Table 2.31: $K_u = .155$, 16by24, two λ gauges

values are washed out. Clearly much better statistics are needed before one can adequately test this approach.

Chapter 3

Effective Quark Mass and Matrix Elements from a Gaugefixed Large N Strong Coupling Expansion

3.1 Motivation and Results

The idea of monitoring chiral symmetry with quark masses goes back to the Gross-Neveu model,[15] where the $N_f \rightarrow \infty$ effective quark mass $M_q = M_c + \pi m_q + \mathcal{O}(m_q^2)$ is a continuous, increasing function of bare mass m_q . Similarly, matching the vacuum expectation value of the operator product expansion of $\psi_{x|\mu} \bar{\psi}_{0|\mu}$ in massless QCD to a free fermion propagator yields, for $N_f = 3$ and Landau gauge,[16]

$$\lim_{\mu^2 > -p^2 \rightarrow \infty} M_q(\mu, p^2) \sim \frac{4g^2(\mu_o)}{p^2} \left(\frac{g(\mu)}{g(\mu_o)} \right)^{8/9} \langle (\bar{\psi}_0 \psi_0)_{|\mu} \rangle. \quad (3.1)$$

Since quarks are confined M_q may depend on gauge. For example, as described in Chapt. 4 in the Schwinger model, where electrons are confined, the effective electron mass depends on the covariant gauge parameter. On $\beta = 5.7$ and 6.0 quenched Wilson lattices in “ λ gauges [4]”

$$\lambda \partial_0 A^0 + \nabla \cdot \vec{A} = 0 , \quad (3.2)$$

effective quark and gluon masses M_q and M_g were evaluated by matching gaugefixed $\vec{p} = 0$ propagators at large t_E to free particle forms

$$\lim_{t_E \rightarrow \infty} \sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \langle V_x \psi_x \bar{\psi}_0 V_0^\dagger \rangle = Z_q^{(\lambda)} \left(\frac{M_q^{(\lambda)} + i\vec{p}}{2E_q^{(\lambda)}} \right) e^{-E_q^{(\lambda)} |t_E|} , \quad (3.3)$$

where the “ (λ) ” superscript allows for λ dependence, $E_q^{(\lambda)}(\vec{p} = 0) \equiv M_q^{(\lambda)}$, and $Z_q^{\text{free}} = 1$. Fits to $M_q^{(\lambda)} = b^{(\lambda)} M_\pi^2 + M_c^{(\lambda)}$, motivated by the CPT_h relation $M_\pi^2 \propto m_q$ between pion and current quark mass, [3] yield $b^{(1)} \sim 2.7(.3) \times 10^{-4}/MeV$ and $M_c^{(1)} \sim 350(40)MeV$ at $\beta \sim 6.0$. At both $\beta = 5.7$ and 6.0 effective quark and gluon masses decrease as λ grows[4] so that, roughly, $M_q^{(2)}/M_q^{(1)} \sim .9$ and $M_q^{(25)}/M_q^{(1)} \sim .75$.

As an independent check on the numerical gauge dependence, this Chapter develops a λ gauge strong coupling expansion on an infinite volume lattice. The gluon does not propagate at $\beta = 0$, where gauge fields oscillate randomly. At $\beta = 0$, $r = 0$, $N \rightarrow \infty$ and within the “trace orthogonality approximation”(to be described), we find that $M_q^{(\lambda)}$ is analytic and linear in m_q as $m_q \rightarrow 0$. Expanding

$$M_q^{(\lambda)} = M_c^{(\lambda)} + B^{(\lambda)} m_q + \mathcal{O}(m_q^2) \quad (3.4)$$

yields

$$M_c^{(\lambda)} = \frac{\sqrt{2D-1}}{2(32D-7)\lambda} \begin{cases} 11 + 14D - 25\lambda^2 & \lambda \leq \frac{1}{2}; \\ \frac{-25+200\lambda+224(D-1)\lambda^2}{4(4\lambda-1)} & \lambda \geq \frac{1}{2}; \end{cases} \quad (3.5)$$

$$B^{(\lambda)} = \begin{cases} \frac{499-1474D+1600D^2+(1376D-751)\lambda^2}{2\lambda(32D-7)^2} & \lambda \leq \frac{1}{2}; \\ \frac{-751+1376D+6008\lambda-11008D\lambda+(-4032-1568D+25600D^2)\lambda^2}{8(32D-7)^2\lambda(4\lambda-1)} & \lambda \geq \frac{1}{2}. \end{cases} \quad (3.6)$$

In the trace orthogonality approximation, only selfavoiding quark paths contribute to the $\beta = 0$ quark propagator. Since $D = 4$ is the critical dimension for when recurrences and selfintersections become unimportant in infinitely long paths—which presumably dominate contributions to the $\vec{p} = 0$ mass pole—our $\beta = 0$ expressions only have a chance of being valid at $D > 4$. However, we find that the qualitative dependence of $M_c^{(\lambda)}$ on λ is the same whether $D \rightarrow \infty$ (where recurrences and selfintersections are negligible) or $D = 4$. This suggests that recurrences and selfintersections are not responsible for the λ dependence and we will generally use the strong coupling formulas at $D = 4$. Note that we could have used the “safer” $D \rightarrow \infty$ formulas without surrendering any of the qualitative features.

Eq. (3.5) mimics the λ dependence of the $\beta \sim 6$ data. If $D = 4$, $M_c^{(0.5)} = 1.3$, $M_c^{(1)} = .77$, $M_c^{(2)} = .60$, $M_c^{(\infty)} = .46$ —the change between $\lambda = 1$ and $\lambda = 2$ is greater than between $\lambda = 2$ and $\lambda \rightarrow \infty$. Slope $B^{(\lambda)}$ is relatively insensitive to λ . $B^{(1)} = 1$ independent of D . If $D = 4$, $B^{(\lambda)}$ decreases with increasing λ but with quickly diminishing slope for $\lambda \geq 1$: $B^{(1)} = 1.0$, $B^{(25)} = .86$, and $B^{(\infty)} = .85$. This agreement between strong coupling and numerical behavior gives us confidence that the numerical λ dependence of M_q is not a finite volume artifact or due to details of how M_q is extracted in the numerical simulations.

For technical reasons we compute the quark propagator dispersion relation not

at $r = 0$ but at $r = 1$, where it is

$$E_q^{(\lambda)^2} = M_q^{(\lambda)^2} + \sum_{i=1}^3 \bar{g}_i^2 \vec{p}_i^2 + \mathcal{O}(a^3), \quad E_q^{(\lambda)} \equiv \pm i p_0, \quad (3.7)$$

$$\bar{g}_i(\lambda) \equiv \begin{cases} \frac{3}{4\lambda} & \lambda \leq \frac{1}{2}; \\ \frac{3\lambda}{4\lambda-1} & \lambda \geq \frac{1}{2}. \end{cases} \quad (3.8)$$

Since $\bar{g}_i(\lambda \neq 1) \neq 1$, the quark dispersion relation is not covariant and, hence, the propagator is not free particle-like except in Landau gauge. $\bar{g}_i$ drops out if $\vec{p} = 0$, where the quark propagator is indistinguishable from the free particle propagator. (However $M_q^{(\lambda)}$ remains dependent on λ .)

In the same limit, matching coefficients for $\bar{s}d$ subtraction of $\Delta I = 1/2$ Rule operators satisfy

$$\alpha_{O_{\pm}[\Gamma_1\Gamma_2]} = (\delta_{\Gamma_1\Gamma_2, SS} \mp \frac{1}{2^{D/2}N} f_{\Gamma_1\Gamma_2}^{SS}) \langle \bar{\psi}_0 \psi_0 \rangle \quad (3.9)$$

where Fierz coefficients $f_{\Gamma_1\Gamma_2}^{\Gamma_3\Gamma_4}$ are given in Eq. (3.133). Since $\langle \bar{\psi}_0 \psi_0 \rangle$ is gauge invariant these $\beta = 0$ matching coefficients, which are directly related to physical continuum decay rates, are gauge invariant. At $\beta = 6.0$, $\alpha_{O_{\pm}[\Gamma_1\Gamma_2]}$ also seems to be λ independent.

3.2 Lattice Gaugefixing

Gaugefixed lattice QCD is given in terms of

$$[\Theta]_{\theta} \equiv z_{\theta}^{-1} \int [d\theta] e^{-S^{\theta}} \Theta, \quad z_{\theta} \equiv \int [d\theta'] e^{-S^{\theta'}}, \quad \theta \in \{f, v, u\}, \quad (3.10)$$

$$S^f \equiv \sum_{x,y} \bar{\psi}_x [\delta_{x,y} + K_B \sum_n (r - \hat{n}) U_{x,n} \delta_{y,x+\hat{n}}] \psi_y, \quad (3.11)$$

$$S^v[\Omega] \equiv - \sum_{\mathbf{x}, n} \frac{J_n}{2} \text{Tr} \Omega_{\mathbf{x}, n}, \quad n \in \pm\{0, \dots, D-1\}, \quad (3.12)$$

$$J_{-n} \equiv J_n \equiv \lambda_n / \xi, \quad \lambda_\mu \equiv (\lambda, \tilde{\lambda}, \dots, \tilde{\lambda}), \quad \Omega_{\mathbf{x}, n} \equiv V_{\mathbf{x}} U_{\mathbf{x}, n} V_{\mathbf{x}+\hat{n}}^\dagger, \quad (3.13)$$

$$S^u[U] = S^u[\Omega] = \beta \text{Re} \sum_{\square} \text{Tr} \square, \quad \langle \Theta \rangle \equiv \left[[[\Theta]_f]_v \right]_u. \quad (3.14)$$

Quarks are quenched by choice. Consistency with the Fadeev-Popov method requires $\{V_{\mathbf{x}}\}$ to be quenched.[6] Since as described in Section 3.3 the hopping expansion turns $[\Theta]_f$ into a composite operator of links and transformations, in this Section we will focus on the gaugefixed expectation value of continuous link chains ${}_y \blacktriangleleft_w$. The strategy is to integrate out $\{V_{\mathbf{x}}\}$ to get an expression for

$$\mathcal{V}_{\alpha\beta;ij}[U] \equiv [(V_y)_{\alpha\beta} (V_w^\dagger)_{ij}]_v. \quad (3.15)$$

This integration is feasible because gauge symmetries relate $\mathcal{V}$ to continuous link chains. Then gaugefixing can be viewed as an operator insertion of $\mathcal{V}$ into

$$\langle V_y ({}_y \blacktriangleleft_w) V_w^\dagger \rangle_{\alpha j} = [({}_y \blacktriangleleft_w)_{\beta i} \mathcal{V}_{\alpha\beta;ij}[U]]_u, \quad (3.16)$$

the usual gauge invariant expectation value.

We will show that in the $\xi \propto 1/N$ and $N \rightarrow \infty$ limit

$$[(V_y)_{\alpha\beta} (V_w^\dagger)_{ij}]_v = \delta_{\alpha j} \sum_{\substack{\mathbf{w} \\ \tilde{\mathbf{w}}_y}} A_{\mathbf{w} \tilde{\mathbf{w}}_y} ({}_w \tilde{\blacktriangleleft}_y)_{i\beta} \quad (N \rightarrow \infty), \quad (3.17)$$

where the sum ranges over a complete set of selfavoiding continuous link chains ${}_w \tilde{\blacktriangleleft}_y$. Orthonormality of the ${}_w \tilde{\blacktriangleleft}_y$ with respect to inner product

$$\langle \Theta_1 | \Theta_2 \rangle \equiv \int [dU] \frac{1}{N} \text{Tr}(\Theta_1^\dagger \Theta_2), \quad [dU] \equiv \prod_{\mathbf{x}, \mu} dU_{\mathbf{x}, \mu}, \quad (3.18)$$

identifies coefficients $A_{\mathbf{w}\tilde{\mathbf{A}}_y}$ with $\beta = 0$ λ -gauge expectation values. In particular,

$$A_{\mathbf{w}\tilde{\mathbf{A}}_y} = \frac{1}{N^2} \int [dU] \text{Tr}(\mathbf{y}\tilde{\mathbf{A}}_{\mathbf{w}}[V_{\mathbf{w}}^\dagger V_{\mathbf{y}}]_{\mathbf{v}}) = \frac{1}{N^2} \text{Tr}\langle V_{\mathbf{y}\mathbf{y}}\tilde{\mathbf{A}}_{\mathbf{w}}V_{\mathbf{w}}^\dagger \rangle_{|\beta=0}. \quad (3.19)$$

If $y = w = x$, the only continuous selfavoiding chain is ${}_x\tilde{\mathbf{A}}_x = 1$. Since $V_x V_x^\dagger = 1$, (3.19) implies $A_{{}_x\tilde{\mathbf{A}}_x} = 1/N$. Hence $[(V_x)_{\alpha\beta}(V_x^\dagger)_{ij}]_{\mathbf{v}}$ is independent of $[U]$ and

$$\langle (V_x)_{\alpha\beta}(V_x^\dagger)_{ij} \rangle = [(V_x)_{\alpha\beta}(V_x^\dagger)_{ij}]_{\mathbf{v}} = \frac{1}{N} \delta_{\alpha j} \delta_{\beta i} = \int dV V_{\alpha\beta} V_{ij}^\dagger. \quad (3.20)$$

Eq. (3.20) implies gaugefixing does not affect closed link loops since

$$\langle (V_x \circ_x V_x^\dagger)_{\alpha j} \rangle = \frac{1}{N} \delta_{\alpha j} [\text{Tr} \circ_x]_{\mathbf{u}} = \langle (\circ_x)_{\alpha j} \rangle. \quad (3.21)$$

3.2.1 Link Operator Definitions

A continuous link chain or randomwalk path may *selfintersect* (touch itself at right angles), *recur* (touch itself at 0° or 180°), or *backtrack* (immediate recurrence as in the sequence " $\dots U_{x,\mu} U_{x+\hat{\mu},-\mu} \dots$ "). A *selfavoiding* link chain or randomwalk path does not selfintersect or recur.

A tree T is a continuous randomwalk path which is nonbacktracking but may be selfintersecting and otherwise recurrent. A branch B is a continuous, possibly selfintersecting and definitely recurring randomwalk loop enclosing a zero area minimal surface. A tip of B is a backtracking subsegment of B . Every B has at least one tip, perhaps more. Every randomwalk path is a sequence of trees and branches.

Continuous link chains extending from y to w , not necessarily straight or self-

avoiding, are denoted by

$${}_y \blacktriangleleft_w \equiv U_{y,\mu} U_{y+\hat{\mu},\nu} \cdots U_{w-\hat{\delta},\delta} , \quad {}_y \blacktriangleleft_w^\dagger = {}_w \blacktriangleleft_y \quad (3.22)$$

where "...” denotes a continuous but not necessarily selfavoiding link chain. Selfavoiding link chains ${}_y \tilde{\blacktriangleleft}_w$ are accented with "~." Examples of recurrent and selfintersecting link chains are

$$\square , \quad \hat{\times} , \quad \square_x^3 \equiv \square_x \square_x \square_x . \quad (3.23)$$

$\square$ is a continuous chain which traces out a unit box and recurs at one side. $\hat{\times}$ selfintersects, traces out a unit box, and selfintersects the original point. Link chain loops ${}_x \blacktriangleleft_x$ are also denoted by $\bigcirc$. The unit square $\bigcirc$ is $\square$, whose trace is the plaquette. If $\bigcirc$ begins and ends at x , then it is denoted $\bigcirc_x$.

If the path traced out by $\bigcirc_x$ is a branch, then it is denoted $\bigcirc_x^B$. Since $U_{x,\mu} U_{x+\hat{\mu},-\mu} = 1$

$$\bigcirc_x^B = 1 \quad \forall \bigcirc_x^B . \quad (3.24)$$

Decomposing link chains as product of tree and branch subsegments yields

$${}_y \blacktriangleleft_w = \bigcirc_y^B {}_y \blacktriangleleft_x^T \bigcirc_x^B \cdots \bigcirc_z^B {}_z \blacktriangleleft_w^T \bigcirc_w^B = {}_y \blacktriangleleft_w^T . \quad (3.25)$$

Hence we will assume that ${}_y \blacktriangleleft_w = {}_y \blacktriangleleft_w^T$ unless specifically noted.

By $SU(N)$ identities such as Schur's lemma

$$\int dU D_{ij}^{(\bar{\nu})}(U) D_{\alpha\beta}^{(\nu')}(U) = \frac{1}{\dim(\nu)} \delta_{i\beta}^{\nu,\nu'} \delta_{j\alpha} , \quad (3.26)$$

continuous link chains are orthonormal with respect to (3.18). However some dis-

connected chains mix with each other. For example, a *nonorthonormal* basis for gauge invariant link operators is

$$\mathcal{B}_o \equiv \{1, \{\text{Tr}\bigcirc\}, \{\text{Tr}\bigcirc \text{Tr}\bigcirc'\}, \dots\} \quad 1, \quad \text{Tr}1 = N \quad (3.27)$$

where $\{\bigcirc\}$ is the set of all nonbacktracking link loops in the lattice. It is possible to orthonormalize $\mathcal{B}_o$ by constructing irreducible irreps combinations out of link operators which mix.

3.2.2 Link Expansion

QCD gauge transformations are

$$U_{x,\mu} \rightarrow Q_x U_{x,\mu} Q_{x+\hat{\mu}}^\dagger, \quad V_x \rightarrow V_x, \quad \psi_x \rightarrow Q_x \psi_x. \quad (3.28)$$

While S^v breaks QCD gauge symmetry, S^u , S^f , and S^v are invariant under

$$U_{x,\mu} \rightarrow R_x U_{x,\mu} R_{x+\hat{\mu}}^\dagger, \quad V_x \rightarrow V_x R_x^\dagger, \quad \psi_x \rightarrow R_x \psi_x. \quad (3.29)$$

We will refer to (3.28) as a “ Q ” transformations and (3.29) as “ R ” transformations. R is a symmetry of $[\Theta]_{\theta|\theta \in \{f,v,u\}}$ (and hence $\langle \Theta \rangle$) whereas Q is a symmetry of only $[\Theta]_{\theta|\theta \in \{f,u\}}$. By Elitzer’s theorem[17] any R -variant operator such as V_x and $U_{x,\mu}$ has zero $\langle \Theta \rangle$ expectation value. R -invariant operators, including Q -variant ones like $\Omega_{x,\mu}$, are not suppressed.

Global transformation “ L_{XY} ” where

$$U_{x,\mu} \rightarrow U_{x,\mu}, \quad V_x \rightarrow L_{XY} V_x, \quad \psi_x \rightarrow \psi_x \quad (3.30)$$

is a symmetry of $[\Theta]_{\theta \in \{f, v, u\}}$. L_{XY} is equivalent to global color symmetry. The XY designation is because in the $\beta \rightarrow \infty$ limit the system approaches the XY model, which has L_{XY} symmetry.

$z_v[U]$ under $U_{x,\mu} \rightarrow U'_{x,\mu} = Q_x U_{x,\mu} Q_{x+\mu}^\dagger$ obeys

$$z_v[U'] = \int [dV] e^{-S^v[\Omega']} = \int [d(VQ^\dagger)] e^{-S^v[\Omega]} = z_v[U]. \quad (3.31)$$

As a Q invariant functional of links z_v , analog of the Fadeev-Popov determinant, is expandable in terms of gauge invariant link structures. Such structures are constructed by joining links together into closed networks with gauge invariant bonds δ_{ij} and $\epsilon_{i_1 \dots i_N}$. Let us take $N \mapsto \infty$ or $U(1)$ and throw out the latter. Then following (3.27), with $\bigcirc \neq 1$ and $\bigcirc' \neq 1$,

$$z_v[U] = (1 + \sum_{\bigcirc} Z_v^\bigcirc \text{Tr } \bigcirc + \sum_{\bigcirc \neq \bigcirc^\dagger} Z_v^{\bigcirc \bigcirc'} \text{Tr } \bigcirc \text{Tr } \bigcirc' + \dots) Z_v^\circ, \quad (3.32)$$

$$Z_v^\circ \equiv Z_v^1 + \sum_{\bigcirc \neq 1} Z_v^{\bigcirc \bigcirc^\dagger} \text{Tr } \bigcirc \text{Tr } \bigcirc^\dagger + \dots \quad (3.33)$$

The reason for factorizing out Z_v° in (3.32) and the reason we know Z_v° can be expanded in terms of zero area (colorless) link structures is because $Z_v^\circ = \int [dU] z_v[U]$. Since it is colorless, Z_v° plays a special role at strong coupling.

If W is in the fundamental representation and $\chi^{(\nu)}$ the irrep ν character, the character expansion of $e^{z \text{ReTr} W}$ is given by

$$b_\nu(z) = b_\nu(z) \equiv \int dW \chi^{(\nu)}(W) e^{\frac{z}{2} \text{Tr}(W+W^\dagger)}, \quad (3.34)$$

$$e^{\frac{z}{2} \text{Tr}(W+W^\dagger)} = \sum_\nu b_\nu(z) \chi^{(\nu)}(W), \quad f_\nu \equiv \frac{b_\nu}{N b_1}. \quad (3.35)$$

Since by (3.12) $e^{-S^*[\Omega]} = \prod_{\mathbf{x}, \mu} e^{J_\mu \text{ReTr} \Omega_{\mathbf{x}, \mu}}$,

$$e^{-S^*[\Omega]} = Z_v^o \sum_{\{\nu_{\mathbf{x}, \mu}\}} \prod_{\mathbf{x}, \mu} N f_{\nu_{\mathbf{x}, \mu}}(J_\mu) \chi^{(\nu_{\mathbf{x}, \mu})}(\Omega_{\mathbf{x}, \mu}) \quad (3.36)$$

where the reason for factoring out Z_v^o will be apparent shortly and where

$$\{\Theta\} \equiv \frac{\int dW \Theta e^{\frac{1}{2}z(\text{Tr}(W+W^\dagger))}}{\int dW' e^{\frac{1}{2}z(\text{Tr}(W'+W'^\dagger))}}, \quad f_N(z)\delta_{ij} \equiv \{W_{ij}\}, \quad (3.37)$$

$$\|f_N(z)\| \leq 1, \quad \lim_{z \rightarrow 0} f_N(z) = \frac{z}{2N}, \quad f_N(\infty) = 1 \quad (\text{finite } N). \quad (3.38)$$

The first inequality in (3.38) follows from $\|\text{Tr} W\| \leq N$; the second from expanding $e^{z \text{ReTr} W}$ in z ; the third since the integrands are dominated by $\text{MaxReTr} W$ when $z \rightarrow \infty$. Integrating both sides of (3.32) over $[dU]$, interchanging integration order $[dU][dV] \rightarrow [dV][dU]$ and changing variables $[dU]$ to $[d\Omega]$ yields

$$Z_v^o = \int [dU][dV] \prod_{\mathbf{x}, \mu} e^{J_\mu \text{ReTr} \Omega_{\mathbf{x}, \mu}} = \prod_{\mathbf{x}, \mu} b_1(J_\mu). \quad (3.39)$$

L_{XY} symmetry implies

$$[(V_y)_{\alpha\beta}(V_w^\dagger)_{ij}]_v = \delta_{\alpha j} t_{i\beta}[U; w, y]. \quad (3.40)$$

Since $t[U; w, y]$ transforms under local R like $[(V_y)_{\alpha\beta}(V_w^\dagger)_{ij}]_v$, at y it transforms like N , at w like $\overline{N}$, and the links away from y and w must be bonded gauge invariantly (either by the $SU(N)$ identity tensor or the completely antisymmetric ϵ tensor).

To appreciate that (3.17) is not a selfevident result—it's valid only in the $N \rightarrow \infty$ limit and orthogonal trace approximation to be described—consider the exactly

solvable $U(1)$ case where

$$U_{x,\mu} \equiv e^{i\phi_{x,\mu}}, \quad V_x \equiv e^{i\theta_x}, \quad S^v = - \sum_{x,\mu} J^\mu \cos(\theta_x - \theta_{x+\mu} + \phi_{x,\mu}). \quad (3.41)$$

The character expansion is given by

$$e^{z \cos \theta} = I_0(z) \sum_{n=-\infty}^{\infty} t_n(z) e^{-in\theta}, \quad \chi^{(n)}(e^{-i\theta}) = e^{-in\theta}, \quad (3.42)$$

$$I_{-n}(z) = I_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{in\theta + z \cos(\theta)}, \quad t_n \equiv I_n/I_0, \quad (3.43)$$

$$\bar{z}_v[\phi] = z_v/Z_v^o = \sum_{\{n_{x,\mu}\}'} \prod_{x,\mu} t_{n_{x,\mu}}(J_\mu) e^{-in_{x,\mu}\phi_{x,\mu}}, \quad (3.44)$$

$$\{n_{x,\mu}\}' \equiv \{\{n_{x,\mu}\} | \sum_m n_{x,m} = 0\}, \quad (3.45)$$

$$[V_y V_w^\dagger]_v = \bar{z}_v^{-1}[\phi] \sum_{\{n_{x,\mu}\}''} \prod_{x,\mu} t_{n_{x,\mu}}(J_\mu) e^{-in_{x,\mu}\phi_{x,\mu}}, \quad (3.46)$$

$$\{n_{x,\mu}\}'' \equiv \{\{n_{x,\mu}\} | \sum_m n_{x,m} = \delta_{x,y} - \delta_{x,w}\}. \quad (3.47)$$

The $U(1)$ identity used to obtain (3.44-3.47) is

$$\int dV_x \prod_m \chi^{(\nu_{x,m})}(\Omega_{x,m}) = \delta(\sum_{m'} \nu_{x,m'}) \prod_m \chi^{(\nu_{x,m})}(U_{x,m} V_{x+m}^\dagger), \quad (3.48)$$

$$\chi^{(\nu_{x,-\mu})}(\Omega_{x,\mu}) \equiv \chi^{(\nu_{x-\hat{\mu},-\mu})}(\Omega_{x-\hat{\mu},\mu}^\dagger). \quad (3.49)$$

Because $U(1)$ coefficients do not satisfy

$$t_n(z) = (t_1(z))^n, \quad (3.50)$$

link loops in the numerator of $[V_y V_w^\dagger]_v$ which touch open chains ${}_w \tilde{\mathbf{A}}_y$ are not cancelled by corresponding z_v loops and (3.17) does not apply to $U(1)$. (3.17) only holds in

systems consistent with both (3.48) and (3.50).

The $SU(N)$ version of (3.48) is complicated because there is more than one way of making a singlet out of $\nu_{-D+1} \otimes \cdots \otimes \nu_{D-1}$. $SU(\infty)$ traces, on the other hand, obey an analog of (3.48):

$$\int dV_x \prod_{m=-D+1}^{D-1} \text{Tr}^{n_{x,m}}(V_x O_{x,m}) \text{Tr}^{l_{x,m}}(O_{x,m}^\dagger V_x^\dagger) \quad (3.51)$$

$$= \delta(\bar{n}_{x,D-1} - \bar{l}_{x,D-1}) \sum_{\sigma \in S_{\bar{n}_{x,D-1}}} \prod_{j=0}^{\bar{n}_{x,D-1}-1} \left[\frac{\text{Tr}(O_{x,M_x(j)} O_{x,M_x(\sigma(j))}^\dagger)}{N} \right],$$

$$O_{x,m} = \begin{cases} U_{x,\mu} V_{x+\hat{\mu}}^\dagger & m = \mu \geq 0; \\ U_{x-\hat{\nu},\nu}^\dagger V_{x-\hat{\nu}}^\dagger & m = -\nu \leq 0; \end{cases} \quad (3.52)$$

$$n_{x,m} = \begin{cases} n_{x,\mu} & m = \mu; \\ l_{x-\hat{\nu},\nu} & m = -\nu; \end{cases} \quad l_{x,m} = \begin{cases} l_{x,\mu} & m = \mu; \\ n_{x-\hat{\nu},\nu} & m = -\nu; \end{cases} \quad (3.53)$$

$$\bar{n}_{x,m} \equiv \sum_{m'=-D+1}^m n_{x,m'}, \quad M_x = \begin{cases} 1-D & 0 \leq j \leq n_{x,1-D}; \\ \vdots & \vdots \\ D-1 & \bar{n}_{x,D-2} < j \leq \bar{n}_{x,D-1}. \end{cases} \quad (3.54)$$

S_p is the permutation group on p elements. Eq. (3.51) says that the numerators and denominators of $t[U; w, y]$ are comprised of chains made by joining links from the trace expansion of $e^{-S'}$ in all possible permutations consistent with $\{n_{x,m}, l_{x,m}\}$ conservation at each site. In Section 3.2.3 we will find that relevant coefficients of the link expansion satisfy (3.50) and that the $SU(\infty)$ analog of (3.44-3.47) is consistent with Eq. (3.17).

3.2.3 Trace Expansion and Orthogonality Approximation

We will call the use of (3.51) the “trace orthogonality approximation.” The reason its use is an approximation even in the $N \rightarrow \infty$ limit is as follows. If $\infty > N > 2$,

$$\int dV \quad V_{i_1 j_1} V_{\alpha_1 \beta_1}^\dagger V_{i_2 j_2} V_{\alpha_2 \beta_2}^\dagger = \frac{1}{N^2 - 1} \left[(\delta_{i_1 \beta_1} \delta_{j_1 \alpha_1} \delta_{i_2 \beta_2} \delta_{j_2 \alpha_2} - \frac{1}{N} \delta_{i_1 \beta_1} \delta_{j_1 \alpha_2} \delta_{i_2 \beta_2} \delta_{j_2 \alpha_1}) + \left(\begin{smallmatrix} \alpha_1 & \leftrightarrow & \alpha_2 \\ \beta_1 & \leftrightarrow & \beta_2 \end{smallmatrix} \right) \right]. \quad (3.55)$$

Since the second term is suppressed by N^{-2} , (3.55) reduces to (3.51) as $N \rightarrow \infty$. However, using (3.55) in place of (3.51) to compute the leading $\mathcal{O}(J_\mu)$ term of $A_{\square}$, the Eq. (3.17) coefficient where $\square$ is a continuous five-link chain which traces out a unit square and recurs at one side, yields

$$A_{\square} \approx \frac{1}{N} \left(\frac{J_\mu}{2N} \right)^5 [1 - 1 - 1] = -\frac{1}{N} \left(\frac{J_\mu}{2N} \right)^5 \neq 0. \quad (3.56)$$

The first two terms of the middle expression come from the numerator of $t[U; w, y]$; the third from a denominator plaquette and a numerator link. Hence, $A_{\square} \neq 0$ if we take $N \rightarrow \infty$ *after* resummation of the trace expansion. On the other hand, as we described below, $A_{\square} = 0$ if one uses Eq. (3.51), valid to leading $\mathcal{O}(1/N)$, to integrate out the $[dV]$; this is tantamount to taking $N \rightarrow \infty$ *before* resummation. We have not been able to improve the trace orthogonality approximation to account properly for coefficients of recurrent chains.

Consider the generating functional

$$Z[L, K] = \int dW \, e^{\frac{1}{2} \text{Tr}(LW + W^\dagger K)}, \quad Z[J, J] = b_1(J). \quad (3.57)$$

Expanding in L and K , taking $N \rightarrow \infty$, imposing

$$\int dW \operatorname{Tr}^n(LW) \operatorname{Tr}^l(W^\dagger K) = n! \left[\frac{\operatorname{Tr}(LK)}{N} \right]^n \delta_{nl} \quad (N \rightarrow \infty) \quad (3.58)$$

and resumming gives

$$\tilde{Z}[L, K] = \sum_{l=0}^{\infty} \sum_{n=0}^l 2^{-l} \int dW \frac{\operatorname{Tr}^n(LW) \operatorname{Tr}^{l-n}(W^\dagger K)}{n! (l-n)!} = e^{\frac{\operatorname{Tr}(LK)}{4N}}. \quad (3.59)$$

(3.58) is a special case of (3.51). We change notation from $Z \mapsto \tilde{Z}$ to emphasize that the interchange of integration and resummation leads to a discrepancy, described below, between $\tilde{Z}$ and the original Z .

Using $\tilde{Z}$ as a generating function yields ($\{\Theta\}$ is defined in (3.37).)

$$F_{\substack{i_1 j_1 \dots i_l j_l \\ k_1 o_1 \dots k_n o_n}}^{(l,n)} \equiv \{W_{i_1 j_1}^\dagger \dots W_{i_l j_l}^\dagger W_{k_1 o_1} \dots W_{k_n o_n}\}, \quad (3.60)$$

$$2 \frac{\partial \log \tilde{Z}}{\partial L_{ji}} \Big|_{\substack{L=J \\ K=J}} = 2 \frac{\partial \log \tilde{Z}}{\partial K_{ji}} \Big|_{\substack{L=J \\ K=J}} = F_{ij}^{(0,1)} = F_{ij}^{(1,0)} = \delta_{ij} f_N(J), \quad (3.61)$$

$$f_N(J) = \frac{J}{2N} \quad (J \ll 2N, N \rightarrow \infty). \quad (3.62)$$

Since following (3.59)

$$\frac{\partial^{n+l} \log \tilde{Z}}{\partial K_{j_1 i_1} \dots \partial K_{j_l i_l} \partial L_{o_1 k_1} \dots \partial L_{o_n k_n}} = 0 \quad (\text{if } n \text{ or } l > 1), \quad (3.63)$$

$$F_{\substack{i_1 j_1 \dots i_l j_l \\ k_1 o_1 \dots k_n o_n}}^{(l,n)}(J) - \prod_{p=1}^{n+l} \delta_{i_p j_p} f_N(J) = 0 \quad (3.64)$$

by induction. Combining (3.61) and (3.64) yields

$$\frac{1}{N^{n+l}} \{\operatorname{Tr}^n W \operatorname{Tr}^l W^\dagger\} = f_N^{n+l}(J) \quad (n \geq 0, l \geq 0). \quad (3.65)$$

(3.65) is the $SU(\infty)$ analog of (3.50). Group integration formulae such as (3.51) can similarly be derived from $\tilde{Z}$.

It must be emphasized that $\tilde{Z}$ is *not* valid to leading $\mathcal{O}(N)$. Rather, following (3.58), it is only asymptotically valid as $N \rightarrow \infty$. To appreciate this, note that formula (3.62) violates the inequality of (3.38) if $J > 2N$. Since (3.59) is analytic in $\text{Tr}(LK)$, the violation implies $\tilde{Z} \neq Z$. This discrepancy arises from applying (3.58) to (3.59) on $n \rightarrow \infty$ (including $n \geq N$) terms before resummation.

An exactly solvable limit is $N \rightarrow \infty$ and $J \propto N$. In this double limit, achieved by identifying the gauge parameter ξ with $\xi \propto 1/N$ so that

$$N \rightarrow \infty, \quad \xi \rightarrow 0, \quad \frac{J_\mu}{2N} \equiv \lambda_\mu, \quad (3.66)$$

the N dependence cancels out leaving a nontrivial λ -dependent solution. The exact generating function is [37]

$$\lim_{N \rightarrow \infty} Z[2N\ell, 2N\kappa] \equiv \int dW e^{N\text{Tr}(\ell W + \kappa W^\dagger)} \equiv e^{N^2 w(\ell, \kappa)}, \quad (3.67)$$

$$w(\ell, \kappa) = \begin{cases} \ell\kappa & \sqrt{\ell\kappa} \leq \frac{1}{2}; \\ 2\sqrt{\ell\kappa} - \frac{3}{4} - \frac{1}{4} \log(4\ell\kappa) & \sqrt{\ell\kappa} > \frac{1}{2}; \end{cases} \quad (3.68)$$

$$\lim_{\substack{N \rightarrow \infty \\ \xi \rightarrow 0}} f_N(J) = \frac{\partial w}{\partial \ell} \Big|_{\ell=\kappa=\lambda} = \begin{cases} \lambda & \lambda \leq \frac{1}{2}; \\ 1 - \frac{1}{4\lambda} & \lambda \geq \frac{1}{2}; \end{cases} \quad (3.69)$$

$$\frac{1}{N^2} \{ \text{Tr}^2 W \} - f_N(J) = \frac{1}{N^2} \frac{\partial^2 w}{\partial \alpha^2} = \mathcal{O}\left(\frac{1}{N^2}\right). \quad (3.70)$$

By induction

$$\frac{1}{N^{n+l}} \{ \text{Tr}^n W \text{Tr}^l W^\dagger \} - f_N^{n+l}(J) = \mathcal{O}\left(\frac{1}{N^2}\right). \quad (3.71)$$

Since the LHS terms in (3.70) and (3.71) are $\mathcal{O}(1)$, property (3.65) is valid when

$J \equiv 2N\lambda$ in the $N \rightarrow \infty$ limit.

The link expansion of $t[U; w, y]$ follows from trace expansion

$$e^{z \text{ReTr} \Omega} = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \tau(n, l; z) \frac{\text{Tr}^n \Omega}{n!} \frac{\text{Tr}^l \Omega^\dagger}{l!} . \quad (3.72)$$

Taylor expansion of $e^{z \text{ReTr} \Omega}$, which defines the exponential operator, implies $\tau(n, l; J) = (J/2)^{n+l}$. In fact, this identification is only asymptotically consistent with using (3.51) when $J \ll 2N$ and $N \rightarrow \infty$. As $J \propto N \rightarrow \infty$, integral formula (3.51) cannot be interchanged with resummation. If we insist on applying (3.51) before resumming the trace expansion, $\tau(n, l; J)$ must be redefined so that correlation functions evaluated using (3.51) and (3.72) are consistent with correlation functions evaluated directly from Z of Eq. (3.67). The matching condition

$$\sum_{n=0}^{\infty} \frac{\tau(n, n+k; J)}{n!} = b_1(J) N^k f_N^k(J) = \sum_{n=0}^{\infty} \frac{\tau(n+k, n; J)}{n!} \quad (3.73)$$

implies when $N \rightarrow \infty$

$$\tau(n, l; J) = N^{l+n} f_N^{l+n}(J) \begin{cases} 1 & J \ll 2N; \\ e^{N^2[w(\lambda, \lambda) - \lambda^2]} & J = 2N\lambda. \end{cases} \quad (3.74)$$

The $SU(\infty)$ analog of (3.50),

$$\tau(n, l+k; J) = \tau(n+k, l; J) = N^k f_N^k(J) \tau(n, l; J) , \quad (3.75)$$

will be referred to as “factorization.”

Applying Eqs. (3.51) and (3.72) leads to the $SU(\infty)$ version of (3.44-3.47). The $[dU]$ integral of Eq. (3.19) for $A_{\mathbf{w} \tilde{\mathbf{m}}_y}$ projects the the numerator and denominator of $t[U; w, y]$ down to the colorless part except along $\mathbf{w} \uparrow_y$. At other sites the denom-

inator contributions cancel the corresponding numerator contributions. Each link of $w \tilde{\mathbf{A}}_y$ coming from a (3.51) integration costs a factor of $1/N$. Additionally there is an overall factor of $\text{Tr}(w \tilde{\mathbf{A}}_y^\dagger w \tilde{\mathbf{A}}_y)/N^2 = 1/N$. Hence

$$\wedge_{\{n,x,m\}} \equiv \sum_{\{n_{x,m}\}} \prod_{\{x,m\}} , \quad p_{x,m} = \begin{cases} 1 & \text{if } \{x,m\} \in w \tilde{\mathbf{A}}_y; \\ 0 & \text{if } \{x,m\} \notin w \tilde{\mathbf{A}}_y; \end{cases} \quad (3.76)$$

$$A_{w \tilde{\mathbf{A}}_y} = \wedge_{\{n,x,m\}} \frac{\tau(n_{x,m}, n_{x,m} + p_{x,m}, J_m)}{n_{x,m}! N^{p_{x,m}}} / \left[N \wedge_{\{l,y,o\}} \frac{\tau(l_{y,o}, l_{y,o}, J_o)}{l_{y,o}!} \right] \quad (3.77)$$

$$= \frac{1}{N} \prod_{\{x,m\} \in w \tilde{\mathbf{A}}_y} f_N(J_m) . \quad (3.78)$$

Let $a(\Theta)$ be the coefficient of link chain Θ in the numerator or denominator of $t[U; w, y]$. Consider again $A_{\square}$ where $\square$ is a link chain which traces out a unit square and recurs at one side. The chain $\square$ does not contribute to $A_{\square}$ in the same manner that $w \tilde{\mathbf{A}}_y$ contributes to $A_{w \tilde{\mathbf{A}}_y}$ because, due to factorization, $a(\square) = a(-) a(\square)$ so that

$$\frac{a(-) + a(\square)}{1 + a(\square)} = a(-) \frac{1 + a(\square)}{1 + a(\square)} = a(-) . \quad (3.79)$$

(3.79) says that the $\square$ contribution to $A_{\square}$ has already been counted in A_{-} . This implies that

$$A_{\substack{\text{recurrent or} \\ \text{selfintersecting chain}}} = 0 . \quad (3.80)$$

As described in the beginning of this subsection, (3.80) is not predicted by the naive J_μ expansion. The problem is that the leading contribution to $A_{\square}$ is *not* from the leading term of (3.55). In (3.56) the middle “−1” comes from the second term of (3.55)—a term which is neglected in (3.51). Hence it does not appear in $A_{\square}$ if (3.51) is applied to do the $[dV]$ integrals.

3.2.4 Wilson Loop Picture of Gaugefixing

Eqs. (3.16), (3.17), and (3.77) imply that within the trace orthogonality approximation

$$\langle V_y(y \blacktriangleleft_w) V_w^\dagger \rangle_{\alpha j} = \delta_{\alpha j} \sum_{w \tilde{\mathbf{A}}_y} A_{w \tilde{\mathbf{A}}_y}(J) \text{Tr}[y \blacktriangleleft_w w \tilde{\mathbf{A}}_y]_u \quad (N \mapsto \infty), \quad (3.81)$$

$$A_{w \tilde{\mathbf{A}}_y} = \frac{1}{N^2} \text{Tr} \langle V_{yy} \tilde{\mathbf{A}}_w V_w^\dagger \rangle_{|\beta=0} = \prod_{\{w, \mu\} \in y \tilde{\mathbf{A}}_w} f_N(J_\mu), \quad (3.82)$$

where f_N is given in (3.62) if J_μ is finite and in (3.69) if $\xi \rightarrow 1/N$. These formulae have the following interpretation:

- By Eq. (3.81) the gaugefixed expectation value of a link chain $y \blacktriangleleft_w$ is the weighted sum over all Wilson loops made by joining to $y \blacktriangleleft_w$ a selfavoiding link segment $w \tilde{\mathbf{A}}_y$. This holds at all β .
- By (3.82), weights $A_{w \tilde{\mathbf{A}}_y}$ are proportional to the $\beta = 0$ gaugefixed expectation value of $w \tilde{\mathbf{A}}_y$. *All the gauge dependence is in these weights.*
- $A_{w \tilde{\mathbf{A}}_y}$ is a path-product of e^{-S^*} trace expansion coefficients. In the orthogonal trace approximation (3.58), nonselfavoiding operators do not contribute to $[V_w^\dagger V_y]_v$ and the $\beta = 0$ expectation value of nonselfavoiding links vanish.
- $\beta = 0$ λ gauge expectation values satisfy factorization. If P_{SA} is a selfavoiding path

$$\langle \prod_{\{y, n\} \in P_{SA}} \Omega_{y, n} \rangle_{|\beta=0} = \prod_{\{y, n\} \in P_{SA}} \langle \Omega_{y, n} \rangle_{|\beta=0}. \quad (3.83)$$

These results permit us to read off the λ gauge heavy quark propagator

$$\lim_{t_B \rightarrow \infty} \frac{1}{N} \text{Tr} \langle V_{(t_B, \vec{y})} \hat{U}_{\vec{y}, \vec{y} + t_B \hat{0}} V_{(0, \vec{y})}^\dagger \rangle = Z_H e^{-V_H^{(\lambda)} |t_B|}, \quad x \equiv (t_E, \vec{0}). \quad (3.84)$$

$V_H^{(\lambda)}$ can be interpreted as $V_H^{(\lambda)} = M_{q,\text{heavy}}^{(\lambda)} - m_q$. At $\beta = 0$ and $N \rightarrow \infty$,

$$Z_H = 1, \quad V_H^{(\lambda)} = -\log[f_N(J_0)]. \quad (3.85)$$

Hence $V_H^{(\lambda)}$ decreases with increasing λ .

3.3 Hopping Expansion and Resummation

In the $N \rightarrow \infty$ hopping expansion of $[\Theta]_f$, the ψ_x operator hops from site to site leaving behind a trail of

$$(r - \not{n}) K_B U_{y,n}, \quad n \in \{\pm 0, \dots, \pm(D-1)\} \quad (3.86)$$

factors to mark its randomwalk path P . Pauli Exclusion is irrelevant because of quenching. Therefore the quenched background field quark propagator is

$$[\psi_x \bar{\psi}_0]_f = \sum_P \prod_{\{y,n\} \in P} (r - \not{n}) K_B U_{y,n} \quad (3.87)$$

where $\{y, n\}$ are the locations and directions along P . (Assume implicit path ordering when appropriate.) Following Eq. (3.25), decompose the sum over P into a sum over trees and branches. The gaugefixed expectation value of branches is trivial by (3.24). $\beta = 0$ expectation values of nonselfavoiding trees, as discussed in Section 3.2.3, are suppressed in the orthogonal trace approximation. Hence, following Eq. (3.82), (3.87) is equivalent to

$$P_q \equiv \langle V_x \psi_x \bar{\psi}_0 V_0^\dagger \rangle = \sum_{\{T_{SA}\}} \left[\prod_{\{y,n\} \in T_{SA}} f_N(J_n) \right] \sum_{\{B_y | y \in T_{SA}\}} \prod_{\{y,n\} \in P} (r - \not{n}) K_B \quad (3.88)$$

where P is the selfavoiding tree T_{SA} going through sites y with a branch B_y at each site.

As there is no technique for summing selfavoiding trees as specified in (3.88), we shall sum all trees T . This approximation is justified if $D > 4$ by the fact that the quark mass pole is determined in the $t \geq L \rightarrow \infty$ limit. In this limit the average number of selfintersections and recurrences,[19]

$$\bar{s} = L \times (\text{path density}) \sim \frac{L^2}{(x^2)^{D/2}} = L^{2-D/2} \quad (3.89)$$

vanishes if $D > 4$ for randomwalk paths. (Nonbacktracking randomwalks are presumably *at least* as selfavoiding.) This suggests that the number of nonselfavoiding paths, being negligible, may not affect the quark mass pole.

Note that whether selfavoidance affects the propagator pole is analogous to whether

$$f(a, x) \equiv \sum_{L=0}^{\infty} a_L x^L, \quad \lim_{L \rightarrow \infty} a_L = a^L. \quad (3.90)$$

guarantees $f(a, x)$ is of the form $f(a, x) \propto \frac{1}{1-ax} + \text{finite}$. A counterexample satisfying (3.90) is $a_L = a^L + 1/L$ in which case a second pole exists at $x = 1$. Hence anomalous quark mass poles or mass shifts stemming from the inclusion of non-selfavoiding paths is a possibility which cannot be ruled out.

There are three effects we wish to capture: (i)spontaneous chiral symmetry breaking, (ii)gauge dependence of the dispersion relation, and (iii)proximity of the effective quark propagator to free particle form. The source of (i) is certainly different from (ii) and (iii) so that we may study them separately. If the dispersion relation is gauge dependent at (r, β) , there is no reason to believe this property would not persist to other (r, β) values since (r, β) are unrelated to gaugefixing. To study (ii) and (iii) we set $(r = 1, \beta = 0)$ where we can solve for the full quark prop-

agator and its dispersion relation. In this limit, spontaneous symmetry breaking is absent. To study the combination of spontaneous symmetry breaking and gauge dependence, we set $(r = 0, \beta = 0)$ where we can solve for $M_q^{(\lambda)}$.

3.3.1 $N \rightarrow \infty, r = 1, \beta = 0$ Quark Propagator

Since $(1 - \gamma_\mu)(1 + \gamma_\mu) = 0$ quarks cannot backtrack and the hopping expansion cannot generate any zero area Wilson loops. Hence, at $\beta = 0$

$$\langle \bar{\psi}_0 \psi_0 \rangle^{\text{cont}} = -2CK_B, \quad C \equiv 2^{D/2}N, \quad K_B \equiv \frac{1}{2(m_q + D)}, \quad (3.91)$$

where $\psi^{\text{cont}} = \sqrt{2K_B}\psi$ and C is due to the Dirac and color traces. The $r = 1$ pion mass obeys[20]

$$\cosh(M_\pi) = 1 + \frac{1 - 4K_B^2(D + 1) + 16DK_B^4}{8K_B^2(1 - 2(D - 1)K_B^2)}, \quad (3.92)$$

which has $K_c = 1/4$ at $D = 4$.

Upon the approximation $T_{SA} \rightarrow T$ in Eq. (3.88), the sum over trees is implemented by solving the recursion relation

$$P_q(x) = -K_B \delta_{x,0} - K_R \sum_n f_N(J_n) \not{n} P_q(x - n). \quad (3.93)$$

Fourier transform $P_q(x) \propto \sum_p e^{ipx} \tilde{P}_q(p)$ produces

$$\tilde{P}_q(p) = -\frac{C}{g_0} \left[\frac{[m_R + \sum_\mu \bar{g}_\mu(1 - \cos p_\mu)] - i \sum_\mu \gamma_\mu \bar{g}_\mu \sin p_\mu}{[m_R + \sum_\mu \bar{g}_\mu(1 - \cos p_\mu)]^2 + \sum_\mu \bar{g}_\mu^2 \sin^2 p_\mu} \right], \quad (3.94)$$

$$g_\mu(\lambda) \equiv f_N(2N\lambda), \quad \bar{g}_\mu \equiv \frac{g_\mu}{g_0}, \quad m_R \equiv \frac{1}{2K_B g_0} - \sum_{\mu=0}^{D-1} \bar{g}_\mu. \quad (3.95)$$

Eq. (3.8) gives the expression for $\bar{g}_i$. $M_q^{(\lambda)}$, the $\vec{p} = 0$ pole of $\tilde{P}_q(p)$, is related to continuum pole m_R by

$$\sinh M_q^{(\lambda)} = m_R . \quad (3.96)$$

In Landau gauge where $\bar{g}_\mu = 1$, P_q reduces to free particle form with $m_R = m_q/g_0$. Otherwise, P_q obeys a noncovariant Dirac equation which in continuum language is

$$(i \sum_\mu \bar{g}_\mu \gamma_\mu \partial_\mu + M_q^{(\lambda)}) P_q(x) = 0 . \quad (3.97)$$

(3.97) corresponds to dispersion relation (3.7).

When $K \rightarrow K_c = 1/4$, $M_q^{(\lambda)}$ is negative—for nonnegative values of $M_q^{(\lambda)}$ there is no massless pion. This reflects the absence of chiral symmetry and spontaneous chiral symmetry breaking at $(r = 1, \beta = 0)$. Thus we turn to $(r = 0, \beta = 0)$ in order to examine the λ dependence of $M_q^{(\lambda)}$ in the presence of spontaneous chiral symmetry breaking.

3.3.2 $N \rightarrow \infty$, $r = \beta = 0$ Quark Mass

When $r = 0$, by virtue of $\gamma_n \gamma_{-n} = 1$, the Dirac matrix of each branch is 1 and the sum over branches in (3.88) is equivalent to renormalizing K_B to [21]

$$K_R \equiv K_B W(K_B) \quad (3.98)$$

where $W(K_B)$ is the weighted sum over all branch configurations at one site. [21, 22]

The number $I(L)$ of length $L \geq 1$ “irreducible” branches, branches with a single

base stem, is

$$I(L) = (2D - 1) \sum_{p=0}^{L-1} \sum_{\substack{\{l_i\} \\ \sum_{i=1}^p l_i = L-1}} \prod_{i=1}^p \left[\frac{2D-1}{2D} I(l_i) \right] \quad (I(0) = 0), \quad (3.99)$$

the sum over all arrangements of its irreducible subbranches. At any site, there is a unique length L irreducible branch so

$$W(K_B) = 1 + \sum_{L=1}^{\infty} (-K_B^2)^L I(L) = \frac{1}{1 - w_I(K_B)}, \quad (3.100)$$

$$w_I(x) \equiv \sum_{L=1}^{\infty} x^L I(L) = \frac{x(2D-1)}{1 - \frac{2D-1}{2D} w_I(x)}. \quad (3.101)$$

Definition (3.101) factorizes the RHS of (3.99) to give the RHS of (3.101). Equating the RHS and LHS of (3.101) gives a polynomial whose solution yields

$$K_R = 1/(m_q + \sqrt{m_q^2 + (2D-1)}). \quad (3.102)$$

The quark propagator of (3.88) reduces to

$$P_q(x) = \sum_{T_{SA}} \prod_{\{n\} \in T_{SA}} (-K_R) \not{n} f_N(J_n). \quad (3.103)$$

($r = 0, \beta = 0$) meson propagators are [21, 23]

$$\langle \bar{\psi}_x \Gamma \psi_x \bar{\psi}_0 \Gamma \psi_0 \rangle = N \sum_T \text{Tr} \left(\Gamma \left[\prod_{\{n\} \in T} K_R \not{n} \right] \Gamma \left[\prod_{\{m\} \in T^\dagger} K_R \not{m} \right] \right) \quad (3.104)$$

where T may be nonselfavoiding since no gaugefixing is involved. Since trees are nonbacktracking—a backtrack makes a branch—its Fourier transform $D_m(p)$ obeys

nonbacktracking randomwalk recursion relation

$$\text{Tr}[D_m \Gamma] = C - K_R^2 \sum_n \text{Tr}[\gamma_m \Gamma \gamma_m (1 - \delta_{n,-m}) D_n e^{-ipm}] \quad (3.105)$$

where $\Gamma = \gamma_5$ for the pion and γ_3 for the rho. The commutation of Γ with the γ_m splits

$$M_\rho^2 = 4 + M_\pi^2 \quad (3.106)$$

from

$$M_\pi^2 = \frac{4(D-1)}{\sqrt{2D-1}} m_q + \mathcal{O}(m^2) . \quad (3.107)$$

While M_ρ does not vanish as $m_q \rightarrow 0$, M_π vanishes and the pion (actually 2^D pions) plays the role of Goldstone boson for chiral symmetry breaking. The same sort of arguments give[22, 23]

$$\langle \bar{\psi}_0 \psi_0 \rangle = -C \frac{D \sqrt{2D-1+m_q^2} - (D-1)m_q^2}{D^2 + m_q^2} , \quad C = 2^{D/2} N \quad (3.108)$$

for the chiral order parameter, which is nonzero when $m_q \rightarrow 0$.

Nonbacktracking Approximation

Since including backtracking overcounts branches, already summed by K_R , we will enforce nonbacktracking but otherwise permit recurrence and selfintersection. The sum over nonbacktracking trees is implemented by

$$P_q(x)_n = -\frac{\langle \bar{\psi}_0 \psi_0 \rangle}{2DC} \delta_{x,0} - K_R f_N(J_n) \not\sim \sum_m (1 - \delta_{m,-n}) P_q(x-n)_m \quad (3.109)$$

where $P_q(x) = \sum_n P_q(x)_n$. Subscript n indicates the direction from which site x was approached. The Fourier transform of Eq. (3.109) produces

$$\bar{P}_q(p)_n = -\frac{\langle \bar{\psi}_0 \psi_0 \rangle}{2DC} + \sum_m \mathcal{M}_{nm}(E, \vec{p}) \bar{P}_q(p)_m, \quad (3.110)$$

$$\mathcal{M}_{nm}[E, \vec{p}] = -K_R f_N(J_n) \not{p} (1 - \delta_{m,-n}) e^{i\vec{p}n}, \quad (3.111)$$

where $\mathcal{M}$ is a $2D \times 2^{[D/2]}$ matrix.

Effective quark mass $M_q^{(\lambda)}$ obeys

$$\det(1 - \mathcal{M}[M_q^{(\lambda)}, \vec{0}]) = 0. \quad (3.112)$$

Explicit solution by Mathematica[©] at $D = 2$ and $D = 4$ reveals that, despite being a $2D$ polynomial in $e^{-M_q^{(\lambda)}t}$, Eq. (3.112) has only two distinct solutions corresponding to $\pm M_q^{(\lambda)}$. The positive energy solution reduces to

$$\sinh M_q^{(\lambda)} = \frac{1 - (2D - 3)\bar{g}^2 K_R^2 - g_0^2 K_R^2 - (2D - 1)\bar{g}^2 g_0^2 K_R^4}{2(1 + \bar{g}^2 K_R^2)g_0 K_R} \quad (3.113)$$

where $\bar{g} \equiv g_i$ (for $i = 1, \dots, D - 1$). In the absence of gaugefixing $g_\mu \rightarrow 0$ and $M_q^{(\lambda)} \rightarrow \infty$. In the absence of renormalization and perfect gaugefixing, $K_R \rightarrow 1/(2m_q)$, $g_\mu \rightarrow 1$ and the free particle relation $\sinh M_q^{(\lambda)} = m_q$ is recovered.

Following Eq. (3.4), the chiral limit is extracted by replacing K_R with m_q according to Eq. (3.102) and expanding about $m_q = 0$. If $d \equiv 2D - 1$,

$$M_c^{(\lambda)} = \frac{1 + 4D(D - 1) - (3 + 4D(D - 2))\bar{g}^2 - d(g_0^2 + \bar{g}^2 g_0^2)}{\sqrt{d}(2d + 2\bar{g}^2)g_0}, \quad (3.114)$$

$$E^{(\lambda)} = \frac{1 + 4D(D - 1) + d(2D\bar{g}^2 + g_0^2) - (d - 2)\bar{g}^4 + (\bar{g}^2 + 3d - 1)\bar{g}^2 g_0^2}{2(d + \bar{g}^2)^2 g_0}. \quad (3.115)$$

The choice $\lambda_i \equiv \bar{\lambda} = 1$ (for $i = 1, \dots, D - 1$) yields (3.5) and (3.6). While $M_c^{(\lambda)}$ is continuous, its second derivative is discontinuous at $\lambda = 1/2$.

$D \rightarrow \infty$ Limit

In the $D \rightarrow \infty$ limit, there is no difference between selfavoiding and random paths. In this limit, the leading $\mathcal{O}(1/D)$ $r = 0$ critical mass is

$$M_c^{(\lambda)} = \frac{7C}{16\langle\bar{\psi}_0\psi_0\rangle g_0}, \quad B^{(\lambda)} = \frac{25}{32g_0} \quad (3.116)$$

$1/g_0(\lambda) \equiv 1/f_N(2N\lambda)$ is a continuous, monotonically decreasing function whose 2nd derivative is discontinuous at $\lambda = 1/2$.

3.3.3 Matching Coefficients

Effective electroweak matrix elements on the lattice involves operators of the form[11]

$$O_{\pm}^{\text{cont}}[LL] \equiv z_{\pm}^J J^{\text{latt}} + \sum_{\Gamma_1\Gamma_2} z_{\pm}^{\Gamma_1\Gamma_2} O_{\pm}^{\text{latt}}[\Gamma_1\Gamma_2], \quad (3.117)$$

$$\Gamma_1\Gamma_2 \in \{LL, SS, PP, TT, LR\}, \quad J(0) \equiv \bar{s}(0)d(0). \quad (3.118)$$

The RHS of (3.117) is the combination of lattice operators required to reproduce S-matrix elements of continuum operator $O_{\pm}^{\text{cont}}[LL]$. “Matching” coefficients $z_{\pm}^{\Gamma_1\Gamma_2}$ are determined by gauge invariant numerical methods. The problem at hand is determining $z_{\pm}^J$. Proportional to $(m_s + m_d)a^{-2}$, the $z_{\pm}^J$ coefficients are beyond weak coupling perturbation theory or the usual lattice methods. Matching condition[4]

$$\langle s | O_{\pm}^{\text{cont}}[LL] | d \rangle = 0, \quad (3.119)$$

the parity even quark-equivalent of the parity odd hadronic renormalization condition that $K \not\rightarrow \text{vac}$, implies that the lattice operators satisfy

$$z_{\pm}^J = - \sum_{\Gamma_1 \Gamma_2} \alpha_{O_{\pm}[\Gamma_1 \Gamma_2]} z_{\pm}^{\Gamma_1 \Gamma_2}, \quad (3.120)$$

$$\langle s | O_{\pm}^{\text{latt}}[\Gamma_1 \Gamma_2] | d \rangle \equiv \alpha_{O_{\pm}[\Gamma_1 \Gamma_2]} \langle s | J^{\text{latt}}(0) | d \rangle. \quad (3.121)$$

Since quarks are not part of the physical S -matrix, Eq. (3.121) may give a value of $\alpha_{O_{\pm}[\Gamma_1 \Gamma_2]}$ different from its physical sector value and, moreover, a value that is gauge variant.

We have two purposes in this Section: (a) to understand how λ independence of $\alpha_{O_{\pm}[\Gamma_1 \Gamma_2]}$ comes about; and (b) to derive a consequence of $\beta = 0$ factorization which can be compared to $\beta = 5.7$ numerical results. This comparison lends insight into why $\beta = 0$ formulae mimic so closely the $\beta = 5.7$ and 6.0 data.

Following (3.121) we are interested in λ gauge quark correlation functions of $O_{\pm}^{\text{latt}}[\Gamma_1 \Gamma_2]$ and $J^{\text{latt}}(0)$. Because of $\beta = 0$ factorization, quark propagators in the fermionic Wick expansion of correlation functions do not interfere with each other. Hence the $\beta = 0$ correlation functions are given by replacing the background field quark propagators in their Wick expansion with the $\beta = 0$ value of $\langle V_x \psi_x \bar{\psi}_0 V_0^{\dagger} \rangle$.

Let $\{a, b\}$ be color and $\{\alpha, \beta\}$ be Dirac indices and define

$$Q^{ab}(t) \equiv \sum_{\vec{x}} \langle (V\psi)_x^a (\bar{\psi} V^{\dagger})_0^b \rangle, \quad \chi_{\alpha\beta}^{ab} \equiv \langle (\psi_0)_\alpha^a (\bar{\psi}_0)_\beta^b \rangle \quad (3.122)$$

where $Q \in \{D, S, C\}$ corresponding to $\psi \in \{d, s, c\}$. If $t_y = -t$, by (3.103)

$$\sum_{\vec{y}} \langle V_0 \psi_0 \bar{\psi}_y V_y^{\dagger} \rangle = \gamma^5 Q^{\dagger}(-t) \gamma^5. \quad (3.123)$$

By R symmetry and (3.108),

$$\chi_{\alpha\beta}^{ab} = -\delta^{ab} \delta_{\alpha\beta} \langle \bar{\psi}_0 \psi_0 \rangle / (2^{D/2} N) . \quad (3.124)$$

In fact, a stronger statement can be made for χ . Whenever $\bigcirc_\bullet$ encloses zero area $\prod_{(y,n) \in \bigcirc_\bullet} U_{y,n} \neq 1$ and the $[dU]$ integral in (3.124) is trivial. Therefore,

$$[(\psi_0)_\alpha^a (\bar{\psi}_0)_\beta^b]_f|_{\beta=0 \text{ graph}} \propto \delta^{ab} \delta_{\alpha\beta} \quad (3.125)$$

where “ $\beta = 0$ graph” refers to any single hopping expansion graph of $[\psi_0 \bar{\psi}_0]_f$ which contributes nontrivially to χ at $\beta = 0$. This color diagonality of $[\psi_0 \bar{\psi}_0]_f$ permits factorization of correlation functions containing $[\psi_0 \bar{\psi}_0]_f$ “bubble” contractions at $\beta = 0$. While $[\psi_0 \bar{\psi}_0]_f$ graphs are not diagonal at $\beta > 0$, at $\beta = 5.7$

$$[(\psi_0)_\alpha^a (\bar{\psi}_0)_\beta^b]_f \propto \delta^{ab} \delta_{\alpha\beta} + 0.1\% \text{ fluctuations.} \quad (3.126)$$

On a typical 16by24 gauge configuration with $\lambda = 1$, $K = .094$, and in Dirac space

$$[\psi_0^1 \bar{\psi}_0^1]_f = \frac{2K}{10^3} \begin{pmatrix} (995., .1) & (.0, .0) & (.0, .0) & (.1, .0) \\ (.0, .0) & (995., .1) & (.1, .0) & (.0, .0) \\ (-.2, .0) & -(1., .3) & (995., -.1) & (.0, .0) \\ (-.1, .3) & (.2, .0) & (.0, .0) & (995., -.1) \end{pmatrix}; \quad (3.127)$$

with $K = .166$,

$$[\psi_0^1 \bar{\psi}_0^1]_f = \frac{2K}{10^3} \begin{pmatrix} (902., .9) & -(1.1, .6) & (-.4, .0) & (-.9, 2.1) \\ (2.8, .9) & (903., 4.2) & -(9, 2.1) & (-1.9, .0) \\ (-2.7, .0) & -(1.1, 4.1) & (902., -.9) & (2.8, -.9) \\ (-1.1, 4.1) & (.7, .0) & (-1.1, .6) & (903., -4.2) \end{pmatrix}. \quad (3.128)$$

In the color off-diagonal sector with $K = .166$,

$$[\psi_0^1 \bar{\psi}_0^2]_f = \frac{2K}{10^3} \begin{pmatrix} -(8.0, 3.6) & (5.7, 2.1) & (11., 5.1) & (.04, 8.4) \\ (2.6, -2.3) & -(9.8, 7.4) & -(20., 8.4) & -(5.3, 8.2) \\ -(2.1, 7.0) & (6.5, 1.1) & (4.0, .00) & (7.3, 5.1) \\ (5.9, 7.6) & (1.4, 11.) & (3.9, -1.2) & (1.3, -13.) \end{pmatrix}. \quad (3.129)$$

In general, (I)fluctuations grow as K increases; (II)in the color singlet sector, scalar components dominate; (III)in the color nonsinglet sector, the *nonscalar* fluctuations are greater than the scalar, sometimes by as much as an order of magnitude; (IV)these statements are valid for all λ and axial gauge configurations with typical variation between configurations of $\sim 20\%$.

The Euclidean Dirac matrices we use are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (3.130)$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad \vec{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \quad (3.131)$$

Useful Fierz relations are

$$\Gamma_1^{\alpha\beta} \Gamma_2^{\gamma\delta} = \sum_{\Gamma_3 \Gamma_4} f_{\Gamma_1 \Gamma_2}^{\Gamma_3 \Gamma_4} \Gamma_3^{\alpha\delta} \Gamma_4^{\gamma\beta}, \quad (3.132)$$

$$f_{SS}^{SS} = f_{PP}^{SS} = \frac{1}{4}, \quad f_{VV}^{SS} = -f_{AA}^{SS} = 1, \quad f_{TT}^{SS} = 3, \quad (3.133)$$

$$[\gamma^\mu(1 \pm \gamma^5)]_{ij} [\gamma^\mu(1 \pm \gamma^5)]_{kl} = -[\gamma^\mu(1 \pm \gamma^5)]_{il} [\gamma^\mu(1 \pm \gamma^5)]_{kj}. \quad (3.134)$$

If $t_y = -t_x \equiv -t$, Wick expansion, (3.124), (3.132), and (3.134) imply (Recall from

Eq. (2.55) that $F_{\Gamma_1\Gamma_2}^{qq} = \bar{s}_0\Gamma_1 d_0 \cdot \bar{q}_0\Gamma_2 q_0$ and $H_{\Gamma_1\Gamma_2}^{qq} = \bar{s}_0\Gamma_1 q_0 \cdot \bar{q}_0\Gamma_2 d_0$.)

$$\mathcal{J} \equiv \sum_{\vec{x}, \vec{y}} \langle V_{\vec{x} s_x} J(0) \bar{d}_y V_y^\dagger \rangle = S(t) \gamma_5 D^\dagger(-t) \gamma^5, \quad (3.135)$$

$$\sum_{\vec{x}, \vec{y}} \langle V_{\vec{x} s_x}^c H_{\Gamma_1\Gamma_2}^{qq} \bar{d}_y V_y^\dagger \rangle = \begin{cases} \sim \text{Tr}(\chi^{ab} L) = 0 & \Gamma_1\Gamma_2 = LL; \\ -\frac{1}{2^{D/2}N} f_{\Gamma_1\Gamma_2}^{SS} \mathcal{J}^{cd} \langle \bar{\psi}_0 \psi_0 \rangle & \text{otherwise;} \end{cases} \quad (3.136)$$

$$\sum_{\vec{x}, \vec{y}} \langle V_{\vec{x} s_x}^c F_{\Gamma_1\Gamma_2}^{qq} \bar{d}_y V_y^\dagger \rangle = \begin{cases} \mathcal{J}^{cd} \langle \bar{\psi}_0 \psi_0 \rangle & \Gamma_1\Gamma_2 = SS; \\ 0 & \text{otherwise.} \end{cases} \quad (3.137)$$

Comparing (3.136) and (3.137) to (3.135) yields

$$\alpha_{F_{\Gamma_1\Gamma_2}^{qq}} = \begin{cases} \langle \bar{\psi}_0 \psi_0 \rangle & \Gamma_1\Gamma_2 = SS; \\ 0 & \text{otherwise;} \end{cases} \quad (3.138)$$

$$\alpha_{H_{\Gamma_1\Gamma_2}^{qq}} = \begin{cases} 0 & LL; \\ -\frac{1}{2^{D/2}N} \langle \bar{\psi}_0 \psi_0 \rangle f_{\Gamma_1\Gamma_2}^{SS} & \text{otherwise.} \end{cases} \quad (3.139)$$

Therefore, $\alpha_{H_{\Gamma_1\Gamma_2}^{qq}}$ is the dominant contribution to $\alpha_{O_{\pm}^{qq}[\Gamma_1\Gamma_2]}$ unless $\Gamma_1\Gamma_2 = SS$, in which case $\alpha_{H_{SS}^{qq}}$ is $1/N$ suppressed relative to $\alpha_{F_{qq}[SS]}$. By (3.133) the $\beta = 0$ ratios are

$$\alpha_{H_{SS}^{qq}} : \alpha_{H_{PP}^{qq}} : \alpha_{H_{VV}^{qq}} : \alpha_{H_{AA}^{qq}} : \alpha_{H_{TT}^{qq}} = 1 : 1 : 4 : -4 : 12. \quad (3.140)$$

As described in Chapt. 2, these $\beta = 0$ ratios are approximately reproduced at $\beta = 5.7$ and 6.0 . Appropriate linear combinations give Eq. (3.9). Since $\langle \bar{\psi}_0 \psi_0 \rangle$ is gauge invariant, $\alpha_{O_{\pm}[\Gamma_1\Gamma_2]}$ is gauge invariant.

Chapter 4

Gaugefixing Ambiguities and Gauge Dependence of Gauge Variant Schwinger Model Correlation Functions

4.1 Toy Model of Gaugefixing

Gaugefixing is of current interest in the lattice community. Gluon[7], quark[3, 4] and photon[8] propagators, effective masses, and wavefunctions[24] have been studied[3] in “ λ -gauges” $\lambda \partial_0 A_0 + \nabla \cdot \vec{A} = 0$. The numerical work has spawned analytical and computational studies of longitudinal[25] and topological[26] gaugefixing ambiguities and their effects on gluon, quark[4] and photon[8] correlation functions. Nonperturbatively solvable models of gaugefixing are needed to guide these projects.

Continuum QED $_{\mathcal{M}}$, the massless Schwinger model, is exactly solved by bosonizing its quark fields, choosing a gauge, and diagonalizing the gaugefixed bosonized ac-

tion. Its physical spectrum consists of a single free massive boson, a quark-antiquark condensate. While analyses exist in both planar($\mathcal{M} = M^2$) and cylindrical($\mathcal{M} = C^2$) Minkowski space,[39, 28] we will focus mainly on $\mathcal{M} = M^2$. The bosonized Schwinger model action is given by[29] $S^g = \int_{\mathcal{M}} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and

$$S^f = \int_{\mathcal{M}} \bar{\Psi}_x (i\not{\partial} - g\not{A}) \Psi_x = \int_{\mathcal{M}} \frac{1}{2} \partial\phi \cdot \partial\phi + \frac{g}{\sqrt{\pi}} \epsilon^{\mu\nu} A_\mu \partial_\nu \phi. \quad (4.1)$$

Bosonization maps Heisenberg quark field Ψ to boson field ϕ according to

$$\Psi^r(x) = i^{r-1} \sqrt{Z_I} : e^{-i\sqrt{\pi}\Theta_r^H(x)} : , \quad (4.2)$$

$$\Theta_r^H(x) = \Theta_r(x) + \frac{g}{\sqrt{\pi}} \int^{\vec{x}} d\vec{x}' A_1(x') \quad (4.3)$$

where Θ_r is defined in (A.44). Reproducing Maxwell equation $(i\not{\partial} - g\not{A})\Psi = 0$ requires

$$A_0(x) = \frac{1}{m} \left[-\partial_1 \phi + \int^{\vec{x}} d\vec{x} \partial_0 (\partial_0 \phi + m A_1) \right] , \quad m \equiv \frac{g}{\sqrt{\pi}} . \quad (4.4)$$

If $g = 0$, Ψ is the free field ψ of Eq. (A.44). Since $\psi_x \longrightarrow e^{i\theta\gamma_5} \psi_x$ under

$$\phi(x) \longrightarrow \phi(x) + \theta/\sqrt{\pi} , \quad (4.5)$$

(4.5) is the boson-equivalent $U_A(1)$ transformation. The corresponding Noether current and its dual, both conserved when $g = 0$, are $J_{\mu 5} = \frac{1}{\sqrt{\pi}} \partial_\mu \phi$ and $J_\mu = -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \phi$. Since $[J^0(x), \psi_x^r] = -\delta(\vec{y} - \vec{x}) \psi_x^r$, J is the $U_V(1)$ current.

If $g \neq 0$, Eq. (4.4) implies $F_{01} = -\square\phi/m$ so that Maxwell equation $\partial^\nu F_{\nu\mu} = gJ_\mu$ requires $(\square + m^2)\phi = \text{constant}$ and $\partial^\mu J_{\mu 5} = g\epsilon^{\mu\nu} F_{\mu\nu}/2\pi$. While $J_{\mu 5}$ is no longer conserved, S^f remains invariant under (4.5). The new Noether current is Chern-

Simons current $K_\mu = J_{\mu 5} - \frac{g}{\pi} \epsilon_{\mu\nu} A^\nu$, whose conservation implies

$$Q_5 \equiv \int_{-\infty}^{\infty} d\vec{x} J_5^0, \quad \partial_0 Q_5 = -\frac{g}{\pi} \int_{-\infty}^{\infty} d\vec{x} F_{01}. \quad (4.6)$$

Effective quark and gluon masses extracted from zero momentum lattice QCD₃₊₁ quark and gluon propagators decrease with increasing λ . [4] It is important to support these numerical results independently. To this end, in Chapt. 3 we showed analytically that effective quark masses decrease with increasing λ in the strong coupling, $N \rightarrow \infty$, $r = 0$ and infinite volume limit. As in QCD₃₊₁, quarks in the massless Schwinger model are confined. [30, 31] In Section 4.2 we show that the covariant gauge effective Schwinger model quark mass is

$$M_q^{\text{eff}} = \lim_{-x^2 \rightarrow \infty} \left[\frac{\log(-x^2 m^2)}{4\sqrt{-x^2}} - \xi_R(\mu) \left(\frac{m}{16\pi} \right) \sqrt{-x^2 m^2} \log(-x^2 \mu^2) \right], \quad (4.7)$$

which is gauge variant. M_q^{eff} diverges to negative infinity unless $\xi_R = 0$, in which case it vanishes. In contrast, the Schwinger model photon is not confined [39] and the anomaly-generated photon mass M_g^{eff} , we verify, is ξ independent.

$S^g + S^f$ is invariant under local $U_V(1)$ transformations

$$A_V^\mu = A^\mu + \frac{i}{g} V_x^\dagger \partial^\mu V_x, \quad \Psi_{x,V} = V_x \Psi_x. \quad (4.8)$$

Gaugefixing seeks to eliminate the nondynamical V_x degrees of freedom. Consider Fadeev-Popov gaugefixing action

$$S^{GF} = -\frac{1}{2\xi} \int_{\mathcal{M}} (h^{\mu\nu} \partial_\mu A_\nu)^2 \quad (4.9)$$

where $h^{00} = \lambda$, $h^{11} = -1$, and $h^{01} = h^{10} = 0$. Special cases are covariant gauges

$[\xi, \lambda = 1]$ and λ -gauges $[\xi \rightarrow 0, \lambda]$. In Appendix A.5 we show that λ -gauge($\lambda \neq 0$) QED $_{M^2}$ is equivalent to Landau($\lambda = 1$) gauge QED $_{M^2}$ with a new nonlocal, anisotropic quark-photon interaction

$$g\bar{\Psi}_x A(x) \Psi_x \longrightarrow \bar{\Psi}_x \gamma_\mu \Psi_x \int_{\mathcal{M}} G^{\mu\nu}(x, y) A_\nu(y), \quad (4.10)$$

$$G(x, y) \equiv g \begin{pmatrix} \lambda^{-\frac{1}{4}} & 0 \\ 0 & -\lambda^{\frac{1}{4}} \end{pmatrix} \delta(t_y - \frac{t_x}{\sqrt{\lambda}}) \delta(\vec{y} - \vec{x}). \quad (4.11)$$

As a corollary, pure ($g = 0$) QED $_{M^2}$ in any λ -gauge is equivalent to Landau gauge.

Nuisances (not responsible for the ξ dependence of M_q^{eff}) are residual gauge invariances $\bar{V}_x$ present after gaugefixing. A $\bar{V}_x$ which is continuously connected over spacetime $\mathcal{M}$ to $\bar{V}_x = 1$ is a “longitudinal” ambiguity. If $\bar{V}_x$ cannot be continuously deformed to $\bar{V}_x = 1$, it is a “Gribov” ambiguity. Gauge equivalent gauge configurations satisfying the gauge condition which cannot be transformed into one another by longitudinal transformations are Gribov[32] copies, connected only by topologically nontrivial transformations. For $SU(N)$ gauge theories defined on an S^4 spacetime, on which $SU(N)$ homotopy groups are not all trivial, Singer[33] proved that continuous gaugefixing conditions which admit no Gribov copies do not exist.

By the Hodge-DeRham theorem any $U(1)$ gauge field $A \equiv d^\dagger \chi + d\Lambda + \omega$ is decomposable into a divergence piece $d^\dagger \chi$, an exact piece $d\Lambda$, and a topological piece ω satisfying $d\omega = d^\dagger \omega = 0$. Consider residual symmetries $\bar{V}_x$ on a background Landau gauge($d^\dagger A = 0$) photon field. Since $\bar{V}_x$ preserves the gauge condition, it generates vector fields of the form $\bar{V}_x^\dagger d\bar{V}_x$ in the kernel of $d^\dagger$. Such vector fields potentially change χ and Λ as well as ω . The $\bar{V}_x$ which change χ or Λ are longitudinal ambiguities; the ω -changing $\bar{V}_x$ are Gribov ambiguities.

There are no Gribov copies in QED_{M^2} . On $\mathcal{M} = C^2$ where space is a circumference $2\pi L$ circle, the λ -gauge ambiguities satisfy

$$h^{\mu\nu} \partial_\mu (\bar{V}_x^\dagger \partial_\nu \bar{V}_x) = 0. \quad (4.12)$$

If the V_x are parametrized as $V_x = e^{-i\Omega(t, \vec{x})}$, dummy parameter Ω need *not* be periodic. For V_x and $V_x^\dagger dV_x$ to be singlevalued and continuous it is sufficient that

$$\Omega(t, \vec{x} + \vec{L}) = \Omega(t, \vec{x}) + 2\pi n, \quad (n \text{ integer}). \quad (4.13)$$

The most general Ω satisfying (4.13) is

$$\Omega(t, \vec{x}) = \frac{2\pi n \vec{x}}{L} + \Omega_o(t, \vec{x}) + \tilde{\Omega}(t, \vec{x}), \quad n \in Z, \quad (4.14)$$

$$h^{\mu\nu} \partial_\mu \partial_\nu \Omega_o = 0, \quad \Omega_o(t, \vec{x} + \vec{L}) = \Omega_o(t, \vec{x}), \quad (4.15)$$

$$h^{\mu\nu} \partial_\mu \partial_\nu \tilde{\Omega} \neq 0, \quad \tilde{\Omega}(t, \vec{x} + \vec{L}) = \tilde{\Omega}(t, \vec{x}). \quad (4.16)$$

Transformations $\tilde{V} \equiv e^{-i\tilde{\Omega}}$ are suppressed by gauge condition (4.12) so they do not generate gauge ambiguities. On the other hand,

$$\bar{V}_{x,o} \equiv e^{-i\Omega_o}, \quad \bar{V}_{x,n} \equiv e^{2\pi i n \vec{x}/L} \quad (4.17)$$

obey (4.12) and generate, respectively, longitudinal ambiguities and Gribov copies.

Gaugefixing ambiguities pose two issues. (A) $SU(N)$ Gribov ambiguities ruin the usual ghosts representation of the Fadeev-Popov Jacobian. This is not a problem in lattice gaugefixing which works around the Fadeev-Popov method[7] or continuum $U(1)$ where the Jacobian decouples. (B) λ independent quantities may not be gauge

independent since, when there are residual gauge symmetries, varying λ does not fully test the gauge dependence. (Of course, even in the absence of gaugefixing ambiguities, λ -independence does not imply gauge independence since λ gauges are not the only possible gauge choices.) In Section 4.3, we try to compute QED_{M^2} operator product expansion coefficients by matching quark correlation functions. We find that, despite being ξ independent, these matching coefficients are generally incorrect due to contamination by longitudinal ambiguities.

4.2 Covariant Gauges on $\mathcal{M} = M^2$

Following Hodge-DeRham, the $\mathcal{M} = M^2$ gauge field is parametrizable as

$$A_\mu \equiv \frac{\sqrt{\pi}}{g} \epsilon_{\mu\nu} \partial^\nu (\chi_2 - \chi_1) + \partial_\mu \Lambda, \quad \square \chi_1 \neq 0, \quad \square \chi_2 = 0. \quad (4.18)$$

Since $\square \chi_2 = 0$ implies $\epsilon_{\mu\nu} \partial^\nu \chi_2 \equiv \partial_\mu \Lambda_o$ where $\square \Lambda_o = 0$, χ_2 corresponds to the longitudinal polarization. χ_1 is the transverse polarization. Λ and Λ_o , both equivalent to gauge transformations, decouple from S^f and S^g . The gaugefixing action

$$S^{GF} = -\frac{1}{2\xi} \int_{M^2} (\square \Lambda)^2 \quad (\square \equiv \partial_\mu \partial^\mu) \quad (4.19)$$

depends only on Λ . Since gauge invariant operators do not depend on Λ , gauge invariant correlation functions do not depend on ξ as expected.

Following the Nakanishi-Lautrup gaugefixing procedure,[34] introduce a regulator mass $\mu \rightarrow 0$ so that

$$S^{GF} = \int_{M^2} \partial B \cdot \partial \Lambda + \frac{1}{2} \xi B^2 - \mu^2 \Lambda B, \quad (4.20)$$

which in $D = 3 + 1$ is the Froissart model. As $\mu \rightarrow 0$, S^{GF} in (4.19) is equivalent to (4.20) because

$$-B\Box\Lambda + \frac{1}{2}\xi B^2 = -\frac{1}{2\xi}(\Box\Lambda)^2 + \frac{\xi}{2}(B - \xi^{-1}\Box\Lambda)^2. \quad (4.21)$$

Equations of motion are

$$\Box\Lambda + \mu^2\Lambda = \xi B, \quad \Box B + \mu^2 B = 0. \quad (4.22)$$

By (4.22), B is a free boson but Λ is not. However,

$$\Lambda_D(x) \equiv \Lambda(x) + \frac{\xi}{2\mu^2} x^\lambda \partial_\lambda B(x) \quad (4.23)$$

satisfies $\Box\Lambda_D + \mu^2\Lambda_D = 0$. Hence, the free particles are B and Λ_D . Canonical commutation relations are $[B^+(y), B^-(x)] = [\Lambda_D^+(y), \Lambda_D^-(x)] = 0$ and

$$[B^+(y), \Lambda_D^-(x)] = [\Lambda_D^+(y), B^-(x)] = \Delta(y - x; \mu) \quad (4.24)$$

where $\Delta(x; \mu)$ is given in Appendix A.4. Eqs. (4.23) and (4.24) imply

$$[\Lambda^+(y), \Lambda^-(x)] = -\frac{\xi}{2\mu^2} (y - x)^\lambda \partial_\lambda \Delta(y - x; \mu) \quad (4.25)$$

$$= -\xi \frac{\partial}{\partial \mu^2} \Delta(y - x; \mu). \quad (4.26)$$

Collecting $S^g + S^f$, A^μ and S^{GF} from Eqs. (4.1), (4.18), and (4.20) and using a trick from the Pais-Uhlenbeck recipe for higher derivative actions[29] begets

$$S^\xi \equiv S^g + S^f + S^{GF} = \quad (4.27)$$

$$\int_{M^2} \frac{1}{2} (\partial\phi' \cdot \partial\phi' - \partial\chi_2 \cdot \partial\chi_2 + \partial\chi_1 \cdot \partial\chi_1 - m^2 \chi_1^2 + 2\partial B \cdot \partial\Lambda + \xi B^2), \quad (4.28)$$

$$m \equiv \frac{g}{\sqrt{\pi}}, \quad \phi \equiv \phi' - \chi_1 + \chi_2. \quad (4.29)$$

The by-parts integrations require

$$\lim_{-\phi^2 \rightarrow \infty} \chi_1 = 0. \quad (4.30)$$

Λ_D , ϕ' and χ_2 are free, massless bosons, the latter of negative metric. χ_1 is a positive metric massive boson. Θ_r^H of (4.3) reduces to

$$\Theta_r^H = \Theta'_r + m\Lambda + (-1)^r (\chi_1 - \chi_2) \quad (4.31)$$

where Θ' is the operator in (A.45) with $\phi \rightarrow \phi'$.

According to S^{GF} of Eq. (4.9), the A_μ equation of motion is $\partial_\mu \square \Lambda = 0$. Consistency with (4.22) and the Euler-Lagrange equations requires

$$\partial_\mu B = 0, \quad (4.32)$$

$$\partial^\mu F_{\mu\nu} - gJ_\nu = \frac{g}{\sqrt{\pi}} \epsilon_{\nu\mu} \partial^\mu (\phi' + \chi_2) \equiv C_\mu = 0. \quad (4.33)$$

Since C_μ and $\partial_\mu B$ cannot vanish as operators without spoiling commutation relations for B , ϕ' and χ_2 , we impose these constraints as Gupta-Bleuler physical state conditions

$$C_\mu^+ |\text{PHYS}\rangle = 0, \quad \partial_\mu B^+ |\text{PHYS}\rangle = 0 \quad (4.34)$$

where “+” denotes the positive frequency (annihilation) parts. This decomposition into creation and annihilation components is well-defined since ϕ' , χ_2 and B are

free fields. Consequently, once imposed at $t = 0$, these conditions are maintained by time evolution.

The vacuum $|\Omega\rangle \equiv |\Omega\rangle_{\phi'} \otimes |\Omega\rangle_{\chi_1} \otimes |\Omega\rangle_{\chi_2} \otimes |\Omega\rangle_{\Lambda_D}$ satisfies (4.34). Physical states are created by acting on $|\Omega\rangle$ with operators which *commute* with both C_μ and $\partial_\mu B$. Since $[\Lambda(y), \partial^0 B(x)]_{E.T.} = i\partial_1 \delta(\vec{y} - \vec{x})$, $[A_1(y), C^0(x)]_{E.T.} = i\partial_1 \delta(\vec{y} - \vec{x})$, and $[\Psi_\nu^r, C^0(x)]_{E.T.} = -g\delta(\vec{y} - \vec{x})\Psi_\nu^r$, $\partial_\mu B$ generates $\bar{\Omega}$ transformations and C_0 generates the Ω_0 ambiguities. Elementary operators which commute with C_μ and $\partial_\mu B$ are $\phi' + \chi_2$, χ_1 , and B . As an auxiliary field, B never comes up in quark and photon matrix elements. Moreover, since the $\phi' + \chi_2$ propagator is the sum of cancelling terms it vanishes. Hence B , ϕ' and χ_2 poles are absent from physical matrix elements, which contain only χ_1 poles. Therefore χ_1 is the only physical particle. In particular,

$$J_{\mu 5} = \frac{1}{\sqrt{\pi}} \partial_\mu (\phi' - \chi_1 + \chi_2), \quad \langle \Omega | J_{\mu 5}(x) J_{\nu 5}(0) | \Omega \rangle = -\frac{1}{\pi} \partial_\mu \partial_\nu \Delta(x; m), \quad (4.35)$$

$$L_\pm \equiv \bar{\Psi}_x (1 \pm \gamma_5) \Psi_x \sim: e^{\pm 2i\sqrt{\pi}(\phi' - \chi_1 + \chi_2)} :, \quad \langle \Omega | L_\pm(x) L_\mp(0) | \Omega \rangle = Z_1^2 e^{4\pi\Delta(x; m)}. \quad (4.36)$$

The gauge variant Chern-Simons current is $K_\mu = +\frac{1}{\sqrt{\pi}}(\partial_\mu \phi' - m\epsilon_{\mu\nu}\partial^\nu \Lambda)$. Accordingly, transformation (4.5) is reproduced by

$$\phi'(x) \longrightarrow \phi'(x) + \theta/\sqrt{\pi} \quad (\chi_1, \chi_2, B \text{ fixed}), \quad (4.37)$$

which preserves boundary condition (4.30). This leads to $|\theta\rangle$ vacua in the ϕ' sector but, since $\partial_\mu \phi'/\sqrt{\pi}$ is a conserved current, θ can be rotated away.[31] This connection between ϕ' and θ is permitted because the value of θ is unphysical when quarks are massless.

4.2.1 Effective Photon Propagator

By Eqs. (4.18), (4.26), and (A.29), the nonperturbative photon propagator is

$$\langle \Omega | A^\mu(x) A^\nu(0) | \Omega \rangle = i \int \frac{d^2 p}{(2\pi)^2} \frac{-\eta^{\mu\nu} + (1 - \xi[1 - (\frac{m}{p})^2])(p^\mu p^\nu / p^2)}{p^2 - m^2 + i\epsilon} e^{ipx}. \quad (4.38)$$

When $-p^2 \gg m^2$, $\langle \Omega | A^\mu(x) A^\nu(0) | \Omega \rangle$ has free particle form. When $-p^2 \ll m^2$, its behavior is dominated by the Λ modes. The component of the photon propagator usually computed on the lattice,[7, 4]

$$\int_{-\infty}^{\infty} d\vec{x} \langle \Omega | A^1(x) A^1(0) | \Omega \rangle = \frac{1}{2m} e^{-m|t_E|}, \quad x \equiv (-it_E, \vec{x}) \quad (4.39)$$

is independent of ξ .

4.2.2 Effective Quark Propagator

Do gauge ambiguities give rise to effective quark masses? Additionally, an OPE analysis[16] in QCD_{3+1} suggests effective quark masses are proportional to $\langle \Omega | \bar{\Psi}_0 \Psi_0 | \Omega \rangle$, which in the Schwinger model is $\langle \Omega | L_+(0) + L_-(0) | \Omega \rangle \neq 0$ due to the anomaly.

Does the anomaly generate effective quark masses?

By Lorentz invariance $\langle \Omega | \Psi_x^\dagger | \Omega \rangle = 0$ so that

$$\langle \Omega | \Psi_x \bar{\Psi}_0 | \Omega \rangle = \langle \Omega | \begin{pmatrix} \Psi_x^1 \Psi_0^{2\dagger} & \Psi_x^1 \Psi_0^{1\dagger} \\ \Psi_x^2 \Psi_0^{2\dagger} & \Psi_x^2 \Psi_0^{1\dagger} \end{pmatrix} | \Omega \rangle \quad (4.40)$$

is intrinsically connected. By (4.2) and (A.41) and since $\langle \Omega | : e^{i\sqrt{\pi}[\Theta_r^H(0) - \Theta_r^H(x)]} : | \Omega \rangle = 1$ the quark propagator components reduce to

$$\langle \Omega | \Psi_x^\dagger \Psi_0^\dagger | \Omega \rangle = (-i)^r (i)^s Z_I e^{\pi[\Theta_r^{H+}(x), \Theta_r^{H-}(0)]}. \quad (4.41)$$

Since the bosons are free, Eq. (4.31) implies

$$\begin{aligned} [\Theta_s^{H+}(x), \Theta_r^{H-}(0)] &= [\Theta_s'^+(x), \Theta_r'^-(0)] + m^2 \langle \Omega | \Lambda(x) \Lambda(0) | \Omega \rangle \\ &\quad + (-1)^{r+s} \Delta(x; m) - (-1)^{r+s} \Delta(x; \mu) \quad (\mu \rightarrow 0). \end{aligned} \quad (4.42)$$

The terms on the RHS of (4.42) come from respectively ϕ' , Λ , χ_1 and χ_2 . Hence $\langle \Omega | \Psi_s^\dagger \Psi_0^\dagger | \Omega \rangle$ factorizes into four pieces:

$$\mathcal{A}_{\phi'}(s, r) \equiv e^{\pi[\Theta_s'^+(x), \Theta_r'^-(0)]} \sim \begin{pmatrix} 0 & (t - \bar{x})/x^2 \\ (t + \bar{x})/x^2 & 0 \end{pmatrix}, \quad (4.43)$$

$$\mathcal{A}_{\chi_1}(s, r) \equiv e^{(-1)^{r+s} \pi \Delta(x; m)} \sim \begin{cases} (-x^2 m^2)^{\frac{(-1)^{r+s}+1}{4}} & +x^2 \rightarrow 0; \\ +1 & -x^2 \rightarrow \infty; \end{cases} \quad (4.44)$$

$$\mathcal{A}_{\chi_2}(s, r) \equiv e^{(-1)^{r+s} \pi \Delta(x; \mu_1)} \sim (-x^2 \mu_1^2)^{\frac{(-1)^{r+s}}{4}}, \quad (4.45)$$

$$\mathcal{A}_\Lambda \equiv e^{m^2[\Lambda^+(x), \Lambda^-(0)]} \sim e^{\frac{\xi_R}{4\pi} \left(\frac{m}{\mu}\right)^2} e^{\frac{\xi_R}{8\pi} (-x^2 m^2)(\gamma - \frac{1}{2})} \left(\frac{1}{2} \mu \sqrt{-x^2}\right)^{\frac{\xi_R}{8\pi} (-x^2 m^2)}. \quad (4.46)$$

We have used Eq. (A.46) for $\mathcal{A}_{\phi'}$ and Eqs. (A.28) and (A.29) for $\mathcal{A}_{\chi_i}$. “ $\sim$ ” means modulo a normalization factor which does not depend on μ , μ_1 or x .

Plugging Eq. (A.29) into Eq. (4.26) yields

$$[\Lambda^+(x), \Lambda^-(0)] = \lim_{\mu \rightarrow 0} \frac{\xi}{4\pi\mu} \sqrt{-x^2} K_1(\mu \sqrt{-x^2}) \quad (4.47)$$

$$\sim \frac{\xi}{4\pi} \left\{ \frac{1}{\mu^2} - \frac{1}{2} \left(\gamma - \frac{1}{2} + \log\left(\frac{\mu}{2} \sqrt{-x^2}\right) \right) x^2 \right\}, \quad (4.48)$$

where K_1 is the modified Bessel function. Eq. (4.48) follows from $K_1(x) = -K_0'(x)$ and $\lim_{\mu \rightarrow 0} \mu^\alpha \log \mu = 0$ for positive α . In order for the x -dependence to be unambiguous when the regulator μ is removed, $\mu^{-c\xi x^2/(8\pi)}$ must be μ -independent as $\mu \rightarrow 0$. This induces a renormalization group flow $\xi = \xi_R(\mu)$. Solving the corre-

sponding RGE yields

$$\xi_R(\mu) = \xi(\mu_o) e^{-\int_{\mu_o}^{\mu} \frac{d\mu}{\mu} \frac{1}{\log \mu}} = \xi(\mu_o) \frac{\log(\mu_o)}{\log(\mu)}. \quad (4.49)$$

The normalization choice

$$Z_I(\mu, \mu_1, \xi_R) \equiv \sqrt{\frac{m}{\mu_1}} e^{-\frac{\xi_R}{4\pi} \left(\frac{m}{\mu}\right)^2} \quad (4.50)$$

produces

$$\langle \Omega | \Psi_x \bar{\Psi}_0 | \Omega \rangle \sim e^{\frac{\xi_R}{8\pi} (-x^2 m^2) (\gamma - \frac{1}{2})} \left(\frac{1}{2} \mu \sqrt{-x^2} \right)^{\frac{\xi_R}{8\pi} (-x^2 m^2)} \frac{\not{x}}{x^2} (-x^2 m^2)^{\frac{1}{4}} e^{\pi \Delta(x; m)}. \quad (4.51)$$

The quark propagator has no scalar component.

Near the lightcone ($x^2 m^2 \rightarrow 0$), the condensate mass is insignificant and effects of χ_1 poles are cancelled by χ_2 ones. The Λ contribution also disappears in this limit. Identity $\lim_{x^2 \rightarrow 0} (x^2)^{\pm \frac{1}{4}} e^{\pm \pi \Delta(x; m)} \sim (m)^{\mp \frac{1}{2}}$ implies that as $x^2 \rightarrow 0$ ($-p^2 \rightarrow \infty$)

$$\lim_{x^2 \rightarrow 0} \langle \Omega | \Psi_x \bar{\Psi}_0 | \Omega \rangle \sim \frac{\not{x}}{x^2}, \quad (4.52)$$

the free, massless quark propagator.

When $-x^2 m^2 \gg 1$ the Λ contribution is dominant. Comparing to the vector component of the massive free quark propagator in Eq. (A.31) yields Eq. (4.7). Therefore, the $U_A(1)$ anomaly and longitudinal ambiguities do not lead to quark masses in covariant gauges.

Physically, free quarks do not exist in the Schwinger model (confinement). Dynamically, [30] dielectric breakdown of the vacuum is responsible for confinement—in two dimensional empty space an electric field $\vec{E}$ is constant and cannot die away.

A quark's electric field produces a, hence, constant stream of quark-antiquark pairs which move at the speed of light away from the original quark with a constant flux. As a result, the amplitude to have just one quark at $t \neq 0$ if there is a $t = 0$ quark vanishes. In Coulomb gauge, which has no unphysical modes such as Λ , ϕ' , or χ_2 , this amplitude is the quark propagator so the effective Coulomb gauge quark mass diverges. In covariant gauges, the presence of Λ *et al* (or, alternatively, Gupta-Bleuler physical-state conditions on the Hilbert space) ruins the identification of the quark propagator with this amplitude.

4.3 Matching Coefficients

To compute correlation functions of continuum operator Θ^{cont} in lattice gauge theory it is necessary to know

$$(\Theta^{\text{cont}})_{\text{rep}}^{\text{latt}} = \sum_i A_i(a) \Theta_i^{\text{latt}}. \quad (4.53)$$

Θ^{latt} denotes the naive lattice transcription of a continuum operator Θ^{cont} while $(\Theta^{\text{cont}})_{\text{latt}}^{\text{latt}}$ refers to the lattice operator which reproduces S -matrix elements of Θ^{cont} . Lattice spacing “ a ” plays the role of inverse momentum cutoff Λ of continuum theories. The coefficients $A_i(a)$ must be such that lattice matrix elements reproduce corresponding continuum S -matrix elements. In practice this expansion defines $(\Theta^{\text{cont}})_{\text{rep}}^{\text{latt}}$ since constructing a closed form expression for $(\Theta^{\text{cont}})_{\text{rep}}^{\text{latt}}$ is impossible (modulo special cases like the correspondence of J_μ to the nonlocal lattice conserved current). Since the Wilson action is nonlocal (It has nearest-neighbor fermion interaction terms.) whereas the continuum action is local, usually $(\Theta^{\text{cont}})_{\text{rep}}^{\text{latt}}$ is nonlocal if Θ^{cont} is local. Eq. (4.53) is an expansion of a nonlocal LHS operator

in terms of local RHS operators—an operator product expansion(OPE).

Consider the Schwinger model analog of the lattice QCD₃₊₁ calculation described in Chapt. 2. Define

$$(\Theta^{\text{cont}})_{|_{\text{rep}}}^{\text{latt}} \equiv \lim_{x \rightarrow 0} \left(L_+(x) L_-(0) - \alpha \frac{x^\mu}{x^2} \partial_\mu \chi_1(0) \right) \quad (4.54)$$

where α is such that $(\Theta^{\text{cont}})_{|_{\text{rep}}}^{\text{latt}}$ does not mix with $x^\mu \partial_\mu \chi_1$ in the $x \rightarrow 0$ limit. This means we want to “subtract” $x^\mu \partial_\mu \chi_1$ from $L_+(x) L_-(0)$. $x^\mu \partial_\mu \chi_1$ is the analog of $\bar{s}d$ and $(\Theta^{\text{cont}})_{|_{\text{rep}}}^{\text{latt}}$ the analog of the 4-fermi electroweak operators in the $\bar{s}d$ subtraction. The analog of the quark matrix element matching condition is

$$\lim_{x \rightarrow 0} \langle \Omega | \Psi_y^\dagger L_+(x) L_-(0) \Psi_w^\dagger | \Omega \rangle_{\text{con}} = \alpha \frac{x^\nu}{x^2} \langle \Omega | \Psi_y^\dagger \partial_\nu \chi_1(0) \Psi_w^\dagger | \Omega \rangle_{\text{con}}. \quad (4.55)$$

“Con” means the direct contraction of Ψ_y with $\Psi_w^\dagger$ is not permitted. (Note that, while “con” is motivated by the correspondence of Ψ_y and $\Psi_w^\dagger$ to different flavors in the lattice $\bar{s}d$ subtraction, Ψ_y and $\Psi_w^\dagger$ are not different flavors. Rather, we have simply designed our matching condition to refer to only the connected part of matrix elements. We are free to do this since we can put forth any matching condition we are interested in.) If $y \equiv -w$, by (4.31), (4.60), (4.55) and (A.43),

$$\begin{aligned} & \lim_{x \rightarrow 0} \langle \Omega | \Psi_y^\dagger L_+(x) L_-(0) \Psi_w^\dagger | \Omega \rangle_{\text{con}} \\ &= Z_1 \frac{2i\sqrt{\pi}e^{-\gamma}}{(x^2 m^2)} \left(\langle \Omega | \Psi_y^\dagger x^\nu \partial_\nu \chi_1(0) \Psi_w^\dagger | \Omega \rangle_{\text{con}} - \epsilon_{\mu\nu} x^\mu \partial_\nu \Delta(w; \mu) \right). \end{aligned} \quad (4.56)$$

The inhomogeneous term in (4.56) is due to contractions of $\phi' - \chi_2$ from Ψ_x with $x^\mu \partial_\mu (\phi' + \chi_2)$ from $L_+(x) L_-(0)$. Since different gauge variant operators have different amounts of $\phi' - \chi_2$ modes, the inhomogeneous term is specific to Ψ . In this case, it prevents us from satisfying (4.55) and we cannot obtain a value for α

in this example.

An example where we can satisfy the matching condition is given by the gauge variant operator

$$O(c) \equiv y^\mu \partial_\mu (\chi_1(y) + c[\phi'(y) - \chi_2(y)]). \quad (4.57)$$

Matching condition

$$\lim_{x \rightarrow 0} \langle \Omega | O(c) L_+(x) L_-(0) | \Omega \rangle = \alpha \frac{x^\nu}{x^2} \langle \Omega | O(c) \partial_\nu \chi_1(0) | \Omega \rangle \quad (4.58)$$

yields

$$\alpha(c) = \left(1 - 2c \frac{\Delta(y; \mu)}{\Delta(y; m)}\right) \alpha(0), \quad (4.59)$$

which depends on c . For example, as $y \rightarrow 0$, $\alpha(c) = (1 - 2c)\alpha(0)$. c is the contribution from the gauge variant part of $O(c)$. Nonetheless, $\alpha(c)$ is independent of gauge parameter ξ_R even if $c \neq 0$.

For comparison's sake, let us consider two manifestly gauge invariant ways one might define α : by an OPE of $L_+(x)L_-(0)$ with no reference to correlation functions; and by matching gauge invariant correlation functions. Eq. (4.36), BCHM formula (A.41), and Taylor expanding the free bosonic fields yields

$$L_+(x)L_-(0) = \left(\frac{Z_1 e^{-\gamma}}{-x^2 m^2}\right) : 1 + 2i\sqrt{\pi} x^\mu \partial_\mu (\phi' + \chi_2 - \chi_1)(0) : + \mathcal{O}\left(\frac{1}{x}\right). \quad (4.60)$$

Therefore the OPE result is

$$\alpha = Z_1 e^{-\gamma} \left(\frac{2i\sqrt{\pi}}{m^2}\right). \quad (4.61)$$

Next, the gauge invariant matching condition

$$\lim_{x \rightarrow 0} \langle \Omega | \partial_\mu \chi_1(y) L_+(x) L_-(0) | \Omega \rangle = \alpha \frac{x^\nu}{x^2} \langle \Omega | \partial_\mu \chi_1(y) \partial_\nu \chi_1(0) | \Omega \rangle \quad (4.62)$$

also retrieves, upon applying BCHM formula (A.43), Eq. (4.61).

The discrepancy between α in (4.61) and $\alpha(c)$ when $c \neq 0$ means matching condition (4.58) is inequivalent to the other definitions of α . However, if $\alpha(c)$ is indeed gauge invariant—and showing gauge invariance would require computing it in all gauges—(4.58) is as physical a matching condition as the manifestly gauge invariant ones. In conclusion, these examples suggest that not all matching conditions can be satisfied. In the satisfiable ones, while different gauge variant matching conditions may lead to different (or similar) values for matching coefficients, and that while the differences may even be due to unphysical gaugefixing modes, the different matching coefficients may nonetheless all be gauge invariant. They correspond to physically inequivalent definitions.

Chapter 5

Central Charge of the Parallelogram Lattice $\beta = 0$ Schwinger Model

5.1 Introduction and Results

In recent years there has been remarkable progress in the classification of two dimensional spin systems and three dimensional topological Yang Mills theories in relation to conformal field theories. [37] Yet, despite considerable activity in lattice gauge theories with fermions [20] and two dimensional lattice toy models, [38] the Wilson lattice transcription of the Schwinger model [39] remains unsolved. The only exactly solved model with Wilson fermions and local gauge invariance is the square lattice strong coupling Schwinger model, whose partition function $Z_{\square}$ at infinite hopping constant equals $Z_{6V}[\frac{1}{2}, \frac{1}{2}, 1]$, a 6-vertex model partition function. [40] By known 6-vertex model features, this mapping reveals that $Z_{\square}$ is critical and its continuum limit is a conformal field theory with central charge $c = 1$.

Since the 6-vertex model has a nontrivial phase structure, it is natural to seek the $Z_\square$ extensions which map to other regions of the $Z_{6V}[A, B, C]$ parameter space. To this end, we put forth a Fierz strong coupling hopping expansion for actions of the form

$$S_F = \sum_{x \in \Lambda} \left(-M \bar{\psi}_x \psi_x + \sum_{\mu=0}^1 \bar{\psi}_{x+\hat{\mu}} T^{(+,\mu)} U_{x,\mu}^\dagger \psi_x + \bar{\psi}_x T^{(-,\mu)} U_{x,\mu} \psi_{x+\hat{\mu}} \right), \quad (5.1)$$

$$\Lambda \equiv \left\{ \sum_{\mu=0}^1 x^\mu \hat{e}_\mu \mid x^\mu \in Z \right\}, \quad M \equiv m + \sum_{\mu=0}^1 (T^{(+,\mu)} + T^{(-,\mu)}). \quad (5.2)$$

S_F is the Wilson fermion action on a square lattice action if $\{\hat{e}_1, \hat{e}_2\}$ are orthonormal and $T^{(\pm,\mu)} = \frac{1}{2}(r \pm \gamma^\mu)$. More generally, on a parallelogram lattice defined by

$$\hat{e}_\mu \equiv \begin{pmatrix} \hat{e}_\mu^{(0)} \\ \hat{e}_\mu^{(1)} \end{pmatrix}, \quad \hat{e}_0 \equiv \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{e}_1 \equiv \lambda' \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad (5.3)$$

$$g \equiv \sum_{a,b=0}^1 \hat{e}^{(a)} \hat{e}^{(b)} \delta_{ab} = \begin{pmatrix} \lambda^2 & \lambda \lambda' \cos \theta \\ \lambda \lambda' \cos \theta & \lambda'^2 \end{pmatrix}, \quad (5.4)$$

the strong coupling partition function is

$$Z_{SC} \equiv \int_F \int \prod_{x,\mu \in \Lambda} dU_{x,\mu} e^{-S_F}, \quad (5.5)$$

$$\int_{F_x} \equiv \int d\psi_x^1 d\bar{\psi}_x^1 d\psi_x^2 d\bar{\psi}_x^2, \quad \int_F \equiv \prod_{x \in \Lambda} \int_{F_x}. \quad (5.6)$$

Assuming nonbacktracking condition

$$T^{(-,\mu)} T^{(+,\mu)} = 0 = T^{(+,\mu)} T^{(-,\mu)}, \quad (5.7)$$

which prevents hopping expansion quarks from backtracking, we will show that

$$Z_{SC} = \int_F e^{-S_{SCF}}, \quad S_{SCF} = \sum_{x \in \Lambda} \left(-M \bar{\psi}_x \psi_x + \sum_{\mu=0}^1 \Theta_x^{(+, \mu 5)} \Theta_{x+\hat{\mu}}^{(-, \mu 5)} \right). \quad (5.8)$$

Commuting operators $\Theta_x^{(\epsilon, \mu 5)}$ are characterized by

$$\int_{F_x} (\bar{\psi}_x \psi_x)^q (\Theta_x^{(\epsilon, \mu 5)})^p = 2\delta_{q,2}\delta_{p,0} \quad (\forall q, p \in \{0, 1, 2, \dots\}), \quad (5.9)$$

$$-\int_{F_x} \Theta_x^{(\epsilon', \nu 5)} \Theta_x^{(\epsilon, \mu 5)} = r^{(\mu)} r^{(\nu)} (\delta_{\mu\nu} \delta_{-\epsilon'\epsilon} + \delta_{\mu 0} \delta_{\nu 1} S_{-\epsilon', \epsilon}^2 + \delta_{\mu 1} \delta_{\nu 0} S_{\epsilon, -\epsilon'}^2), \quad (5.10)$$

where $r^{(\mu)}$ and $S_{\epsilon, \epsilon'}$ are functions (given below) of λ , λ' and θ . (Under the Euclidean analog of hermitian conjugation, $\psi_E^\dagger \equiv \bar{\psi} \gamma_5$, $\bar{\psi}_E^\dagger = \gamma_5 \psi$, $(S_F)^\dagger_E = S_F$, and $(\Theta_x^{(\epsilon, \mu 5)})^\dagger_E = \Theta_x^{(\epsilon, \mu 5)}$.) Since $\Theta_x^{(+, \mu 5)} \Theta_x^{(-, \nu 5)}$ completely saturates $\int_{F_x}$, $\Theta_x^{(\epsilon, \mu 5)}$ hops along selfavoiding paths. Monomers $(M \bar{\psi}_x \psi_x)^2$ fill in unhopped sites. We call this the Fierz hopping expansion because the " $\Theta_x^{(+, \mu 5)} \Theta_{x+\hat{\mu}}^{(-, \mu 5)}$ " form in (5.8) is achieved using Fierz identities. As described in Section 5.3, the Fierz hopping expansion also applies to backtracking models.

Following (5.8), $\Theta_x^{(\epsilon, \mu 5)}$ hopping amplitudes are identified in Section 5.2 with Boltzmann weights of two-state vertex models. Nonperturbative solution of these vertex models (by Bethe ansatz or whatever) is tantamount to resummation of the hopping expansion. In this way we identify Z_{SC} with Z_{8V} , the 8-vertex model partition function with Boltzmann weights

$$\omega_1 = M^2, \quad \omega_2 = 0, \quad \omega_3 = (\lambda' \sin \theta)^{-2}, \quad \omega_4 = (\lambda \sin \theta)^{-2}, \quad (5.11)$$

$$\omega_5 = \omega_6 = \frac{1}{4\lambda\lambda'} \left(\frac{1}{\cos \frac{\theta}{2}} \right)^2, \quad \omega_7 = \omega_8 = \frac{1}{4\lambda\lambda'} \left(\frac{1}{\sin \frac{\theta}{2}} \right)^2. \quad (5.12)$$

The $\{\omega_i\}$ are defined in, for example, p. 347 of Lieb and Wu [41]: ω_1 corresponds

to the empty vertex; ω_3 and ω_4 to vertical and horizontal lines; ω_5 and ω_6 to lower-right and upper-left corners; and ω_7 and ω_8 to lower-left and upper-right corners. Setting $\theta = \pi/2$ and $r^{(\mu)} = 1$ recovers the square lattice result of Ref. [40].

The 8-vertex model is not solved in general for the Boltzmann weights given in (5.11)-(5.12). A solvable case is $M = 0$ and $\lambda' = \lambda$. In this subspace, define

$$D \equiv \omega_1 = \omega_2 = 0, \quad C \equiv \omega_3 = \omega_4, \quad A \equiv \omega_5 = \omega_6, \quad B \equiv \omega_7 = \omega_8. \quad (5.13)$$

Then 8-vertex model symmetries [41] imply

$$Z_{SC} = Z_{8V}[D, C, A, B] = Z_{8V}[A, B, C, D] = Z_{6V}[A, B, C]. \quad (5.14)$$

In the thermodynamic limit, these 6-vertex models fall on the critical line characterized by Lieb parameter [41]

$$\Delta \equiv \frac{A^2 + B^2 - C^2}{2AB} = -1. \quad (5.15)$$

Varying skewness angle θ from 0 to π spans the critical line between the antiferroelectric and disorder phases of the 6-vertex model.

While the points on this critical line all have central charge $c = 1$, [42] to our knowledge it has not been demonstrated whether they are the same $c = 1$ conformal field theory or different ones. Renormalization group arguments suggest that the subset of points $A = B = \frac{C}{2}$, the so-called F-model, corresponding to $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ can be identified with free massless boson field theory. [43]

5.2 Fierz Strong Coupling Hopping Expansion

In the finite coupling ($\beta > 0$) hopping expansion [20] ψ hops from site-to-site with weight $T^{(\epsilon, m)} U_{x, m}$. At strong coupling such quark motions are suppressed. Define

$$D_{il, jk}^{(\mu)} \equiv T_{ij}^{(-, \mu)} T_{kl}^{(+, \mu)}, \quad \Xi(x, y; D) \equiv \bar{\psi}_x^i \psi_x^l D_{il, jk} \psi_y^j \bar{\psi}_y^k. \quad (5.16)$$

Since $(\Xi)^3 = 0$ in two dimensions, integrating out $U(1)$ links $\{U_{x, \mu}\}$ using

$$\int dU \exp(aU^\dagger + bU) = \sum_{k=0}^{\infty} (ab)^k / (k!)^2 \quad (5.17)$$

yields

$$Z_{SC} = \int_F e^{M \sum_x \bar{\psi}_x \psi_x} \prod_{x, \mu} [1 + \Xi(x, x + \hat{\mu}; D^{(\mu)}) + \frac{1}{4} \Xi^2(x, x + \hat{\mu}; D^{(\mu)})] \quad (5.18)$$

$$= \int_F e^{\sum_x [M \bar{\psi}_x \psi_x + \sum_{\mu} (\Xi(x, x + \hat{\mu}; D^{(\mu)}) - \frac{1}{4} \Xi^2(x, x + \hat{\mu}; D^{(\mu)})]} \quad (5.19)$$

Let $n, m \in \{\pm 0, \pm 1\}$, $T^{(\epsilon, -m)} \equiv T^{(-\epsilon, m)}$, and $k_{y, -n} \equiv k_{y - \hat{n}, n}$. Expanding

$$\exp(M \bar{\psi}_x \psi_x) = \sum_{s_x=0}^2 (M \bar{\psi}_x \psi_x)^{s_x} / s_x! \quad (5.20)$$

reveals that $\psi \otimes \bar{\psi}$ hops with amplitude $D^{(m)}$ such that $s_x + \sum_{m=-1}^1 k_{x, m} = 2$ for each $x \in \Lambda$. Thus only three events are possible at each site: (5.21) $s_x = 2$, $k_{x, m} = 0$; (5.22) $s_x = 1$, $k_{x, m} = 1$; or (5.23) $s_x = 0$, $k_{x - \hat{n}, n} = k_{x, m} = 1$. The associated amplitudes are

$$\int_F \frac{1}{2} (M \bar{\psi}_x \psi_x)^2 = M^2, \quad (5.21)$$

$$\int_{F_x} M \bar{\psi}_x \psi_x \Xi(x, x + \hat{n}; D^{(m)}) = M \text{tr}(T^{(+,m)} T^{(-,m)} \psi_{x+\hat{n}} \bar{\psi}_{x+\hat{n}}), \quad (5.22)$$

$$\int_{F_x} \Xi(x - \hat{n}, x; D^{(n)}) \Xi(x, x + \hat{n}; D^{(m)}) = \Xi(x - \hat{n}, x + \hat{n}; \bar{D}^{(n,m)}), \quad (5.23)$$

$$\bar{D}_{il,jk}^{(n,m)} \equiv (T^{(-,n)} T^{(-,m)})_{ij} (T^{(+,m)} T^{(+,n)})_{kl} - (T^{(-,n)} T^{(+,n)})_{il} (T^{(+,m)} T^{(-,m)})_{kj}. \quad (5.24)$$

A backtracking $\psi \otimes \bar{\psi}$ pair, $k_{x,m} = 2$ or $n = -m$ in (5.23), saturates $\int_F$ at the two sites it occupies and, hence, makes a dimer. Since 2-state vertex models cannot model dimer-loop mixtures, Z_{SC} is not a 2-state vertex model unless backtracking is forbidden. In this Section we adopt (5.7), which sets $\bar{D}^{(n,-n)} = 0$. Then $\psi \otimes \bar{\psi}$ worldlines comprise a selfavoiding loop gas with Boltzmann weight M^2 for unoccupied sites and weights $\bar{D}^{(n,m)} = T^{(-,n)} T^{(-,m)} \otimes T^{(+,m)} T^{(+,n)}$ along loops.

We now recast the problem so that the Boltzmann weights are more succinctly related to hopping amplitudes. Define $\gamma_5 \equiv i\gamma_0\gamma_1$,

$$\gamma_0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^\mu = \sum_{\nu,a} g^{\mu\nu} \hat{e}_\nu^{(a)} \gamma_a, \quad (5.25)$$

$$T^{(\pm,\mu)} \equiv \frac{1}{2}(r^{(\mu)} \pm \Gamma^\mu), \quad r^{(0)} = (\lambda \sin \theta)^{-1}, \quad r^{(1)} = (\lambda' \sin \theta)^{-1}. \quad (5.26)$$

Wilson regulators $\{r^{(\mu)}\}$ are chosen so that, in addition to (5.7),

$$T_{ij}^{(\epsilon,\mu)} T_{kl}^{(-\epsilon,\mu)} = (T^{(\epsilon,\mu)} \gamma_5)_{il} (T^{(-\epsilon,\mu)} \gamma_5)_{kj} \quad (\mu \text{ fixed}), \quad (5.27)$$

$$T^{(\epsilon,\mu)} T^{(\epsilon',\mu)} = \delta^{\epsilon\epsilon'} r^{(\mu)} T^{(\epsilon,\mu)}, \quad r^{(1)} T^{(\epsilon,0)} = r^{(0)} S^\dagger T^{(\epsilon,1)} S, \quad (5.28)$$

$$S = \begin{pmatrix} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \\ -\cos \frac{\theta}{2} & \sin \frac{\theta}{2} \end{pmatrix} \equiv \begin{pmatrix} S_{+,+} & S_{+,-} \\ S_{-,+} & S_{-,-} \end{pmatrix}. \quad (5.29)$$

Fierz identity (5.27) implies

$$\Xi(x, x + \hat{m}, D^{(m)}) = -\Theta_x^{(-, m^5)} \Theta_{x+\hat{m}}^{(+, m^5)}, \quad \Theta_x^{(\epsilon, \mu^5)} \equiv \bar{\psi}_x T^{(\epsilon, \mu)} \gamma_5 \psi_x \quad (5.30)$$

and transforms (5.19) to (5.8). Following (5.8), $\Theta_x^{(\epsilon, \mu^5)}$ hops from direction $n \equiv \text{sign}(n)\nu$ to direction $m \equiv \text{sign}(m)\mu$ with amplitude (5.10) where $\epsilon' = \text{sign}(n)$ and $\epsilon = -\text{sign}(m)$. Boltzmann weights $\{\omega_i\}$ can be read off from (5.10). Straight $n = m$ vertical or horizontal hops, corresponding to ω_3 and ω_4 , have weight $(r^{(n)})^2$. If x is approached from $n = 0$ and exited in the $m = 1$ direction, $\omega_6 = r^{(0)}r^{(1)}S_{-,-}^2$. Similarly, $n = -1$ and $m = 0$ yields $\omega_7 = r^{(0)}r^{(1)}S_{+,-}^2$. Eqs. (5.11)-(5.12) list the remaining vertices.

5.3 Backtracking

To study backtracking effects when (5.7) is removed, we work on a square lattice with $T^{(\pm, \mu)} \equiv \frac{1}{2}(r \pm \gamma^\mu)$. Fixed- μ Fierz transformations are

$$T_{ij}^{(\epsilon, \mu)} T_{kl}^{(-\epsilon, \mu)} = (T^{(\epsilon, \mu)} \gamma_5)_{il} (T^{(-\epsilon, \mu)} \gamma_5)_{kj} + \left(\frac{r^2 - 1}{8} \right) \sum_{\sigma} (-1)^{\sigma} \gamma_{il}^{\sigma} \gamma_{kj}^{\sigma}, \quad (5.31)$$

where $\gamma^{\sigma} \in \{\gamma^s \equiv 1, \gamma_5, \gamma_0, \gamma_1\}$, $(-1)^{\sigma} = -1$ for γ_5 , and $(-1)^{\sigma} = +1$ otherwise.

Hermitian operators

$$\Theta_x^{\sigma} \equiv \begin{cases} \bar{\psi}_x \gamma^{\sigma} \psi_x & \text{if } \gamma^{\sigma} \in \{\gamma^s, \gamma_5, T^{(\epsilon, \mu)} \gamma_5\}; \\ i \bar{\psi}_x \gamma^{\sigma} \psi_x & \text{if } \gamma^{\sigma} \in \{\gamma_0, \gamma_1\}; \end{cases} \quad (5.32)$$

are characterized by

$$\int_{F_x} (\Theta_x^s)^2 = 2, \quad (\Theta_x^s)^3 = 0, \quad (5.33)$$

$$\Theta_x^{\sigma'} \Theta_x^{\sigma} = (-1)^{\sigma} \delta_{\sigma'\sigma} (\Theta_x^s)^2 \quad (\sigma \in \{s, 5, 0, 1\}), \quad (5.34)$$

$$\Theta_x^\sigma \Theta_x^{(\epsilon, \mu 5)} = \left[\frac{1}{4} (\epsilon \epsilon' \delta_{\mu\nu} - r^2) \delta_{\sigma(\epsilon', \nu 5)} - \frac{r}{2} \delta_{\sigma 5} + \frac{\epsilon}{2} \epsilon^{\mu\sigma} \right] (\Theta_x^\sigma)^2. \quad (5.35)$$

Following (5.19)

$$S_{SCF} = S'_{SCF} + \sum_{x, \mu \in \Lambda} \left[\Theta_x^{(-, \mu 5)} \Theta_{x+\hat{\mu}}^{(+, \mu 5)} + \left(\frac{1-r^2}{8} \right) \sum_{\sigma=5,0,1} \Theta_x^\sigma \Theta_{x+\hat{\mu}}^\sigma \right], \quad (5.36)$$

$$S'_{SCF} \equiv \sum_{x \in \Lambda} \left[-M \Theta_x^\sigma + \frac{1}{32} \sum_{\mu=0}^1 \left((r^2 - 1) \Theta_x^\sigma \Theta_{x+\hat{\mu}}^\sigma + 2 \right)^2 \right]. \quad (5.37)$$

This system is currently under investigation. We have not been able to identify it with solved models. Note that S'_{SCF} contains dimers.

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Appendix A

Appendix

A.1 Free Fermion Matrix Elements

The Euclidean Dirac matrices we use are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (\text{A.1})$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad \vec{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \quad (\text{A.2})$$

Two useful Fierz relations

$$\Gamma_1^{\alpha\beta} \Gamma_2^{\gamma\delta} = \sum_{\Gamma_3 \Gamma_4} f_{\Gamma_1 \Gamma_2}^{\Gamma_3 \Gamma_4} \Gamma_3^{\alpha\delta} \Gamma_4^{\gamma\beta} \quad (\text{A.3})$$

are

$$[\gamma^\mu(1 \pm \gamma^5)]_{ij} [\gamma^\mu(1 \pm \gamma^5)]_{kl} = - [\gamma^\mu(1 \pm \gamma^5)]_{il} [\gamma^\mu(1 \pm \gamma^5)]_{kj} \quad (\text{A.4})$$

and

$$f_{SS}^{SS} = f_{PP}^{SS} = \frac{1}{4}, \quad f_{VV}^{SS} = -f_{AA}^{SS} = 1, \quad f_{TT}^{SS} = 3. \quad (\text{A.5})$$

In Euclidean momentum space, the Dirac equation and free particle wavefunctions are

$$(i\not{p} - m_q)U_p^{(r)} = 0 = \overline{U}_p^{(r)}(i\not{p} - m_q), \quad (\text{A.6})$$

$$p = \begin{cases} (-iE_q, \vec{p}) & t \geq 0; \\ (+iE_q, \vec{p}) & t < 0; \end{cases} \quad p^2 + m_q^2 = 0, \quad (\text{A.7})$$

$$U_p^{(r)} = \mathcal{N} \begin{pmatrix} \chi^{(r)} \\ \frac{i\vec{\sigma} \cdot \vec{p}}{p^0 - im_q} \chi^{(r)} \end{pmatrix}, \quad \chi_j^{(i)} = \delta_{ij}, \quad |\mathcal{N}|^2 = ip^0 + m_q. \quad (\text{A.8})$$

Gordon decomposition with $r \equiv p + p'$ and $q \equiv p - p'$ is

$$\overline{U}_p^{(r)} \gamma^\mu U_{p'}^{(s)} = \frac{i}{2m_q} \overline{U}_p^{(r)} [\tau^\mu + i\frac{g}{2} q_\nu \sigma^{\mu\nu}] U_{p'}^{(s)} \quad (\text{A.9})$$

where $g = 2$ is the free particle magnetic moment. In terms of continuum free particle spinors U ,

$$\langle 0 | \psi_x | q_p^{(r)} \rangle = \sqrt{Z_q} e^{i\phi_q} U_p^{(r)} e^{-ipx}, \quad Z_q^{\text{free}} = 1 \quad (\text{A.10})$$

where ϕ_q is an undetermined phase and Z_q is a real, positive number. The normalization of quark states, both free and nonperturbative, is fixed by

$$\langle q_p^{(r)} | q_{p'}^{(s)} \rangle = (2\pi)^3 2E_q \delta_{r,s} \delta(\vec{p} - \vec{p}'), \quad p^2 = (p')^2. \quad (\text{A.11})$$

If $\mathcal{O}(0)$ is any color singlet operator, inserting Eq. (2.7) into Eq. (2.8) yields in the $t_x \equiv -t_y \equiv t \rightarrow \infty$ limit

$$\sum_{\vec{x}, \vec{y}} e^{i\vec{p} \cdot \vec{x} - i\vec{p}' \cdot \vec{y}} \langle \psi_x \mathcal{O} \overline{\psi}_y \rangle = \frac{\sqrt{Z_q Z_{q'}}}{4E_q E_{q'}} e^{-(E_q + E_{q'})t_B} \sum_{r,s} U_p^{(r)} \langle q_p^{(r)} | \mathcal{O} | q_{p'}^{(s)} \rangle \overline{U}_{p'}^{(s)}. \quad (\text{A.12})$$

A.2 Numerical Lattice Miscellany

" Σ " gauge configurations are thermalized Wilson link configurations which have not been gaugefixed. To gaugefix, produce an ensemble of importance sampling gauge configurations $\{\mathcal{U}_1, \mathcal{U}_2, \dots\}$ for the Gibbs average $[\dots]_u$. For each $\mathcal{U}_i$, generate an ensemble of importance sampling gauge transformations $\{\mathcal{V}_1^{(i)}, \mathcal{V}_2^{(i)}, \dots\}$ for $[\dots]_v$. The RHS of Eq. (2.16) may be viewed either as

$$[[\Theta(U, V)]_v]_u = [([\Theta(U, V)]_v)]_u \quad (\text{A.13})$$

or as

$$[[\Theta(U, V)]_v]_u = ([[\Theta(U, V)]_v])_u. \quad (\text{A.14})$$

Following (A.13), $\Theta(U, V)$ in each background field $\mathcal{U}_i$ is transformed into $\{\mathcal{V}_1^{(i)}, \mathcal{V}_2^{(i)}, \dots\}$ gauges. This yields an ensemble of $[\Theta(U, V)]_v$ in background gauge fields whose importance sum yields $\langle \Theta \rangle$. (A.14) says to transform each link configuration $\mathcal{U}_i$ into an ensemble of fixed gauge link configurations using $\{\mathcal{V}_1^{(i)}, \mathcal{V}_2^{(i)}, \dots\}$. This makes a master ensemble

$$\mathbf{U} \equiv \{\mathcal{U}_1^{\mathcal{V}_1^{(1)}}, \mathcal{U}_1^{\mathcal{V}_2^{(1)}}, \dots, \mathcal{U}_2^{\mathcal{V}_1^{(2)}}, \mathcal{U}_2^{\mathcal{V}_2^{(2)}}, \dots\} \quad (\text{A.15})$$

$\Theta(U, V)$ is computed on each $\mathcal{U}_i^{\mathcal{V}_j^{(i)}}$ and summed over $\mathbf{U}$ to get $\langle \Theta \rangle$.

For λ gauges the ensemble $\{\mathcal{V}_1^{(i)}, \dots, \mathcal{V}_n^{(i)}\}$ for each configuration collapses to the subset of gauge transformations globally minimizing $S^v[\Omega]$ with respect to $\{V_x\}$. If there are Gribov ambiguities, then $n > 1$. For our numerical results presented below, $n = 1$ is assumed. Upon computing $\mathcal{V}^{(i)}$, one either transforms each background field quark propagator into λ gauge according to

$$(\psi_x \bar{\psi}_y)^{(\lambda)} = V_x (\psi_x \bar{\psi}_y)^{(\Sigma)} V_y^\dagger \quad (\text{A.16})$$

or link configuration according to

$$U_{x,\mu}^{(\lambda)} = V_x U_{x,\mu}^{(\Sigma)} V_{x+\hat{\mu}}^\dagger. \quad (\text{A.17})$$

In the latter alternative, the quark propagator is computed on each λ gauge link configuration. In both alternatives, average over the importance sampled propagators for the λ gauge quark propagator.

1 vs 2	P_R	M_D	M_R
$\epsilon = .002$ vs $\epsilon = .0002$	.965(.073)	-.0025(.020)	.994(.045)
sc vs vec	1.01(.040)	.014(.015)	1.03(.030)

Table A.1: ϵ independence of quark propagator

Numerically, $\mathcal{V}^{(i)}$ is determined by maximizing[45] S^v site by site. Repeated sweeps through the lattice until $\{V_y\}$ stabilizes yields the $\mathcal{V}^{(i)}$ which minimizes S^v for that background link configuration. The absolute minimum is not unique and, moreover, there is evidence that sometimes it the sweeps converge to a local minimum.[44] Our measure of convergence is

$$\epsilon(\lambda, \mathcal{U}) \equiv \frac{1}{N_{\text{sites}}} \sum_{\mu, x} \text{Tr}(\lambda_\mu \partial_\mu A_\mu)^2, \quad (\text{A.18})$$

which in λ gauge is down by a factor of 10^4 from Σ gauge.[3] Tables A.1 and 2.6 show that the Landau gauge quark propagator is stable to an additional order of magnitude decrease of $\epsilon(1, \text{Landau})$.

Note that Coulomb gauge permits residual gauge fluctuations

$$A_0(x) \longrightarrow A_0^\Lambda(x) = A_0(x) + \partial_0 \Lambda(t). \quad (\text{A.19})$$

Since $\Psi_y \bar{\Psi}_x \longrightarrow e^{+i[\Lambda(t_y) - \Lambda(t_x)]} \Psi_y \bar{\Psi}_x$ under (A.19), these fluctuations potentially de-

stroy effective electron propagators. In perturbation theory the implicit boundary condition

$$\lim_{-x^2 \rightarrow \infty} A_\mu^\Lambda(x) \longrightarrow 0 \quad (\text{A.20})$$

fixes $\Lambda(t_y) = \Lambda(t_x)$. In a finite volume lattice periodicity replaces (A.20). In $\lambda \neq 0$ gauges, let “ v ” denote a unit $D - 1$ dimensional cube inside timeslice V and $\int_v \equiv \int_v d^{D-1} \vec{x}$. Then gauge condition (2.6) is equivalent to the set of constraints

$$\int_{\partial v} d\vec{S} \cdot \vec{A} = \lambda \frac{\partial}{\partial t} \int_v A_0^\Lambda, \quad (\forall v \in V). \quad (\text{A.21})$$

The number of such constraints is proportional to the timeslice volume V .

When $\lambda = 0$, (A.21) is no longer constrains A_0^λ . Motivated by the fact that periodic boundary conditions imply $\int_V \vec{\nabla} \vec{A} = \int_{\partial V} d\vec{S} \cdot \vec{A} = 0$, in Reference [3] we adopted supplementary Coulomb gaugefixing condition

$$\frac{\partial}{\partial t} \int_V A_0^\Lambda(x) = 0. \quad (\text{A.22})$$

In fact (A.22) is insufficient. Since (A.20) or (A.21) consists of a timeslice of constraints, effective lattice quark propagators in $\lambda = 0$ gauge supplemented by (A.22) vanish inversely proportional to timeslice volume V .

We don’t have a way of simulating Coulomb gauge. Instead, we define Coulomb gauge as the $\lambda \rightarrow 0$ limit of $\lambda > 0$ gauges and hope that the limit equals the limitpoint. This identification is supported at $\beta = 0$, where $\lim_{\lambda \rightarrow 0} M_q^{(\lambda)} = \infty$ in agreement with numerical simulations.

A.3 Chiral Improvement Algorithm

The algorithm used to implement Section 2.2.8 is as follows: (1) Partition the set $\{SS, PP, TT, LR\}$ into $\binom{4}{2} = 6$ pairs $p_j \equiv \{\Gamma_1^{i_j} \Gamma_2^{i_j}, \Gamma_1^{2j} \Gamma_2^{2j}\}$. (2) Select a fresh pair p_j and compute

$$\langle\langle O_{\pm}^{uu'}[LL] \rangle\rangle = z_{\pm}^L \langle\langle \sum_{i,j=1}^2 r_{\pm}^{\Gamma_1^{i_j} \Gamma_2^{i_j}} O_{\pm}^{uu'}[\Gamma_1^{i_j} \Gamma_2^{i_j}] + O_{\pm}^{uu'}[LL] \rangle\rangle_{\text{lat}} \quad (\text{A.23})$$

Requiring $\langle\langle O_{\pm}^{uu'}[LL] \rangle\rangle^{\text{chiral}} = \gamma_{\pm}^j \cdot p$ and assuming a quadratic form for

$$\langle\langle O_{\pm}^{uu'}[\Gamma_1^{i_j} \Gamma_2^{i_j}] \rangle\rangle_{\text{lat}} \quad (\text{A.24})$$

and $\langle\pi^+(K')^- | O_{\pm}^{uu'}[\Gamma_1^{i_j} \Gamma_2^{i_j}] | 0\rangle$, $i \in \{1, 2\}$, determines $\gamma_{\pm}^j$ and its jackknife variances $\delta_{\gamma_{\pm}^j}$. (3) Repeat starting from step (2) until all pairs p_j are done. (4) Compute $\rho_{\pm}^j \equiv \delta_{\gamma_{\pm}^j}^2 / (\sum_k \delta_{\gamma_{\pm}^k}^2)$, weighted averages $\bar{\gamma}_{\pm} = \frac{1}{6} \sum_{j=1}^6 \rho_{\pm}^j \gamma_{\pm}^j$ and their systematic errors $\Delta_{\pm} \equiv \sqrt{\sum_{j=1}^6 \rho_{\pm}^j (\gamma_{\pm}^j - \bar{\gamma})^2 / 5}$. (5) Repeat starting from step (1) to determine jackknife errors for $\bar{\gamma}_{\pm}$, $\Delta_{\pm}$, and $\bar{\gamma}_{-}/\bar{\gamma}_{+}$.

A.4 Free $D = 2$ Propagators

The metric tensor is $\eta_{00} = -\eta_{11} = 1$ and $\eta_{01} = \eta_{10} = 0$. Useful identities are

$$\epsilon_{\mu\lambda} \epsilon^{\lambda\nu} = \delta_{\mu}^{\nu}, \quad \epsilon^{\mu\delta} \epsilon^{\nu\rho} \partial_{\delta} \partial_{\rho} = -\eta^{\mu\nu} \square + \partial^{\mu} \partial^{\nu} \quad (\text{A.25})$$

where $\epsilon_{01} = -\epsilon_{10} = +1$. Dirac matrices are

$$\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 \equiv \gamma^0 \gamma^1. \quad (\text{A.26})$$

In Minkowski space M^2 the free scalar propagator $[\phi^+(x), \phi^-(0)]$ is [36]

$$\Delta(x; m) \equiv \int \frac{d^2 p}{(2\pi)^2} \frac{ie^{ipx}}{p^2 - m^2 + i\epsilon} = \frac{1}{2\pi} K_0(m\sqrt{-x^2}) \quad (\text{A.27})$$

$$= \frac{e^{-m\sqrt{-x^2}}}{\sqrt{8m\pi\sqrt{-x^2}}} \quad (-x^2 \rightarrow \infty) \quad (\text{A.28})$$

$$= -\frac{1}{4\pi}(\gamma + \log(-x^2\mu^2)) \quad (m \equiv \mu \rightarrow 0) \quad (\text{A.29})$$

where K_0 (no relation to Chern-Simons current K_μ) is the modified Bessel function and γ is the Euler-Mascheroni number. The free quark propagator is

$$(+i\not{\partial} + m)\Delta(x; m) = \frac{m}{2\pi} \left[\frac{i\not{x}}{\sqrt{-x^2}} K_1(m\sqrt{-x^2}) + K_0(m\sqrt{-x^2}) \right] \quad (\text{A.30})$$

$$= \left[\frac{m}{8\pi\sqrt{-x^2}} \right]^{\frac{1}{2}} \left[1 + \frac{i\not{x}}{\sqrt{-x^2}} \right] e^{-m\sqrt{-x^2}} \quad (-x^2 \rightarrow \infty) \quad (\text{A.31})$$

$$= \frac{1}{2\pi i} \frac{\not{x}}{x^2} \quad (m \rightarrow 0). \quad (\text{A.32})$$

A.5 Relation Between $\lambda \neq 1$ and $\lambda = 1$ Gauge

In this Appendix, we prove Eqs. (4.10) and (4.11) on planar $D = 1 + 1$ spacetime M^2 . Under coordinate transformation $x \rightarrow w$ where

$$w^0 \equiv x^0/\sqrt{\lambda}, \quad w^1 \equiv x^1, \quad d^2 x = d^2 w \sqrt{\lambda} \quad (\text{A.33})$$

the electromagnetic field becomes $\mathcal{F}_{01}(w) = \sqrt{\lambda} F_{01}(x(w))$ and

$$S^g = - \int d^2 w \sqrt{\lambda} \frac{1}{4} \mathcal{F}^{\mu\nu}(w) \mathcal{F}_{\mu\nu}(w) = + \frac{1}{2\sqrt{\lambda}} \int d^2 w \mathcal{F}_{01}(w) \mathcal{F}_{01}(w). \quad (\text{A.34})$$

w -coordinate indices are raised and lowered by the metric tensor $\bar{\eta}^{00} = \lambda^{-1}$, $\bar{\eta}^{01} = \bar{\eta}^{10} = 0$, and $\bar{\eta}^{11} = -1$. Equality (A.34) follows from

$$\mathcal{F}^{\mu\nu} = -\mathcal{F}_{\mu\nu}/\lambda, \quad (\text{A.35})$$

which is valid only in two spacetime dimensions.

The vector potential in w coordinates is

$$\begin{pmatrix} \mathcal{A}_0(w) \\ \mathcal{A}_1(w) \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda} A_0(x(w)) \\ A_1(x(w)) \end{pmatrix}. \quad (\text{A.36})$$

An additional photon field rescaling

$$\bar{\mathcal{A}}_\mu(w) \equiv \lambda^{-1/4} \mathcal{A}_\mu(w), \quad \bar{\mathcal{F}}_{\mu\nu} \equiv \partial_\mu \bar{\mathcal{A}}_\nu - \partial_\nu \bar{\mathcal{A}}_\mu \quad (\text{A.37})$$

does not affect physics and gets rid of the $1/\sqrt{\lambda}$ prefactor in Eq. (A.34). *Defining*

$$\bar{\mathcal{F}}_{01} \equiv \mathcal{F}_{01}, \quad \bar{\mathcal{F}}^{01} \equiv \eta^{00} \eta^{11} \bar{\mathcal{F}}_{01} \quad (\text{A.38})$$

and renaming dummy variable w in (A.34) to y recovers

$$S^g = - \int d^2 y \frac{1}{4} \bar{\mathcal{F}}^{\mu\nu}(y) \bar{\mathcal{F}}_{\mu\nu}(y) |_{\text{Mink}}, \quad (\text{A.39})$$

which does not refer to $\bar{\eta}$, λ or w . This sequence of maneuvers turns S^{GF} of (4.9) into

$$S^{GF} = -\frac{\lambda}{2\xi} \int d^2 y (\eta^{\mu\nu} \partial_\mu \bar{\mathcal{A}}_\nu)^2 |_{\text{Mink}}. \quad (\text{A.40})$$

Eqs. (A.39) and (A.40) show that *pure* QED $_{\mathcal{M}}$ in an arbitrary $[\xi \rightarrow 0, \lambda \neq 0]$ gauge is equivalent to Landau gauge.

The quark fields Ψ_a are fixed by hypothesis in the original coordinate system under the transformations above. Consequently the quark-photon interaction transforms into (4.10) where $\bar{A}$ has been renamed A .

For $\text{QCD}_{\mathcal{M}}$ one must rescale coupling constant g while rescaling $\mathcal{A}$ to $\bar{\mathcal{A}}$ to get the desired effect of Eq. (A.37). Otherwise the argument goes through unaltered for any $SU(N)$. If $D > 2$, however, identity (A.35) is violated since $\mathcal{F}^{ij} = \mathcal{F}_{ij}$.

A.6 Free Quark Propagator

The Baker-Campbell-Hausdorff-Mandelstam(BCHM) identity,[35] if $[A^+, B^-]$ is a c -number, is

$$:e^{\alpha A}::e^{\beta B}: = e^{\alpha\beta[A^+, B^-]} :e^{\alpha A + \beta B}:. \quad (\text{A.41})$$

α and β derivatives of (A.41) combined with associativity

$$(:e^A::e^B:):e^C:= :e^A: (:e^B::e^C:) \quad (\text{A.42})$$

generate a hierarchy of relations. Useful examples are $[A, :e^B:] = [A, B] :e^B:$ and

$$:e^A: B :e^C:= \quad (\text{A.43})$$

$$([A^+, B^-] + [B^+, C^-])e^{[A^+, C^-]} :e^{A+C}: + e^{[A^+, C^-]} :Be^{A+C}:.$$

Bosonized free, massless Dirac fields[35] are

$$\psi_a^r \equiv i^{r-1} \sqrt{Z} :e^{-i\sqrt{\pi}\Theta_r(x)}: \quad (r = 1, 2), \quad (\text{A.44})$$

$$\Theta_r(x) \equiv (-1)^{r-1} \phi(x) + \int^{\vec{x}} d\vec{x}' \partial_0 \phi(x'). \quad (\text{A.45})$$

Boundary conditions on M^2 are $\lim_{\vec{x} \rightarrow \pm\infty} \phi(x) = 0$. Appendix A.4 gives the commutator $\Delta(x; \mu)$ between positive(negative) frequency components $\phi^+(\phi^-)$. μ is the infrared regulator and normal ordering is with respect to $\phi^\pm$. Applying Eq. (A.41) yields

$$\langle \Omega | \psi_{\vec{x}} \bar{\psi}_0 | \Omega \rangle \sim \frac{Z}{\mu} \begin{pmatrix} -i\mu & (t - \vec{x})/x^2 \\ (t + \vec{x})/x^2 & +i\mu \end{pmatrix}. \quad (\text{A.46})$$

The normalization $Z \sim 1$ is forbidden by parity. The normalization $Z \sim \mu$ reproduces the $D = 1 + 1$ free particle propagator $1/\not{x}$.

A.7 Figures

Figure A.1: Compare $M_q^{(\lambda)}$ to Strong Coupling λ -Dependence

Figure A.2: Configuration Dependence, Gluon Mass

Figure A.3: Rho/Quark Mass Ratio

Figure A.4: $M_q^{(\lambda)}/M_q^{(1)}$ Quark Mass Ratio

Figure A.5: $\lambda = 2$ to $\lambda = 1$ Propagator Ratio

Figure A.6: Steps in $\bar{3}d$ Subtraction Procedure

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