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Long-Term Stability for Particle Orbits

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and Jürgen Moser

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LONG-TERM STABILITY FOR PARTICLE ORBITS

This report contains the lecture delivered by Dr. L. J. Laslett at a one-day conference on The Mathematical and Computational Aspects of Accelerator Design, and a discussion by Dr. J. Moser of some of the points raised by Dr. Laslett. Two appendices contain some further comments by Dr. Laslett and Dr. E. McMillan. In addition to the talks reported here, the Conference also heard a report by Dr. G. Parzen on the iterative techniques used to calculate the eigenfrequencies in recent cavities in connection with linear accelerator design. He reported on the difficulties caused by shapes that consist of two nearly disconnected domains of unequal size, and of methods for overcoming these difficulties. Dr. Symon reported¹ on non-linear resonance phenomena due to the coupling of two particles, and Dr. E. Courant reported² on the effect of coupling in many particles. Dr. A. Garren described some sophisticated features of automation in codes used in accelerator design.

We thank all participants of the Conference for their contribution, and hope that this report will stimulate mathematicians to concern themselves with aspects of long-range stability.

¹ See H. Meier and K. R. Symon, "Analytical and Computational Studies on the Interaction of a Sum and a Difference Resonance," Proc. of the Intern. Conf. on High-Energy Accelerators and Instrumentation - CERN 1959

² See E. D. Courant and A. M. Sessler, "Transverse Coherent Resistive Instabilities of Azimuthally Bunched Beams in Particle Accelerators," The Review of Scientific Instruments, Vol. 37, No. 11, 1579-1588, Nov., 1966.

LONG-TERM STABILITY FOR PARTICLE ORBITS

L. Jackson Laslett *

The three accelerator papers on the program for this afternoon will be concerned with types of dynamical problems that arise in the study of particle accelerators, although the same mathematical problems may well arise also in other contexts. The accelerators with which we are concerned presumably will be of the "alternating-gradient" type, in which -- for very good reasons -- the functional character of the focusing force experienced by an individual displaced particle changes periodically as the particle proceeds on its way. Dr. Symon will review some of the quite striking effects that can develop from "coupling resonances," when the equations of motion are non-linear, and Dr. Courant will summarize the complications that develop with beam intensities sufficiently great that inter-particle forces must be considered.

My own paper is intended to report results on some computer experiments that I had hoped might cast light on the question of long-term stability -- specifically, in my work, for motion with only one spatial degree of freedom. For most accelerator projects this question may not be one of as immediate importance as the questions to be discussed by my colleagues this afternoon, but the answers could be vital for the proton storage-ring devices now being considered

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Berkeley, California

or under construction. The general problem appears to be a very interesting and quite difficult one, and I would look forward to comments from our mathematician friends concerning what one can say at this time in simple terms with respect to this problem.

Thus, for my own part, I would like this afternoon to call to your attention this question of long-term stability and report on a few elementary computational experiments that I performed last summer for my own orientation and amusement. As for motivation, the first Figure indicates some of the numbers that may be relevant for characterizing modern accelerators with respect to the interval over which one would wish them to exhibit stability of the particle motion. We note in particular that a representative particle may be called upon to traverse some 10^7 periods of the alternating-gradient (A-G) focusing structure. These parameters are indeed essentially the same as the number of oscillations or periods successfully experienced by the beams of existing high-energy synchrotrons, and so we have the opportunity of concluding from experience that treacherous long-term instabilities need not be troublesome in a conventional A-G design.

PRESENT AND FUTURE A-G SYNCHROTRONS

Oscillation Wavelength: One to Several Hundred Meters

Period of Structure: 10 to 100 m.

Time Interval: ~ 1 sec.

No. of Periods Traversed:

$$\frac{(3 \times 10^8 \text{ m/sec})(1 \text{ sec})}{(10 \text{ to } 100 \text{ m})} \sim 10^7$$

Fig. 1

It would not be out of the question, moreover, to think of computational experiments that would subject particles with a limited number of selected initial conditions to algebraic transformations that could simulate passage through some 10^7 periods of the A-G structure, although one would have to be attentive to the possibility of obtaining misleading results as a consequence of round-off or truncation errors -- especially if the structure of the problem is such that the distinction between the initial conditions for stable vs. unstable motion is very fine grained.

A storage ring, on the other hand, might be of comparable

dimensions -- or at least involve a similar number of oscillations per second and number of periods traversed per second -- but would be intended to retain particles for hours. Thus, as we see on Fig. 2, one's interest in long-range stability becomes extended to intervals some 10^4 times as great -- i.e., possibly to 10^{11} periods, or even to 5×10^{11} periods of the structure.

PRESENT AND FUTURE A-G SYNCHROTRONS

Oscillation Wavelength: One to Several Hundred Meters

Period of Structure: 10 to 100 m.

Time Interval: ~ 1 sec.

No. of Periods Traversed:

$$\frac{(3 \times 10^8 \text{ m/sec})(1 \text{ sec})}{(10 \text{ to } 100 \text{ m})} \sim 10^7$$

STORAGE RINGS

Time Interval: ~ 3 hrs. or $\sim 10^4$ sec.

No. of Periods Traversed: $\sim 10^{11}$

Fig. 2

Here past experience is not particularly helpful, since the storage rings that have been operated to date have been electron rings (or electron-positron rings) in which radiation damping can helpfully play a dominant role that is effectively absent (by a relative factor of ten orders-of-magnitude or more) for protons.

Cosmologically, the universe is supposed to be only some 5×10^9 years old, corresponding to no more than that of many planetary periods at the present rate, so astronomical evidence may provide little assurance to the storage-ring designer unless we are prepared to be relatively restrained in our non-linearities, and unless we have some theoretical scaling laws that then could be applied to relate the long-term stability of our proposed device to that of another system that observationally has appeared to be stable.

I could illustrate the mathematical form in which one might pose this problem in our application by considering explicitly the case of a single spatial degree of freedom -- such as the transverse particle displacement, in the median plane of the accelerator, measured from the equilibrium orbit (Fig. 3).

In the ideal case of an A-G accelerator that is perfectly constructed and aligned, the linearized equations of motion would be a simple equation of the form of Hill's equation (to which the Floquet theory would apply) -- as shown at the top of the Figure, where $F(\theta)$ is periodic in θ and in practice

might be represented simply by one of the two forms indicated.

LINEARIZED EQUATIONS

$$\frac{d^2 u}{d\theta^2} + F(\theta)u = 0$$

$$\text{where } F(\theta) = \begin{cases} \text{Periodic Square Wave} \\ \text{or} \\ a + b \cos N\theta \end{cases}$$

TYPICAL NON-LINEAR EQUATIONS

One degree of Freedom

$$\frac{d^2 u}{d\theta^2} + (a+b \cos \theta)u + \alpha(\sin N\theta)u^2 - \beta(\cos N\theta)u^3 = 0$$

Two coupled degrees of Freedom

$$\frac{d^2 u}{d\theta^2} + (a+b \cos N\theta)u + \alpha(\sin N\theta)(u^2 - v^2) - \beta(\cos N\theta)(u^3 - 3uv^2) = 0$$

$$\frac{d^2 v}{d\theta^2} + (a' - b \cos N\theta)v - 2\alpha(\sin N\theta)uv + 3\beta(\cos N\theta)u^2 v = 0$$

Fig. 3

In actuality, some non-linearity of the restoring (focusing) force will be present, either deliberately or inadvertently introduced. My own interest in the effect of non-linearities arose in connection with designs for fixed-field alternating-gradient (FFAG) accelerators, as proposed by Symon and others

of the Midwestern Universities Research Association (M.U.R.A.), in which non-linearities necessarily would be prominent at the amplitudes of interest. Representative terms that could be added to account for quadratic and cubic components to the force would, in that application, be of the form shown in the middle of the Figure, and below it one sees the Hamiltonian pair of equations that one could take to describe the corresponding motion in two spatial degrees of freedom.

The inclusion of a periodic θ -dependent coefficient for the linear terms in these equations is not significant, since that feature can be transformed away by an explicit, well-behaved transformation. What is important is that we do not have linear equations, to which the well-behaved properties of the Floquet solutions apply, nor do we have a θ -independent Hamiltonian that could be taken as a constant of the motion. (As is of course known, Dr. Moser has investigated the possibility of systematically introducing a series of transformations for the working variables such that the θ -dependence of the Hamiltonian function becomes displaced to higher and higher order -- this is done with the expectation, as I understand it, that the new Hamiltonian functions will progressively become better approximations to a constant of the motion, as in fact certainly seems to be the case in limited applications of this technique to the study of solutions over moderately long intervals.)

One has the impression that the essential features of solutions to equations of this form can be exhibited by suitably

constructed non-linear algebraic transformations whose short-term phase-plane characteristics would be qualitatively similar to those implied by the differential equations, and for which computational studies would be both faster and less affected by numerical error. Such transformations can, in fact, be regarded as special cases of the differential equations, with the periodic coefficients becoming periodic delta functions of the independent variable (discrete, localized lenses). What is required of the transformation is simply that the iterated values of the coordinate and conjugate-momentum variables, as functions of the previous values of these variables, satisfy the Poisson-bracket conditions -- or any other of the several alternative forms in which one can express the necessary and sufficient conditions for a system to be Hamiltonian. In one spatial degree of freedom, this condition can be simply stated as requiring that the transformation be area preserving in the two-dimensional q, p phase plane.

A simple transformation in which I became interested some ten years ago at M.U.R.A. pertained to motion in one spatial degree of freedom and was believed to simulate the solution to the radial motion in the median plane of a spirally-ridged FFAG accelerator if the coordinate and momentum for this case are plotted at one-period intervals of the structure. This transformation is shown in Fig. 4, in which A is a constant and I regard x and y respectively as a coordinate

and its canonically-conjugate momentum.

$$x' = Ax \pm (1-A^2)y + (1-A)[x \pm (1-A)y]^2$$

T:

$$y' = \mp x + Ay \pm [x \pm (1-A)y]^2$$

If $T(x_1, y_1) = (x_2, y_2)$, then $T^{-1}(x_1, -y_1) = (x_2, -y_2)$

Sketch for $A = -5/8$

$$\cos^{-1}A \cong (0.35745)(2\pi)$$

Area $\cong 5.5 \times 10^{-3}$
(would shrink to zero
if $A \rightarrow -1/2$)

$$F_1: \left(-\frac{\sqrt{41-5}}{26}, 0\right) = (-0.053966\ldots, 0)$$

$$F_{2,3}: \left(\frac{1}{26}, \pm \frac{4\sqrt{41-16}}{169}\right) = (0.03846\ldots, \pm 0.05688\ldots)$$

Fig. 4

The upper and lower signs can be taken as referring to the forward or inverse transformation, respectively. The constant A can be interpreted as the cosine of the phase advance per iteration that would be exhibited by solutions to the linearized transformation, and A normally would be taken to have an absolute value less than unity. It also turns out that to

avoid being exactly at a one-third resonance when the quadratic terms are present, one likewise should avoid the value $A = -1/2$ (for which $\cos^{-1}A = 2\pi/3$).

This transformation, as written, was intended to simulate the solutions in a spirally-ridged FFAG accelerator when viewed at a suitable point of symmetry in the structure (and at homologous points, spaced at intervals, $2\pi/N$, equal to one period of the structure). In this form the transformation (T) has the great convenience of exhibiting a symmetry about the x-axis, as has been noted on the Figure and as is evident from inspection of the transformation equations. The phase diagram, as revealed by short-term computations -- or roughly by application of Moser's methods or in some respects by more simple analytic considerations -- looks somewhat as sketched on the Figure for $A = -5/8$, $\cos^{-1}A \cong 0.35745(2\pi) \cong 128.7 \text{ deg.}7, and in this respect the proposed transformation appears consistent with trajectories computed for some typical FFAG structures in which one would plan to employ the greater part of the apparently stable region that is situated within the roughly triangular boundary shown.$

What appears to be a stable area is bounded by the curves that I have drawn through the three unstable fixed points of order 3 (F_1, F_2, F_3), that are rigorous fixed points whose coordinates can be explicitly obtained from the roots of a quadratic equation. In calling your attention to transformations of this form (and to the analogous non-linear differential

equations). I do not mean to imply that present accelerator or storage-ring projects visualize such pronounced excursions toward the boundaries of regions so obviously influenced by non-linear resonant effects, but smaller non-linearities may well be present in practice and future accelerator concepts may again involve strongly non-linear restoring forces. I also would not wish to imply that the most important problems to pursue in this connection are those that involve only one spatial degree of freedom. It may well be that results of some computational experiments -- and theoretical work by Kolmogorov and others that I hope Dr. Moser will review for us -- can set the designer's mind at rest with respect to possible long-range instability in one degree of freedom and that emphasis should be given in this context to the much more difficult problems that arise with systems having two (or more) spatial degrees of freedom.

In any event, I thought last spring that for me it would prove instructive to examine the performance of the transformation just shown, using the CDC-6600 computer in double precision, since my previous look at this problem had been with the earlier IBM-704. This work was undertaken at Berkeley with the programming assistance of Mrs. Levine and with helpful advice from Eric Beals and Loren Meissner. By using double precision, we had available some 96 bits -- although truncation rather than true rounding was used in the arithmetic operations -- and it was possible to investigate

spurious fluctuations and drifts both by computational experiments and/or by a rough analysis.

In investigating the behavior of an algebraic transformation such as the one just proposed, one of course has a wide choice of working variables. Thus, if desired, new variables could be introduced that would result in a transformation having symmetry about the $+45^\circ$ diagonal, or one could select variables that would reduce the number of computational steps (and hence improve the speed and accuracy of the work) required to perform each iteration. One may merely regard the introduction of such new variables as a mathematical operation, but, in some instances, use of the new variables rather easily can be interpreted as viewing the dynamics at some other reference point within the lattice of the physical structure.

For my own computational experiments, it was convenient to introduce new ("working") variables defined in Fig. 5 for the direct or inverse transformations, respectively. In terms of these variables the transformation assumed the form shown at the bottom of the Figure.

By introduction of the working variables

$$q = x \mp (1-A)y$$

$$p = \pm y$$

for the direct transformation T or its inverse,

$$q' = Cp + q$$

$$p' = (q'-1)q' + p$$

with

$$C = 2(1-A)$$

Fig. 5

This transformation seemed particularly suitable for repeated iteration, but phase diagrams plotted in terms of these working variables (q,p) do not show any apparent symmetry and it therefore was convenient to express results in terms of the original x,y variables whenever output data were printed. The parameter A ($= -5/8$, a binary fraction) and derived parameters [such as $(1-A) = 13/8$ and $C = 2(1-A) = 13/4$] were stored exactly in the computer memory, and double-precision octal I-O was available if required for resumption of any particular run.

An initial attempt to reveal erratic or stochastic behavior gave negative results, in that for runs of as many as 10^7 forward iterations,[†] starting with $x = +0.026 = 0.85 x_I$, there appeared to be no computationally significant anomalous

[†] 941.169 sec of CP time were required for a run consisting of 10^7 forward iterations followed by 10^7 reverse iterations.

drift -- e.g., after correction for computational drift and allowance for computational fluctuations, the +x intercept of the apparent invariant phase curves could be said to remain constant to an absolute accuracy of about 10^{-27} , or a relative accuracy of about 4×10^{-26} in this example.

A positive, and hence more dramatic, effect can be obtained, however, by examining in some detail the character of the apparent separatrix. For this purpose we note that the transformation T^3 will return a phase point to any one of the unstable third-order fixed points (F_1, F_2, F_3), and that the "separatrices" drawn through the fixed points can represent at these points the directions of outgoing and ingoing eigenvectors for the transformation T^3 when the latter is linearized about the fixed points. If a true separatrix F_2F_3 exists, a line segment formed of points situated very near to F_2 and that lie on a curve whose slope is that of the appropriate (outgoing) eigenvector through that point should iterate under T^3 so as to approach F_3 along a similar eigenvector direction and, in the process, should generate a single smooth curve (F_2F_3). The symmetry of the present transformation (with respect to the x-axis) would imply, therefore, that the curve so generated cross the x-axis with a vertical slope.

Numerically, forward iteration of a line segment from the neighborhood of F_2 down toward the x-axis should be favorable with respect to unavoidable computational errors,

and detection of a non-vertical crossing would imply the non-existence of a firm separatrix between F_2 and F_3 (and, by application of T and T^2 or T^{-1} , correspondingly also would imply the absence of firm separatrices F_3F_1 and F_1F_2 .)

Our work with double-precision arithmetic made it unambiguously clear that extension of the eigenvector direction at F_2 led to non-vertical crossing of the x-axis (at crossing, slope $\cong +10^6$), although if single-precision arithmetic had been employed the results might have been ambiguous in this case. Checks of the computational process could be made by use of different compiler systems, writing the algebraic statements in a different order or form, influencing the truncation errors (as by introducing intermediate steps such as "times 3" followed by "divide by 3" into one or more equations for each iteration), and by use of varied starting conditions. Although some of these steps affected noticeably the location of individual iterates (x,y) near the x-axis, they had no significant effect on the curve that was generated in this process.7

As is indicated in Fig. 6, the situation thus is such that a line segment QRS originating at F_2 with the eigenvector slope will transform, after repeated iterations of T^3 , to generate the S-curve shown just below the x-axis (initially with an eigenvalue, for T^3 , $\lambda_I \cong 2.102$).

Fig. 6

A similar sequence of inverse iterations from F_3 will generate the dashed curve, and forward iteration would cause points on the dashed curve to approach F_3 in the absence of computational errors. Thus only particular points (e.g.: Q , R , and their iterates) that lie on both curves can originate arbitrarily close to F_2 and ultimately approach F_3 . The area 1 transforms to the equal area 3 (unit Jacobian) and areas 1 and 2 are equal by virtue of the symmetry of the transformation, so all the loops shown -- and their iterates under the transformation T -- are equal in this example.

It will be recognized that points, such as Q' and R' , that approach F_3 ultimately do so in smaller and smaller

steps ($\lambda_{II} = \lambda_I = 0.48$). Conservation of the areas mentioned thus requires that the curve bounding the transforms of the areas 2 and 3 correspondingly must become increasingly elongated laterally, and (as suggested by Fig. 7) the solid curve Q'R'S'Q" will become increasingly sinuous.

Fig. 7

The areas shown (transforms of regions 2 and 3) each have the approximate magnitude 4.8×10^{-11} , or roughly $1/10^8$ times the area of the entire superficially-stable region $F_1F_2F_3$. The uniqueness of the transformation precludes that a finger such as that shown extending to the left will be intersected by one of its iterates, although such an iterate could well cross the dashed line. If the transformation continues to be applied, these fingers thus cannot remain

at all times within the finite area that at first might have been supposed to be a stable region, and an observed means of escape is that in which an iterate of this finger does cross the dashed line to enter a loop extending to the right.

It will be appreciated that the evolution of these fingers necessarily will become quite complex, and certainly they will make inroads into the ostensible "interior" region, but at this point it is by no means clear that the entire "interior" region need be consumed in this way. What is clear is that the non-smooth joining of the forward and backward eigenvector directions from F_2 and F_3 , respectively -- either by a disparity of slope (as in the present example) or by a disparity of some higher derivative -- will imply that these curves do not generate a true separatrix and the truly stable area (if one exists) consequently is somewhat smaller.

In the "interior," closer to the "stable" fixed point at the origin, other (higher-order) fixed-point systems may be found. It can become increasingly difficult in such cases, however, to examine whether smooth or non-smooth intersection of their extended eigenvectors occurs, and a computational investigation would offer little hope of establishing absolute smooth joining.

It is interesting and somewhat informative to examine other area-preserving transformations in these respects. Professor deVogelaere, who is now at Berkeley and has been very generous in discussion of these points, has for some time

been directing attention to a very simple transformation that requires little calculational precision to exhibit the failure of eigenvectors, when extended, to join smoothly. In its original form this transformation was of the form shown at the top of Fig. 8,

De Vogelaere's Transformation

$$\begin{aligned}x' &= y + x^2 \\y' &= -x + x'^2 \\ \frac{1}{2} \text{Tr.} &= 0.\end{aligned}$$

Modified De Vogelaere Transformation

$$\begin{aligned}x' &= y + Tx + (1-T)x^2 \\y' &= -x + Tx' + (1-T)x'^2 \\ \frac{1}{2} \text{Tr.} &= T.\end{aligned}$$

Fig. 8

and can be shown to possess the same symmetry property with respect to the x-axis that was noted for the transformation I mentioned originally. This particular form may be somewhat distressing to a physicist, since the trace of the matrix for the linearized transformation vanishes (phase advance = $\pi/2$), and one would be situated at a "quarter-integral resonance." A simple generalization of this transformation can be made, however, without affecting the interesting features by adopting the form shown at the bottom of the Figure. For either

transformation, an unstable fixed point, of order 1, occurs at the point (1,0), and the eigenvector directions extended from this point show a gross failure to join smoothly on crossing the negative x axis. This behavior is indicated on Fig. 9.

Although this transformation may not be representative in any evident way of accelerator-orbit behavior, its study can be informative concerning features that can arise in non-linear problems. One can find, for example with $T = -1/8$, systems of unstable fixed points for which the eigenvalues are negative and that do not appear to have associated with them a corresponding system of stable fixed points.

E.g., for $T = -1/8$: Order 4/1, with eigenvalues -3.197
and -0.3128 at $x = 0.5615983$, $y = 0$;

Order 8/2, with eigenvalues -8.369
and -0.1195 at $x = 0.4562733$, $y = 0.7$

Also, with $T > 0$ (e.g., $+1/8$), one finds for example a system of unstable and stable fixed points of order 19/4 in the "interior" that roughly exhibits the classical "island structure" but for which the extended eigenvectors very easily can be shown not to intersect smoothly (both on the inner and on the outer island boundaries). One member of the stable and of the unstable fixed-point systems of this family will be found respectively at $x = -0.38429776$, $y = 0$ and $x = +0.50736937$, $y = 0$. The eigenvalues for the latter are $\lambda_I = 1/\lambda_{II} \cong 3.2217$

Fig. 9

Somewhat closer to the origin, however, one finds a similar system of order $9/2$ (indicated on Fig. 10), for which the intersections appeared completely smooth to the limit of my computational accuracy [fixed-points at $x = -0.29000009$, $y = 0$ and at $x = +0.32176070$, $y = 0$ (with $\lambda_I = 1/\lambda_{II} \approx 1.179$)].

Fig. 10

Experience with each of the transformations mentioned could be said to be not inconsistent with the view that some of the curves commonly regarded as stability boundaries are not firm and imply a sort of stochastic behavior of phase points in their neighborhood, but, so far as we can tell, others may be perfect barriers and indeed may have true invariant phase curves nearby (ignoring, of course, the fluctuations of the parameters that in practice would be present as a result of "noise" in a physical system). The designer of particle accelerators -- or, more particularly, of storage rings --

of course would like to know whether any regularity of behavior can be anticipated in this regard, and whether quantitative or reasonably-accurate semi-quantitative estimates can be made of diffusion rate in a region that possibly is characterized by stochastic behavior. It may be, of course, that such detailed and subtle features of behavior are highly sensitive to the exact values of the physical parameters that determine the transformation, and as such would be beyond the precise physical control of the experimenter.

(I might add that the systems of unstable fixed points that have been found for both my transformation and for that of Professor deVogelaere appear to show what to me is a quite surprising regularity: I believe that deVogelaere noticed empirically that for his transformation, linearized about the fixed points of a certain class, the quantity $\frac{1}{2} \text{Trace} - 1$ showed an exponential dependence on the fixed-point order. It may be that other interesting features of such transformations can also be found to exhibit regularities -- such as that suggested by Mel'nikov in regard to the dimensions of the areas that are developed by the failure of separatrices to join smoothly -- and that some of these regularities might permit one to draw inferences that would have practical application.)

An interesting class of transformations has been proposed by Dr. McMillan, for which it is possible to establish that the eigenvectors through the fixed point or fixed points do form

a true impenetrable separatrix. He finds it convenient in this connection to consider the transformations to be written in a form showing symmetry about the $+45^\circ$ axis (rather than about the x-axis) -- a form in which each of the transformations previously mentioned can be written by a suitable change of variables. This form is taken to be, for the forward transformation, that which will be seen on Fig. 11 in the upper left-hand corner. Such a transformation has a simple physical

Fig. 11

interpretation in terms of a structure formed simply from a periodic sequence of identical non-linear lenses $\Delta y = -x + f(x)$ placed with unit separation just before the measurement planes and with linear lenses $\Delta y = -x$ situated just after each of these planes. For a fixed point of order one, the trace of the matrix for the transformation linearized about that point is just the derivative of f at that point, and stability of the linearized system requires $\left| \frac{1}{2} f' \right| < 1$.

The transformation in the form shown can be executed by a simple graphical construction, in which one first mirrors the initial point about the diagonal and then vertically reflects the resulting y -value about the curve $y = \frac{1}{2} f(x)$. A fixed point of order 1 is synonymous with the intersection of the diagonal with this latter curve. Referring to Fig. 12, McMillan has pointed out [UCRL-17795] that if the function $f(y)$ can be expressed as $\phi(y) + \phi^{-1}(y)$, where ϕ^{-1} is the inverse of ϕ , then $x = \phi(y)$ will be an invariant curve. The phase diagram shown in the Figure is for an example in which

$$x = \phi(y) = a - \frac{1}{y + a} \quad \text{and} \quad f(y) = \frac{2y}{a^2 - y^2},$$

with fixed points at $x = y = \pm \sqrt{a^2 - 1}$ ($|a| > 1$).

Example in which

$$f(y) = \phi(y) + \phi^{-1}(y)$$

Fig. 12

An interesting and simple, although perhaps artificial, example of this class is one suggested by Dr. Judd, for which, as indicated on Fig. 13, one employs a step-wise linear $f(x)$.

Fig. 13

Here the broken-line separatrix shown is a rigorous barrier. Also, of course, motion of sufficiently small amplitude to remain within the dotted square will trace the perfect elliptical invariant phase trajectories that are characteristic of truly linear motion. Outside of this dotted square, moreover, stable fixed points can be found [For example, with $a = 0.6$:

an order-6 FP at $x = y = 9/13$ and an order-7 FP at $-x = y = 378/6837$ that are surrounded by an elliptical boundary within which the motion is again truly linear (and truly stable "in the small"). In Fig. 14 one such region is shown as the hatched area in the upper right-hand portion of the diagram -- the motion being controlled in such a case in a periodic way by definite values of f' at successive

Fig. 14

iterations. Other points, including those neighboring unstable fixed points of the same family, however, show the failure of extended eigenvector directions to intersect smoothly and evidently are characterized by a sort of stochastic behavior that is confined to a finite region.[†] Nonetheless, the motion within the diamond is not completely ergodic, since, as we have seen, there exist in this area regions of limited extent within which the motion follows true simple invariant phase trajectories.

In presenting this material I do not mean to imply that it contains much information that is new to the mathematicians who have been following recent developments of the theory. I am afraid that some of the computational results fail, as might be expected, to give definitive experimental answers to some of the significant questions, and my own work has neglected entirely the more important (and more difficult) problems that arise with motion in more than one spatial degree of freedom. I also would not wish to claim that the questions of long-range stability are of quite as immediate importance to the accelerator designer as those, for example,

[†] In this transformation (and also in a similar transformation with $f(y)$ analytic), some cases were found in which the eigenvectors, when extended, generated a double-S (or double-Z), rather than a single S (or Z) between successive iterates.

that concern collective motion. I would like to suggest, however, that the properties of such transformations as the one I mentioned at the beginning suggest questions that inherently are quite fascinating, and that it would be quite helpful for future design work if our mathematical colleagues could assist us in obtaining a better -- and hopefully somewhat quantitative -- understanding of these properties.

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Appendix A

I would like to call specifically to your attention a situation we encountered in which an area-preserving algebraic transformation, when linearized about certain high-order fixed points, exhibited negative eigenvalues. This phenomenon arose in connection with some runs that I made using a modification of Professor deVogelaere's transformation, namely

$$T: \begin{cases} x' = y + Tx + Cx^2 \\ y' = -x + Tx' + Cx'^2 \end{cases}$$

with $C = 1 - T$ and $T = -1/8$.

This transformation, whose phase plots are symmetrical about the x-axis, has an unstable fixed point of order 1 at the point (1,0) and the eigenvector-directions extended from this point develop loops in the familiar way (as qualitatively sketched in Fig. 1). In seeking higher-order fixed points in the "interior" of this diagram, the following families of unstable fixed points appeared for which the eigenvalues are negative:

Family 1 ("Tune" = $1/4$):

<u>x</u>	<u>y</u>
0.5615983	0
0.2846170	-0.5060428
-0.4504872	0
0.2846170	0.5060428

Eigenvalues: $\lambda = -3.197, -0.3128$

Fig. 1

Family 2 ("Tune" = 2/8):

<u>x</u>	<u>y</u>
0.4562733	0
0.1771744	-0.4431055
-0.4299377	0.0845201
0.3462146	0.5215085
0.6130793	0
0.3462146	-0.5215085
-0.4299377	-0.0845201
0.1771744	0.4431055

Eigenvalues: $\lambda = -8.369, -0.1195$

Phase trajectories formed by extending the eigenvector directions for the system described as Family 2 are shown in Fig. 2 in the region near the points

$$(0.6130793, 0) \quad \text{and} \quad (0.4562733, 0).$$

It does not appear necessary to have an associated family of stable fixed points in such a negative- λ case, and none was found.

With reference to Fig. 2, we may consider a point such as A_n , that lies at the intersection of a phase trajectory that approaches the right-hand one of the two fixed points shown and a trajectory that is directed away from the left-hand point in the figure. After 8 applications of the transformation, this point moves so as to lie closer to the right-hand fixed point (but on the extended eigenvector segment that lies on the opposite side of this fixed point, since

Fig. 2

$\lambda \cong -0.1195 < 0$), while it also comes to lie more remotely from the left-hand fixed point (and on the extension of the eigenvector segment on the opposite side of this fixed point, since $\lambda \cong -8.369 < 0$). This new point is designated A_{n+8} on the drawing.

The point B_n lies on the intersection of outward- and inward-directed eigenvectors from the left-hand fixed point. Its T^8 iterate is seen to lie on another such intersection -- closer in on the eigenvector for which $|\lambda| < 1$ and more remotely on the extension of the vector for which $|\lambda| > 1$. This performance may suggest that it is not necessary for a loop such as that shown between B_n and the left-hand fixed point to enclose another (e.g., stable) fixed point of this rotation number. I thus wonder whether the behavior of such fixed points (with λ real and negative) may be qualitatively distinct from that expected in cases for which $\lambda > 0$ (and for which one expects to find a "string of islands" such that one then inquires concerning the smoothness or lack of smoothness with which eigenvector directions, when extended, intersect). The type of phase trajectories indicated in Fig. 2 may be most readily adaptable to fit in with the rather gross loops that develop in this region of the phase plot from the eigenvectors extended from the order-1 fixed point at (1,0). [To avoid confusion on the drawing of Fig. 2, I have not shown there the order-4 unstable fixed point (0.5615983, 0), of Family 1, that falls in this region of the diagram. This point does not

lie on the intersection of any of the Family-2 trajectories shown in the Figure -- it lies on the x-axis some three-and-one-half or four millimeters to the right of the point A_n . The features illustrated in this diagram can be exhibited without any exceptional computational accuracy, and I have found it convenient to ad lib runs by sitting at a readily-programmed Olivetti-Underwood "101 Programma".]

As a second point, I neglected to mention on Tuesday, during the discussion concerning "my" transformation, that this transformation also has an order-1 fixed point at (1,0). The transformation in question had the form

$$\begin{cases} x' = Ax + (1 - A^2)y + (1 - A)[x + (1 - A)y]^2 \\ y' = -x + Ay + [x + (1 - A)y]^2. \end{cases}$$

The extensions of the eigenvectors from this order-1 fixed point also develop gross loops, as might be expected. This behavior is sketched in Fig. 3, where the hatched area is the triangular area (with order-3 fixed points at its vertices) to which I directed attention in my talk. As before, the constant A has been given the value $-5/8$.

This same transformation, but with $A = -1/4$, shows a quite striking approximate constancy of the "tune" with respect to amplitude. With this value of A ($A = -1/4$) the small-amplitude tune corresponds to about 0.2902 (2π) radians phase advance per iteration, and a system of 7 stable and 7 unstable fixed points ($\text{tune} = 2/7 \doteq 0.2857$) develops at a

$$A = -\frac{5}{8}$$

Fig. 3

rather large amplitude. [The x-coordinates of the fixed points that lie on the x-axis are given below:

Stable F.P.: -0.123211 58766 12485 36429 040273;

U.S. F.P.: +0.2 (exact), (half trace + 1.05078125).

Reasonably-appearing phase curves can be drawn inside and outside this system (Fig. 4).]

As a final point, you may recall that during my talk I mentioned in passing that certain regularities appeared to be present in the eigenvalues (or, more simply, in the matrix trace) for the successive families of fixed points of a particular class. You might be interested in the results for fixed points of tune $\frac{m}{4m + 1}$ for Professor deVogelaere's transformation (the transformation T, with the parameter T set equal to zero). These results, from the CDC-6600, are given in the attached Table, titled "TRACE", and apply to this transformation after linearization about the fixed points in question.

It will be noted that the amount by which $\frac{1}{2}$ Trace differs from unity is closely the same in absolute value for the stable and unstable fixed-point families of the same order, particularly for those with a tune having a large denominator. As Professor deVogelaere has pointed out, a plot of $\log|1 - \frac{1}{2} \text{Trace}|$ vs this denominator is highly linear through many decades. Somewhat similar regularities also appeared with my transformation (taking $A = -5/8$).

Fig. 4

FIXED POINTS OF ORDER 7
(Tune = $\frac{2}{7}$)

⊙ Stable

+ Unstable

$$A = -\frac{1}{4}$$

TRACE -- with Characteristic Exponent or Eigenvalue

Rotation		$\left 1 - \frac{1}{2} \text{Trace} \right $	μ (radians and degrees) or λ_I
4/17	Stable	1.4425105233511725112475209	2.0291926112 rad. = 116.26417 deg.
	Unstable	1.43969121690663246936442793	3.66502113263853697319254146
5/21	Stable	5.7813842878779832846897813 x 10 ⁻¹	1.1352987670 65.04783
	Unstable	5.8036264715679285844257943 x 10 ⁻¹	1.80410530659480794010475635
6/25	Stable	2.1759794076036170323557400 x 10 ⁻¹	0.6722827659 38.51897
	Unstable	2.1788566763236692069200659 x 10 ⁻¹	9.1304428221204405827208083 x 10 ⁻¹
7/29	Stable	7.829060692954532623645218 x 10 ⁻²	0.3983316626 22.82272
	Unstable	7.831797137551201910136569 x 10 ⁻²	4.8176517658506550100501700 x 10 ⁻¹
8/33	Stable	2.728644682248974445483034 x 10 ⁻²	0.2341429028 13.41540
	Unstable	2.728895164746296854366273 x 10 ⁻²	2.6249649867124260506989502 x 10 ⁻¹
9/37	Stable	9.29792427831269351629599 x 10 ⁻³	0.1364724784 7.81930
	Unstable	9.29815526758626295733878 x 10 ⁻³	1.459830711381850496853179 x 10 ⁻¹
10/41	Stable	3.11733627966553849389897 x 10 ⁻³	0.0789804692 4.52525
	Unstable	3.11735776570024232911098 x 10 ⁻³	8.21390854792845376708780 x 10 ⁻²
11/45	Stable	1.03289297698264626685582 x 10 ⁻³	0.0454548311 2.60437
	Unstable	1.03289498665285647336435 x 10 ⁻³	4.64955922818988299371776 x 10 ⁻²
12/49	Stable	3.3928393819586502370257 x 10 ⁻⁴	0.0260500720 1.49256
	Unstable	3.3928412689282888627246 x 10 ⁻⁴	2.6390836240047607322398 x 10 ⁻²
13/53	Stable	1.1073874588761127191517 x 10 ⁻⁴	0.0148822572 0.85269
	Unstable	1.107387636632479404429 x 10 ⁻⁴	1.4993271827637053389945 x 10 ⁻²

TRACE -- Continued

14/57	Stable Unstable	3.597507701874426428512 3.597507869862235287326	$\times 10^{-5}$ $\times 10^{-5}$	0.0084823691 8.518395228840448345675	0.48600 $\times 10^{-3}$
15/61	Stable Unstable	1.164736575122263454806 1.164736591051033262679	$\times 10^{-5}$ $\times 10^{-5}$	0.0048264663 4.838123045135810082937	0.27654 $\times 10^{-3}$
16/65	Stable Unstable	3.76188982802960206947 3.76188984318697930160	$\times 10^{-6}$ $\times 10^{-6}$	0.0027429518 2.74671537799954849551	0.15716 $\times 10^{-3}$
17/69	Stable Unstable	1.21302583785923977869 1.21302583930697271578	$\times 10^{-6}$ $\times 10^{-6}$	0.0015575789 1.55879228547447310154	0.08924 $\times 10^{-3}$
18/73	Stable Unstable	3.9073431514620690060 3.9073431528502623898	$\times 10^{-7}$ $\times 10^{-7}$	0.0008840072 8.843979646269819787	0.05065 $\times 10^{-4}$
19/77	Stable Unstable	1.2579061979687540848 1.2579061981024041544	$\times 10^{-7}$ $\times 10^{-7}$	0.0005015788 5.017045535711146920	0.02874 $\times 10^{-4}$
20/81	Stable Unstable	4.048890866188331620 4.048890866317540574	$\times 10^{-8}$ $\times 10^{-8}$	0.0002845660 2.846065075505638042	0.01630 $\times 10^{-4}$
21/85	Stable Unstable	1.303401892994049370 1.303401893006593882	$\times 10^{-8}$ $\times 10^{-8}$	0.0001614560 1.614690281164473223	0.00925 $\times 10^{-4}$
22/89	Stable Unstable	4.19742353638860984 4.1974235364008412	$\times 10^{-9}$ $\times 10^{-9}$	0.0000916234 9.1627595583722791	0.00525 $\times 10^{-5}$
23/93	Stable Unstable	1.35249829376673101 1.3524982937679288	$\times 10^{-9}$ $\times 10^{-9}$	0.0000520096 5.2010934205562950	0.00298 $\times 10^{-5}$
24/97	Stable Unstable	4.3612513115999777 4.3612513116011556	$\times 10^{-10}$ $\times 10^{-10}$	0.0000295340 2.9534319420601064	0.00169 $\times 10^{-5}$

STABLE FIXED POINTS -- Coordinates of Fixed Points on x-axis

Rotation	x_0
4/17	-0.405787495533527606179670330
5/21	-0.3898327790576272411210745517
6/25	-0.376256564258982703826351991
7/29	-0.364375876292784753973866988
8/33	-0.353849991897727178101212375
9/37	-0.344454467825082214779278635
10/41	-0.3360136625106348627695919205
11/45	-0.328382766046255557958214443
12/49	-0.321441972573218211407301229
13/53	-0.315092612636619870929695716
14/57	-0.309253536442459300884027290
15/61	-0.303857827005765909029891050
16/65	-0.298850010528990438538647972
17/69	-0.294183791444198337338681611
18/73	-0.289820254666633978578222174
19/77	-0.285726451504977016300760846
20/81	-0.281874288628402645655758688
21/85	-0.27823965234541592668327308
22/89	-0.274801714559297280796278695
23/93	-0.271542379067610302194439393
24/97	-0.268445836701300885933631152

UNSTABLE FIXED POINTS -- Coordinates and Eigenvector-Slopes of Fixed Points on x-axis

Rotation	x_0		Eigenvector Slope
4/17	0.496506084637672571858047332	+	1.19087319776336072103372725
5/21	0.462456818522159807417166287	$\pm$	1.98025640724266507001182499
6/25	0.434919899202634068763276322	$\pm$	3.6448303715138160554898096
7/29	0.4129712543411046777982053091	$\pm$	6.71149665458175416010959
8/33	0.395143181201283923562103066	$\pm$	12.13194932674224587711334
9/37	0.38028378479061446738668478	$\pm$	21.65256874244943270658525
10/41	0.367615693398945789225052500	$\pm$	38.3974228188253509576353
11/45	0.356618501511893262559214861	$\pm$	67.899195745219753844038
12/49	0.346933293897475164157287819	$\pm$	119.930944670210285412930
13/53	0.338303540926722169037576515	$\pm$	211.74390558804250663261
14/57	0.330539854025521833310535365	$\pm$	373.77790810525522489253
15/61	0.323498535361983972920093128	$\pm$	659.72253985475741663301
16/65	0.31706804409355469021833772	$\pm$	1164.23127759143813146135
17/69	0.311160132509971989083355634	$\pm$	2054.08906983771121440622
18/73	0.30570384083580695724148746	$\pm$	3623.0142160560248067746
19/77	0.300641307899366821915148229	$\pm$	6387.950458569018040897
20/81	0.295924774533113647798361329	$\pm$	11258.14336841298826259
21/85	0.29151439360049880578482880	$\pm$	19831.80650990849558229
22/89	0.287376599445981434654903827	$\pm$	34916.28816384722538324
23/93	0.283482873883316580042953742	$\pm$	61439.461613242624541
24/97	0.279808798677119570429068096	$\pm$	108045.428556521895602

UNSTABLE FIXED POINTS -- Coordinates and Eigenvector-Slopes of Fixed Points near Negative x-axis

Rotation	x_0	y_0	Eigenvector Slopes, from upper F.P., for $\lambda \geq 1$
4/17	-0.374963547975391591167940018	$\pm 0.1059206297719378880335235985$	0.915486568783131693553300896 2.643243716380710957237796760
5/21	-0.3709792222736135823472685995	$\pm 0.076240725638902658433412233$	1.29658202546409954392340036 2.69794367031100718218121146
6/25	-0.363360430000666384442227247	$\pm 0.057124516632160203007080502$	1.62198974578660483326242039 2.59545656091817951642492881
7/29	-0.354799885030699916627188833	$\pm 0.044662298494267490408919155$	1.8831866989852810751563540 2.5294029554032894752569088
8/33	-0.346384947641954153109214825	$\pm 0.036155601696951380707060071$	2.0958394793051152906058737 2.5203670128578479810549472
9/37	-0.338456350459898304615153470	$\pm 0.030063055808040875997004950$	2.2735196256654867498100389 2.5501109094473573751463494
10/41	-0.331089352530756057848620544	$\pm 0.02552113867392451901906386$	2.425483218694276943146054 2.603884480416832690781204
11/45	-0.324270647171535629173200926	$\pm 0.022025303003541669253251617$	2.5584286643518998960738116 2.6722903765866525976589024
12/49	-0.317958929780800613936981251	$\pm 0.019264829387199782429528874$	2.6774112341632897363909809 2.7493648481699475750699039
13/53	-0.312106262884832988816311803	$\pm 0.017038966471621903572171853$	2.786211224111726315951442 2.831281990313632622526412

UNSTABLE FIXED POINTS -- near negative x-axis -- Continued

14/57	-0.306665846880174961760996189	$\pm 0.015212653456478366785589701$	2.887575959286764751494648 ₅ 2.915592163216093996261229 ₇
15/61	-0.301594745181758972978848546	$\pm 0.013691912060098667750220235$	2.98344770709623458485973 3.00074835801797030168621
16/65	-0.29685461990371039921986437	$\pm 0.012409479227135966967667072$	3.075172816638866086836259 3.085796178029668174989598
17/69	-0.292411664472703019119058302	$\pm 0.011316046543926678118726977$	3.163674983099432211082672 3.170166636821707266037392
18/73	-0.288236227231789099079304058	$\pm 0.010374715612948833543028581$	3.2495872938313320205429 3.2535376525569821477697
19/77	-0.284302340581867119271190549	$\pm 0.009557375155113915940587013$	3.33334713195506387450208 3.33574239886664091863436
20/81	-0.280587249052701960844405559	$\pm 0.008842267851511150343049067$	3.4152617043102799591962 3.4167095355863915233517
21/85	-0.27707097468107878695467275	$\pm 0.00821231666554353688336967$	3.4955520696750023864899 3.4964248555929535977352
22/89	-0.27373593262040534223520233	$\pm 0.007653949101572963481466505$	3.574382187359892100404 3.575907087079015179917
23/93	-0.27056659828187875545358686	$\pm 0.00715625567931681763591264$	3.6518778642681330455689 3.6521928967283763252765
24/97	-0.267549222310577829183807391	$\pm 0.006710377458137833432946976$	3.728139039631599455481 3.728327777110349067290

DISCUSSION FOLLOWING LASLETT'S PAPER

Jürgen Moser*

I should like to add some comments about some recent theoretical work on the problems of the type discussed by Laslett as well as mention some numerical experiments conducted by other people.

It is hard to believe that these questions have their origin in the theory of accelerator design. In fact, originally the basic principle for the orbit stability of the Alternating Gradient Synchrotron depends to a large extent on the linear theory. However, in the course of deeper studies of the particle orbits in an accelerator, the nonlinear effects became more and more important and a source of concern. It was crucial to have some guarantee that the nonlinear terms would not destroy the stability which was so carefully provided for by the linear theory. This led to very delicate and difficult theoretical problems which also were approached with numerical experiments conducted on simplified model equations as we have just learned from Dr. Laslett's presentation.

First, to mention some numerical studies of quadratic measure preserving mappings, I want to refer to the interesting work of Hénon, "Numerical Study of Quadratic Area-Preserving Mappings", 1967, to be published. He studied simple quadratic mappings of the same nature as Laslett mentioned toward the end of his talk (transformation by Dr. McMillan). The main

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results are the following:

In the neighborhood of an elliptic fixed point, one finds well organized curve patterns which disintegrate at larger distances. Immediately the problem arises to determine the size of this "stability region" as well as determine this curve pattern possibly by analytic procedure. Hénon restricted himself to the description of the computations and comparison of his results with asymptotic series obtained by G. D. Birkhoff. The remarkable fact is that the agreement between the numerical results and these asymptotic series is tremendously close. However, no clear indication for the size of this stability region is in sight.

It would require more space to refer to various theoretical studies which have been made in this direction. First of all, it is to be mentioned that the curve pattern observed in most of these problems in general does not exist in the mathematically rigorous sense as a continuous family of invariant curves. This fact was known to Poincaré and is closely related to his nonexistence proof of integrals. Of course, one can easily construct examples for which such family of closed curves does exist; however, this is an exceptional situation. For a rigorous proof of this last statement we refer to Rüssmann¹. The situation is actually

1 H. Rüssmann, "Über die Existenz einer Normalform inhaltstreu elliptischer Transformationen," Math. Ann. 137, 1967, pp. 64-67.

very delicate insofar as arbitrarily small changes of the mapping lead to a distribution of the curve pattern. On the other hand, by changes of sufficiently high order terms, one can force the existence of a family of smooth closed curves. The situation is somewhat reminiscent of the distribution of rational numbers on the real line. Both the rational numbers on the real line. Both the rationals as well as the irrationals are dense yet the irrationals form the majority.

To continue this analogy: it is rather easy to establish the existence of irrational numbers and, similarly, it is not too hard to establish the existence of mappings which do not possess such a continuous curve pattern. It is usually a more difficult problem to decide whether a preassigned number, say π is irrational. Therefore, one would expect it should be difficult to decide whether a given mapping of this sort belongs to one class or the other. In this connection, I want to refer to a paper of mine² in which it is shown rigorously that even the simple polynomial mappings do not possess a family of closed curves. This includes, in particular, the example by McMillan.

² J. Moser, "On the Integrability of Area-preserving Cremona Mappings near an Elliptic Fixed Point," Bol. Soc. Mat. Mexicana, 1961, pp. 176-180.

At this point, it seems confusing that the numerical evidence does not match the theoretical prediction of non-existence of such a curve pattern. It turns out that this paradox can be resolved: even though there is no continuous family of closed invariant curves there does exist a large set of closed curves which, however, do not join together into a continuous family. They are interrupted by infinitely many gaps so that one obtains a "Cantor set" of closed curves. This was proven in my paper³ as well as in Arnold's paper⁴. It is to be emphasized that the closed curves so obtained are by no means pathological, they are differentiable, and, in fact, analytic curves. However, their distribution is pathological. The crucial point of these investigations is that they insure stability of the elliptic fixed point since points starting in the interior of such curves are forced to remain there.

What happens in the gaps left by the invariant curves mentioned above is only partially understood. These regions contain, in general, infinitely many fixed points of the

³ J. Moser, "On Invariant Curves of Area-preserving Mappings of an Annulus," Nachr. Akad. Wiss., Göttingen, Math. Phys. Kl. IIa, Nr. 1, 1962, pp. 1-20.

⁴ V. I. Arnold, "Proof of A. N. Kolmogorov's Theorem on the Preservation of Quasi-periodic Motions under Small Perturbations of the Hamiltonian", Uspekhi Mat. Nauk U.S.S.R. 18, Ser. 5 (113), 1963, pp. 13-40.

mapping or one of its higher iterates. However, the behavior of the consecutive iterates in these regions is quite erratic and irregular, in particular near unstable fixed points. These domains are often referred to as "regions of instability". For an instructive schematic picture we refer to the paper by Arnold⁵. There are conjectures, more or less informal, that these regions of instability contain open invariant sets in which the mappings are ergodic. Nothing of this sort has been proven and there is some numerical evidence, especially in the work of John M. Greene, Forrestal Research Center, Princeton,⁶ for extremely ergodic behavior in these domains. In fact, he expressed the opinion that there are invariant domains in which the unstable fixed points are dense. Since this situation is typical for ergodicity, one may take this as an indication for the existence of ergodic domains.

Comparing the numerical and theoretical results, one has to be very careful: the theoretical results are strictly applicable only in extremely small neighborhoods of elliptic fixed points (in some paper, Henon estimates a radius of validity of the proof as about 10^{-300} !). This is due to the

⁵ V. I. Arnold, "Small Divisor and Stability Problems in Classical and Celestial Mechanics," *Uspekhi Mat. Nauk*, 18, No. 6 (114), 1963, pp. 81-192.

⁶ Private communication.

shortcoming of proofs which are very crude in the details of the estimate. One would have to resort to numerical experiments, such as Laslett presented here, to obtain the feeling for the actual size of the stability region. In all these numerical studies one finds, in fact, a sizeable neighborhood covered to a large extent by invariant curves. On which parameters or properties the size of this stability region depends seems still to be quite in the dark. For this reason, I consider the closer study of numerical experiments extremely worthwhile and instructive.

ADDENDUM TO UCRL 17795, "SOME THOUGHTS ON STABILITY IN
NONLINEAR PERIODIC FOCUSING SYSTEMS", Sept. 5, 1967.

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1.) Introduction

In UCRL 17795, it was shown that curves in the x, y plane having reflection symmetry about the positive diagonal are invariant under the transformation:

$$\begin{aligned}x' &= y \\ y' &= -x + f(y),\end{aligned}\tag{1}.$$

where $f(y)$ is the sum of the two values of x corresponding to the given y . It is required that there be just two values, but the two branches on which they occur are not required to have a common analytic form. An example given was the pair of rectangular hyperbolas $y = 1 - a/(x+1)$ and $y = -1 + a/(1-x)$, with $f(y) = 2ay/(1-y^2)$, mentioned in paragraph 3 and illustrated in Fig. 1. The question whether there are other invariant curves belonging to the same $f(y)$ was left open.

This question was answered by John M. Greene in a letter to L. Jackson Laslett (March 8, 1968). He pointed out that all curves of the form $(1-x^2)(1-y^2) + 2axy = \text{const.}$ are such invariants. If the constant has the value $2a-a^2$, the equation factors into two equations representing the rectangular hyperbolas, which are now seen to be simply the separatrices

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of a family of invariant curves. In the course of checking the invariance of "Green's function" by the methods of UCRL 17795, I found that it is a special case of a broader class, which can be called "double quadratic" curves.

2.) "Double quadratic" curves

Any equation which is quadratic in x can be solved explicitly for x . If x and y occur in it symmetrically, it represents a curve with the required symmetry about the positive diagonal. The most general equation with these properties is:

$$Ax^2y^2 + B(x^2y+xy^2) + C(x^2+y^2) + Dxy + E(x+y) + F = 0, \quad (2)$$

whose solution is:

$$x = \frac{1}{2(Ay^2 + By + C)} \left[- (By^2 + Dy + E) \right. \\ \left. \pm \sqrt{(By^2 + Dy + E)^2 - 4(Ay^2 + By + C)(Cy^2 + Ey + F)} \right] \quad (3).$$

The sum of the two values of x gives $f(y)$:

$$f(y) = - \frac{By^2 + Dy + E}{Ay^2 + By + C} \quad (4).$$

Since $f(y)$ does not depend on F , all members of the family generated by giving different values to F are invariant under the transformation (1), with $f(y)$ given by (4).

We thus have the remarkable result that an $f(y)$ which is the ratio of any two quadratic functions of y leads to a family of invariant curves, with the single restriction that the coefficients of y^2 in the numerator and of y in the denominator must be of equal magnitude and opposite in sign.

The first order fixed points, if they exist, are at $f(y) = 2y$, and are therefore the solutions of:

$$2 Ay^3 + 3 By^2 + (2C+D)y + E = 0 \quad (5).$$

The number of parameters in (4) is easily reduced; E can be eliminated by a coordinate displacement along the positive diagonal, either A or B can be made equal to D or E by a change of scale, and any one of the remaining parameters can be set equal to unity. Thus we have a two-parameter system. Some interesting cases are:

$$(1) \quad A = 1, B = 0, C = -1, D = 2a, E = 0, F = c.$$

$$x^2 y^2 - x^2 - y^2 + 2a x y + c = 0. \quad (\text{"Green's function"})$$

$$f(y) = \frac{2 a y}{1 - y^2}$$

The first order fixed points are at $y = 0, \pm \sqrt{1-a}$.

The separatrices are displaced rectangular hyperbolas, as pointed out above.

$$(2) \quad A = 1, B = 0, C = 1, D = -2a, E = 0, F = c.$$

$$x^2 y^2 + x^2 + y^2 + 2a x y + c = 0.$$

$$f(y) = \frac{2 a y}{1 + y^2}$$

The first order fixed points are at $y = 0, \pm \sqrt{a-1}$.

The separatrix is the curve given by setting $c = 0$.

In cases (1) and (2), if a is negative, the curve is rotated by 90° , and the first order fixed points (except the one at $x = 0$) become second order fixed points. (See paragraph 6 and Fig. 3b of UCRL 17795)

$$(3) \quad A = 0, B = 1, C = -1, D = 0, E = 0, F = c.$$

$$x^2 y + x y^2 - x^2 - y^2 + c = 0.$$

$$f(y) = \frac{y^2}{1 - y}$$

The first order fixed points are at $y = 0, \frac{2}{3}$.

The separatrices are the curve given by setting $c = \frac{8}{27}$, the line $x+y+2 = 0$, and the curve $xy - x - y + 2 = 0$.

(I thank Dr. Laslett for finding the last two of these.)

$$(4) \quad A = 1, B = -2, C = 1, D = 0, E = 0, F = c.$$

$$x^2 y^2 - 2 (x^2 y + x y^2) + x^2 + y^2 + c = 0.$$

$$f(y) = \frac{2y^2}{(1-y)^2}$$

The first order fixed points are at $y = 0, \frac{1}{2} (3 \pm \sqrt{5})$.

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