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SCALE TRANSFORMATIONS. THE ENERGY-MOMENTUM TENSOR.
AND THE EQUATION OF STATE*

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ABSTRACT

The Equation of State (EOS) relates diagonal elements of the energy-momentum tensor $\theta_{\mu\nu}$. The first moment of the energy-momentum tensor generates scale transformations. The virial theorem, a consequence of the behavior of the energy density under scale transformations, allows one to eliminate the kinetic energy in terms of the potential terms. The trace theorem for the energy-momentum tensor expresses $\epsilon - 3p$ in terms of ensemble averages of scale-breaking operators, allowing a new approach to the EOS.

I. THE VIRIAL THEOREM IN QUANTUM THEORY

Consider a single-particle nonrelativistic bound state $\psi_n(\mathbf{r})$ with potential $V(\mathbf{r}) = A\mathbf{r}^{-d}$ ($A > 0$ for $d < 0$) and $A < 0$ for $d > 0$). The virial operator

$$D \equiv \mathbf{x} \cdot \mathbf{p} \rightarrow -i \mathbf{r} \cdot \frac{\partial}{\partial \mathbf{r}} \quad (1.1)$$

is not conserved (here $T = \mathbf{p}^2/2m$ is the usual kinetic energy);

$$i[D, H] = -2T - dV \quad (1.2)$$

but its average in the eigenstate does vanish, since both D and H are Hermitian (a consequence of the absence of surface terms);

$$\begin{aligned} \langle \psi_n | [D, H] | \psi_n \rangle &= 0 \\ \langle T \rangle_{nn} &= -\frac{d}{2} \langle V \rangle_{nn} \end{aligned} \quad (1.3)$$

This simplest example of the "virial theorem" allows us to re-express the kinetic energy contribution to E_n in terms of the average potential:

$$\begin{aligned} E_n &= \langle T \rangle_{nn} + \langle V \rangle_{nn} \\ &= \left(1 - \frac{d}{2}\right) \langle V \rangle_{nn} \end{aligned} \quad (1.4)$$

Historically the most famous example is the $1/r$ potential (gravity or Coulomb potential) for which $\langle T \rangle = -1/2 \langle V \rangle$, $\langle E \rangle = 1/2 \langle V \rangle$.

Note that D measures the dimension of the components of H (see Eq. (1.2)). The coefficients 2 and d are just the dimensions of ∇^2 and r^{-d} in the usual sense. (Here and later we use "mass dimension" instead of "length dimension.") Thus D has a simple geometric meaning. This result, generalized to many-particle and field-theoretic systems, allows one to simplify the computation of the equation of state by eliminating the kinetic energy in favor of potentials or polynomials of fields whose coefficients are dimension-dependent.

In the one-particle problem just reviewed, the dimensions of the bound system are fixed by the dynamics. As long as the system has a (radius)³ much smaller than the confining volume Ω , it is not appropriate to ask about pressure, much less temperature.

Next consider a degenerate Fermi gas of N neutral non-interacting spin- $1/2$ particles confined to a volume Ω . As is well-known, when Ω is decreased, the average kinetic energy increases, so that one speaks of the "Fermi pressure." Computing the volume dependence of the energy gives for $-\partial E/\partial \Omega$ the zero-temperature equation of state

$$p = A \rho^{5/3} \quad (1.5)$$

where $\rho = N/\Omega$ is the density.

This simple case provides a nice example of subtleties connected with surface effects. Naive application of (1.2) would indicate that the kinetic energy also vanishes. Close investigation shows that the assumed Hermiticity used in (1.3) is equivalent to the vanishing of a surface term. This is automatically satisfied for bound states but not for the Fermi gas. The failure of formal Hermiticity is in fact a very common occurrence. The most familiar example¹ is the orbital angular-momentum component $L_z = -i\hbar \partial/\partial \phi$ satisfying $[\phi, L_z] = i\hbar$. Taking the diagonal matrix element in the angular-momentum eigenstate $|m\rangle$, Hermiticity of L_z would give $\langle m | [\phi, L_z] | m \rangle = 0$, generating a paradox. The resolution can be expressed as follows: L_z is Hermitian only in the space of periodic functions. The azimuth ϕ is continuous, not periodic, for the form of the assumed commutator. A calculation designed to verify the Hermiticity of L_z will generate a non-vanishing surface term which resolves the "paradox."

Now consider a system of N identical particles interacting through two-particle potentials $v(r_{ij})$. The Hamiltonian is

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{i < j} v_{ij} \quad (1.6)$$

where a, b run from 1 to N , and the N -particle virial operator is

$$D = \sum_i \mathbf{x}_i \cdot \mathbf{p}_i \quad (1.7)$$

D is not conserved (i.e., the system is not invariant under a scale transformation), as shown by

$$i[D, H] = -2T + \left[\sum_i \mathbf{x}_i \cdot \nabla_i \right] \sum_{k < l} v_{kl} \quad (1.8)$$

In order to simplify the potential energy, we use the identity

$$\sum_i \mathbf{x}_i \cdot \nabla_i = \frac{1}{N-1} \sum_{i<j} d_{ij} \quad (1.9)$$

$$d_{ij} = \mathbf{x}_i \cdot \nabla_i + \mathbf{x}_j \cdot \nabla_j$$

We can regard d_{ij} as the dilation operator for the pair ij . The contributing terms to $\sum d_{ij} \sum v_{kl}$ have either $(i,j) = (k,l)$ or one of i,j equal to one of k,l . Noting that $(d_{ij} + d_{ji})v_{ij}$ equals $d_{ij} v_{ij}$, we easily find

$$\begin{aligned} \frac{1}{N-1} \sum_{i<j} d_{ij} \sum_{k<l} v_{kl} &= \sum_{i<j} d_{ij} v_{ij} \\ &= \sum_{i<j} \mathbf{r}_{ij} \cdot \frac{\partial v}{\partial \mathbf{r}_{ij}} \end{aligned} \quad (1.10)$$

Taking an ensemble average (e.g., with density matrix $\rho \propto \exp(-\beta H)$), we can write (1.8) as

$$2\langle T \rangle = \sum_{i<j} \mathbf{r}_{ij} \cdot \frac{\partial v(\mathbf{r}_{ij})}{\partial \mathbf{r}_{ij}} + \left\langle \frac{dD}{dt} \right\rangle \quad (1.11)$$

If v is a simple power, as assumed in Eq. (1.2), we obtain

$$2\langle T \rangle = -d\langle V \rangle + \left\langle \frac{dD}{dt} \right\rangle \quad (1.12)$$

for the N -particle system. If the interactions are attractive, it is possible to have a bound N -particle system, in which case $\langle T \rangle = -d\langle V \rangle/2$ results. For a thermalized system, the temperature must also be sufficiently low so that continuum states (representing breakup) are not important, if the term $\langle dD/dt \rangle$ is to be dropped.

The energy of the system is

$$\langle H \rangle = \langle T \rangle + \langle V \rangle ;$$

$$E = \left\langle \sum_{i<j} \left(1 + \mathbf{r}_{ij} \cdot \frac{\partial}{\partial \mathbf{r}_{ij}} \right) v(\mathbf{r}_{ij}) \right\rangle + \frac{1}{2} \left\langle \frac{dD}{dt} \right\rangle \quad (1.13)$$

As for the one-particle problem, (1.13) is not an equation of state. However, if we enclose the system in a volume Ω , and adjust Ω so that interactions with the container influence the wave functions $\Psi_n(\Omega)$, (1.13) contains all the ingredients of the equation of state. It is necessary to include the interaction of the particles with the "walls" of the container by a term like $\sum_i V(r_i)$ (for a static wall). As shown by the example of the Fermi gas, careful evaluation of the term $\langle \dot{D} \rangle$ is necessary to obtain a correct $E(\Omega)$ and corresponding pressure $p = -\partial E/\partial \Omega$.

II. SCALING AND DILATIONS

In Section I we saw that the virial operator $\mathbf{x} \cdot \mathbf{p}$ measures the dimension of operators which are powers of x or p . A component of $\mathbf{x}(p)$ counts as dimension $-1(1)$

and a product $x^m p^n$ has dimension $n-m$. Such objects exhibit pure scaling behavior (in contrast to a function like $\exp(-x)$, for example). We now establish the necessary machinery² to describe scale transformations of functions and field operators.

As an example, consider the parabola $f(x) = x^2$. The value of the function f depends on the arbitrary convention about the scale, or ruler, used to label points on the x axis. Now suppose that we examine the same function using a ruler with spacings one-half the former. The same point now has (on the new ruler) the value $x' = 2x$ and the function $f'(x') = (x')^2 = 4x^2$. Although this gedanken measurement may seem like a hard way to express the idea that the dimension of x^2 is -2 , it leads naturally to the definition of a scaled function.

Suppose that the function $f(x)$ (tacitly implying a ruler defining the convention for the values of x) is observed using a new ruler whose values describing the same fixed point in space are $x' = \lambda x$. The scaled function observed in the primed system is defined by

$$f'(x') = \lambda^{-d} f(x) \quad (2.1)$$

In order to compare the transformed function f' to f at the same value of their arguments, we write

$$f'(x) = \lambda^{-d} f(\lambda^{-1} x) \quad (2.2)$$

Consider now the infinitesimal form of the transformation $f'(x) = U^{-1}(\lambda)f(x)U(\lambda)$, where $U(\lambda) = \exp(iD \ln \lambda)$. Setting $\lambda = 1 + \epsilon$, we find for $\epsilon \rightarrow 0$

$$\delta f = f'(x) - f(x) = i\epsilon[f(x), D] ;$$

$$\delta f = -\epsilon(d + x \cdot \partial)f ;$$

$$[f(x), D] = i[d + x \cdot \partial]f \quad (2.3)$$

Hence the local change in the form of the function induced by the scale transformation is composed of two parts: d comes from the multiplicative change of scale and $x \cdot \partial$ from the change in argument on the right-hand side of (2.2). By the symbol x we mean a vector, and $x \cdot \partial$ is the scalar $x \cdot \partial / \partial x$ with whatever metric is appropriate.

The particular structure (2.2) is the most natural in field theory, since unitary transformations of the form $\phi(x) \rightarrow \phi'(x) = U^{-1}\phi(x)U$ are most natural. (For conserved generators, U is independent of x .)

As is well-known,^{2,3} the Poincaré group generators are constructed from various moments of the energy-momentum tensor $\theta_{\mu\nu}$. (θ_{00} is the energy density, $\theta_{0i} = \theta_{i0}$ is the momentum density, θ^{ij} , $i, j = 1, 2, 3$ stresses. Our metric is $\text{diag } g_{\mu\nu} = (1, -1, -1, -1)$ for $\mu, \nu = 0, 1, 2, 3$.) The geometric scale transformation operator, although not conserved in general, is also a moment:^{3,4}

$$\begin{aligned} D &= \int d^3x \, x^\mu \theta_{\mu 0} \\ &= tH + \int d^3x \, x^i \theta_{i0} \end{aligned} \quad (2.4)$$

The final term in (2.4) is just the virial for a momentum density, and could be guessed by generalizing Eq. (1.7). The explicit time dependence of D is given by the trace θ^μ_μ

$$\begin{aligned}
\frac{dD}{dt} &= \int d^3x \left[\theta_{00} - x^m \frac{\partial \theta_{m0}}{\partial t} \right] \\
&= \int d^3x \left[\theta_{00} - x^n \frac{\partial}{\partial x^m} \theta_{mn} \right] \\
&= \int d^3x \left[\theta_{00} - \sum_n \theta_{nn} \right] .
\end{aligned} \tag{2.5}$$

using successively the conservation law $\partial_\mu \theta^{\mu\nu} = 0$ and integrating by parts. Hence we have

$$\frac{dD}{dt} = \int d^3x \theta^\mu_\mu(x) . \tag{2.6}$$

showing that the non-conservation of the dilation operator depends directly on the trace of the energy-momentum tensor. The field operators ϕ transform exactly as (2.3);

$$[\phi(x), D] = i(d + x \cdot \mu)\phi(x) . \tag{2.7}$$

In general, D is not conserved, so that $D(t)$ is at the same time as the variable $x = (t, \mathbf{x})$.

We remark that the form (2.4) depends on choosing $\theta_{\mu\nu}$ to be the "improved energy-momentum tensor," which has particularly advantageous renormalization properties. We give simple examples of canonical (i.e., prior to renormalization) dimension. Since $\int d^4x i\bar{\psi}\partial\psi$ and $\int d^4x \phi^4$ are dimensionless contributors to Lagrangians, we expect (correctly) that the Dirac field ψ has dimension $3/2$ and the spinless Bose field ϕ has dimension 1. ϕ^2 , ϕ^3 , and $\phi\bar{\psi}\psi$ have, respectively, dimensions 2, 3, and 4. Note that a mass term $m\bar{\psi}\psi$ (or $m^2\phi^2$) has scaling dimension 3 (or 2), even though the "engineering" dimension is 4. When the theory is renormalized,⁴ the dimensions shift from their canonical values and are called anomalous. In this paper we shall only use canonical dimensions.

III. CONNECTION BETWEEN THE EQUATION OF STATE AND THE ENERGY-MOMENTUM TENSOR

An equation of state, say $p = p(\epsilon)$ with p the pressure and ϵ the energy density, relates components of the expectation values of the energy-momentum tensor. If ρ is the density matrix (most simply $\propto e^{-\beta H}$), the observed energy-momentum tensor $T_{\mu\nu}$ is

$$T_{\mu\nu}(x) \equiv \text{Tr}(\rho \theta_{\mu\nu}(x)) \equiv \langle \theta_{\mu\nu}(x) \rangle . \tag{3.1}$$

The diagonal components are

$$\begin{aligned}
\epsilon &= T_{00} = \langle \theta_{00}(x) \rangle \\
p_{xx} &= T_{xx} = \langle \theta_{xx}(x) \rangle \\
p_{yy} &= T_{yy} = \langle \theta_{yy}(x) \rangle \\
p_{zz} &= T_{zz} = \langle \theta_{zz}(x) \rangle .
\end{aligned} \tag{3.2}$$

Most current work assumes isotropy ($p_{xx} = p_{yy} = p_{zz}$), although this approximation is probably wrong in applications to heavy ion collision geometries.

In order to compare (3.2) with the notations of relativistic fluid mechanics, we imagine the expressions (3.2) to be evaluated in the local rest frame. The four-velocities themselves are determined⁵ from $T^{\mu\nu}$. The classical tensor for a perfect relativistic fluid is

$$T_{\mu\nu}^{cl} = (\epsilon + p) u_\mu u_\nu - p g_{\mu\nu} \quad (3.3)$$

where u_μ is the four-velocity ($u^\mu u_\mu = 1$) and ϵ, p are Lorentz scalar densities, i.e., by definition $\epsilon'(x') = \epsilon(x)$, $p'(x') = p(x)$ for Lorentz transformations $x' = \Lambda x$ to the primed frame. The expression

$$\epsilon \equiv u^\mu T_{\mu\nu}^{cl} u^\nu \quad (3.4)$$

exhibits both the scalar character of ϵ and $\epsilon = T_{\alpha\alpha}^{cl}$ in the local rest frame. Similarly the pressure field is given by projecting $T_{\mu\nu}$ on directions transverse to u_μ :

$$\begin{aligned} p_{\mu\nu} &\equiv \Delta_{\mu\rho} T_{\alpha\beta}^{cl} \Delta_{\sigma\nu} \\ \Delta_{\mu\nu} &= g_{\mu\nu} - u_\mu u_\nu \\ u^\mu \Delta_{\mu\nu} &= 0 \\ \Delta_{\mu\lambda} \Delta_\nu^\lambda &= \Delta_{\mu\nu} \end{aligned} \quad (3.5)$$

In the local rest frame $\Delta_{\mu\nu} = g_{ij} = -\delta_{ij}$ ($i, j = 1, 2, 3$ only) and zero otherwise. Hence only p_{ii} terms survive. The pressure scalar is then

$$p = -\frac{1}{3} p_\mu^\mu \quad (3.6)$$

The trace of Eq. (3.3) is

$$T_\mu^\mu = \epsilon - 3p \quad (3.7)$$

The "perfect" EOS.

$$p = \frac{1}{3} \epsilon \quad (3.8)$$

easily proven for free relativistic particles (masses $\rightarrow 0$), is seen to be equivalent to the vanishing of the trace. We shall extend this result to assert that scale invariance implies the perfect EOS (3.8) even in the presence of interactions. Unfortunately, we shall find that genuine scale invariance is hard to come by because of trace anomalies.

Evaluating (3.1) in the local rest frame (assuming this concept makes sense) gives

$$T_\mu^\mu = \epsilon - 3p \quad (3.9)$$

If we can evaluate this equation as a function of temperature, for example, the corrections to (3.8) are all contained in T_μ^μ . Notice that formal scale invariance $\theta_\mu^\mu = \theta_{00} - \sum_i \theta_{ii} = 0$ is an operator identity version of (3.8).

Here we restrict our attention to LTE (local thermodynamic equilibrium), although attempts have been made⁵ to obtain closure without an LTE EOS. We especially stress that conceptually, a perfect EOS (Eq. (3.8)) does not imply the absence (or weakness) of interactions. The simplest example is the scalar field with

mass m and ϕ^4 interaction. The Lagrangian density is $1/2(\partial\phi)^2 - 1/2 m^2\phi^2 + \lambda\phi^4$. The trace theorem (see Eq. (3.12) below) gives

$$\langle\theta_{00}\rangle = \sum_i \langle\theta_{ii}\rangle = m^2\langle\phi^2\rangle \quad (3.10)$$

Note that this expression does not depend on the (scale invariant) coupling $\lambda\phi^4$. We can let $m^2 \rightarrow 0$, obtaining the perfect EOS (3.8) with $\lambda \neq 0$.

In order to generalize and to put the foregoing in context, we next explain the "trace theorem," which expresses the trace in terms of scale-breaking operators (i.e., those with dimension not equal to 4). We refer to Refs. 2 and 4 for the nontrivial details. The procedure is as follows: decompose $\theta_{\mu\nu}$ into $\bar{\theta}_{\mu\nu}$ plus other terms. $\bar{\theta}_{\mu\nu}$ is obtained from $\theta_{\mu\nu}$ by discarding terms with $d \neq 4$ and using the equations of motion to eliminate time derivatives. $\bar{\theta}_{\mu\nu}$ is not a true tensor and the dimensional analysis is done with the frame-dependent operator $D(t)$. The trace theorem states⁶ that if the energy density has the form

$$\theta_{00} = \bar{\theta}_{00} + \sum_d w_d \quad (3.11)$$

where w_d are scalars of dimension d , then the trace is

$$\theta^\mu_\mu = \sum_d (4-d) w_d + \text{anomalies} \quad (3.12)$$

This formula (perhaps senza anomalies) shares with the virial theorem of Section I the elimination of kinetic energy in terms of the interaction terms. The anomaly term is non-canonical (coming from fermion loops) and must be added for QCD.

Consider some examples of (3.12). In QED, the mass term $w_3 = m\bar{\psi}\psi$ and $\theta^\mu_\mu = (4-3)w_3 = m\bar{\psi}\psi$. For a scalar field theory with $\alpha = 1/2(\partial\psi)^2 - 1/2 m^2\phi^2 + g\phi^3 + \lambda\phi^4$, we find $w_2 = 1/2 m^2\phi^2$, $w_3 = -g\phi^3$ and $\theta^\mu_\mu = m^2\phi^2 - g\phi^3$. In the case of interest here, QCD, we have⁷

$$\theta^\mu_\mu = \sum_q m_q \bar{q}q - \frac{b\alpha}{8\pi} F^\alpha_{\mu\nu} F^{\alpha\mu\nu} \quad (3.13)$$

The anomaly term⁷ has $\alpha = g^2/4\pi$, $b = 11/3 N_c - 2/3 N_f$; $F^\alpha_{\mu\nu}$ is the usual field tensor and $F^2 = (E^2 - B^2)$. N_c , N_f are the number of quark colors and flavors. Equation (3.13) shows that even when the Lagrangian is scale invariant, the trace does not vanish. Of course, a length has crept into the theory via the renormalization process.

Suppose that we apply (3.13) to an equilibrium QCD plasma. The equation of state is

$$\epsilon - 3p = \sum_q \langle\bar{q}q\rangle m_q - k \langle F^\alpha_{\mu\nu} F^{\alpha\mu\nu} \rangle \quad (3.14)$$

where $k = b\alpha/8\pi$. First consider the mass term (as in QED where there is only $m\bar{\psi}\psi$ on the right-hand side). If we evaluate $\langle\bar{\psi}\psi\rangle$ in lowest order, we easily find

$$\epsilon - 3p = \frac{m^2}{\pi^2} \int_m^\infty \frac{d\sqrt{x^2 - m^2}}{\exp(\beta(x - \mu) + 1)} \quad (3.15)$$

where $\beta = 1/kT$ and μ is the chemical potential. Equation (3.15) is easily derived from the free-particle thermodynamic expressions for ϵ and p .

For u and d quarks, the right-hand side of (3.15) is negligibly small when $(m_q\beta)^2 = (m_q/T)^2 \ll 1$. We do not know the value of $\langle F^2 \rangle$, however. Clearly (3.14) can be evaluated in perturbation theory, the right-hand side being expressed as a function of temperature.

In Ref. 8 we argued that the anomaly term might be small, thereby justifying a nearly ideal EOS and hydrodynamic sound velocity $1/\sqrt{3}$. Presently we are not confident that this is correct. Comparison with recent lattice gauge theory calculations is under way.

The equation of state allows closure of the equations of motion for relativistic hydrodynamics. Since LTE cannot be expected for all space-time domains of relativistic heavy ion collisions, it is worth exploring⁵ the relation of $T_{\mu\nu}$ to the phase-space distributions of relativistic field theory.⁹⁻¹⁰ Since much of the dynamical content of the hydrodynamic equations is local energy-momentum conservation, much of the behavior should survive the loss of LTE. Thus far, the problem has been how to obtain closure of generalized kinetic-hydrodynamic equations. Luckily there is a lot of interesting theoretical work to be done on these problems.

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