MASTER

CONF-820447--2

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36.

LA-UR--82-1092

DE82 014042

TITLE: COORDINATE SYSTEM CONTROL: ADAPTIVE MESHES

AUTHOR(S): J. U. Brackbill

DISCLAIMER

This document contains information which has been classified as CONFIDENTIAL. It is the property of the United States Government and is loaned to you. It and its contents are not to be distributed outside your agency. If you are not an authorized recipient, please return this document to the person from whom you received it. If you are an authorized recipient, please return this document to the person from whom you received it when it is no longer needed. This document is not to be used for the purpose of reproducing or transmitting information in any form or by any means, electronic or mechanical, including photocopying, recording, or by any information storage and retrieval system, without prior written permission from the United States Government.

SUBMITTED TO: Numerical Generation of Curvilinear Coordinate Systems Symposium,
Nashville, TN, April 1982

NOTICE

PORTIONS OF THIS REPORT ARE ILLEGIBLE.
It has been reproduced from the best available
copy to permit the broadest possible availability.

MN ONLY

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

By acceptance of this article, the publisher recognizes that the U.S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this contribution or to allow others to do so, for U.S. Government purposes.

The Los Alamos National Laboratory requests that the publisher identify this article as work performed under the auspices of the U.S. Department of Energy.

Los Alamos Los Alamos National Laboratory
Los Alamos, New Mexico 87545

COORDINATE SYSTEM CONTROL: ~~ADAPTIVE MESHES~~

J. U. BRACKBILL

Applied Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

INTRODUCTION

In the numerical solution of transient fluid flow problems, the flow variables are calculated at a finite number of points. Typically, the points form ordered arrays which are joined together in some systematic way to form a computation mesh. On this mesh, the flow equations are approximated by finite differences, which are then marched in time.

It is intuitively obvious that the greater the number of points in the computation mesh, the more accurate the numerical solution will be. For one thing, the accuracy of the difference equations depends on the fineness of the mesh. As importantly, the accuracy of the solution depends on the resolution of flow gradients. Where the flow gradients are largest, the absolute error in the difference approximation to derivatives is largest.¹ In addition, more numerical diffusion must be introduced to maintain sufficient smoothness of the solution for nonlinear stability.²

When the flow gradients vary from place to place and the mesh spacing is constant, the numerical errors are largest where flow gradients are least well resolved. When mesh points can be added, errors are reduced most efficiently by adding points only in regions of strong gradients.³ Similarly, when the number of mesh points is fixed, greater overall accuracy can often be obtained by concentrating mesh points where gradients are strong and dispersing them where gradients are weak.¹

An algorithm which concentrates and disperses points automatically based on the numerical solution of the flow equations is described as adaptive. Here, an adaptive algorithm is derived from a minimum principle, and its application to transient flow problems in two dimensions is described.

The major topics discussed are the variational formulation of a mesh generator with interior control, its use in adaptive, time-dependent problems, and the solution of the inverse problem to allow adaptive zoning of arbitrary initial meshes. Some of the discussion is reproduced from Ref. 1, but the application to time-dependent problems is emphasized.

REFINEMENT VARIATIONAL FORMULATION OF THE MESH GENERATOR

For the solution of finite difference equations on a computation mesh, the data is typically stored in ordered arrays of numbers, $\phi(i,j)$ in which the indices $i = 1, \dots, M$; $j = 1, \dots, N$, give not only the location of the data in computer memory, but also the physical relationship between the data at one vertex $x(i,j)$ and another, $x(i',j')$.

In formulating the mesh generator problem mathematically, it is useful to view the mesh, whose vertices are $x(i,j)$, as the image of a mapping $x(\xi, \eta)$ in which the points corresponding to integer values of the natural coordinates, ξ and η , are the mesh vertices. The image of the mesh in natural coordinates is a uniform, rectilinear mesh with mesh spacing $\Delta\xi = \Delta\eta = 1$. A mesh generator determines the mapping $x(\xi, \eta)$.

Obviously, the differential properties of the mapping are reflected in the properties of the computation mesh. For example, $[(\partial x / \partial \xi)^2 + (\partial y / \partial \xi)^2]^{1/2}$ along a level curve of η is related to mesh spacing between vertices with the same index j . Similarly, the volume of computational cells is related to the Jacobian, J , of the mapping,

$$J = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} ,$$

and the orthogonality of the mesh is related to the scalar, $\nabla \xi \cdot \nabla \eta$, which is zero when conjugate lines of the mesh are orthogonal.

Integrals over the computation mesh can be written which measure these properties of the mapping. The global smoothness of the mapping (the variation in mesh spacing along level curves of ξ and η) is measured by the integral,

$$I_0 = \int_D [(\nabla \xi)^2 + (\nabla \eta)^2] dv .$$

The orthogonality of the mapping is measured by,

$$I_0' = \int_D (\nabla \xi \cdot \nabla \eta)^2 J^3 dv$$

and the weighted volume variation is measured by,

BEGIN HERE

$$I_V = \int_D wJ \, dV ,$$

where $w = w(x,y)$ is given.

The smoothest mapping can be obtained by minimizing I_S , the most orthogonal mapping by minimizing I'_0 , and the mapping with specified variation of J by minimizing I_V .^{1,4}

A useful mesh generator results when a combination of I'_0 , I_V and I_S is minimized as in the penalty method.⁵ That is, the integral I is minimized,

$$I = I_S + \lambda_V I_V + \lambda'_0 I'_0 ,$$

where λ_V and λ'_0 are positive constants of $O(1)$.

SOLUTION OF THE VARIATIONAL PROBLEM IN TWO DIMENSIONS

To derive the Euler equations for the variational problem formulated in the preceding section, it is first convenient to interchange dependent and independent variables using the relations.

$$\xi_x = \frac{+y_\eta}{J} , \quad \xi_y = \frac{-x_\eta}{J} , \quad \eta_x = \frac{-y_\xi}{J} , \quad \eta_y = \frac{+x_\xi}{J} .$$

After interchanging variables, the smoothness measure can be written,

$$I_S = \int_1^M \int_1^N d\xi d\eta \frac{(x_\xi^2 + x_\eta^2 + y_\xi^2 + y_\eta^2)}{J} ,$$

for which the corresponding Euler equations are,

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \xi} \frac{\partial}{\partial x_\xi} - \frac{\partial}{\partial \eta} \frac{\partial}{\partial x_\eta} \right) \left[\frac{(x_\xi^2 + x_\eta^2 + y_\xi^2 + y_\eta^2)}{J} \right] = 0 ,$$

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial \xi} \frac{\partial}{\partial y_\xi} - \frac{\partial}{\partial \eta} \frac{\partial}{\partial y_\eta} \right) \left[\frac{(x_\xi^2 + x_\eta^2 + y_\xi^2 + y_\eta^2)}{J} \right] = 0 .$$

The measure of volume variation, after interchanging dependent and independent variables, can be written,

$$I_v = \int_1^M \int_1^N dx dy w J^2 ,$$

for which the Euler equations are,

$$\frac{\partial w}{\partial x} J^2 - 2 [J(w_{\xi} y_{\eta} - w_{\eta} y_{\xi}) + w(J_{\xi} y_{\eta} - J_{\eta} y_{\xi})] = 0 ,$$

and

$$\frac{\partial w}{\partial y} J^2 - 2 [J(x_{\xi} w_{\eta} - x_{\eta} w_{\xi}) + w(x_{\xi} J_{\eta} - x_{\eta} J_{\xi})] = 0 .$$

Similarly, the orthogonality measure, I_o , can be written

$$I_o = \int_1^M \int_1^N dx dy (x_{\xi} x_{\eta} + y_{\xi} y_{\eta})^2 ,$$

for which the Euler equations are

$$b_{01} x_{\xi\xi} + b_{02} x_{\xi\eta} + b_{03} x_{\eta\eta} + a_{01} y_{\xi\xi} + a_{02} y_{\xi\eta} + a_{03} y_{\eta\eta} = 0 ,$$

and

$$a_{01} x_{\xi\xi} + a_{02} x_{\xi\eta} + a_{03} x_{\eta\eta} + c_{01} y_{\xi\xi} + c_{02} y_{\xi\eta} + c_{03} y_{\eta\eta} = 0 .$$

The mesh generator equations can be written in a form convenient for computation by collecting the coefficients of the highest derivatives of x and y with respect to ξ and η and writing,

$$b_1 x_{\xi\xi} + b_2 x_{\xi\eta} + b_3 x_{\eta\eta} + a_1 y_{\xi\xi} + a_2 y_{\xi\eta} + a_3 y_{\eta\eta} = -J^2 \frac{1}{2w} \frac{\partial w}{\partial x} , \quad (1)$$

$$a_1 x_{\xi\xi} + a_2 x_{\xi\eta} + a_3 x_{\eta\eta} + c_1 y_{\xi\xi} + c_2 y_{\xi\eta} + c_3 y_{\eta\eta} = -j^2 \frac{1}{2w} \frac{w}{y} \quad (2)$$

The coefficients are reproduced here from Ref. 1 for completeness:

$$a = a_0 + \lambda_v a_v + \lambda_o a_o \quad ,$$

$$b = b_0 + \lambda_v b_v + \lambda_o b_o \quad , \text{ and}$$

$$c = c_0 + \lambda_v c_v + \lambda_o c_o \quad .$$

$$a_{01} = -Ax \quad ; \quad b_{01} = Bx \quad ; \quad c_{01} = Cx$$

$$a_{02} = 2AB \quad ; \quad b_{02} = -2BB \quad ; \quad c_{02} = -2CB$$

$$a_{03} = -AY \quad ; \quad b_{03} = BY \quad ; \quad c_{03} = CY$$

where

$$A = x_\xi y_\xi + x_\eta y_\eta \quad , \quad B = y_\xi^2 + y_\eta^2 \quad , \quad C = x_\xi^2 + x_\eta^2 \quad .$$

and

$$a = \frac{x_\eta^2 + y_\eta^2}{j^3} \quad , \quad b = \frac{x_\xi x_\eta + y_\xi y_\eta}{j^3} \quad , \quad c = \frac{x_\xi^2 + y_\xi^2}{j^3} \quad .$$

$$a_{v1} = \dots \quad , \quad b_{v1} = y_\eta^2 \quad , \quad c_{v1} = x_\eta^2 \quad .$$

$$a_{v2} = x_\xi^2 x_\eta + x_\eta y_\xi \quad , \quad b_{v2} = -2y_\xi y_\eta \quad , \quad c_{v2} = -2x_\xi x_\eta \quad .$$

$$a_{v3} = -x_\xi y_\xi \quad , \quad b_{v3} = y_\xi^2 \quad , \quad c_{v3} = x_\xi^2 \quad .$$

$$a_{01} = x_n y_n, \quad b_{01} = x_n^2, \quad c_{01} = y_n^2.$$

$$a_{02} = x_\xi y_\eta + x_\eta y_\xi, \quad b_{02} = 2(x_\xi x_\eta + x_\eta^2), \quad c_{02} = 2(x_\xi x_\eta + 2y_\xi y_\eta).$$

The generator equations form a set of coupled, quasi-linear, partial differential equations of second order which are only slightly more complex than in the original Winslow algorithm.

Algebraic equations at each node result from the substitution of the differences for derivatives. The system of equations is solved by a Jacobi iteration in which the values of $x_{i,j}$ and $y_{i,j}$ are treated as parameters. The iterative solution of these equations will give new values of $x_{i,j}$ and $y_{i,j}$ which satisfy the Euler equations above. A more complete discussion of the numerical solution of the Euler equations is given in Refs. 1 and 6.

THE ADAPTIVE MESH

By choosing an appropriately defined weight function, $w(x,y)$, the mesh generator can be made part of an algorithm to adapt a computation mesh dynamically to data generated by the solution of finite difference equations.

The weight function is chosen in such a way that numerical errors are reduced. An obvious approach is to minimize the truncation error for a particular difference approximation. While this may be optimal, detailed analysis of a given difference equation usually yields a very complex expression for the truncation error whose minimization may have no clear connection to the variational forms above.

However, a simpler approach yields a practical, yet useful, algorithm. As has been noted already, numerical errors are reduced significantly when flow gradients are well resolved. When there are sufficient mesh points in each gradient length of the function being represented, the smoothness of the function assures that truncation errors and numerical diffusion necessary for non-linear stability are reasonable.

The choice of weight function, w , is simplified by noting that as λ_v becomes large, the solution to the mesh generator can be written,

$$wJ^2 = \text{const.}$$

Where ϕ , the function being represented, is positive, a weight function with correct dimensionality is given by,

$$w = \left(\frac{1}{J} \nabla \phi \right)^4$$

(3)

so that wJ^2 is dimensionless. Thus, where w is large, corresponding to steep gradients, J will be small, corresponding to increased resolution.

Although the choice of weight function is straightforward, there are a number of problems of a practical nature which must be solved. The gradient of ϕ is calculated from the numerical data by finite differences and may accentuate the roughness of the data. Variations in mesh size occur in a discrete way in moving from one cell to the next and may increase truncation error. Finally, there are constraints on maximum and minimum zone sizes. The minimum zone size determines the time step, and thus the cost of a calculation. The maximum zone size cannot be greater than the entire mesh.

To address these problems, the algorithm must be modified. The weight function calculated from the data is smoothed to spread the influence of a single data point over a region of the mesh. The coefficient, λ_v , is typically chosen to be $O(1)$ so that the influence of the smoothness integral results in a smooth variation in cell size from zone to zone. Finally, w is scaled between maximum and minimum values corresponding to prescribed maximum and minimum zone sizes.

A NUMERICAL EXAMPLE OF ADAPTIVE ZONING

A simple example illustrating the effect of adaptive zoning in two dimensions is afforded by considering the steady solution of a convection-diffusion equation in two dimensions,

$$\nabla \cdot (\phi \underline{U}) - \mu \nabla^2 \phi = 0$$

where $\underline{U} = \underline{U}(r)$ is given, and μ is a positive constant. On an infinite domain, the first integral of the equation is simply,

$$\underline{U} - \mu \nabla \phi = 0$$

One can see clearly that when μ is small, where $\underline{U}$ is finite $\nabla \phi / \phi$ will be large. Choosing $\underline{U}(r)$ to be a function which is non-zero only in an annulus of width δr at r_0 will result in the formation of a boundary layer at r_0 .

To demonstrate the increased accuracy of difference equations with adaptive zoning the solution ϕ is given analytically, and the error is defined as

value of the RHS of the first integral when ϕ is approximated by a finite difference.

The result of a numerical calculation on a square domain, $0 \leq x \leq 1$, $0 \leq y \leq 1$, with $r = 0$ at $x = y = 1/2$, $r_0 = 1/4$, and $U_0 = 12$ is shown in Fig. 1. Where there is rapid variation of ϕ , the cells in the adapted mesh are small in the figure.

An important result is shown in Fig. 2a where the variation of I_V , I_S and the error, ϵ , with λ_V is shown.

Clearly, minimizing I_V reduces the error since both I_V and ϵ decrease together as λ_V increases. The decrease in I_V results in a small increase in I_S corresponding to a departure from Winslow's mesh. The decrease in ϵ as λ_V increases from 0 to 1 is dramatic; from 25% to 3.5%. The conclusion is that with increasing adaptivity, substantial increases in accuracy are obtained. The effect of varying the size of the mesh with and without adaptivity, shown in Fig. 2b, indicates that equivalent accuracy is obtained with the adaptive mesh with one tenth as many cells as in the nonadaptive mesh.

ADAPTIVE ZONING OF TIME-DEPENDENT PROBLEMS

Time dependent flow calculations can be adaptively zoned simply and cheaply by the following algorithm. At the beginning of each time step, the weight function is calculated from the numerical data and processed as described above. The coordinates of the mesh are stored, and new coordinates are calculated from the mesh generator and separately stored. Either by interpolation or by the solution of transport equations in conservation form,⁷ the data are transferred from the old mesh to the new one, and the new mesh replaces the old mesh in memory.

The algorithm is explicit. The new mesh is generated with data from the previous time step and does not anticipate changes which may occur during the course of the time step. This splitting of time advancement of the flow equations and time advancement of the mesh is more accurate when the solution changes slowly from time step to time step, as expressed by the inequality,

$$\frac{1}{\delta} \frac{\partial \phi}{\partial t} \Delta t < 1$$

One might object that this condition may be violated locally, especially in singular perturbation problems. For example, a shock may move a distance comparable to its width in a time step resulting in large changes in the solution in the neighborhood of the shock. With this in mind, one can understand

where $\frac{1}{W} \frac{\partial W}{\partial t} \Delta t < 1$. The value of smoothing the weight function so that the fine zoning induced by strong gradients in the shock front extends some distance to either side of the shock front. Thus, a more relevant condition for the time step is given by the rate of change of the weight function,

$$\frac{1}{W} \frac{\partial W}{\partial t} \Delta t < 1.$$

When the weight function is varying too rapidly, additional smoothing of the data is often required (with some consequent loss in adaptivity).

When the inequality above is satisfied, the displacement of the mesh over a time step is relatively small, and the mesh at the beginning of the time step very nearly satisfies the mesh generator equations. After only two or three iterations, the solution is usually sufficiently good that further improvement is unnecessary. There is also a physical argument why this is so. The change in the flow solution over a time step at a given point depends only on the flow variables at neighboring points. So that the mesh generator can keep on the same time scale as the flow solution, the domain of dependence for each point in the mesh must be at least as large. Since each iteration of the mesh generator increased the domain of dependence by one cell in every direction, two or three iterations are sufficient.

The value of adaptive gridding in time-dependent flow problems can be identified by considering the problem of numerical diffusion. In all flow calculations, the approximation to convective transport introduces numerical diffusion. The amount of diffusion can be estimated by truncation error analysis.² For example, when convective transport in one dimension is approximated by an implicit difference equation,⁸

$$n+1 \phi_i = n \phi_i - \frac{\delta t}{\Delta x} (n+1 (\phi u)_{i+1/2} - n+1 (\phi u)_{i-1/2}),$$

where $n+1 \phi_i = \phi(x_i, (n+1)\delta t)$, the differential equation approximated by the difference equation is

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x} (\phi u) = \kappa' \frac{\partial^2 \phi}{\partial x^2},$$

where κ' is the numerical diffusion,

$$\kappa' = \frac{\delta t}{2} u^2 - \frac{\delta x^2}{2} \frac{\partial u}{\partial x}.$$

When it is required for accuracy's sake that $u \delta t \ll \delta x$, κ' can be written,

$$\kappa' = \frac{1}{2} \frac{\delta x^2}{\delta t} \left(1 - \frac{\delta x}{u} \frac{\partial u}{\partial x} \right).$$

Clearly, when flow gradients are not resolved and $\delta x (\partial u / \partial x) > u$, κ' is negative and numerical instability results.² Analysis of other, standard difference approximations yields similar results.

There are a number of approaches to maintaining stability. Historically, an explicit numerical viscosity has been added to the flow equations to smooth gradients and make them resolvable. However, the consequent reduction of maximum, representable Reynolds numbers is currently considered unacceptable. Several other approaches which do not introduce excessive additional diffusion are now available. One can calculate the negative diffusion locally from truncation error analysis, and compensate by adding a comparable positive diffusion. One can control diffusion locally using flux corrected transport. Alternatively, one can improve the resolution of flow gradients through adaptive gridding. In this case adaptive gridding not only increases the accuracy of numerical calculations, but also the stability. Of course, adaptive gridding is compatible with flux corrected transport.⁹

The effect of adaptive gridding is illustrated by the numerical solution of Fisher's equation in a moving frame,

$$\frac{\partial \phi}{\partial t} + \underline{U} \cdot \nabla \phi = \kappa \nabla^2 \phi + \phi(1 - \phi)/\tau,$$

where $\underline{U}$ is the frame velocity, κ the diffusion coefficient, and τ the reaction time. This single equation contains many of the effects modelled by a system of equations describing diffusion of chemically reacting materials.

In one dimension, the solution to Fisher's equation corresponds to a steady front moving with speed $v = (\kappa/\tau)^{1/2}$. Across the front, whose thickness is proportional to $\kappa^{1/2}$, ϕ increases from zero to one.¹⁰

Since Fisher's equation will be solved on a moving grid, it is convenient to integrate the equation over a volume whose boundary is moving with local velocity $\underline{U}$.

nature of the ~~numerical diffusion~~, the reduction results from better resolution of the flow gradients.

ADAPTIVITY FOR ARBITRARY MESHES: THE INVERSE PROBLEM

To apply adaptivity to a non-uniform mesh, one must solve the following inverse problem: consider a mesh given by the solution of the mesh generator equations with $\lambda_v = 0$. This mesh satisfies the generator equations with $\lambda_v \neq 0$ only for some particular $w(x,y)$, which must be calculated.

In a typical problem, one has no a priori knowledge of the location of boundary layers or shocks. However, the boundaries of the domain or embedded structures may be represented most conveniently using the mesh generator with $\lambda_v = 0$, or by a simple functional mapping $x(\xi, \eta)$, $y(\xi, \eta)$. To adaptively zone a calculation using this mesh as the initial mesh, one must calculate the corresponding $w_1(x,y)$.

The weight function $w_1(x,y)$ can be calculated from the Euler equations most conveniently when they are written in a form which explicitly displays their dependence on w and its derivatives. This form can be derived from Eqs. (1) and (2) by collecting coefficients of w , $\partial w / \partial x$ and $\partial w / \partial y$ to obtain,

$$S_1 + wR_1 + \lambda_v J^2 \frac{\partial w}{\partial x} = 0 \quad ,$$

and

$$S_2 + wR_2 + \lambda_v J^2 \frac{\partial w}{\partial y} = 0 \quad .$$

where S_1 , S_2 , R_1 and R_2 are easily evaluated using the definitions of the coefficients and are functions of the given mesh coordinates.

These equations are easily integrated along level curves of the computation mesh. Noting that w_F and w_n are given by the chain rule, the derivatives can be written,

$$w_F = -\{(S_1 + wR_1)x_F + (S_2 + wR_2)y_F\} / \lambda_v J^2 \quad , \quad (4)$$

and

$$w_n = -\{(S_1 + wR_1)x_n + (S_2 + wR_2)y_n\} / \lambda_v J^2 \quad . \quad (5)$$

On level curves of the mesh where ξ is constant, one integrates the equation for w_n , and vice versa for mesh lines where n is constant. So that the resulting w is single valued, one must require that the difference equations satisfy the condition,

$$\langle w_{\xi n} \rangle - \langle w_{n\xi} \rangle = 0,$$

where the brackets denote difference equations.

A simple example taken from a recent calculation of magnetic reconnection in the earth's magnetosphere is shown in Fig. 7. The mesh is generated in the upper half plane with level curves of ξ corresponding to nested ellipses whose eccentricity increases from zero at the center to some large value on the outside. In addition, the distance between ellipses increases geometrically from the center outward.

With $\lambda_v \neq 0$ and w calculated from Eqs. (4) and (5) above, the mesh which satisfies the generator equations is as shown. With w equal to a constant, the mesh would collapse onto the center of curvature.⁶

ADAPTIVE ZONING FOR MOVING BOUNDARY CALCULATIONS

In many moving boundary calculations, the ability to redistribute points along the boundary increases the accuracy and reliability of the calculations. For example, in the modeling the Rayleigh-Taylor instability at the interface between two fluids regions of large curvature may develop which require additional points to resolve. When the total number of points along the interface is fixed, the points may be redistributed along the curve using the adaptive mesh generator.

Where the curve is parameterized by s , so that $x = x(s)$, $y = y(s)$ where $s = s(\xi)$, the analogue of the smoothness integral along the curve may be written,

$$I_s = \int \xi_s^2 ds.$$

To alter the distribution of points along the boundary, the integral,

$$I_R = \int w s_\xi^2 d\xi,$$

is minimized with given weight function $w(s)$. When w is proportional to the curvature

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \underline{F} = - \int (\underline{h} \cdot \underline{F}) d\mathbf{s} + \int (\phi(1-\phi)/\tau) dV,$$

where $\underline{U}$ is the displacement of the grid over a time step and where $\underline{F}$, the transport across the surface of the moving control volume due to convection and diffusion is given by,

$$\underline{F} = \{(\underline{U}_0 - \underline{U}) - \kappa \nabla\} \phi$$

As noted by Thompson, et al.,⁶ conservation form is preserved in natural coordinates,

$$\nabla \cdot \underline{F} = \{(F_1 \eta_n - F_2 \xi_n)_\xi + (F_1 \xi_\eta + F_2 \eta_\xi)_\eta\} / J$$

where F_1 and F_2 are the x and y components of $\underline{F}$. In this form, difference equations in natural coordinates are easily derived.

Using an algorithm which combines donor and interpolated donor cell differencing in the correct proportion to guarantee positive diffusion,⁷ Fisher's equation is solved on the domain $0 \leq x \leq L$, $0 \leq y \leq L$ with $\phi = \epsilon > 0$ in a small neighborhood of $(x,y) = (0.2L, 0.2L)$ initially, and $\underline{U} = (v,v)$, where $v = (\kappa/\tau)^{1/2}$, a circular front expands to eventually fill the domain. With $(\kappa/\tau)^{1/2} = 10^{-2}$, the thin front shown in Fig. 4 results at $t = 0.41/(\kappa/\tau)^{1/2}$.

With an Eulerian mesh, the results with a mesh of 625 cells shown in Fig. 3 are typical of a calculation with significant numerical diffusion. The speed of the front is greatest toward the lower left corner, and the front actually overtakes the incoming flow and stops only at the boundary. When the mesh size is increased to 5625 cells, the results shown in Fig. 4 are satisfactory. The speed of propagation is approximately correct and the shape of the front is circular.

With an adaptive mesh and weight function given by Eq. (1), the results with 625 cells are shown in Figs. 5 and 6, and are similar to the finely zoned Eulerian calculation with nearly ten times as many cells. The speed of propagation is nearly correct, and the shape is roughly circular. (With finer zoning, there is further improvement as one might expect.)

With the same number of zones, evidently the adaptively zoned calculation introduces less numerical diffusion than the Eulerian one. Because of the

$$\frac{1}{R^2} = \left(\frac{\partial^2 x}{\partial s^2} \right)^2 + \left(\frac{\partial^2 y}{\partial s^2} \right)^2$$

points along the curve are concentrated in regions of greatest curvature.

The Euler equations for combined smoothing and curvature resolution terms can be written,

$$10 = s_{\xi\xi} + \frac{2\lambda_R}{R^2} \left(s_{\xi\xi} - \frac{1}{R} R_{\xi} s_{\xi} \right)$$

A simple numerical example of the effect of adaptive zoning on an interface is shown in Fig. 8. In this example, points are distributed at equal intervals along the curve $y = \sin^4 x$ when $\lambda_R = 0$, and are concentrated where the curvature is largest when $\lambda_R = 100$. Note that the area under the curves is the same. Displacements of points along the curves are constrained to preserve the area under the curve.

CONCLUSIONS

The discussion of the formulation and properties of the adaptive mesh algorithm is as complete as space permits. Some additional details and examples are given in Refs. 1 and 9. Other aspects of the method, especially regarding the use of adaptive gridding in realistic calculations, are described in Refs. 11 and 12.

As should be clear from the discussion, adaptive gridding is most useful for singular perturbation problems where localized regions with large gradients develop spontaneously. Often, choosing the correct weight function is not trivial, and, sometimes, it is appropriate to cause the mesh to respond to structure in several dependent variables simultaneously.

An interesting new direction for adaptive meshes is described in Ref. 12. There, an adaptively zoned, particle-in-cell method is presented. Numerical diffusion of material properties is eliminated, and changes in scale are accommodated. The method appears to have significant potential for multi-material calculations.

ACKNOWLEDGEMENTS

Work performed under the auspices of the US Department of Energy.

REFERENCES

1. J. U. Brackbill and J. S. Saltzman, Adaptive Zoning for Singular Problems in Two Dimensions, *J. Comp. Phys.*, to be published.
2. C. W. Hirt, *J. Comp. Physics.*, **2**, 339 (1966).
3. I. Babuska and W. Rheinboldt, *SIAM J. Num. Anal.*, **15** (1978).
4. N. N. Yanenko, V. M. Kovenya, V. D. Lisejkin, V. M. Fonin, and E. V. Vorozhtsov, *Proc. 6th Int'l. Conf. on Num. Meth. in Fluid Dynamics, Lecture Notes in Physics*, **90**, 565 (1979).
5. R. Courant, *Trans. Amer. Math. Soc.*, **50**, 40 (1941).
6. Joe F. Thompson, Frank C. Thames and C. Wayne Mastin, *J. Comp. Phys.*, **15**, 299 (1974).
7. C. W. Hirt, A. A. Amsden and J. L. Cook, *J. Comp. Phys.*, **14**, 227 (1974).
8. J. U. Brackbill, *Meth. Comp. Phys.*, **16**, 1 (1976).
9. J. J. Saltzman and J. Brackbill, Applications and Generalizations of Variational Methods for Generating Adaptive Meshes, these proceedings.
10. J. Canosa, *IBM Journal of Research and Development*, **17**, 307 (1973).
11. R. D. Milroy and J. U. Brackbill, Numerical Studies of a Field Reversed Theta Pinch Plasma, *Phys. of Fluids* (to appear).
12. J. U. Brackbill and H. M. Ruppel, A Method for Adaptively Zoned, Particle-in-Cell Calculations in Two Dimensions, submitted *J. Comp. Phys.*

Fig. 1. In the adaptive mesh, the zones are concentrated in an annulus where there is the greatest variation in the solution.

Fig. 2. The error in the finite difference approximation and the integral of weighted volume decrease together in (a) with increasing adaptivity. The error in an adaptive mesh shown in (b) is much smaller than in a fixed Eulerian mesh, especially for Re greater than one. Re , the grid Reynolds number, is the ratio of the average mesh spacing to smallest gradient length.

Fig. 3. The solution of Fisher's equation on a 625 zone Eulerian mesh is depicted. Contours of constant ϕ are concentrated in the front across which ϕ increases from 0 outside to 1 inside. The distortion of the front from the correct, circular shape is evident.

Fig. 4. With a 5625 zone Eulerian mesh, a solution with a more circular front and correct propagation speed is obtained.

Fig. 5. With a 625 zone adaptive mesh, a comparable solution to one with a 5625 zone Eulerian mesh is obtained. The shape is nearly circular and the propagation speed is approximately correct.

Fig. 6. The mesh corresponding to Fig. 5 is shown. Note the concentration of zones in the front.

Fig. 7. A computation mesh for a simulation of reconnection in the earth's magnetosphere is shown. The grid results from the solution of the inverse problem with adaptivity implemented.

Fig. 8. In (a), points are equally spaced along the curve. In (b), the points are concentrated in regions of high curvature.