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THRESHOLDS FOR THE ONSET OF FLUID AND MAGNETOFLUID TURBULENCE*

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Hydrodynamic stability theory has focussed on a few simple test cases to obtain the sharpest possible confrontations between theory and experiment. Six of these are briefly reviewed: plane Poiseuille and Couette flow, pipe flow, rotating Couette flow, thermally-driven Bénard convection, and the Blasius laminar boundary layer. Linear perturbation theory seems inadequate in the first three cases, and satisfactory in the last three. Insufficient information, experimental or theoretical, exists in magnetohydrodynamics to make any comparably decisive tests.

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1. Introduction

In plasma physics, it is almost always possible to do another linear stability calculation. No matter what the geometry or the parameter ranges, the available mathematical descriptions are invariably rich enough that an equilibrium can be found which seems to contain many of the essential features of the situation at hand. The equations of motion can be linearized about this equilibrium and their solutions can be more or less systematically analyzed. It is a game which can go on forever, and it almost has.

It can be lost sight of that the model for this program, which has been in effect since the 1950s, was the theory of the hydrodynamic stability of neutral fluids. Pivotal books like those of Lin [1] and Chandrasekhar [2] were new at about the time modern plasma physics (controlled fusion and space research) was going into high gear. Their assimilation by plasma physicists was perhaps not as thorough as it might have been, but the commitment to them was wholehearted, and these books have motivated a great deal of what has happened in plasma physics since then.

Much could be said about the successes and failures of the linear stability program in plasma physics. However, the intent here is more limited. What is intended is a brief inquiry into how the hydrodynamic part of the program has been faring. How well does it, in fact, work on its home ground?

Hydrodynamics has several advantages over plasma physics as a test of a theoretical development. (1) It is a mature subject in an engineering sense, one in which most of the pressing technical problems have already been solved, so that the theory is free from pressures to show how to remove serious limitations

standing in the way of potential applications. It has the luxury of being able to concentrate on doing things right rather than fast. (2) Good diagnostics are available for the important dynamical variables, so that far fewer long chains of inference are required to compare theory with experiment. (3) The theory has had time to formalize itself around a small number of agreed-upon "universal" situations (flow between parallel planes, down cylindrical pipes, between concentric cylinders, etc.) which are thought to be sharp tests against which theoretical predictions must be measured.

We remark here upon six of these much-studied situations. All have the property that the fluid goes from laminar to turbulent at critical values of some dimensionless number, such as the Reynolds number or the Rayleigh number. The question is, At what threshold values of the dimensionless number do the unstable motions set in, and can these thresholds be predicted by a linear analysis of the stability of the laminar state as the threshold is approached from the stable side?

Of the six examples chosen, the situation seems to be that the linear stability analysis seems to be satisfactory in just half the cases. The three cases in which it clearly seems to fail are:

- (1) Plane Poiseuille Flow
- (2) Plane Couette Flow
- (3) Cylindrical Pipe Flow (Hagen-Poiseuille Flow).

The situations in which the linear stability analysis seems to give predictions which are essentially correct are:

- (4) Rotating Couette Flow (Flow Between Rotating Cylinders)
- (5) Thermally-Driven Convection (Bénard Instability)
- (6) Instability of Laminar Boundary Layers (Tollmien-Schlichting Waves).

It is important to stress that in the three cases in which the linear theory fails, it fails not because of some subtle error in the calculation, but because the instabilities which show up in the real world are fundamentally different from those the conventional linear theory can find. It is the work of S. A. Orszag [3,4,5,6,7] and collaborators who, more than anyone else, have finally pinned down the nature of these instabilities. It is their work and that of others that is being remarked upon; very little in this article is original. There may, however, be some lessons for plasma physics in these hydrodynamic advances that have not been appreciated. Paying some attention to a few well-documented hydrodynamic examples may be of as much value as adding one more to the (already many thousands of) difficult-to-test plasma predictions. In the final section, we attempt to draw some implications for MHD stability theory from the examples considered.

2. Plane Shear Flows (Poiseuille and Couette)

The Navier-Stokes fluid in the incompressible limit is governed by [8,9]

$$\frac{\partial \underline{y}}{\partial t} + \underline{y} \cdot \nabla \underline{y} = -\nabla(p/\rho) + \nu \nabla^2 \underline{y} \quad (1)$$

$$\nabla \cdot \underline{y} = 0 \quad (2)$$

Here $\underline{y} = \underline{y}(\underline{x}, t)$ is the fluid velocity as a function of position $\underline{x}$ and time t , p/ρ is the pressure-to-density ratio, ν is the kinematic viscosity, and in the case of present interest, $\rho = \rho_0$ is a uniform constant. p/ρ_0 is determined from solving the Poisson equation which results from taking the divergence of (1) and using (2) to get rid of the $\partial/\partial t$ term. This solution is plugged back into (1) to express $\partial \underline{y}/\partial t$ as a function of $\underline{y}$ alone.

Equations (1) and (2) may be rewritten in dimensionless units, scaling all velocities by U_0 , a typical flow speed; all the lengths by L , a typical macroscopic length scale determined by the boundary conditions; and all the times by L/U_0 . Equations (1),(2) then keep the same form, except that the kinematic viscosity $\nu \rightarrow 1/R$, where $R \equiv U_0 L/\nu$ is the Reynolds number. Reynolds number is then the only physical number left in the problem.

We are interested in the stability of plane shear flows, which are unidirectional flows parallel to two parallel plane boundaries which are assumed to extend to infinity. If the planes are normal to the z axis, say, the equilibrium flow velocity depends only upon z and may be chosen to lie in the x direction: $\underline{v}^{(0)} = (U_0(z), 0, 0)$. Linearized about the equilibrium, writing $\underline{v} = \underline{v}^{(0)} + \underline{v}^{(1)}$ and discarding all products of first-order quantities, Eqs. (1), (2) become, in the dimensionless variables

$$\frac{\partial \underline{v}^{(1)}}{\partial t} + \underline{v}^{(1)} \cdot \nabla \underline{v}^{(0)} + \underline{v}^{(0)} \cdot \nabla \underline{v}^{(1)} = -\nabla(p/\rho_0)^{(1)} + \frac{\nabla^2 \underline{v}^{(1)}}{R} \quad (3)$$

$$\nabla \cdot \underline{v}^{(1)} = 0 \quad (4)$$

The linear stability theory revolves around the solutions to Eqs. (3) and (4). The boundary conditions are that the total $\underline{v}$ shall have the value of the velocity (if any) of the planes which, in the dimensionless units, may be put at $z = \pm 1$.

Plane Poiseuille flow has stationary planes at the boundaries and is driven by a uniform pressure gradient in the x direction, so that the velocity profile is parabolic:

$$\underline{v}^{(0)}(\underline{x}) = (1-z^2, 0, 0); \quad p^{(0)}(\underline{x})/\rho_0 = -2\nu x, \quad (\text{plane Poiseuille}) \quad (5)$$

where now ν and $1/R$ mean the same thing in the dimensionless units.

For plane Couette flow, there is no pressure gradient, but the bounding planes move relative to each other at a constant relative velocity ($2U_0$, say), so that the velocity profile is linear:

$$\underline{y}^{(0)}(\underline{x}) = (z, 0, 0); \quad p^{(0)}(\underline{x})/\rho_0 = 0 \quad (\text{plane Couette}). \quad (6)$$

The coefficients in the linear system (3) and (4) are independent of x , y and t , so without loss of generality, we may seek solutions in which all perturbed quantities vary as $\exp[i\alpha x + i\beta y - i\omega t]$ times an as yet unknown function of z , with arbitrary real α and β . In general ω will be complex, and a positive imaginary part to ω for some α, β will signal linear instability.

Seeking components $\underline{y}^{(0)} = (u, v, w)$, the components of Eqs. (3), (4) may be manipulated to lead to a single, scalar, fourth-order ordinary differential equation for w , the Orr-Sommerfeld [1] equation:

$$\left(\frac{d^2}{dz^2} - \alpha^2 - \beta^2\right)^2 w = iR[(\alpha U_0 - \omega)\left(\frac{d^2}{dz^2} - \alpha^2 - \beta^2\right)w - \alpha U_0'' w] \quad (7)$$

where $\underline{y}^{(0)} = (U_0(z), 0, 0)$, $R = 1/\nu$, the primes denote differentiation with respect to z , and the boundary conditions are that $w = 0$ and $dw/dz = 0$ at $z = \pm 1$.

Note that the fact that the viscous terms are present is what has led to a fourth-order equation. Dividing out an R in Eq. (7) and letting $R \rightarrow \infty$ recovers the "ideal" case, which has virtually nothing to say about the two flows at hand. The difference between the cases R very large and $\nu = 1/R \equiv 0$ is enormous, and there is no sense in which the results of the former case reduce to those of the latter. Following the parallel program of ideal MHD stability theory, summarized in the "energy principle" [10,11], one would miss all the instabilities of the Orr-Sommerfeld equation for non-inflectional profiles.

There is a theorem called Squire's theorem [1] which guarantees that if one is interested in threshold values of R for which $\text{Im } \omega$ first becomes positive for some solution of (7), it is sufficient to consider the two-dimensional case $\beta = 0$. The first instability which appears, as R is raised, is always two-dimensional.

Much lore has accumulated concerning the solutions of Eq. (7) and in particular, the very accurate methods of Orszag [3,4] have established that the threshold critical Reynolds number R_c for plane Poiseuille flow is 5772.22, and for plane Couette flow, infinity. These results, as far as I am aware, are universally accepted, and should be regarded as far more precise than any plasma theory prediction that has ever been made.

The trouble is that they are wholly at variance with the facts. A long series of careful experiments have shown (see Orszag and Kells [5] for a summary) that both flows are typically unstable above a threshold $R = R_c$ of about 1000, though it has occasionally been reported [e.g., 12] that the onset of instability can be delayed to significantly higher values of R , by careful noise reduction.

The situation is rendered more complex by the fact that finite-amplitude, two-dimensional computations, starting directly with Eqs. (1) and (2), show Poiseuille nonlinear 2D instabilities setting in at about $R_c \approx 2800$, well below the threshold for the Orr-Sommerfeld predictions to be relevant, but still well above the experimentally determined values.

What seems to be the true situation has been sorted out by Orszag and Kells [5] and Orszag and Patera [6,7]. Proceeding from spectral-method numerical solutions of Eqs. (1) and (2), it was established that:

(1) Finite-amplitude two-dimensional solutions to Eqs. (1) and (2) exist, well below $R \approx 2800$ which, in an appropriate coordinate frame, are either stationary or very slowly damped in time. These solutions are periodic in x .

(2) These finite-amplitude two-dimensional solutions are strongly unstable to small three-dimensional disturbances, down to values of R of the order of 1000. Even round-off errors on two-dimensional initial conditions, which contribute three-dimensional noise of the order of 1 part in 10^{14} if not suppressed, are enough to trigger it within a few hundred time steps.

Experimentally, the presence of the stable two-dimensional finite-amplitude disturbances must be regarded as a somewhat dark event: part of the inevitable noise that forms the background of any experiment. It is not entirely clear as to what extent they may be tailored or suppressed, but it seems clear that the threshold in R which they may define will necessarily be less sharp than that defined by Orr-Sommerfeld theory[7].

For the first time, it appears to be intuitively clear what is happening in the transition to turbulence in plane shear flows. The intricate reasoning which sorted it out [5,6,7] is one of the great classic detective stories of applied mathematics, one that could probably never have been written without the capacity to solve Eqs. (1) and (2) numerically to high accuracy.

3. Pipe Flow (Hagen-Poiseuille flow)

Pipe flow is historically the most important single example in the study of the transition to turbulence.[13,14] It is difficult to imagine a more central situation in fluid mechanics. It is roughly analogous to the plasma situation in which a cylindrical column of plasma or magnetofluid carries an

axial current. It has been studied probably as much as any other continuum mechanics problem, but it is not yet fully understood and is currently an active and controversial area of research.

The one-dimensional laminar solution analogous to Eq. (5) is, in cylindrical coordinates (r, θ, z) ,

$$\underline{v}^{(0)}(\underline{x}) = (0, 0, 1-r^2) \quad (8)$$

where U_0 is now the axial component of the flow velocity at the axis $r = 0$, and L is the tube radius, which in the dimensionless units lies at $r = 1$. There is a uniform pressure gradient down the axis of the pipe, which is assumed to be infinitely long.

Equations (3) and (4) are again suitable for discussing the linear stability of the flow (8). The consensus of many investigators [9,15,16,17,18,19] is that the parabolic profile (8) is linearly stable at all Reynolds numbers up to $R = \infty$. The results are not beyond question, but only slightly so. Again, more care and precision have been brought to bear on the problem than have ever been expended on any theoretical plasma problem.

However, the indisputable fact is that pipe flow is far from stable experimentally. Under normal conditions, the threshold values above which instabilities set in are of the order of 2000 [9]. However, it has been reported that, by careful attention to symmetry and noise suppression, laminar flow has been maintained up to $R \approx 10^5$ [9,20]. We have the unpleasant situation that reliable experimenters have been able to report critical values which differ by as much as a factor of 50, none of it predictable by linear theory!

Many subtle effects appear to be involved, the most important of which concerns the entry length of the pipe [21]. When fluid enters a pipe at low

Reynolds numbers, the axial velocity profile typically does not have the parabolic shape (8), but rather something much flatter [9,22], which only becomes parabolic as the fluid moves downstream. At entry, the velocity goes from zero to its mean interior value over a radial boundary layer (see Sec. 6, below) which is localized near the wall. As the distance z down the pipe increases, the boundary layer widens, and roughly speaking, the profile becomes parabolic when the boundary layer has widened comparably to the tube radius.

As this widening occurs, the Reynolds number associated with the boundary layer width gradually increases. If the critical Reynolds number for the instability of the boundary layer is crossed, boundary layer modes present in the noise can begin to amplify in the boundary layer. These disturbances are typically swept downstream. For each such mode there is characteristically an unstable band of local Reynolds numbers, i.e., a maximum Reynolds number above which the mode becomes again stable. As the disturbance is swept downstream, it may pass out of this band of unstable local Reynolds numbers before it is amplified into the strongly nonlinear regime. If so, it shows up as a wavy disturbance propagating downstream that, at a fixed point in space, has a quasi-periodicity in time, but does not cause the full column to burst into turbulence. If not, then turbulence can result at the point where the amplified noise goes fully nonlinear.

Pipe flow turbulence, as it appears first, forms in patches or "plugs" which are of finite extent in z and which are swept downstream, lengthening as they go. They are separated by laminar regions which, as the turbulence develops, get fewer and farther between. An observer at a fixed point sees a highly intermittent velocity field, with a lot of intense activity as the plugs pass by.

. At first glance, the similarities between pipe flow and plane Poiseuille flow described in Sec. 3 would suggest a common approach. Weakly damped or neutrally stable quasi-equilibria, periodic in z , might be unstable to three-dimensional perturbations, but the recent results of Patera and Orszag [19] throw doubt on the existence of axisymmetric quasi-equilibria of an appropriate kind. Recently Garg and Gupta [23,24] have concentrated on the linear stability of different non-periodic entry-length profiles and have found critical Reynolds numbers of the order of 10^4 .

The situation remains confused. Any or all of the following may be crucial in the onset of turbulence in pipe flow: (1) two-dimensionality of entry-length profiles; (2) nonlinear instabilities; (3) nonaxisymmetry. What seems virtually certain is that further ingenious linear investigation of the idealized profile of Eq. (8) is unlikely to predict anything real. Perhaps the lesson for plasma physics is the factor of 50 variation in critical Reynolds number experimentally observable, and the lack of theoretical success at predicting this variation from lengthy mathematical computations. A second lesson may concern the impossibility of reaching the factual conclusions that have been reached if the only diagnostics available had been external ones of the kind most current plasma measurements must rely upon.

4. Rotating Couette Flow

We consider flow between infinitely-long, coaxial, rotating cylinders of radii a_1 and a_2 , $a_1 < a_2$. The inner cylinder rotates with constant angular velocity Ω_1 and the outer one with constant angular velocity Ω_2 ; Ω_1 and Ω_2 can be in the same sense or opposite senses. A time-independent solution to (1) can

be found with $\underline{v}^{(0)} = (0, v_\theta, 0)$ in cylindrical coordinates (r, θ, z) , where $v_\theta = Ar + B/r$, and A and B are chosen so that v_θ becomes $a_1\Omega_1$ at the inner cylinder and $a_2\Omega_2$ at the outer one:

$$A = \frac{\Omega_2 a_2^2 - \Omega_1 a_1^2}{a_2^2 - a_1^2}, \quad B = \frac{(\Omega_1 - \Omega_2) a_1^2 a_2^2}{a_2^2 - a_1^2}. \quad (9)$$

The linear stability analysis of this equilibrium flow was carried out by Taylor [25], extended by Chandrasekhar [26], and has been rather spectacularly confirmed by experiment [27,28]. For fixed Ω_2 , positive or negative, there is always a critical $\Omega_{1c} > 0$ such that the flow is unstable whenever $\Omega_1 > \Omega_{1c}$.

At first glance, it would appear that rotating Couette flow ought to bear some similarities to plane Couette flow, at least when $(a_2 - a_1)/a_2 \ll 1$. If one read only the results presented in the literature of rotating Couette flow [1,2], one would get no inkling of the rather tortuous path that had to be followed in order to get a theoretical grip on the planar case.

5. Thermally-Driven Convection (Bénard)

If two rigid, infinite parallel planes are maintained at temperatures T_1 (below) $>$ T_2 (above), under gravity, there is a heat flow, and the possibility of thermally-driven convection. Equations (1) and (2) have to be modified to encompass the phenomena, in that a temperature equation must be added to (1) and (2), and the possibility of density stratification [2,9]. The density stratification is due to possible variations of density with temperature. A gravitational field perpendicular to the planes is a key ingredient. The possibility of motion produced by having the lighter fluid below is restrained by the

action of viscosity and thermal conductivity. The dimensionless number which signals the onset of instability is the Rayleigh number R_a :

$$R_a \equiv g\alpha\Delta T d^3/\nu\kappa, \quad (10)$$

where g is the acceleration of gravity, $\alpha = -\rho_0^{-1}d\rho_0/dT$ is the coefficient of expansion of the fluid, ΔT is the temperature difference maintained across the plate separation d , ν is the kinematic viscosity, and $\kappa \equiv k/\rho_0 C_p$ is the thermal diffusivity ($k \equiv$ thermal conductivity, $\rho_0 \equiv$ density, and $C_p \equiv$ specific heat at constant pressure). The critical value of R_a , above which motion of the fluid sets in, is about 1700, experimentally and theoretically. This agreement, as in the case of rotating Couette flow, is regarded as a triumph of the linear stability theory.

Secondarily to our purposes here, the problem of thermally driven Bénard convection just above the critical Rayleigh number lies at the heart of one of the most active areas of modern physics, that of "dynamical systems": nonlinear systems of a few degrees of freedom whose behavior exhibits striking qualitative changes as certain thresholds are crossed in R_a . The spatial structures associated with the modes which first become excited are various: "rolls", which seem to depend on the geometry of the containing vessel, particularly the width-to-height ratio, or "aspect ratio". The frequency behavior exhibits a curious "period doubling" [29] of certain discrete frequencies which dominate the spectrum for awhile until it becomes broad-band [30,31]. This behavior has occupied so much attention that in some quarters it has begun to be alluded to synonymously with "turbulence". Like many other sallies into the field, one suspects that this fascinating behavior may in time come to seem somewhat less

universal than was initially hoped, but the limits of the behavior have not been reached. Related considerations have been discussed in connection with rotating Couette flow [32].

6. Instability of Laminar Boundary Layers [9,33]

A fluid flowing past any solid surface subject to no-slip boundary conditions always generates a boundary layer. In a boundary layer, there is a concentration of vorticity occasioned by the need of the fluid velocity to change abruptly from its interior value to the value $\underline{v} = 0$ at the boundary. If flow velocities are low enough, the relevant solutions of the Navier-Stokes equation become time independent and stable. As the flow speed increases, the boundary layer becomes unstable, and can become turbulent. This behavior is a virtually ubiquitous aspect of fluid flow. As in the other idealized examples previously described, the simplest geometry in which the phenomenon can be exhibited appears as the most likely candidate for study.

Much of the boundary layer lore has been formalized around the case of the semi-infinite, infinitesimally-thin, flat plate. Suppose the plate occupies that part of the xz plane for which $x \geq 0$, and that, far from the plate, the flow velocity $\underline{v} \rightarrow U_0 \hat{e}_x$. The solution for $\underline{v}$ which $\rightarrow 0$ at the surface of the plate is the celebrated Blasius profile; in Cartesian coordinates,

$$\underline{v} = (u, v, 0)$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$\psi = U_0 \Delta f(y/\Delta)$$

$$\Delta = (\nu x / U_0)^{1/2}$$

and the solution to

$$ff'' + 2f''' = 0$$

(11)

must be obtained numerically [8]. The characteristic thickness of the profile is $\delta \sim \sqrt{\nu x/U_0}$, which gradually widens downstream, and decreases with increasing U_0 at a given x .

The local Reynolds number based on the width of the Blasius boundary layer, $U_0 \delta/\nu$, gradually increases downstream. It may reach a value at which the flow is locally unstable. The critical value, taking δ as defined by $u/U_0 = 0.99$, is $R \approx 1200$. If only a narrow band of wave numbers is locally unstable, the disturbance may develop an oscillatory wave structure in space, propagating away from the region of instability. For a given wavenumber, there is typically a maximum Reynolds number for instability as well as a minimum, and the wavenumber component may be swept out of the unstable region before it has a chance to amplify into a fully nonlinear disturbance. These unstable propagating waves are called Tollmien-Schlichting waves, and represent one of the more readily visualizeable instabilities when artificially excited so that an injected sharp frequency may be amplified [9].

Though the theory and experiment are not as sharply correlated as they are for rotating Couette flow and Bénard convection, the situation still seems to be one in which the linear stability analysis can be called a satisfactory prediction of the onset of instability.

7. The Magnetohydrodynamic Program

If one were to consider only the final three examples, it would be easy to conclude that hydrodynamic stability can be reliably treated by means of the linearized equations of motion. However, considering only the first three would perhaps lead to the opposite conclusion: that linear stability theory is of

little use in predicting critical Reynolds numbers for the onset of unstable motions. It is not easy to see any common quality that otherwise distinguishes the first three cases from the last three. The only clear lesson to be learned may be the platitudinous one of not jumping to conclusions.

Assuming the value of doing a linearized stability analysis, what else can be noticed that might bear on the parallel effort in plasma physics to analyze the stability of equilibria using the linearized MHD equations? Perhaps a few things are worth noting, since hydrodynamics is after all a special case of MHD.

(1) In every case, spatial nonuniformity was of the essence in determining instability thresholds. The length scales of the nonuniformity were neither much greater nor much less than the length scales associated with the first unstable mode. It was never possible to eliminate all the space dependences by simply assuming a complex exponential space dependence $\sim \exp(ikx)$ in all three coordinates. At least one non-trivial spatial dependence remained in the eigenmode equation which required numerical analysis. Perusing the plasma literature, one might conclude that, except for occasional matching procedures associated with "singular layers", a great deal can usefully be inferred from problems in which the space dependence of the eigenmodes is effectively sidestepped. The reason for this hope is not hard to locate historically: there are a great many velocity-space Vlasov instabilities, which have no counterpart in hydrodynamics, in which the spatial eigenfunction dependence is trivially sine-wave. It is very unlikely that this fact has any great relevance for MHD stability, however. MHD appears as a natural generalization of neutral-fluid stability theory much more than it bears on anything to do with kinetic theory.

- (2) None of the instabilities in any of the six cases was "ideal".

Finite values of the dissipation coefficients were necessary, and the thresholds, the order of the ultimate differential equations, and the behavior of the eigenfunctions would have been totally different if the dissipation coefficients, however small, had been omitted. The dimensionless numbers such as R and R_a , which determined the thresholds in stability, always contained the dissipation coefficients multiplicatively, in the numerator or denominator, and would thus always be zero or infinity in the "ideal approximation".

(3) Partially because of the characteristic space dependences noted in (1), boundary conditions determine the stability properties of a hydrodynamic configuration, and can never be ignored. There are no "local" instabilities among the six, in the plasma sense. In MHD, we are only beginning to arrive at a set of boundary conditions that we can live with that are simple enough to be tractable mathematically and yet do not do too much violence to the very complex physics that goes on at a plasma-solid wall interface.

(4) From the point of view of doing good science, the interest has centered around situations which are as highly idealized as possible. These "simple" classic situations have taxed the ingenuity of the best minds in the subject for a century. Our program of plunging the raw theoretical troops into the most tangled engineering battles of confinement devices can seem naive in the extreme. Trying to "calculate" the stability of a stellarator or a tokamak is a program about as audacious as trying to calculate the stability of a river or an internal combustion engine.

(5) There are highly simplified MHD equilibria analogous to plane shear flows, pipe flows, boundary layer flows, etc. (with the current density, roughly

speaking, taking over the role of the fluid velocity) which could and should form the focus for a serious theory of MHD stability. The Lundquist number can be imagined as a parameter analogous to the Reynolds number as having a central role in determining the onset of instability for static equilibria. For most of these, we do not at present have a clue as to what these thresholds might be, or how they might be affected by such complications as external dc magnetic fields, tensor viscosities and resistivities, etc.

It is not clear that a theoretical study of MHD stability theory can be justified for its engineering applications: most fluid-mechanical applications seem to have preceded, rather than followed, their mathematical elucidation. But it can be suggested that if the task is worth doing, it is worth doing right. What that may amount to seems to have been best exemplified by what has happened in hydrodynamics.

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