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THE EFFECT OF PROMPT NEUTRONS ON THE
DIRECTION AND TIME OF FLIGHT OF
FISSION FRAGMENTS

R. W. Lide

OAK RIDGE NATIONAL LABORATORY

operated by

UNION CARBIDE CORPORATION

for the

U.S. ATOMIC ENERGY COMMISSION

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R. W. Lide

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THE EFFECT OF PROMPT NEUTRONS ON THE DIRECTION AND
TIME OF FLIGHT OF FISSION FRAGMENTS

R. W. Lide*

ABSTRACT

Calculations have been made of the effect of prompt neutrons on the deflection and time of flight of fission fragments. This has been done as an aid to the analysis of data from fission experiments using time-of-flight techniques.

Assuming that the energy distribution function of prompt neutrons is that given by Watt, an expression is obtained for the probability distribution in the angle between the final velocities of the two fragments as a function of the number of prompt neutrons emitted and the masses of the fragments.

Under the same assumption, it is shown that the probability distribution for the flight time of a given fragment over a given flight path is very nearly Gaussian. This distribution is (to a close approximation) centered about the flight time for no neutron emission with a variance which depends on the number of neutrons emitted and the mass of the fragment.

*Research Participant and Consultant from the University of Tennessee.

THE EFFECT OF PROMPT NEUTRONS ON THE DIRECTION AND TIME OF FLIGHT OF FISSION FRAGMENTS

1. Introduction

This study was made in conjunction with a time-of-flight experiment on fission fragments.

In analyzing an accurate experiment of this type, one must know the variance in the time of flight of the fragments that arises from the emission of prompt neutrons.

In designing such an experiment, one needs to know the probable deflection of the fission fragments by prompt neutrons in order to choose a geometry that will minimize the loss of fission-fragment coincidences as a result of this deflection. In analyzing the data it may be desirable to correct for this loss, otherwise the results will be weighted in favor of those cases where the fewest prompt neutrons are emitted.

It is the purpose of this report to derive expressions for the deflection and the variance in the time of flight of fission fragments due to prompt neutron emission. The expression for the variance in flight time should be useful in itself. The expression for the deflection should be useful for calculating the loss of fission fragment coincidences in a particular experiment.

PROBABILITY DISTRIBUTION FOR THE MAGNITUDE OF THE
VECTOR SUM OF PROMPT-NEUTRON MOMENTA

Single Neutron

Watt⁽¹⁾ has shown that one may fit the observed prompt-neutron spectra obtained in the fission of U^{235} by assuming that the neutrons are emitted in the center of mass system with an energy distribution given by the relation

$$n(E) = K\sqrt{E} \exp \left\{ \frac{-E}{T} \right\}, \quad (1a)$$

where E is the neutron energy in the center of mass system and T is a constant. Transforming to the laboratory system one obtains (assuming isotropic neutron emission in the c.m. system)

$$n(E') = \text{const} \exp \left\{ \frac{-E'}{T} \right\} \sinh \frac{2}{T} \sqrt{\epsilon E'}, \quad (1b)$$

where E' is the neutron energy in the laboratory system and ϵ is the kinetic energy per nucleon in the fragment prior to neutron emission, i.e.,

$$\epsilon = \frac{m}{M} E_f,$$

where m is the neutron mass, and E_f and M are respectively the mass and energy of the fragment prior to neutron emission.

By assuming an "average" fragment (i.e., by picking a value for ϵ and T) one is able to fit the observed neutron spectrum with equation (1b). Leachman⁽²⁾ has found that putting

$$T = 0.965 \text{ Mev}$$

into formula (1b) gives the best fit to the observed spectrum. We shall then use this value of T along with equation (1a) in what follows.

The distribution in momentum can be obtained from the distribution in energy from the relation

$$n(p) = n(E) \left| \frac{dE}{dp} \right|$$

which leads to

$$n(p) = K \sqrt{\frac{p^2}{2m^3}} \exp \left\{ -\frac{p^2}{2\sigma^2} \right\}, \quad (2a)$$

where

$$\sigma = mT, \quad (2b)$$

m being the neutron mass.

We obtain K by integrating (2a) to give the total number of neutrons,

$$N = \int_0^\infty n(p) dp = \frac{K}{\sqrt{2m^3}} \int_0^\infty p^2 \exp \left\{ -\frac{p^2}{2\sigma^2} \right\} dp.$$

Using equation 861.7 in Dwight⁽³⁾, we obtain

$$K = \frac{2N}{\sigma^3} \sqrt{\frac{m^3}{\pi}}.$$

Then

$$n(p) = N \sqrt{\frac{2}{\pi}} \frac{p^2}{\sigma^3} \exp \left\{ -\frac{p^2}{2\sigma^2} \right\}. \quad (3)$$

Any Number of Neutrons

Let k = number of prompt neutrons emitted

$\vec{p}_k$ = vector sum of momenta of k neutrons

$n(p_k)$ = distribution function for magnitude of p_k .

We will show that

$$n(p_k) = N \sqrt{\frac{2}{\pi}} \frac{p_k^2}{k^{3/2} \sigma^3} \exp \left\{ -\frac{p_k^2}{2k\sigma^2} \right\}. \quad (4)$$

We will do this by showing that if (4) holds for $(k - 1)$ it also holds for k . Then since it holds for $k = 1$ it must hold for all k . We will assume that the neutrons are emitted isotropically in the center of mass system of the fragment and we will refer everything to this system. The momentum of k neutrons may be obtained by adding the momentum of a single neutron, $\vec{p}$, to the vector sum of $(k - 1)$ neutrons (see Fig. 1). Thus

$$\vec{p}_k = \vec{p}_{k-1} + \vec{p},$$

and

$$p_k^2 = p_{k-1}^2 + p^2 + 2pp_{k-1} \cos \psi .$$

First, consider a fixed p_{k-1} and a fixed p and consider only the variations in p_k with the angle ψ . Since p is isotropic, the probability per solid angle is a constant, or

$$n(\Omega) = \frac{dN}{d\Omega} = \frac{N}{4\pi},$$

where $d\Omega = 2\pi \sin \psi \, d\psi$.

Then the distribution function in ψ is given by

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Fig. 1. Schematic diagram showing addition of momenta of prompt neutrons.

$$n(\psi) = n(\Omega) \left| \frac{d\Omega}{d\psi} \right| = \frac{N}{2} \left| \sin \psi \right|.$$

Also

$$n(p_k) = n(\psi) \left| \frac{d\psi}{dp_k} \right| = N p_k / 2 p_{k-1} p.$$

Hence distribution is linear in p_k between upper limit U and lower limit L where

$$U = p_{k-1} + p,$$

$$L = |p_{k-1} - p|.$$

Now consider $n(p_k)$ for a distribution of p_{k-1} and p . Let the probability distribution for a given p_k be $W(p_k)$, then

$$W(p_k) = \frac{n(p_k)}{N} = \iint \frac{p_k}{2 p_{k-1} p} W(p_{k-1}) dp_{k-1} W(p) dp,$$

but

$$W(p_{k-1}) = \frac{n(p_{k-1})}{N}, \quad W(p) = \frac{n(p)}{N}.$$

Hence

$$n(p_k) = \frac{p_k}{2N} \int_0^{\infty} \frac{n(p_{k-1})}{p_{k-1}} dp_{k-1} \int_{|p_k - p_{k-1}|}^{p_k + p_{k-1}} \frac{n(p)}{p} dp \quad (5)$$

We know that $n(p)$ is given by (2), and we assume that

$$n(p_{k-1}) = N \sqrt{\frac{2}{\pi(k-1)^3}} \frac{p_{k-1}^2}{\sigma^3} \exp \left\{ - \frac{p_{k-1}^2}{2(k-1)\sigma^2} \right\}.$$

The region of integration is shown in Fig. 2. We first integrate over p between the limits $|p_k - p_{k-1}|$ and $(p_k + p_{k-1})$ holding p_{k-1} constant. Then the first integral in (4),

$$I_1 = \frac{N}{\sigma^3} \sqrt{\frac{2}{\pi}} \int_{|p_k - p_{k-1}|}^{p_k + p_{k-1}} p \exp \left\{ -\frac{p^2}{2\sigma^2} \right\} dp.$$

Performing the integration one obtains

$$I_1 = \frac{N}{\sigma} \sqrt{\frac{2}{\pi}} \left[\exp \left\{ -\frac{(p_k - p_{k-1})^2}{2\sigma^2} \right\} - \exp \left\{ -\frac{(p_k + p_{k-1})^2}{2\sigma^2} \right\} \right].$$

We now integrate over p_{k-1} . We have

$$n(p_k) = \frac{p_k}{2N} \left[I_2' - I_2'' \right],$$

where

$$I_2' = \frac{2N^2}{\pi\sigma^4(k-1)^{3/2}} \int_0^\infty dp_{k-1} p_{k-1} \exp \left\{ -\frac{(k-1)(p_k - p_{k-1})^2 - p_{k-1}^2}{2(k-1)\sigma^2} \right\},$$

and

$$I_2'' = \frac{2N^2}{\pi\sigma^4(k-1)^{3/2}} \int_0^\infty dp_{k-1} p_{k-1} \exp \left\{ -\frac{(k-1)(p_k + p_{k-1})^2 - p_{k-1}^2}{2(k-1)\sigma^2} \right\}.$$

First consider I_2' . In the numerator of the exponential function we have

$$\begin{aligned} & - \left[k p_{k-1}^2 - 2(k-1) p_k p_{k-1} + (k-1) p_k^2 \right] \\ & = - \left\{ k \left[p_{k-1}^2 - \frac{2(k-1)}{k} p_k p_{k-1} + \left(\frac{k-1}{k} \right)^2 p_k^2 \right] + (k-1) p_k^2 - \frac{(k-1)^2}{k} p_k^2 \right\} \\ & = - \left\{ k \xi^2 + \frac{(k-1)}{k} p_k^2 \right\}, \end{aligned}$$

where we have let

$$\xi = p_{k-1} - \frac{(k-1)}{k} p_k.$$

Then

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Fig. 2. The p, p_{k-1} plane showing the region of integration for obtaining $n(p_k)$, the distribution in p_k .

$$I_2' = \frac{2 N^2}{\pi \sigma^4 (k-1)^{3/2}} \exp \left\{ -\frac{p_k^2}{2k\sigma^2} \right\} \int_{-(\frac{k-1}{k})p_k}^{\infty} d\xi \left[\xi + \frac{(k-1)}{k} p_k \right] p_k \exp \left\{ -\frac{k \xi^2}{2(k-1)\sigma^2} \right\}$$

Similarly transforming I_2'' we have

$$I_2'' = \frac{2 N^2}{\pi \sigma^4 (k-1)^{3/2}} \exp \left\{ -\frac{p_k^2}{2k\sigma^2} \right\} \int_{(k-1)p_k/k}^{\infty} d\xi' \left[\xi' - \frac{(k-1)}{k} p_k \right] p_k \exp \left\{ -\frac{k \xi'^2}{2(k-1)\sigma^2} \right\},$$

where $\xi' = p_{k-1} + \frac{(k-1)}{k} p_k$

Combining I_2' and I_2'' , we get

$$n(p_k) = \frac{p_k N}{\pi \sigma^4 (k-1)^{3/2}} \exp \left\{ -\frac{p_k^2}{2k\sigma^2} \right\} \left[\int_{-(\frac{k-1}{k})p_k}^0 d\xi \xi + \int_0^{\infty} d\xi \xi + \int_{-(\frac{k-1}{k})p_k}^0 d\xi \xi + \int_0^{\infty} d\xi \xi \exp \left\{ -\frac{k \xi^2}{2(k-1)\sigma^2} \right\} \right. \\ \left. + \int_{-(\frac{k-1}{k})p_k}^0 d\xi \xi + \int_0^{\infty} d\xi \xi + \int_{-(\frac{k-1}{k})p_k}^0 d\xi \xi + \int_0^{\infty} d\xi \xi \frac{(k-1)p_k}{k} \exp \left\{ -\frac{k \xi^2}{2(k-1)\sigma^2} \right\} \right]$$

Integrations marked with a \ cancel because the integrand is an odd function of ξ . Integrations marked with a // cancel because the integrand is an even function of ξ . We thus have

$$n(p_k) = \frac{2 p_k N}{\pi \sigma^4 (k-1)^{3/2}} \exp \left\{ -\frac{p_k^2}{2k\sigma^2} \right\} \frac{(k-1)p_k}{k} \int_0^{\infty} d\xi \exp \left\{ -\frac{k \xi^2}{2(k-1)\sigma^2} \right\}.$$

The integral is given in Dwight's integral tables ⁽³⁾ number 861.3, p. 200.

Putting in the value of the integral and simplifying, one obtains

$$n(p_k) = \frac{N \sqrt{2}}{\sqrt{\pi}} \frac{p_k^2}{k^{3/2} \sigma^3} \exp \left\{ -\frac{p_k^2}{2k\sigma^2} \right\} \quad \text{Q.E.D.}$$

We have thus shown that if the distribution function for the magnitude of the vector sum of the momenta of the neutrons is given by this formula for $(k-1)$ neutrons it is also given for k neutrons. But it is given by this formula for one neutron. Hence, this formula holds for any number of neutrons.

We should also point out that the vector sum of the momenta will have a random (isotropic) orientation in the center of mass frame if (as we assume) the individual momenta have a random orientation in this frame.

FRAGMENT DEFLECTION PRODUCED BY PROMPT NEUTRONS

Definition of Terms: (see Fig. 3)

Let

u = initial velocity of fragment before emission of neutrons,

V_k = final velocity of fragment in center-of-mass frame, after emission of k neutrons

V_k' = final velocity of fragment in lab frame

P_i = initial momentum of fragment

P_k = final momentum of fragment, c.m. frame, after emission of k neutrons

P_k' = final momentum of fragment lab frame after emission of k neutrons.

To simplify equations, we go to the dimensionless quantities,

$$\alpha_k = \frac{V_k}{u} = \frac{P_k}{(M-km)} \cdot \frac{M}{P_i} = \left(\frac{M}{M-km} \right) \frac{P_k}{P_i}, \quad (6)$$

$$\alpha_k' = \frac{V_k'}{u} = \left(\frac{M}{M-km} \right) \frac{P_k'}{P_i}, \quad (7)$$

where

M = Mass of fragment before neutron emission,

m = mass of neutron.

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Fig. 3. Diagram showing change in fission fragment momentum and velocity due to prompt neutron emission.

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Fig. 4. Diagram showing fragment deflection as related to the initial and final velocities of the fragment.

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Fig. 5. Figures 3 and 4 redrawn with dimensionless quantities.

Consider a fragment traveling along the axis of the tube before emitting any neutrons. The prompt neutrons are emitted in a time less than 4×10^{-14} sec. The velocity of the fragment is of the order of 10^9 cm/sec. Hence the distance traveled before emission of neutrons is less than 10^{-4} cm. Thus the fragment may be considered as receiving all its deflection at the source. In traveling a distance L it will be deflected from the center a distance y (see Fig. 4) where

$$y = L \tan \phi.$$

We define the dimensionless quantity

$$z = \frac{y}{L} = \tan \phi. \quad (8)$$

Figures 3 and 4 can now be combined and redrawn as is shown in Fig. 5 in terms of the dimensionless quantities, α_k , α'_k and z . In the solution of this problem we will use the quantity $\mathcal{R}_k$ defined as follows:

$$\mathcal{R}_k = 2mT \left[\frac{M}{(M - km)P_i} \right]^2 = 2 \left[\frac{M\sigma}{(M - km)P_i} \right]^2. \quad (9)$$

One Fragment Deflected

Deflection for a given recoil momentum

Consider a coincidence experiment with a source and one detector with infinitesimal geometry (see Fig. 6). Consider the case where only the fragment which hits the large detector emits prompt neutrons. We wish to calculate the probability distribution for the deflection of this fragment.

Consider first the case where the magnitude P_k of the recoil momentum

Fig. 6. Schematic diagram of source and detectors where the finite sizes of the source and first detector are neglected.

Fig. 7. Diagram showing fast and slow components of a given type of fragment for a given deflection angle, ϕ .

is given and average over the directions (see Fig. 7). We see that for $\alpha < 1$ there are two angles θ_1 and θ_2 that give the same deflection z for a given relative recoil velocity α_k . From similar triangles in Fig. 7, we see that

$$z = \frac{\alpha \sin \theta}{1 + \alpha \cos \theta} . \quad (10)$$

For constant α_k but varying θ , the distribution in z is given by the relation

$$n(z) = n(\theta) \left| \frac{d\theta}{dz} \right| = \frac{n(\theta)}{\left| \frac{dz}{d\theta} \right|} .$$

Since the vector sum of the neutron momenta (and hence the recoil momentum of the fragment) is isotropic in the center of mass system we have

$$n(\theta) = n(\Omega) \left| \frac{d\Omega}{d\theta} \right| = \frac{N}{2} \left| \sin \theta \right| . \quad (11a)$$

Then

$$n(z) = \frac{N}{2} \left| \sin \theta \frac{d\theta}{dz} \right| . \quad (11b)$$

From equation (10) one obtains

$$\alpha^2(1+z^2)\cos^2\theta + 2\alpha z^2\cos\theta + (z^2 - \alpha^2) = 0.$$

Solving by means of the quadratic formula, one gets

$$\cos \theta = \frac{-z^2 \pm \sqrt{\alpha^2(1+z^2) - z^2}}{\alpha(1+z^2)}. \quad (12)$$

Differentiation gives

$$\sin \theta \frac{d\theta}{dz} = \frac{2z}{\alpha(1+z^2)^2} \left\{ 1 \pm \frac{1 + \alpha^2(1+z^2) - z^2}{2\sqrt{\alpha^2(1+z^2) - z^2}} \right\}.$$

Note that the positive sign corresponds to θ_1 and the negative sign corresponds to θ_2 , or

$$\sin \theta_1 \frac{d\theta_1}{dz} = \frac{2z}{\alpha(1+z^2)^2} \left\{ 1 + \frac{1 + \alpha^2(1+z^2) - z^2}{2\sqrt{\alpha^2(1+z^2) - z^2}} \right\},$$

$$\sin \theta_2 \frac{d\theta_2}{dz} = \frac{2z}{\alpha(1+z^2)^2} \left\{ 1 - \frac{1 + \alpha^2(1+z^2) - z^2}{2\sqrt{\alpha^2(1+z^2) - z^2}} \right\}.$$

By inspection of Fig. 7, it can be seen that whereas both $\sin \theta_1$ and $\sin \theta_2$ are positive, $d\theta_1/dz$ is positive and $d\theta_2/dz$ is negative. Hence

$$\left| \sin \theta_1 \frac{d\theta_1}{dz} \right| = \sin \theta_1 \frac{d\theta_1}{dz} = \frac{2z}{\alpha(1+z^2)^2} \left\{ 1 + \frac{1 + \alpha^2(1+z^2) - z^2}{2\sqrt{\alpha^2(1+z^2) - z^2}} \right\}, \quad (13a)$$

$$\left| \sin \theta_2 \frac{d\theta_2}{dz} \right| = -\sin \theta_2 \frac{d\theta_2}{dz} = \frac{2z}{\alpha(1+z^2)^2} \left\{ -1 + \frac{1 + \alpha^2(1+z^2) - z^2}{2\sqrt{\alpha^2(1+z^2) - z^2}} \right\}. \quad (13b)$$

From equation (11), we have

$$n(z) = \frac{N}{2} \left\{ \left| \sin \theta_1 \frac{d\theta_1}{dz} \right| + \left| \sin \theta_2 \frac{d\theta_2}{dz} \right| \right\}.$$

Substituting in equations (13) we obtain for the case where $\alpha < 1$

$$n(z) = \frac{N z}{\alpha(1+z^2)^2} \frac{[1 + \alpha^2(1+z^2) - z^2]}{\sqrt{\alpha^2(1+z^2) - z^2}} \quad (14)$$

This is the number of fragments out of a total of N which will strike the detector in a ring of inner radius $y = L z$ and outer radius $y + dy = L(z + dz)$. Also of interest is the number of fragments per unit area striking a small area of the target. We will define this quantity as

$$\rho(z) = \frac{dN}{dA} = \frac{dN}{2\pi y dy} = \frac{dN}{2\pi L^2 z dz} = \frac{n(z)}{2\pi L^2 z}.$$

Hence

$$\rho(z) = \frac{N}{2\pi L^2 \alpha(1+z^2)^2} \frac{[1 + \alpha^2(1+z^2) - z^2]}{\sqrt{\alpha^2(1+z^2) - z^2}} \quad (15)$$

In experiments where α and the maximum value of z are small in comparison to unity [they are $\sim .04$ in the experiment being planned] we have as a first approximation

$$n(z) = \frac{Nz}{\alpha} \frac{1}{\sqrt{\alpha^2 - z^2}}, \quad (16)$$

$$\rho(z) = \frac{N}{2\pi L^2 \alpha} \frac{1}{\sqrt{\alpha^2 - z^2}}. \quad (17)$$

For small values of z , one sees from Fig. 7 that

$$z \simeq \alpha \sin \theta,$$

so that

$$\rho(z) = \rho_0 \sec \theta, \quad (18)$$

where

$$\rho_0 = \frac{N}{2\pi L^2 \alpha}.$$

In Fig. 8 ρ/ρ_0 is plotted as a function of z/α for the case where α is small.

If we let

$$Z = R/L$$

where R is the radius of the large detector, than the fraction of fragments detected is

$$\frac{1}{N} \int_0^Z n(z) dz = 1 - \cos(\arcsin \frac{Z}{\alpha}). \quad (19)$$

The percentage of fragments detected is plotted as a function of (Z/α) in Fig. 9.

To make things more specific, consider the case where only one neutron is emitted by one of the fragments and this neutron has a known energy but unknown direction. Take the case where the fragments have masses 140 and 96 mass units. From Table 11.22, p. 205 of E. K. Hyde's report⁽⁴⁾ we have

$$E_H = 67 \text{ Mev}, E_L = 98 \text{ Mev} \quad (20)$$

for the energies of the heavy and light fragments, respectively. From this we can calculate the initial momentum of the fragments. We obtain

$$P_i = 4188 \text{ Mev/c.} \quad (21)$$

We then use equation (6) to calculate α_H for the case where the heavy fragment emits the neutron and α_L where the light fragment emits the neutron. We obtain Table I.

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Fig. 8. Probability density for fission fragment detection as a function of the tangent of the deflection angle for the case where the fragment recoils from prompt neutron emission with a given momentum parameter α defined by equation (6) of text.

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Fig. 9. Percentage of sister fragments detected by the large detector (for each fragment detected by the small detector) as a function of the ratio of the radius of the large detector to the product of its distance from the source and the recoil momentum parameter α (assuming a point source, a point small detector, and a known recoil momentum).

Table I

RECOIL MOMENTUM PARAMETER, α , FOR DIFFERENT
NEUTRON ENERGIES

Neutron Energy Mev	α_H	α_L
1	1.043×10^{-2}	1.046×10^{-2}
2	1.474	1.479
5	2.331	2.339
10	3.294	3.308
15	4.037	4.051

Using equation (19) and the mean value of α_H and α_L for α we can calculate the percent of the fragments detected for different values of Z .

The results are shown plotted in Fig. 10.

In the foregoing we have considered the case where $\alpha \leq 1$, i.e., $V \leq u$. This is certainly the region of most interest since a single neutron would have to have an energy around 10 Gev in order to cause the fragment to recoil with a center-of-mass velocity equal to its initial laboratory velocity. However, because we will want to integrate our momentum distribution to infinity, we might consider the case $\alpha > 1$.

In this case there is only one angle θ_1 that results in scattering in the forward direction. The angle θ_2 gives backward scattering. See Fig. 11.

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Fig. 10. Percentage of sister fragments detected by the large detector (for each fragment detected by the small detector, assumed to be a point) for one prompt neutron (of various energies as shown) as a function of the ratio of the radius of the large detector to its distance from the point source.

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Fig. 11. Diagram showing geometric relations between fragment deflection angles for $\alpha > 1$, (α defined by equation (6) in text).

If we retain as our definition of z

$$z = \tan \phi,$$

then we see that for backward scattering z is negative. This can be interpreted geometrically as shown in Fig. 11. The dotted lines in Fig. 11 are an aid in seeing the geometrical relationships.

It is easy to show that the same relationships hold for $\alpha > 1$ as for $\alpha < 1$ through equation (13a). We have again

$$\begin{aligned} \sin \theta_2 \frac{d\theta_2}{dz} &= \frac{2z}{\alpha(1+z^2)^2} \left\{ 1 - \frac{1 + \alpha^2(1+z^2)^2 - z^2}{2\sqrt{\alpha^2(1+z^2)^2 - z^2}} \right\}, \\ &= \frac{-z}{\alpha(1+z^2)^2 \sqrt{\alpha^2(1+z^2)^2 - z^2}} \left[\sqrt{\alpha^2(1+z^2)^2 - z^2} - 1 \right]^2. \end{aligned}$$

However, since z is now negative we have

$$\left| \sin \theta_2 \frac{d\theta_2}{dz} \right| = \frac{|z|}{\alpha(1+z^2)^2 \sqrt{\alpha^2(1+z^2) - z^2}} \left[\sqrt{\alpha^2(1+z^2) - z^2} - 1 \right]^2$$

$$= \frac{-z}{\alpha(1+z^2)^2 \sqrt{\alpha^2(1+z^2) - z^2}} \left[\sqrt{\alpha^2(1+z^2) - z^2} - 1 \right]^2 \quad (22)$$

In place of (14) we now have for $\alpha > 1, z > 1$

$$n(z) = \frac{Nz \left[\sqrt{\alpha^2(1+z^2) - z^2} + 1 \right]^2}{2\alpha(1+z^2)^2 \sqrt{\alpha^2(1+z^2) - z^2}}, \quad (23)$$

and for $\alpha > 1, z < 1$

$$n(z) = -\frac{Nz \left[\sqrt{\alpha^2(1+z^2) - z^2} - 1 \right]^2}{2\alpha(1+z^2)^2 \sqrt{\alpha^2(1+z^2) - z^2}}. \quad (24)$$

Probable deflection for a distribution in recoil momenta

We now consider the case where the magnitude of the recoil momentum is not fixed but given by a probability distribution.

We have

$$\vec{P}_k = -\vec{p}_k,$$

$$|\vec{p}_k| = |\vec{P}_k| = \frac{(M - km)}{M} P_i \alpha_k.$$

Thus

$$n(P_k) = n(p_k),$$

and

$$n(\alpha_k) = n(P_k) \frac{dP_k}{d\alpha_k} = n(P_k) \left(\frac{M - km}{M} \right) P_i.$$

Making use of equations (4), (6) and (9) we obtain

$$n(\alpha_k) = \frac{4N}{\sqrt{\pi}} \frac{\alpha_k^2}{(k \alpha e_k)^{3/2}} \exp \left\{ -\frac{\alpha_k^2}{k \alpha e_k} \right\}. \quad (25)$$

To obtain the distribution in z we replace N in equations (14) and (23) by $n(\alpha_k)$ and integrate over α_k . For a given z there is a minimum value of α_k that will contribute. From Fig. 12 we see that

$$\alpha_{k \min} = \frac{z}{\sqrt{1+z^2}}.$$

Then making use of equations (14), (23), and (25) we have for positive z :

$$\begin{aligned} n(z) = & \frac{4 N z}{\sqrt{\pi}(1+z^2)^2 (k \mathcal{L}_k)^{3/2}} \int_{\alpha_{k \min}}^1 \frac{[1 + \alpha_k^2 (1+z^2) - z^2]}{\sqrt{\alpha_k^2 (1+z^2) - z^2}} \exp \left\{ - \frac{\alpha_k^2}{k \mathcal{L}_k} \right\} \alpha_k d\alpha_k \\ & + \frac{2 N z}{\sqrt{\pi}(1+z^2)^2 (k \mathcal{L}_k)^{3/2}} \int_1^{\infty} \frac{[\sqrt{\alpha_k^2 (1+z^2) - z^2} + 1]^2}{\sqrt{\alpha_k^2 (1+z^2) - z^2}} \exp \left\{ - \frac{\alpha_k^2}{k \mathcal{L}_k} \right\} \alpha_k d\alpha_k. \end{aligned}$$

Making the change in variables

$$y = \sqrt{\alpha_k^2 (1+z^2) - z^2},$$

we obtain

$$\begin{aligned} n(z) = & \frac{4 N}{\sqrt{\pi}(k \mathcal{L}_k)^{3/2}} \frac{z}{(1+z^2)^3} \int_0^1 (1+y^2) \exp \left\{ - \frac{(y^2+z^2)}{k \mathcal{L}_k(1+z^2)} \right\} dy \\ & + \frac{2 N}{\sqrt{\pi}(k \mathcal{L}_k)^{3/2}} \frac{z}{(1+z^2)^3} \int_1^{\infty} (y+1)^2 \exp \left\{ - \frac{(y^2+z^2)}{k \mathcal{L}_k(1+z^2)} \right\} dy, \end{aligned}$$

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Fig. 12. Geometrical construction showing the minimum α_k for a given z (α_k defined by equation (6) of text).

$$n(z) = \frac{4 N}{\sqrt{\pi(k \mathfrak{A}_k)^3}} \frac{z}{(1+z^2)^3} \exp \left\{ \frac{-z^2}{k \mathfrak{A}_k (1+z^2)} \right\} \left[\int_0^{\infty} (1+y^2) \exp \left\{ \frac{-y^2}{k \mathfrak{A}_k (1+z^2)} \right\} dy \right. \\ \left. - \frac{1}{2} \int_1^{\infty} (y-1)^2 \exp \left\{ \frac{-y^2}{k \mathfrak{A}_k (1+z^2)} \right\} dy \right] \quad (26).$$

The first integral in the above expression can be evaluated easily by reference to Dwight⁽³⁾ equations 861.3 and 861.7, p. 200. One obtains

$$\int_0^{\infty} (1+y^2) \exp \left\{ \frac{-y^2}{k \mathfrak{A}_k (1+z^2)} \right\} dy = \frac{\sqrt{\pi}}{2} \sqrt{k \mathfrak{A}_k (1+z^2)} + \frac{\sqrt{\pi}}{4} [k \mathfrak{A}_k (1+z^2)]^{3/2} \\ = \frac{1}{2} \sqrt{\pi k \mathfrak{A}_k (1+z^2)} \left[1 + \frac{k \mathfrak{A}_k (1+z^2)}{2} \right] \quad (27)$$

Consider the second integral

$$\int_1^{\infty} (y-1)^2 \exp \left\{ \frac{-y^2}{B} \right\} dy = \int_1^{\infty} y^2 e^{-y^2/B} dy - 2 \int_1^{\infty} y e^{-y^2/B} dy \\ + \int_1^{\infty} e^{-y^2/B} dy, \quad (28)$$

where $B = k \mathfrak{A}_k (1 + z^2)$.

The first integral of the right-hand member of the above equation may be integrated by parts

$$\begin{aligned} \int_1^{\infty} y^2 e^{-y^2/B} dy &= -\frac{B}{2} \left[y e^{-y^2/B} + \frac{B}{2} \int_1^{\infty} e^{-y^2/B} dy \right] \\ &= \frac{B}{2} e^{-1/B} + \frac{B}{2} \int_1^{\infty} e^{-y^2/B} dy. \end{aligned}$$

The second integral of equation (28) is directly integrable,

$$-2 \int_1^{\infty} y e^{-y^2/B} dy = -2 \left[-\frac{B}{2} e^{-y^2/B} \right]_1^{\infty} = B e^{-1/B}.$$

We have then that

$$\int_1^{\infty} (y-1)^2 \exp \left\{ \frac{-y^2}{B} \right\} dy = -\frac{B}{2} e^{-1/B} + \frac{(B+2)}{2} \int_1^{\infty} e^{-y^2/B} dy.$$

In the integral on the right of the above equation let

$$t = \frac{y}{\sqrt{B}},$$

then

$$\begin{aligned} \int_1^{\infty} e^{-y^2/B} dy &= \sqrt{B} \int_{1/\sqrt{B}}^{\infty} e^{-t^2} dt \\ &= \frac{\sqrt{\pi B}}{2} \left[\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_0^{1/\sqrt{B}} e^{-t^2} dt \right] \\ &= \frac{\sqrt{\pi B}}{2} \left[1 - \operatorname{erf} \left\{ \frac{1}{\sqrt{B}} \right\} \right]. \end{aligned}$$

In the above we have made use of entries 861.3 and 590. in Dwight's tables⁽³⁾ p. 200 and p. 129. Thus

$$\int_1^{\infty} (y-1)^2 \exp \left\{ \frac{-y^2}{B} \right\} dy = -\frac{B}{2} e^{-1/B} + \frac{(B+2)}{4} \sqrt{\pi B} \left[1 - \operatorname{erf} \left(\frac{1}{\sqrt{B}} \right) \right]. \quad (29)$$

Substituting (27) and (29) into (26) one obtains for positive z

$$n(z) = \frac{2N}{k \mathfrak{A}_k} \frac{z}{(1+z^2)^{5/2}} \exp \left\{ \frac{-z^2}{k \mathfrak{A}_k (1+z^2)} \right\} \times$$

$$\left[\left\{ 1 + \frac{k \mathfrak{A}_k (1+z^2)}{2} \right\} + \frac{\sqrt{k \mathfrak{A}_k (1+z^2)}}{2\sqrt{\pi}} \exp \left\{ \frac{-1}{k \mathfrak{A}_k (1+z^2)} \right\} \right.$$

$$\left. - \frac{1}{2} \left\{ 1 + \frac{k \mathfrak{A}_k (1+z^2)}{2} \right\} \left\{ 1 - \operatorname{erf} \left(\frac{1}{\sqrt{k \mathfrak{A}_k (1+z^2)}} \right) \right\} \right] \quad (30)$$

In terms of the angle ϕ we have from equation (8)

$$n(\phi) = n(z) \frac{dz}{d\phi} = n(z) \sec^2 \phi.$$

Substituting into equation (30)

$$n(\phi) = \frac{2N}{k \mathfrak{A}_k} \sin \phi \cos^2 \phi \exp \left\{ -\frac{\sin^2 \phi}{k \mathfrak{A}_k} \right\} \times$$

$$\left[\left\{ 1 + \frac{k \mathfrak{A}_k \sec^2 \phi}{2} \right\} + \frac{\sqrt{k \mathfrak{A}_k \sec^2 \phi}}{2\sqrt{\pi}} \exp \left\{ -\frac{\cos^2 \phi}{k \mathfrak{A}_k} \right\} \right.$$

$$\left. - \frac{1}{2} \left\{ 1 + \frac{k \mathfrak{A}_k \sec^2 \phi}{2} \right\} \left\{ 1 - \operatorname{erf} \left(\frac{\cos \phi}{\sqrt{k \mathfrak{A}_k}} \right) \right\} \right] \quad (31)$$

We can use the expansion of the error function in a power series in $k \mathfrak{A}_k (1+z^2)$ given as formula 592. in Dwight's tables⁽³⁾ to obtain the expansion

$$n(z) = \frac{2N}{k \mathfrak{A}_k} \frac{z}{(1+z^2)^{5/2}} \exp \left\{ \frac{-z^2}{k \mathfrak{A}_k (1+z^2)} \right\} \times$$

$$\left[\left\{ 1 + \frac{k \mathfrak{A}_k (1+z^2)}{2} \right\} - \frac{\sqrt{k \mathfrak{A}_k (1+z^2)}}{2\sqrt{\pi}} \exp \left\{ \frac{-1}{k \mathfrak{A}_k (1+z^2)} \right\} \sum_{n=0}^{\infty} \frac{(n+1)(2n)!}{n! 2^{2n}} \left\{ k \mathfrak{A}_k (1+z^2) \right\}^{n+1} \right] \quad (32)$$

This expansion is useful where $k\alpha_k (1 + z^2)$ is small.

Consider the case where the fragments are 96 and 140 mass units and three neutrons are emitted by the light fragment, then from equation (20) $P_i = 4188 \text{ Mev/c}$. We can then calculate $k\alpha_k$, using equation (9). We obtain $k\alpha_k = 4.418 \times 10^{-4}$. Now $z_{\max} = R/\bar{L}$ where R is the radius of the large detector and $\bar{L}$ is the flight path. For $R = 6 \text{ in}$ and $L = 12 \text{ ft}$, $z_{\max} = 1/24 = .0417$ and $z_{\max}^2 = 1.736 \times 10^{-3}$. Hence

$$k\alpha_k (1 + z_{\max}^2) = 4.426 \times 10^{-4}.$$

Thus we make an error of only .04% in neglecting this term in comparison to unity. Thus we can write

$$n(z) = \frac{2N}{k\alpha_k} \frac{z}{(1+z^2)^{5/2}} \exp \left\{ -\frac{z^2}{k\alpha_k(1+z^2)} \right\}, \quad (33a)$$

$$\rho(z) = \frac{N}{\pi L^2 k\alpha_k} \frac{1}{(1+z^2)^{5/2}} \exp \left\{ -\frac{z^2}{k\alpha_k(1+z^2)} \right\}. \quad (33b)$$

In terms of the angle ϕ we have to the same approximation

$$n(\phi) = \frac{2N}{k\alpha_k} \sin \phi \cos^2 \phi \exp \left\{ -\frac{\sin^2 \phi}{k\alpha_k} \right\}. \quad (34)$$

Further we make a maximum error of only 0.2% in neglecting z^2 in comparison to unity.*

*In integrating $n(z)$ we make an even smaller error since $n(z)$ is largest for small values of z where the error is the least.

Making this approximation we have

$$n(z) = \frac{2N}{k \partial e_k} z \exp \left\{ - \frac{z^2}{k \partial e_k} \right\}, \quad (35a)$$

$$\rho(z) = \frac{N}{\pi L^2 k \partial e_k} \exp \left\{ - \frac{z^2}{k \partial e_k} \right\}. \quad (35b)$$

Both Fragments Deflected

Consider the case where the source and small detector have infinitesimal geometry. (Correction for finite geometry must be done separately). Call the fragment hitting the small detector the first fragment and the fragment hitting the large detector the second fragment. Let ϕ_1 be the angle of deflection of the first fragment due to neutron emission and ϕ_2 the same thing for the second fragment (see Fig. 13). Let k_1 and k_2 be the number of neutrons emitted by the first and second fragments, respectively.

Fig. 14. Drawing showing three-dimensional relationship of geometrical parameters for the case where both fragments emit neutrons.

Let $\vec{L}$ be a vector along the line of centers of the two detectors directed from the source toward the large detector. Then from Fig. 14,

$$\begin{aligned}\cos \gamma &= \frac{1}{P L} \vec{P}_f \cdot \vec{L} \\ &= \frac{1}{P L} (P_{fx} L_x + P_{fy} L_y + P_{fz} L_z).\end{aligned}$$

Take Z axis along $\vec{P}_i$ and Y axis in plane of $\vec{P}_i$ and $\vec{L}$. Thus

$$\begin{aligned}P_{fx} &= P_f \sin \phi_2 \sin C, & L_x &= 0, \\ P_{fy} &= P_f \sin \phi_2 \cos C, & L_y &= L \sin \phi_1, \\ P_{fz} &= P_f \cos \phi_2, & L_z &= L \cos \phi_1.\end{aligned}$$

Then

$$\cos \gamma = \sin \phi_1 \sin \phi_2 \cos C + \cos \phi_1 \cos \phi_2. \quad (36)$$

First consider a given ϕ_1 and ϕ_2 and get the distribution in γ by varying the angle C. Then

$$n(\gamma) = n(C) \left| \frac{dC}{d\gamma} \right| = \frac{N}{2\pi} \left| \frac{dC}{d\gamma} \right|,$$

since we expect the distribution in C to be random (isotropic). Taking the differential of equation (36) one obtains

$$\frac{dC}{d\gamma} = \frac{\sin \gamma}{\sin \phi_1 \sin \phi_2 \sin C}.$$

We want to eliminate C from this expression. We have

$$\begin{aligned} \sin^2 C &= 1 - \cos^2 C = 1 - \frac{[\cos \gamma - \cos \phi_1 \cos \phi_2]^2}{\sin^2 \phi_1 \sin^2 \phi_2} \\ &= \frac{\sin^2 \phi_1 \sin^2 \phi_2 - \cos^2 \gamma + 2 \cos \gamma \cos \phi_1 \cos \phi_2 - \cos^2 \phi_1 \cos^2 \phi_2}{\sin^2 \phi_1 \sin^2 \phi_2}. \end{aligned}$$

The numerator of this last expression can be written

$$\begin{aligned} &(\sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2)(\sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2) \\ &\quad + 2 \cos \gamma \cos \phi_1 \cos \phi_2 - \cos^2 \gamma \\ &= [\cos(\phi_1 - \phi_2) - \cos \gamma] [\cos \gamma - \cos(\phi_1 + \phi_2)]. \end{aligned}$$

Hence

$$\sin C = \pm \sqrt{\frac{[\cos(\phi_1 - \phi_2) - \cos \gamma] [\cos \gamma - \cos(\phi_1 + \phi_2)]}{\sin \phi_1 \sin \phi_2}}.$$

Thus

$$n(\gamma) = \frac{N}{2\pi} \frac{2 \sin \gamma}{\sqrt{[\cos(\phi_1 - \phi_2) - \cos \gamma] [\cos \gamma - \cos(\phi_1 + \phi_2)]}}. \quad (37)$$

The factor 2 in the numerator arises from the fact that there are two values of the angle C that give the same γ namely C_1 and $C_2 = 2\pi - C_1$.

From the way γ , ϕ_1 , ϕ_2 are defined each one must lie between 0 and π . They have no meaning outside this interval. Therefore, for a given ϕ_1 and ϕ_2

$$\gamma_{\min} = |\phi_1 - \phi_2| ,$$

$$\gamma_{\max} = \text{smaller of } (\phi_1 + \phi_2) \text{ and } (2\pi - \phi_1 - \phi_2).$$

For a distribution in ϕ_1 and ϕ_2 we replace N in equation (37) by $n(\phi_1) \times n(\phi_2)/N$ and integrate over ϕ_1 and ϕ_2 . Then

$$n(\gamma) = \frac{\sin \gamma}{\pi N} \int_0^\pi d\phi_2 \int_{L_1}^U d\phi_1 \frac{n(\phi_2)n(\phi_1)}{\sqrt{[\cos(\phi_1 - \phi_2) - \cos \gamma][\cos \gamma - \cos(\phi_1 + \phi_2)]}} \quad (38)$$

$$\text{where } L = |\gamma - \phi_2| ,$$

$$U = \text{smaller of } (\gamma + \phi_1) \text{ and } (2\pi - \gamma - \phi_1) .$$

By means of some algebraic and trigonometric manipulation one can show that

$$\begin{aligned} & [\cos(\phi_1 - \phi_2) - \cos \gamma] [\cos \gamma - \cos(\phi_1 + \phi_2)] \\ &= [\cos(\gamma - \phi_2) - \cos \phi_1] [\cos \phi_1 - \cos(\gamma + \phi_2)] . \end{aligned} \quad (39)$$

Substituting this and the expression for $n(\phi_1)$ given by equation (34) into equation (38) one obtains

$$n(\gamma) = \frac{2 \sin \gamma}{\pi k_1 \varpi_{k_1}} \int_0^\pi d\phi_2 n(\phi_2) \int_L^U \frac{d\phi_1 \sin \phi_1 \cos^2 \phi_1 \exp\{-\sin^2 \phi_1 / k \varpi_k\}}{\sqrt{[\cos(\gamma - \phi_2) - \cos \phi_1] [\cos \phi_1 - \cos(\gamma - \phi_2)]}} k.$$

This expression does not seem to be readily integrable in closed form as it stands. However, if we make use of the fact that $k \varpi_k$ is of the order of 4×10^{-4} for three neutrons, we see that the integrand in the above expression is appreciable only for small values of ϕ_1 , ϕ_2 , and γ . Thus let

$$\sin \beta \rightarrow \beta, \quad \cos \beta \rightarrow 1 - \frac{\beta^2}{2}$$

where $\beta = \phi_1, \phi_2, \gamma$. Then

$$n(\gamma) \approx \frac{2 \sin \gamma}{\pi k_1 \varpi_{k_1}} \int_0^\pi d\phi_2 n(\phi_2) \int_{|\gamma - \phi_2|}^{\gamma + \phi_2} d\phi_1 \frac{2 \phi_1 (1 - \phi_1^2)}{\sqrt{[\phi_1^2 - (\gamma - \phi_2)^2][(\gamma + \phi_2)^2 - \phi_1^2]}} \exp\left\{-\frac{\phi_1^2}{k_1 \varpi_{k_1}}\right\}. \quad (41)$$

Let I_1 be the integral over ϕ_1 in the above equation. Make the change of variables in I_1 to

$$\xi = \phi_1^2 / k_1 \varpi_{k_1}.$$

Then

$$I_1 = \int_{L'}^{U'} d\xi \frac{(1 - k_1 \varpi_{k_1} \xi)}{\sqrt{(U' - \xi)(\xi - L')}} e^{-\xi}, \quad (42)$$

where

$$U' = \frac{(\gamma + \phi_2)^2}{k_1 \varpi_{k_1}}, \quad L' = \frac{(\gamma - \phi_2)^2}{k_1 \varpi_{k_1}}$$

Make the further change of variables

$$\eta = \xi - \left(\frac{U' + L'}{2} \right),$$

and let

$$L_2 = (U' - L')/2.$$

Then

$$I_1 = \exp \left\{ - \frac{(U' + L')}{2} \right\} \int_{-L_2}^{+L_2} d\eta \frac{e^{-\eta}}{\sqrt{L_2^2 - \eta^2}} \quad (43)$$

where we have neglected a term of the integrand in $k x_k$. The above expression resembles the integral form of a Bessel function. From Wood's "Advanced Calculus"⁽⁵⁾, p. 283, we have that

$$J_n(x) = \frac{x^n}{1 \cdot 3 \cdots (2n-1)\pi} \int_{-1}^{+1} e^{ixt} (1-t^2)^{n-1/2} dt \quad (44)$$

where $J_n(x)$ is the Bessel function of argument x and order n where n is an integer. To put I_1 in this form, make the change of variables

$$\xi = \eta/L_2.$$

Then

$$I_1 = \exp \left\{ - \frac{(U' + L')}{2} \right\} \int_{-1}^{+1} d\xi \frac{e^{-L_2 \xi}}{\sqrt{(1 - \xi^2)^n}},$$

or

$$I_1 = \exp \left\{ - \frac{(U' + L')}{2} \right\} \pi J_0(iL_2). \quad (45)$$

From the definitions of U' , L' , and L_2 we see that

$$\frac{(U'+L')}{2} = \frac{(\gamma^2 + \phi_2^2)}{k_1 \mathcal{A}_{k_1}},$$

$$L_2 = \frac{2 \gamma \phi_2}{k_1 \mathcal{A}_{k_1}}.$$

Putting this into (45) we have

$$I_1 = \exp \left\{ - \frac{(\gamma^2 + \phi_2^2)}{k_1 \mathcal{A}_{k_1}} \right\} \pi J_0 \left(i \frac{2 \gamma \phi_2}{k_1 \mathcal{A}_{k_1}} \right). \quad (46)$$

Putting this into equation (41) we have

$$n(\gamma) = \frac{2 \sin \gamma}{\pi k_1 \mathcal{A}_{k_1}} \pi \exp \left\{ - \frac{\gamma^2}{k_1 \mathcal{A}_{k_1}} \right\} \int_0^\pi d\phi_2 n(\phi_2) J_0 \left(\frac{i 2 \gamma \phi_2}{k_1 \mathcal{A}_{k_1}} \right) \exp \left\{ - \frac{\phi_2^2}{k_1 \mathcal{A}_{k_1}} \right\}. \quad (47)$$

Using equation (34) for $n(\phi_2)$

$$n(\gamma) = \frac{2 \sin \gamma}{k_1 \mathcal{A}_{k_1}} \exp \left\{ - \frac{\gamma^2}{k_1 \mathcal{A}_{k_1}} \right\} \int_0^\pi d\phi_2 \frac{2N}{\phi_2 \mathcal{A}_{k_2}} \sin \phi_2 \cos^2 \phi_2 \exp \left\{ - \frac{\sin^2 \phi_2}{k_2 \mathcal{A}_{k_2}} \right\} \\ \times J_0 \left(\frac{i 2 \gamma \phi_2}{k_1 \mathcal{A}_{k_1}} \right) \exp \left\{ - \frac{\phi_2^2}{k_1 \mathcal{A}_{k_1}} \right\}.$$

Since $k_2 \mathcal{A}_{k_2}$ is small, only small values of ϕ_2 will contribute appreciably to the integral. Thus $\sin \phi_2 \rightarrow \phi_2$, $\cos \phi_2 \rightarrow 1$. We will also neglect terms in $k_2 \mathcal{A}_{k_2}$. Then

$$n(\gamma) = \frac{4N \sin \gamma}{k_1 \mathcal{A}_{k_1} k_2 \mathcal{A}_{k_2}} \exp \left\{ - \frac{\gamma^2}{k_1 \mathcal{A}_{k_1}} \right\} \int_0^\pi d\phi_2 \phi_2 J_0 \left(\frac{i 2 \gamma \phi_2}{k_1 \mathcal{A}_{k_1}} \right) \exp \left\{ - \frac{\phi_2^2}{D} \right\}, \quad (48)$$

$$\text{where } \frac{1}{D} = \frac{1}{k_1 \mathfrak{e}_{k_1}} + \frac{1}{k_2 \mathfrak{e}_{k_2}}, \quad D = \frac{k_1 \mathfrak{e}_{k_1} k_2 \mathfrak{e}_{k_2}}{k_1 \mathfrak{e}_{k_1} + k_2 \mathfrak{e}_{k_2}}. \quad (49)$$

$$\text{Let } \eta = \frac{\phi_2^2}{D}.$$

Then

$$n(\gamma) = \frac{2ND \sin \gamma}{k_1 \mathfrak{e}_{k_1} k_2 \mathfrak{e}_{k_2}} \exp \left\{ -\frac{\gamma^2}{k_1 \mathfrak{e}_{k_1}} \right\} \int_0^\infty d\eta \quad e^{-\eta} J_0 \left(\frac{i2\gamma}{k_1 \mathfrak{e}_{k_1}} \sqrt{D\eta} \right). \quad (50)$$

Here we have replaced the upper limit of π^2/D by ∞ because π^2/D is large and the integrand is vanishingly small from π^2/D to ∞ . We now make use of a series expansion for $J_0(ix)$. From Woods', Advanced Calculus⁽⁵⁾, p. 283, we have

$$J_0(ix) = \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m} (m!)^2}.$$

Hence

$$J_0 \left(\frac{i2\gamma}{k_1 \mathfrak{e}_{k_1}} \sqrt{D\eta} \right) = \sum_{m=0}^{\infty} \frac{\gamma^{2m} D^m \eta^m}{(k_1 \mathfrak{e}_{k_1})^{2m} (m!)^2}.$$

Thus

$$n(\gamma) = \frac{2 \sin \gamma}{k_1 \mathfrak{e}_{k_1} + k_2 \mathfrak{e}_{k_2}} \exp \left\{ -\frac{\gamma^2}{k_1 \mathfrak{e}_{k_1}} \right\} \sum_{m=0}^{\infty} \left(\frac{\gamma}{k_1 \mathfrak{e}_{k_1}} \right)^{2m} \frac{D^m}{(m!)^2} \int_0^\infty \eta^m e^{-\eta} d\eta \quad (51)$$

But

$$\int_0^\infty \eta^m e^{-\eta} d\eta = \Gamma(m+1) = m!.$$

So

$$\begin{aligned}
 n(\gamma) &= \frac{2 N \sin \gamma}{k_1 \alpha_{k_1} + k_2 \alpha_{k_2}} \exp \left\{ -\frac{\gamma^2}{k_1 \alpha_{k_1}} \right\} \sum_{m=0}^{\infty} \left(\frac{\gamma}{k_1 \alpha_{k_1}} \right)^{2m} \frac{D^m}{(m!)} \\
 &= \frac{2 N \sin \gamma}{k_1 \alpha_{k_1} + k_2 \alpha_{k_2}} \exp \left\{ -\frac{\gamma^2}{k_1 \alpha_{k_1}} \right\} \exp \left\{ \frac{D \gamma^2}{(k_1 \alpha_{k_1})^2} \right\}.
 \end{aligned}$$

Combining the two exponential functions leads to

$$n(\gamma) = \frac{2 N \sin \gamma}{(k_1 \alpha_{k_1} + k_2 \alpha_{k_2})} \exp \left\{ -\frac{\gamma^2}{k_1 \alpha_{k_1} + k_2 \alpha_{k_2}} \right\}. \quad (52)$$

(This is about what one would have expected from the addition of random angles).

Letting $z = \frac{y}{L} = \tan \gamma \simeq \gamma$,

$$n(z) = n(\gamma) \frac{d\gamma}{dz} = n(\gamma) \frac{1}{\sec^2 \gamma} \simeq n(\gamma),$$

we then arrive at the following approximate expressions:

$$n(z) = \frac{2 N z}{k_1 \alpha_{k_1} + k_2 \alpha_{k_2}} \exp \left\{ -\frac{z^2}{k_1 \alpha_{k_1} + k_2 \alpha_{k_2}} \right\}, \quad (53a)$$

$$\rho(z) = \frac{N}{\pi L^2 (k_1 \alpha_{k_1} + k_2 \alpha_{k_2})} \exp \left\{ -\frac{z^2}{k_1 \alpha_{k_1} + k_2 \alpha_{k_2}} \right\}. \quad (53b)$$

Calculations for a Particular Case

Consider the case where the U^{236} divides into two fragments of initial mass 96 m.u. and 140 m.u. ($M_H/M_L = 1.458$). From Table 11.22,

p. 205 of E. K. Hyde's review⁽⁴⁾ we have

$$E_H = 67 \text{ Mev}, \quad E_L = 98 \text{ Mev}, \quad E_{\text{tot}} = 165 \text{ Mev}$$

where E_H , E_L , are the energies of the heavy and light fragments, respectively, and E_{tot} is the total energy.

For this case

$$P_i^2 = 2M_H E_H = 2M_L E_L = 17.54 \times 10^6 \left(\frac{\text{Mev}}{c} \right)^2$$

and

$$P_i = 4.19 \times 10^3 \frac{\text{Mev}}{c}.$$

We can then calculate the velocities v_H and v_L of the heavy and light fragments, respectively,

$$\begin{aligned} v_H &= \frac{P_i}{M_H} = 0.0321 c \\ &= 9.63 \times 10^8 \text{ cm sec}^{-1}, \end{aligned}$$

$$\begin{aligned} v_L &= \frac{P_i}{M_L} = 0.0469 c \\ &= 14.06 \times 10^8 \text{ cm sec}^{-1}. \end{aligned}$$

Also

$$k \alpha_k = 2k_n T \left[\frac{M}{(M-k_n) P_i} \right]^2 - \left(\frac{2 m T}{P_i^2} \right) k \left(1 - \frac{k_n}{M} \right)^{-2}$$

where

$$\left(\frac{2 m T}{P_i^2} \right) = 1.382 \times 10^{-4}$$

for this case.

We can then prepare a table of $k \mathcal{E}_k$ as a function of k and M .

This is done for the case considered here in Table II.

Table II

Values of $k \mathcal{E}_k$

	(140 mu)	(96 mu)
k	Heavy Fragment	Light Fragment
1	1.049×10^{-4}	1.055×10^{-4}
2	2.13×10^{-4}	2.16×10^{-4}
3	3.24×10^{-4}	3.30×10^{-4}
4	4.38×10^{-4}	4.50×10^{-4}
5	5.56×10^{-4}	5.75×10^{-4}
6	6.77×10^{-4}	7.06×10^{-4}
7	8.02×10^{-4}	8.42×10^{-4}

Let us now calculate the number of fragments hitting the large detector in a circle of radius R when we have a point source and a point small detector. Let

$$Z = z_{\max} = \frac{R}{L} .$$

$$\text{Let } k_1 \mathcal{E}_{k1} + k_2 \mathcal{E}_{k2} = 2\sigma^2 . \quad (54)$$

Then equation (53a) may be written

$$n(z) = \frac{Nz}{\sigma^2} \exp \left\{ -\frac{z^2}{2\sigma^2} \right\} .$$

Let N' be the number detected. Then the fraction detected is given by the relation

$$\begin{aligned}\frac{N'}{N} &= \frac{1}{N} \int_0^Z n(z) dz \\ &= \frac{1}{\sigma^2} \int_0^Z z \exp \left\{ -\frac{z^2}{2\sigma^2} \right\} dz.\end{aligned}$$

The above expression is readily integrated to give

$$\frac{N'}{N} = 1 - \exp \left\{ -\frac{Z^2}{2\sigma^2} \right\} \quad (55a)$$

$$= 1 - \exp \left\{ -\frac{R^2}{2L^2\sigma^2} \right\} . \quad (55b)$$

The fraction detected is plotted as a function of Z/σ in Fig. 15 which is a universal curve applying for any number of neutrons emitted by either fragment.

In order to make equations (55) and Fig. 15 applicable to specific cases we need to use equation (54) to calculate σ for specific cases. The results of such a calculation for the two extreme cases where all the neutrons are emitted by either the light or heavy fragment are given in Table III.

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Fig. 15. Fraction of sister fragments detected by the large detector (for each fragment detected by the small detector) as a function of Z/σ .

Table III

<u>σ_k for k Prompt Neutrons</u>		
<u>k</u>	<u>Heavy Fragment Emits All</u>	<u>Light Fragment Emits All</u>
1	0.00720	0.00723
2	0.01027	0.01034
3	0.01266	0.01279
4	0.01473	0.01494
5	0.01659	0.01688
6	0.01831	0.01870
7	0.01993	0.02042

Making use of Table III we can plot the percentage of fragment pairs detected as a function of the ratio of the radius of the large detector R to the distance of the source from the large detector L. This has been done in Fig. 16a and Fig. 16b for one through six neutrons all emitted by the heavy fragment and for one and six neutrons all emitted from the light fragment. (As can be seen the per cent detected is not very sensitive to which fragment emits the neutrons. When part of the neutrons are emitted by one fragment and the rest are emitted by the other fragment, the curve will be between the curve for all neutrons from the heavy fragment are the curve for all neutrons from the light fragment.)

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Fig. 16a. Percentage of sister fragments detected by the large detector (for each fragment detected by the small detector) as a function of the ratio of the radius of the large detector to its distance from the source (assuming a point source, a point small detector, and a Watt energy distribution for the prompt neutrons).

Fig. 16b. Expanded section of Fig. 16a.

Tables IV and V have been prepared for the specific conditions planned in the proposed experiment at ORNL.

It should be kept in mind that none of these tables or curves take into account the finite sizes of the source and small detector.

Table IV

Percent of fragments detected by a large detector of 6 inch radius located at a distance of 12 feet from the source ($M_H = 140$ m.u., $M_L = 96$ m.u.)

No. of Neutrons <u>k</u>	All Neutrons Emitted by	
	<u>Heavy Fragment</u>	<u>Light Fragment</u>
1	~ 100.00	~ 100.00
2	99.97	99.97
3	99.53	99.48
4	98.10	97.88
5	95.60	95.10
6	92.30	91.45
7	88.52	87.28

Table V

Radius of circle (in inches) containing 99.0% of the deflected fragments at a distance of 12 ft from the source ($M_H = 140$ m.u., $M_L = 96$ m.u.)

No. of Neutrons <u>k</u>	All Neutrons Emitted by	
	<u>Heavy Fragment</u>	<u>Light Fragment</u>
1	3.19	3.20
2	4.54	4.58
3	5.61	5.66
4	6.51	6.61
5	7.34	7.46
6	8.10	8.35
7	8.81	9.03

DISTRIBUTION IN TIME OF FLIGHT OF FISSION FRAGMENTS DUE TO PROMPT NEUTRON EMISSION

We now consider the distribution in time of arrival of fragments of a given mass and energy. This distribution is caused by the emission of prompt neutrons. To simplify the discussion we break it down into two cases.

Initial Velocity Along Axis

Consider the case where we have a point source, a point small detector, and where only the fragment which hits the large detector emits neutrons. In order to be in coincidence this fragment must have an initial velocity along the axis of the time-of-flight tube (see Fig. 6, Fig. 4, and Fig. 5).

The transit time of the fragment, if no neutrons are emitted, is (see Fig. 4)

$$t_o = \frac{L}{u} \quad (56)$$

When k neutrons are emitted the transit time will be (see Fig. 4)

$$t_k = \frac{L}{u + V_k \cos \theta} = \frac{t_o}{1 + \alpha_k \cos \theta} \quad (57)$$

For a given V_k or α_k the distribution function in t_k is related to the distribution function in θ by the relation

$$n(t_k) = n(\theta) \left| \frac{d\theta}{dt_k} \right|$$

As we have shown previously (equation 11a),

$$n(\theta) = \frac{N}{2} \sin \theta.$$

From equation (57)

$$\frac{d\theta}{dt_k} = \frac{t_o}{\alpha_k t_k^2 \sin \theta}.$$

Thus

$$n(t_k) = \frac{N t_o}{2 \alpha_k t_k^2} \quad (58)$$

within the range

$$\frac{t_o}{1 + \alpha_k} \leq t_k \leq \frac{t_o}{1 - \alpha_k}. \quad (59)$$

Outside this range $n(t_k)$ is zero.

For a distribution in α_k , we replace N by $n(\alpha_k)$ and integrate over α_k .

Using the distribution in α_k as given by equation (25), we have

$$n(t_k) = \frac{2 N t_o}{\sqrt{\pi (k \alpha_k)^3}} \frac{1}{t_k} \int_{\alpha_{min}}^{\infty} \alpha_k \exp \left\{ - \frac{\alpha_k^2}{k \alpha_k} \right\} d\alpha_k \quad (60)$$

The α_{min} arises from the fact that for a given difference between t_k and t_o there is a minimum recoil velocity which is necessary as shown by the limits (59). Examining these limits we see that

$$\alpha_{min} = \frac{|t_k - t_o|}{t_k} \quad (61)$$

Performing the integration indicated in equation (60) with the lower limit as given by equation (61), one obtains

$$n(t_k) = \frac{N t_o}{\sqrt{\pi k x_k}} \frac{1}{t_k^2} \exp \left\{ - \frac{(t_k - t_o)^2}{k x_k t_k^2} \right\}. \quad (62)$$

In terms of the quantity $\epsilon = \delta t_k / t_k = (t_k - t_o) / t_k$, we have

$$n(\epsilon) = n(t_k) \left| \frac{dt_k}{d\epsilon} \right| = n(t_k) \frac{t_k^2}{t_o}.$$

Hence

$$n(\delta t_k / t_k) = \frac{N}{\sqrt{\pi k x_k}} \exp \left\{ - \frac{(\delta t_k / t_k)^2}{k x_k} \right\}. \quad (63)$$

Initial Velocity Not Necessarily Along Axis - First Approximation

Because both fragments may emit neutrons, the initial velocity will not necessarily be along the axis of the flight tube in the general case. From Fig. 13 and Fig. 17 we see that the velocity along the axis of the flight tube is $V' \cos \gamma$. Hence the transit time is now

$$t_k = \frac{L}{V' \cos \gamma} = \frac{t_o}{\alpha_k} \sec \gamma. \quad (64)$$

In the proposed experiment with the large detector having a 6 in radius and being located 12 ft from the source, the maximum value of γ that will result in detection has

$$\sec \gamma = 1.00114.$$

Fig. 17. Schematic diagram showing geometrical parameters necessary for calculating fission fragment flight times.

Similarly the fragments reaching the small detector are defined by a 5" x 2" slit at a distance of 6 ft from the source. The maximum value of γ that a fragment can have and still pass through the slit has

$$\sec \gamma = 1.00109.$$

Thus we see that in either case we make an error of only about 0.1% in neglecting $\sec \gamma$ with respect to unity. We thus take

$$t_k \approx \frac{t_o}{\alpha'_k}.$$

Then

$$n(t_k) = n(\alpha'_k) \left| \frac{d\alpha'_k}{dt_k} \right| = \frac{t_o}{t_k^2} n(\alpha'_k). \quad (65)$$

For a given V or α_k

$$n(\alpha'_k) = n(\theta) \left| \frac{d\theta}{d\alpha'_k} \right| = \frac{N}{2} \sin \theta \left| \frac{d\theta}{d\alpha'_k} \right|.$$

From Fig. 5 we see that

$$(\alpha'_k)^2 = \alpha_k^2 + 1 + 2\alpha_k \cos \theta.$$

Thus

$$\left| \frac{d\theta}{d\alpha'_k} \right| = \frac{\alpha'_k}{\alpha_k \sin \theta},$$

and

$$n(\alpha'_k) = \frac{N\alpha'_k}{2\alpha_k}, \quad (66)$$

for the range

$$|1 - \alpha_k| \leq \alpha'_k \leq (1 + \alpha_k).$$

For a distribution in α_k , we replace N by $n(\alpha_k)$ and integrate over α_k .

Using equation (25) for the distribution in α_k we have

$$n(\alpha'_k) = \frac{2N}{\sqrt{\pi(k \alpha_e)_k}^3} \alpha'_k \int_{|\alpha'_k-1|}^{\alpha'_k+1} \alpha_k \exp \left\{ -\frac{\alpha_k^2}{k \alpha_e_k} \right\} d\alpha_k.$$

Integrating, we obtain

$$n(\alpha'_k) = \frac{N}{\sqrt{\pi k \alpha_e_k}} \alpha'_k \left[\exp \left\{ -\frac{(\alpha'_k-1)^2}{k \alpha_e_k} \right\} - \exp \left\{ -\frac{(\alpha'_k+1)^2}{k \alpha_e_k} \right\} \right].$$

Letting

$$\alpha'_k = 1 + \delta.$$

We have

$$n(\alpha'_k) = \frac{N}{\sqrt{\pi k \alpha_e_k}} \alpha'_k \left[\exp \left\{ -\frac{\delta^2}{k \alpha_e_k} \right\} - \exp \left\{ -\frac{(2+\delta)^2}{k \alpha_e_k} \right\} \right]. \quad (67)$$

Now $k \alpha_e_k$ is very small ($\sim 4.4 \times 10^{-4}$ for 3 neutrons) so the first term in the square brackets of equation (67) is appreciable only for small values of δ and the second term is negligible. Thus we write

$$n(\alpha'_k) \simeq \frac{N}{\sqrt{\pi k \alpha_e_k}} \alpha'_k \exp \left\{ -\frac{(\alpha'_k-1)^2}{k \alpha_e_k} \right\}. \quad (68)$$

Substituting (68) into equation (65) we have

$$n(t_k) = \frac{N}{\sqrt{\pi k \alpha_k}} \left(\frac{t_o^2}{t_k^3} \right) \exp \left\{ - \frac{(t_k - t_o)^2}{k \alpha_k t_k^2} \right\}. \quad (69)$$

In terms of $\epsilon = \delta t_k / t_k$ we have

$$n(\epsilon) = \frac{N}{\sqrt{\pi k \alpha_k}} \left(\frac{t_o}{t_k} \right) \exp \left\{ - \frac{\epsilon^2}{k \alpha_k} \right\}. \quad (70)$$

Since $k \alpha_k$ is so small ($\sim 4 \times 10^{-4}$ for 3 neutrons) the exponential term is only important when t_k is very nearly equal to t_o . Hence, we may write (69) and (70)

$$n \left(\frac{t_k - t_o}{t_o} \right) \approx \frac{N}{\sqrt{\pi k \alpha_k}} \exp \left\{ - \frac{(t_k - t_o)^2}{k \alpha_k t_o^2} \right\}. \quad (71)$$

Calculations For a Particular Case

Consider again the case where the U^{238} nucleus divides into a mass 140 fragment and a mass 96 fragment.

The time to reach a detector 12 feet from the source is, for the heavy fragment,

$$t_{OH} = \frac{L}{v_H} = 3.80 \times 10^{-7} \text{ sec.},$$

and the time for the light fragment is

$$t_{OL} = \frac{L}{v_L} = 2.60 \times 10^{-7} \text{ sec.}$$

We can then calculate the standard deviation in the time of arrival

$$\sigma(t_k - t_0) = \sqrt{k \alpha_k} \frac{t_0}{2} \quad (72)$$

The percentage standard deviation is then just that given by Table III, p. 45. The absolute Standard deviation in flight time is given in Table IV below.

Table IV

Standard Deviation in the Fission Fragment Transit Time Over a 12 ft Flight Path as a Function of the Number of Prompt Neutrons Emitted by the Fragment, for the Case $M_H = 140$ mu, $M_L = 96$ mu, Assuming a Watt Distribution in Neutron Momenta.

<u>k</u>	<u>Heavy Fragment</u> n.s.	<u>Light Fragment</u> n.s.
1	2.74	1.88
2	3.90	2.69
3	4.81	3.33
4	5.60	3.88
5	6.31	4.39
6	6.96	4.86

COMPARISON WITH OTHER CALCULATIONS

Longitudinal Component of Neutron Recoil

Milton and Fraser⁶ have calculated the probability distribution for the change in the longitudinal component of the fragment velocity due to recoil from neutron emission.

Instead of fitting the center-of-mass neutron spectrum with equation (1a), Milton and Fraser have used a two-component distribution, i.e., one of the form

$$n(E) = \frac{K}{T_1^2} E \exp \left\{ \frac{-E}{T_1} \right\} + \frac{(1-K)}{T_2^2} E \exp \left\{ \frac{-E}{T_2} \right\}, \quad (73)$$

where T_1 and T_2 are the "temperatures" of the two components and K is a parameter which is adjusted to give the best fit. Because of this different initial assumption, the resulting distribution in longitudinal velocity differs in form from that obtained here and the two can only be compared in numerical results.

Milton and Fraser have calculated the full width at half maximum of the relative change in the longitudinal velocity of the fragment to be

$$\left(\frac{\Delta V_f}{V_f} \right)^{1/2} = (\text{const})(kT)^{1/2} = .026 \quad (74)$$

for the slow neutron fission of U^{235} . Let us define this relative change in longitudinal velocity as ξ , i.e.,

$$\xi = \frac{\Delta V_f}{V_f} = \alpha \cos \theta. \quad (75)$$

Using equation (25) and assuming isotropic neutron emission in the center of mass system, one easily derives the distribution in ξ ,

$$n(\xi) = \frac{N}{\sqrt{\pi k \alpha}} \exp \left\{ -\frac{\xi^2}{k \alpha} \right\}. \quad (76)$$

Hence the standard deviation in ξ is

$$\sigma(\xi_k) = \frac{1}{2} (k \alpha)^{1/2} \quad (77)$$

for the case where the number of neutrons emitted by the fragment is known. It is important to note that k here refers to the number of neutrons emitted by the fragment in question, not to the total number ν of neutrons emitted in the fission event. If k is not known, we can use the probabilities $p(k)$ of a single fragment emitting k neutrons to obtain

$$\sigma^2(\xi) = \langle \xi^2 \rangle = \sum_k p(k) \sigma^2(\xi_k). \quad (78)$$

The probabilities $p(k)$ for the emission of k neutrons from a single fragment may be obtained from the probabilities $P(\nu)$ of emitting a total of ν neutrons in the fission event by making use of the relation

$$P(\nu) = \sum_{k=0}^{\nu} p(k)p(\nu - k). \quad (79)$$

Using values of $P(\nu)$ obtained by Divcn, et al.,⁷ the $p(k)$ found to give the best fit to equation (79) were

$$\begin{aligned} p(0) &= 0.160 \pm 0.012, \\ p(1) &= 0.499 \pm 0.047, \\ p(2) &= 0.278 \pm 0.127, \\ p(3) &= 0.063 \pm 0.027, \\ p(4) &= 0.000 \pm 0.005. \end{aligned} \quad (80)$$

The probabilities for all k greater than four were taken to be zero.

For the case where $M_H = 140$ m.u. and $M_L = 96$ m.u., equation (79) gives

$$\sigma(\mathfrak{E}) = 0.00812. \quad (81)$$

From this the full width at half maximum is

$$\mathfrak{E})_{1/2} = 2.355 \sigma(\mathfrak{E}) = 0.0191. \quad (82)$$

This is not in very good agreement with the results of Milton and Fraser (equation 74); however, there is some evidence that they used the probabilities for the total number of neutrons emitted per fission rather than the probabilities for the neutrons emitted per fragment in equation (78): For Cr^{252} they used $(k^{1/2})_{\text{eff}} = 1.80$. The average number of neutrons per fission $\langle \nu \rangle$ is equal to 3.82 for Cr^{252} which would give $\langle \nu \rangle^{1/2} = 1.95$ whereas $\langle k \rangle^{1/2} = 1.38$. They state further that putting $(\nu^{1/2})_{\text{eff}}$ in equation (74) gives a width a little narrower than would be obtained by using $\langle \nu \rangle$.

If one uses the $P(\nu)$ instead of the $p(k)$ in equation (78) for our calculation, one obtains

$$\mathfrak{E})_{1/2} = \left. \frac{\Delta \nu}{\nu_f} \right)_{1/2} = .027 \quad (83)$$

in good agreement with the value .026 obtained by Milton and Fraser.

If Milton and Fraser have used the total number of neutrons per fission rather than the number of neutrons per fragment, then their calculated mass resolutions due to neutron emission should be divided by a factor equal roughly to the square root of two.

Fragment Deflection Due to Neutron Emission

Milton and Fraser have used the neutron distribution (73) to calculate the probable lateral deflection of fragments from Cf^{252} . They then used a Monte Carlo method to calculate the efficiency of different detector combinations for detecting a Cf^{252} fragment pair. They assumed a point source.

The present method will be used to calculate the efficiency for one of their cases and the result compared with their result. The case considered will be that where a total of four neutrons are emitted by fragments which are to be detected by two 4-inch diameter detectors each situated a distance of 71.8 inches from the source.

One may fit the neutron spectrum of Cf^{252} with a Watt formula of the form⁴

$$n(E') = (\text{const}) \exp(-0.88 E') \sinh \sqrt{2.0 E'}. \quad (84)$$

This corresponds to

$$T = 1.14 \text{ Mev.} \quad (85)$$

From page 205 of Hyde's report,⁴ the most probable division of Cf^{252} corresponds to

$$\begin{aligned} M_H &= 144, & M_L &= 108, \\ E_H &= 80 \text{ Mev}, & E_L &= 105 \text{ Mev.} \end{aligned}$$

For this case we have an initial momentum

$$P_i = 4.61 \times 10^3 \text{ Mev/c.}$$

A reasonable average value of α averaged over the various combinations in which the four neutrons may be emitted is

$$\langle \mathcal{R} \rangle = 1.045 \times 10^{-4},$$

and

$$\langle 2 \sigma^2 \rangle = v \langle \mathcal{R} \rangle = 4.180 \times 10^{-4}.$$

For 4-inch diameter detectors at 71.8 inches we have

$$\left(\frac{R_1}{L_1} \right)^2 = \left(\frac{R_2}{L_2} \right)^2 = 7.76 \times 10^{-4},$$

where R_1 and R_2 are the radii of the first and second detectors, respectively, and L_1 and L_2 are their respective distances from the source.

In Appendix I equations for the efficiency of a detector arrangement are derived assuming a point source and correcting for the solid angles of the detectors. Using the above values to calculate the $S_n(x)$ of equation (91) and substituting into equation (96) we obtain for the fraction detected

$$f(x) = .601. \tag{86a}$$

Milton and Fraser obtain for this case

$$f(x) = .66. \tag{86b}$$

This is a difference of ten percent.

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APPENDIX I. Solid Angle Correction

In deriving equations (53a,b) we considered only a point source and a point first detector (see Fig. 13). Let us now consider a finite first detector and a finite source. Call the fragment hitting the first detector fragment 1 and the fragment hitting the second detector fragment 2. If fragment 1 arises from an off-center point of the source and/or strikes an off-center point of the first detector the probability distribution will, in general, no longer be centered about the center of the first detector but will be centered about the point Q (see Fig. 18) which is the interception of the backward projection of the flight path of the first fragment with the plane of the second detector. We can now write the probability per unit area by replacing r in equation (53b) by s , the distance from point Q, that the second fragment strikes the second detector. Thus

$$P(s) = \frac{\rho(s)}{N} = \frac{1}{2\pi L_2^2 \sigma^2} \exp \left\{ -\frac{s^2}{2\pi L_2^2 \sigma^2} \right\}, \quad (87)$$

where

$$2\sigma^2 = k_1 x_1 + k_2 x_2,$$

and L_2 is the distance from the source to the second detector. In terms of r and β ,

$$P(s) dA = \frac{1}{2\pi L_2^2 \sigma^2} \exp \left\{ -\frac{(\lambda^2 + r^2 + 2\lambda r \cos \beta)}{2L_2^2 \sigma^2} \right\} r dr d\beta. \quad (88)$$

To obtain the probability of detection $W(\lambda, R, \sigma)$ for this off-center case we integrate over the face of the second detector of radius R_2 . Then

$$W(\lambda, R_2, \sigma) = \frac{1}{\pi L_2^2 \sigma^2} \int_0^{R_2} r dr \exp \left\{ -\frac{(\lambda^2 + r^2)}{2L_2^2 \sigma^2} \right\} \int_0^\pi \exp \left\{ -\frac{\lambda r \cos \beta}{L_2^2 \sigma^2} \right\} d\beta. \quad (89)$$

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Fig. 18. Diagram showing the geometrical relationships for the case of an off-center distribution of fragments hitting the second detector.

The integral over β in equation (89) can be obtained in terms of the Bessel function,

$$J_0(i x) = \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m}(m!)^2}.$$

Thus one obtains

$$W(\lambda, R_2, \sigma) = \frac{1}{L_2^2 \sigma^2} \int_0^{R_2} r dr J_0\left(\frac{i \lambda r}{L_2 \sigma}\right) \exp\left\{-\frac{(\lambda^2 + r^2)}{2 L_2^2 \sigma^2}\right\}.$$

Expanding the Bessel function and integrating in series, one obtains finally

$$W(\lambda, R_2, \sigma) = \exp\left\{-\frac{\lambda^2}{2 L_2^2 \sigma^2}\right\} \sum_{m=0}^{\infty} \frac{1}{m!} \left\{\frac{\lambda^2}{2 L_2^2 \sigma^2}\right\} S_n(x_2), \quad (90)$$

where

$$S_n(x) = 1 - e^{-x} \sum_{m=0}^n \frac{x^m}{m!}, \quad (91)$$

and

$$x_2 = \frac{1}{2} \left(\frac{R_2}{L_2 \sigma}\right)^2. \quad (92)$$

The S_n are usually negligibly small except for a few of the lower values of n .

The efficiency of the detector system for intercepting both of a fragment pair if one of the fragments is intercepted by the first detector is given by the relation

$$f(R_1, R_2, \sigma) = \int_0^{\infty} P(\lambda) W(\lambda, \sigma) d\lambda, \quad (93)$$

where $P(\lambda)$ is the probability that the backward extension of the first fragment's flight path intersects the plane of the second detector a distance λ from the center of the face of the second detector.

In general, $P(\lambda)$ is a complicated function of the source and detector geometry and equation (93) must be integrated numerically. However, in the case of a point source and circular detectors $P(\lambda)$ is comparatively simple. In this case, it can be seen by projecting the face of the first detector through the point source onto the face of the second detector that

$$P(\lambda)d\lambda = 2 L_1^2 \lambda d\lambda / L_2^2 R_1^2, \quad (94)$$

where L_1 is the distance from the source to the first detector. Then using equations (90) and (94) in (93) we have

$$f(R_1, R_2, \sigma) = \sum_{n=0}^{\infty} S_n(x_2) \frac{1}{x_1 n!} \int_0^{x_1} q^n e^{-q} dq$$

where we have let

$$q = \lambda^2 / 2 L_2^2 \sigma^2, \quad (95a)$$

$$x_1 = R_1^2 / 2 L_1^2 \sigma^2. \quad (95b)$$

Performing the above integration we obtain

$$f(R_1, R_2, \sigma) = \sum_{n=0}^{\infty} S_n(x_1) S_n(x_2) / x_1. \quad (96)$$

To obtain the probability of detecting a given fission event oriented at random, we multiply $f(R_1, R_2, \sigma)$ by twice the fractional solid angle of the first detector, thus

$$P(R_1, R_2, \sigma) = \sigma^2 \sum_{n=0}^{\infty} S_n(x_1) S_n(x_2). \quad (97)$$

It is to be noted that this expression is symmetric in x_1 and x_2 as one would expect.

APPENDIX II. Summary of Notation

(If a page number is not given, that particular notation is used throughout the report)

A	area in plane of detector face, p. 16.
B	parameter defined on p. 27.
C	azimuthal angle, (see Fig. 14).
D	parameter defined by equation (49), p. 39.
E	neutron kinetic energy in center of mass system.
E'	neutron kinetic energy in laboratory system.
E_f	fragment kinetic energy prior to neutron emission, p. 2.
E_H, E_L	kinetic energies of heavy and light fragments, respectively.
$f(x)$	fraction of fragment pairs detected provided one fragment is detected by first detector,
I_1, I'_2, I''_2	integrals, pp. 7, 9, 36.
$J_n(x)$	Bessel function of order n and argument x .
K	proportionality constant.
k	number of prompt neutrons emitted by a single fragment.
k_1, k_2	number of prompt neutrons emitted by the fragments hitting the first and second detectors, respectively.
k_H, k_L	number of prompt neutrons emitted by the heavy and light fragments, respectively.
L	fragment flight path, general case.
L_1, L_2	distances from source to first and second detectors, respectively.

$\vec{L}$	vector of magnitude L along line of centers of detectors, (see Fig. 13 and Fig. 14).
L, L'	lower limits of integration, pp. 6, 35, 36.
L_2	limit of integration, p. 37.
M	mass of fragment prior to neutron emission.
M_H, M_L	masses of heavy and light fragments, respectively.
m	neutron mass.
m	index of summation, pp. 30, 39.
N	total number of events
N'	number of fission events where both fragments are detected, p. 43.
$n(x)$	distribution function in any variable x .
n	index of summation, p. 29.
P_i	initial momentum of fragment
P_k, P_k'	final momenta after emitting k neutrons in center of mass and laboratory systems, respectively.
$P(v)$	probability of emitting a total of v neutrons in a fission event.
$P(s)$	probability per unit area that a fragment will strike the second detector a distance s from the point Q where the backward projection of the path of its sister fragment intercepts the plane of the second detector (see Fig. 18).
$P(\lambda)$	probability that the backward projection of the flight path of the fragment hitting the first detector will

	intercept the plane of the second detector a distance λ from the center of the second detector.
p	neutron momentum.
$\vec{p}_k$	vector sum of the momenta of k prompt neutrons.
$p(k)$	probability of a single fragment emitting k neutrons.
q	dimensionless parameter proportional to λ^2 , defined by equation (95a).
R	radius of detector face.
r	distance from center of second detector face that fragment strikes detector.
$S_n(x)$	function used to correct for finite geometry of a detector, defined by equation (91).
s	distance from the point where a fragment strikes the second detector and the point where the backward projection of the path of its sister fragment intercepts the plane of the second detector (see Fig. 18).
T	"temperature" of nucleus, a parameter used in fitting neutron spectra with the Watt formula (see equation (1), p. 2).
T_1, T_2	"temperature" parameters used in fitting neutron spectra with a two-component formula (see equation (73), p. 57).
t	variable of integration, pp. 28, 37.
t_0	fragment transit time from source to detector if no neutrons are emitted.
t_k	fragment transit time if k neutrons are emitted.
U, U'	upper limits of integration, pp. 6, 35, 36.

u	initial velocity of fragment before emission of neutrons.
V_f	fragment velocity.
V_k	fragment velocity in center of mass system after emitting k neutrons.
V'_k	fragment velocity in laboratory system after emitting k neutrons.
v_H, v_L	initial velocity of heavy and light fragments, respectively, p. 55.
$W(p_k)$	probability distribution in p_k .
$W(\lambda, R_2, \sigma)$	probability of a fragment striking the face of the second detector if the backward projection of the flight path of its sister fragment intercepts the plane of the second detector a distance λ from its center.
x	any variable.
x_1	dimensionless variable related to R_1 defined by equation (95b).
x_2	dimensionless function related to R_2 defined by equation (92).
y	transverse deflection of fragment, (see Fig. 4).
y	variable of integration, pp. 25-28.
Z	ratio of detector radius to distance from source to detector.
z	ratio of transverse deflection to flight path of fragment (see Fig. 4).
α_k	ratio of final fragment velocity in c.m.s. to initial fragment velocity, defined by equation (6) p. 10.

α'_k	ratio of final fragment velocity in laboratory system to initial fragment velocity, defined by equation (7) p. 10.
α_H, α_L	α_k for the heavy and light fragment, respectively.
β	any small angle, p. 36.
β	an angle on the detector face (see Fig. 18).
$\Gamma(x)$	the gamma function of x, p. 39.
γ	angle between the final momentum of the second fragment and the line of centers of the source and detectors (see Figs. 13 and 14).
ΔV_f	the change in the longitudinal velocity of the fission fragments due to prompt neutron emission.
δ	ratio of the increase in the velocity of the fragment to its original velocity, p. 54.
δt	small change in flight time, pp. 51, 55.
ϵ	kinetic energy per nucleon in fragment prior to neutron emission.
ϵ	$\delta t_k / t_k$, pp. 51, 55.
S, S'	variables of integration, pp. 7, 9, 36.
η	variable of integration, p. 37.
θ	angle between original velocity of fragment and direction of recoil in c.m.s. (see Fig. 3).
θ_1, θ_2	center-of-mass deflection angles for the fast and slow components of fragments of a given mass having a given deflection (see Fig. 7) p. 15, pp. 20-24.

$\mathcal{R}$	parameter related to T , P_i and less sensitively to k and M , defined by equation (9).
λ	summation index.
λ	distance from center of second detector to the point where the backward projection of the flight path of the fragment striking the first detector intercepts the plane of the second detector (see Fig. 18).
ν	$= k_H + k_L$, total number of neutrons emitted in a fission event.
η	variable of integration, p. 37.
η	longitudinal component of α , defined by equation (75).
$\rho(z)$	number of fragments per unit area of detector.
ρ_0	$= \rho(0)$, number of fragments per unit area hitting center of detector.
σ	$= mT$, pp. 3-9.
σ^2	$= k_1 \mathcal{R}_1 + k_2 \mathcal{R}_2$, p. 42 ff.
σ_k	$= \sqrt{k\mathcal{R}}$, p. 45.
ϕ	fragment deflection angle in laboratory system (see Fig. 3 and 4).
ϕ_1, ϕ_2	laboratory deflection angles of fragments hitting first and second detectors, respectively, (see Figs. 13 and 14).
ψ	angle between neutron momenta (see Fig. 1).
Ω	solid angle.

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