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CONVECTIVE EQUILIBRIUM AND MIXING-LENGTH THEORY
FOR STELLARATOR REACTORS

By

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CONVECTIVE EQUILIBRIUM AND MIXING-LENGTH THEORY
FOR STELLARATOR REACTORS

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ABSTRACT. In high β stellarator and tokamak reactors, the plasma pressure gradient in some regions of the plasma may exceed the critical pressure gradient set by ballooning instabilities. In these regions, convective cells break out to enhance the transport. As a result, the pressure gradient can rise only slightly above the critical gradient and the plasma is in another state of equilibrium -- "convective equilibrium" -- in these regions. Although the convective transport cannot be calculated precisely, it is shown that the density and temperature profiles in the convective region can still be estimated. A simple mixing-length theory, similar to that used for convection in stellar interiors, is introduced in this paper to provide a qualitative description of the convective cells and to show that the convective transport is highly efficient. A numerical example for obtaining the density and temperature profiles in a stellarator reactor is given.

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1. INTRODUCTION

Stellarator and tokamak reactors will most likely operate near the highest possible β [$\beta \equiv P/(B^2/8\pi)$, the ratio of plasma kinetic pressure to the magnetic pressure] set by some limiting modes, i.e., ballooning instabilities [1-3]. In some regions of the plasma, the plasma pressure gradient will be below the critical pressure gradient determined by the ballooning modes and the entire particle and energy fluxes will be carried by neoclassical diffusion (e.g., in stellarators). In other regions, the plasma will be "marginally unstable," i.e., the plasma pressure gradient exceeds the critical gradient by a small amount, and the fluxes will be carried partially by neoclassical diffusion and partially by pressure-driven convective cells. Thus, ballooning instabilities in some regions of the plasma may lead to another state of equilibrium -- "convective equilibrium" -- in which convective motions occur throughout the region.

A similar situation exists in stars which have convective zones where energy is transported by both convection and radiation. In spite of the fact that the convective transport cannot be calculated precisely, it is remarkable that it is still possible to construct detailed models of stellar interiors. In this paper, the method for determining the interior structure of stars is modified to determine the structure of stellarator reactors, i.e., the radial profiles of plasma density and temperature [4]. Stellarators, rather than tokamaks, are the focus of discussion throughout this paper. The reason is that in stellarators, the particle and energy fluxes due to anomalous processes (other than convection induced by MHD instabilities) are likely to be small compared to neoclassical transport, the behavior of which is well known. A method similar to the one described in this paper has been employed by Connor et al. [5] to study the deterioration of tokamak plasma confinement

due to the onset of MHD ballooning instabilities. In their paper, nonconvective transport is assumed to satisfy a specific model derived for anomalous transport in the regions where ballooning modes are stable.

A rudimentary mixing-length theory, similar to that applied to convective zones in stellar interiors, is given here to describe the qualitative behavior of convective motion in reactors. This simple theory shows that convection is so efficient that a slight excess of the pressure gradient above the critical pressure gradient can lead to as much convective particle flux as is necessary to make up the difference between the total particle flux and the neoclassical flux, and similarly for the energy flux.

This paper is organized as follows. The convective equilibrium hypothesis is introduced in Sec. 2. In Sec. 3, the method for determining interior structure of stars is modified to determine the plasma density and temperature profiles in a stellarator reactor. A mixing-length theory describing the convective transport is presented and the character of convection is discussed in Sec. 4. Using the method outlined in Sec. 3, a sample calculation of the density and temperature profiles for a stellarator reactor is given in Sec. 5. The conclusion is given in Sec. 6. In the Appendix, the velocity of a typical convective cell is estimated by using a linear MHD stability analysis.

2. CONVECTIVE EQUILIBRIUM HYPOTHESIS

Local ballooning instabilities in stellarators may determine β_c (the limiting β value) and set a critical pressure gradient $\partial P / \partial r|_c$. When the particle and energy sources are low, the plasma pressure gradient is below the critical pressure gradient everywhere in the plasma. Hence, transport is dominated by the neoclassical diffusion due to particles trapped in local

magnetic wells caused by stellarator external helical windings [6]-[8]. As the strength of the sources increase, the pressure gradient in some region of the plasma will approach the critical gradient. Once above this critical gradient, the local ballooning stability condition is violated. Consequently, pressure drive convective cells will break out throughout the region where the pressure gradient tends to increase above the critical gradient. A convective cell moves mass and energy from the unstable region to some outer region where the surrounding pressure is less than that of the cell. Once the cell is dissolved, the magnetic line of force bounces back to its original position and brings along with it from the outer region an element that is deficient in both mass and energy. Hence, the net effect of the outward and the inward moving elements results in transporting particles and energy outward.

The convective transport increases the total transport in the region where the pressure gradient tends to exceed the critical gradient. Consequently, the pressure gradient can rise only slightly above the critical gradient and the plasma is marginally unstable. This condition is used in the following section to determine the density and temperature profiles in the convective region and is justified by a mixing-length theory presented in Sec. 4. The strength of the convective fluxes is determined by the steady-state conditions. That is, the neoclassical particle and energy fluxes and the corresponding convective fluxes added together have reached values which satisfy exactly the particle and energy steady-state conditions. Therefore, local instabilities caused by excess pressure gradients in some regions of the plasma lead to another state of equilibrium -- convective equilibrium -- in which mass and energy are transported by both neoclassical diffusion and convective cells.

For stellarators, island structure of surfaces may also enhance transport

near rational surfaces. However, the convective transport described here may still occur in regions between the rational surfaces where good surfaces exist.

3. DETERMINATION OF RADIAL DENSITY AND TEMPERATURE PROFILES

The method for determining the interior structure of stars is reviewed briefly in this section. Then, this method is modified to determine the density and temperature profiles in a stellarator reactor.

In stellar interiors where the temperature gradient is less than the "adiabatic gradient" (the condition for stability against convective motions [10]), the temperature is obtained by equating the radiative flux to the total energy flux (this condition gives the radiative energy transport equation). If the temperature gradient exceeds the adiabatic gradient, then convection occurs. The convective transport is so efficient that a slight excess of the temperature gradient above the adiabatic gradient can lead to as much energy flux as is necessary to make up the difference between the total flux and the radiative flux. Thus, the temperature gradient can be set equal to the adiabatic gradient. These conditions, together with the two equations for hydrostatic equilibrium, the equation for thermal equilibrium, and the appropriate boundary conditions, enable one to construct a model for the stellar structure throughout the bulk of the star. (Only in a narrow region near the surface of the star where convection is inefficient does this method fail.) To decide if convection is present or not, one can first calculate the temperature gradient by using the radiative energy transport equation. If this temperature gradient exceeds the adiabatic temperature gradient, then the temperature gradient must be determined from the adiabatic gradient. Note that setting the temperature gradient to the adiabatic gradient is the

condition that allows one to study the stellar convective zone in spite of the fact that a precise calculation of the convective flux cannot be obtained directly.

The stellar method will now be modified to determine the radial density and temperature profiles for the plasma in a stellarator reactor. The analogy to the radiative flux in a star is the neoclassical flux in a stellarator. From Ref. [8], the electron neoclassical particle flux Γ_{neo} and energy flux Q_{neo} (for the large aspect ratio approximation) are:

$$\Gamma_{\text{neo}}(r) = -(2\epsilon_h)^{3/2} \frac{T_4^{7/2}}{B_4^2 R_m^2} \left[14.4 \left(\frac{1}{n_{14}} \frac{\partial n_{14}}{\partial r_m} + \frac{e}{T_4} \frac{\partial \Phi}{\partial r_m} - \frac{3}{2} \frac{1}{T_4} \frac{\partial T_4}{\partial r_m} \right) + 71.2 \frac{1}{T_4} \frac{\partial T_4}{\partial r_m} \right] (10^{19} \text{ cm}^{-2} \cdot \text{sec}^{-1}), \quad (1)$$

$$Q_{\text{neo}}(r) = -(2\epsilon_h)^{3/2} \frac{T_4^{9/2}}{B_4^2 R_m^2} \left[71.2 \left(\frac{1}{n_{14}} \frac{\partial n_{14}}{\partial r_m} + \frac{e}{T_4} \frac{\partial \Phi}{\partial r_m} - \frac{3}{2} \frac{1}{T_4} \frac{\partial T_4}{\partial r_m} \right) + 423.4 \frac{1}{T_4} \frac{\partial T_4}{\partial r_m} \right] (10^{20} \text{ keV} \cdot \text{cm}^{-2} \cdot \text{sec}^{-1}). \quad (2)$$

Here, ϵ_h is the depth of a helical well, the major radius R_m of the torus and the minor radius coordinate r_m are in meters, magnetic field B_4 is in 10 kG, density n_{14} is in 10^{14} cm^{-3} , and temperature T_4 and the potential energy of the ambipolar electric field, $e\Phi$, are in 10 keV. Equation (1) applies to either electrons or ions since the quasineutrality condition must be satisfied. Furthermore, it has been shown in Ref. [8] that the neoclassical energy confinement time is approximately the same for electrons as for ions. Consequently, in order to simplify the calculation, Eq. (2) will be applied to electrons as well as to ions, although in principle, the equation for the ion neoclassical energy flux should be used. Thus, in the regions where

convection is not present, Eqs. (1) and (2) fully determine the density and temperature profiles if the fluxes are given.

The analogy to the condition for stability against convection in stars is the critical pressure gradient condition against ballooning instabilities, i.e.,

$$\left. \frac{\partial p}{\partial r} \right|_c = F(r) , \quad (3)$$

where $F(r)$ is obtained from MHD stability analysis (for example, see Ref. [11]). The energy flux in a reactor is determined by α -particle heating (and, perhaps, external heating).

There is a complication in reactors, however, that is not present in stars. In stars, there is only an energy flux, but, in reactors, there is a particle flux as well as an energy flux. Particle flux can be determined from the rate of injection of pellets. However, unlike the stellar case, steady-state conditions in a reactor do not provide a unique condition for determining the pressure. That is, if the total particle and energy fluxes are given, then there are three equations [one for particle flux, one for energy flux, and the other one is Eq. (3)] for two unknowns (n and T). Thus, another condition is required. To obtain this condition, we postulate a relation between the convective energy flux Q_{conv} and the convective particle flux Γ_{conv} . The precise relation involves a detailed understanding of convection, which we do not have at present. Nevertheless, as shown in the next section, it is plausible that

$$Q_{\text{conv}}(r) = CT(r)\Gamma_{\text{conv}}(r) , \quad (4)$$

where C is a constant of order unity. Since in the convective region, the total flux is equal to the convective flux plus the neoclassical flux, Eq. (4) can be written as

$$Q_{\text{total}}(r) - Q_{\text{neo}}(r) = CT(r) [\Gamma_{\text{total}}(r) - \Gamma_{\text{neo}}(r)]. \quad (5)$$

If we assume that the convective fluxes are the same for electrons and ions, then Eq. (5) can be applied to either species. This equation together with Eq. (3) fully determines the density and temperature in the convective region. Thus, as in the stellar case, detailed knowledge of convective transport in a reactor is not required in order to determine the density and temperature if the constant C is known.

In the present study, the steady-state particle source $\Gamma_{\text{total}}(r)$ and the power source $Q_{\text{total}}(r)$ are assumed to be given. Starting at the edge of the plasma, with given edge density and temperature, the coupled Eqs. (1) and (2) are numerically integrated toward the plasma center. At each step, the pressure gradient is checked to determine if it is above the critical gradient. If not, the numerical integration is continued using Eqs. (1) and (2). If the critical gradient is exceeded, then this minor radius represents the outer boundary of the convective region. From this radius inward, the equations that describe the density and temperature in the convective region, Eqs. (3) and (5), are used and these coupled equations are numerically integrated toward the plasma center. At each step, the neoclassical particle and energy fluxes are checked to determine if they exceed the total particle and energy fluxes. If not, the numerical integration is continued using Eqs. (3) and (5). If so, this minor radius represents the inner boundary of the convective region. From this radius inward, plasma transport is again

dominated by neoclassical diffusion and Eqs. (1) and (2) are used again. The procedure described here is illustrated in Fig. 1. Using this method, an example for calculating the density and temperature profiles for a stellarator reactor is given in Sec. 5.

Note that the way the procedure is described here is not self-consistent, i.e., the thermonuclear power source $Q_{\text{total}}(r)$ is assumed given instead of determined self-consistently from the plasma density and temperature. In the present example of the procedure, the total input thermonuclear power source is chosen in such a way that it matches with the total thermonuclear power generated from the density and temperature profiles calculated from the chosen input power source. In principle, however, a fully self-consistent method can be constructed since in the similar case of the determination of stellar structure with the presence of convection, a fully self-consistent iteration method exists [10].

4. THE MIXING-LENGTH THEORY AND CHARACTER OF CONVECTION

A mixing-length theory, similar to that used for stellar convection, is proposed in this section to justify Eq. (4), to show that the pressure gradient in the convective region in reactors is very close to the critical gradient [Eq. (3)], and to give a qualitative description of the character of convection.

The convective particle and energy fluxes per unit area can be expressed as [12]

$$\Gamma_{\text{conv}} \sim \langle n v_{\text{conv}} \rangle H(\hat{\beta} - 1), \quad (5)$$

$$Q_{\text{conv}} = \langle \rho v_{\text{conv}} \left(\frac{v_{\text{conv}}^2}{2} + \frac{1}{\gamma - 1} \frac{P}{\rho} + \frac{P}{\rho} \right) \rangle H(\hat{\beta} - 1), \quad (6)$$

where ρ and $[1/(\gamma - 1)](P/\rho)$ are the mass density and the internal energy per unit mass of plasma, respectively, and the Heaviside function

$$H = \begin{cases} 1 & \hat{\beta} > 1 \\ 0 & \hat{\beta} < 1 \end{cases},$$

with $\hat{\beta}$ a parameter defined by Eq. (A8). $\hat{\beta} > 1$ means that the plasma pressure gradient exceeds the critical gradient, and convection occurs. The velocity of a typical convective cell, v_{conv} , can be assumed to be normal to the magnetic surfaces (see discussion in the Appendix) and is given by Eq. (A9). It is shown later in this section that v_{conv} is much smaller than the ion thermal velocity, and thus the kinetic energy term $v_{\text{conv}}^2/2$ in Eq. (6) can be ignored. The energy flux due to the $\vec{B} \times (\vec{\nabla} \times \vec{B})$ term in the MHD energy equation has not been included in Eq. (6) since the direction of this term is tangential to the magnetic surfaces. The symbol $\langle \dots \rangle$ denotes the ensemble average. Since we are only considering statistically stationary turbulence, the "ergodic" assumption [13] applies, and thus the ensemble average is identical to the time average.

It is now appropriate to introduce the average mixing-length l . This length represents the average distance, in the minor radius direction, which a convective cell moves before it bounces back or dissolves into the surroundings. (The dissolving of the cell may occur for a number of reasons, e.g., viscosity, neoclassical transport, or microinstabilities.) The average distance which a cell moves at an arbitrary time can be represented, in terms of the mixing-length, by [8]

$$\bar{\xi} \approx \frac{\lambda}{2} . \quad (7)$$

Thus, the convective fluxes across a unit area in the minor radius direction can be approximated by (for $\gamma = 5/3$)

$$\Gamma_{\text{conv}} \approx \left\langle \frac{\partial n}{\partial r} \bar{\xi}_r v_{\text{conv}} \right\rangle H(\hat{\beta} - 1) ,$$

$$Q_{\text{conv}} \approx \left\langle \frac{5}{2} \frac{\partial P}{\partial r} \bar{\xi}_r v_{\text{conv}} \right\rangle H(\hat{\beta} - 1) .$$

Substituting Eq. (A9) into these equations yields

$$\Gamma_{\text{conv}} \approx \left[\frac{4v_{Ti}^2}{RP} \left(\frac{\partial P}{\partial r} - \frac{\partial P}{\partial r} \Big|_c \right) \right]^{1/2} \frac{\partial n}{\partial r} \frac{\lambda^2}{4} H(\hat{\beta} - 1) , \quad (8)$$

$$Q_{\text{conv}} \approx \frac{5}{2} \left[\frac{4v_{Ti}^2}{RP} \left(\frac{\partial P}{\partial r} - \frac{\partial P}{\partial r} \Big|_c \right) \right]^{1/2} \frac{\partial P}{\partial r} \frac{\lambda^2}{4} H(\hat{\beta} - 1) , \quad (9)$$

where v_{Ti} is the ion thermal velocity. If $\partial n/\partial r$ and $\partial P/\partial r$ are approximated by n/a and P/a (a is the minor radius), respectively, then Eqs. (8) and (9) allow one to write

$$Q_{\text{conv}} \approx \frac{5}{2} T \Gamma_{\text{conv}} .$$

This is just Eq. (4) with $C = 5/2$. If the current dissipation ($\vec{J} \cdot \vec{E}$ term in the MHD energy equation) is included, the coefficient in Eq. (4) may be reduced to some value less than $5/2$. Note that in deriving Eqs. (8) and (9), the mixing length for convective particle transport is assumed to be the same as that for convective energy transport; although in reality, they may not

necessarily be the same. Also note that the effect of magnetic shear has not been included in Eqs. (8) and (9).

The amount that the plasma pressure gradient exceeds the critical gradient in the convective region will now be estimated using Eq. (9). Let the size of the mixing length be

$$l \sim \frac{a}{10}.$$

To estimate the amount of convective energy flux per unit area, Q_{conv} , let the density and temperature profiles be parabolic with peak density $n_0 = 6 \times 10^{14} \text{ cm}^{-3}$ and peak temperature $T_0 = 10 \text{ keV}$. If $a = 1.6 \text{ m}$, then the total plasma energy per unit length inside $r = a/2$ is $0(10^{20}) \text{ keV} \cdot \text{cm}^{-1}$. If the major radius of the reactor $R = 10 \text{ m}$ and other reactor parameters are those given in Table I, then the typical neoclassical energy confinement time (i.e., with neoclassical diffusion being the only loss mechanism) is on the order of 0.5 sec (see Ref. [6]). Hence, the amount of energy flux per unit area at $r = a/2$ required to keep the reactor in a steady-state condition is $0(10^{18}) \text{ keV} \cdot \text{cm}^{-2} \cdot \text{sec}^{-1}$. If we use this energy flux to approximate the convective energy flux at $r = a/2$, then Eq. (9) gives

$$1 - \left(\frac{\partial P}{\partial r} \right)_{\text{c}} / \left(\frac{\partial P}{\partial r} \right) = 0(10^{-7}).$$

Therefore, for all practical purposes, we can set

$$\frac{\partial P}{\partial r} = \left(\frac{\partial P}{\partial r} \right)_{\text{c}}.$$

This result is generally true for reactors with convective regions. It is important to note that unlike the stellar case, this result will still hold even if the convective region occurs near the plasma edge.

Using the reference reactor parameters given in Table I, the typical velocity of a convective cell can be estimated using Eq. (A9) and the result is $v_{\text{conv}} = 0(10^4) \text{ cm} \cdot \text{sec}^{-1}$. The average lifetime of a convective cell, t_{conv} , may now be estimated by noting that

$$t_{\text{conv}} \approx \frac{l}{v_{\text{conv}}},$$

which has a value of 10^{-3} sec for the reference reactor. Since this time is short compare to the plasma confinement time (which is on the order of 0.1 - 1.0 sec), the convection is fully developed in a steady-state reactor. Note that v_{conv} is smaller than the ion thermal velocity (on the order of $10^8 \text{ cm} \cdot \text{sec}^{-1}$ for a 10 keV plasma) by four orders of magnitude. Since pressure is proportional to velocity squared, the hydrodynamic effect must be smaller than the plasma pressure by about eight orders of magnitude. Also note that the motion of a convective cell will be stopped by nonlinear mechanisms before it can move a distance large compared to the minor radius. In the meantime, magnetic lines of force tend to slip back to their original positions due to the tension of the field line and finite resistivity [9]. Thus, convective motion may not be able to twist the field line into a very chaotic state.

Unlike the stellar convective zones which have spherical symmetry, convective regions in stellarators and tokamaks will tend to be localized in the "bad" curvature region (i.e., the outer region of the torus where the radius of the field line is pointing towards the plasma). However, the mean density and temperature can still be assumed to be constant on magnetic

surfaces since the density and temperature on the outer region of the torus will even out along a line of force rapidly.

5. NUMERICAL EXAMPLE

The density and temperature profiles in the reference stellarator reactor are calculated in this section. To simplify the calculation, we assume that $(1/n)(\partial n/\partial r)$, $(1/T)(\partial T/\partial r)$, and $(e/T)(\partial \Phi/\partial r)$ in both Eqs. (1) and (2) are approximately equal. We further assume that the neoclassical particle flux is proportional only to $(1/n)(\partial n/\partial r)$, and thus we can replace $(e/T)(\partial \Phi/\partial r)$ and $(1/T)(\partial T/\partial r)$ by $(1/n)(\partial n/\partial r)$ in Eq. (1). Similarly, the neoclassical energy flux is assumed to be proportional only to $(1/T)(\partial T/\partial r)$ and thus $(e/T)(\partial \Phi/\partial r)$ and $(1/n)(\partial n/\partial r)$ can be replaced by $(1/T)(\partial T/\partial r)$ in Eq. (2). As a result, Eqs. (1) and (2) now have the form

$$\Gamma_{\text{neo}}(r) = -78.4 (2\xi_h)^{3/2} \frac{T_4^{7/2}}{B_4^2 R_m^2} \frac{\partial n_{14}}{\partial r_m} (10^{19} \text{ cm}^{-2} \cdot \text{sec}^{-1}), \quad (10)$$

$$Q_{\text{neo}}(r) = -459.0 (2\xi_h)^{3/2} \frac{T_4^{7/2}}{B_4^2 R_m^2} \frac{\partial T_4}{\partial r_m} (10^{20} \text{ keV} \cdot \text{cm}^{-2} \cdot \text{sec}^{-1}). \quad (11)$$

The particle source per unit volume, s_p , is assumed to be a constant in minor radius. Hence, $\Gamma_{\text{total}}(r) = s_p (\int_0^r dr' 2\pi r') / 2\pi r$. The thermonuclear input power source per unit volume is assumed to have a centrally peaked profile, i.e., $s_e \exp[-14.0(r/a)^2]$ where s_e is a constant. Hence, $Q_{\text{total}}(r) = s_e [\int_0^r dr' 2\pi r' \exp(-14.0(r'/a)^2)] / (2\pi r)$. In this example, we choose $s_p = 1.7 \times 10^{15} \text{ cm}^{-3} \cdot \text{sec}^{-1}$ and $s_e = 2.8 \times 10^{17} \text{ keV} \cdot \text{cm}^{-3} \cdot \text{sec}^{-1}$. The function $F(r)$ is assumed to be proportional to $(a - r)$ with the constant of proportionality adjusted so that $\beta = 30\%$ at $r = 0$. The boundary conditions are chosen to be

$n(a) = 10^{12} \text{ cm}^{-3}$ and $T(a) = 2 \text{ keV}$. The density and temperature profiles of the reference reactor are calculated using the method described in the previous section. The results are shown in Fig. 2. With the chosen functional form of $F(r)$, the convective region starts from the center of the plasma column and expands toward the plasma edge as the input particle and energy sources increase. From the density and temperature profiles given in Fig. 2, the average plasma β can be shown to be approximately 12% and the average density $\langle n \rangle$ is $4 \times 10^{14} \text{ cm}^{-3}$. If the energy confinement time is defined by

$$\tau_E = \frac{\int_0^a dr (2\pi r) (3/2) n T}{Q_{\text{total}}(a)}, \quad (12)$$

then it can be shown that $\tau_E \approx 0.6 \text{ sec}$, and $\langle n \rangle \tau_E \approx 2.4 \times 10^{14} \text{ cm}^{-3}$. Note that using the data given in Fig. 2, we find that the particle confinement time is less than the energy confinement time. However, neoclassical theory shows that the energy confinement time should be less than that of particles [6]. The discrepancy may be due to the simplified assumptions used in obtaining Eqs. (10) and (11).

With the specified value of s_E used for this example, the steady-state burning plasma is thermally stable. That is, if the value of s_E is increased slightly (hence the total input power Q_{total} is increased), the increase in the total energy flux calculated from the density and temperature profiles obtained with this value of s_E is less than the increase in the total input power corresponding to the increase of s_E . If the value of s_E is decreased slightly, then the decrease in the total energy flux calculated from the density and temperature profiles is less than the decrease in the total input power corresponding to the decrease of s_E . If the increase (or decrease) of

s_e and the energy flux is considered to be the result of a temperature perturbation, then the above conditions mean that the plasma temperature is stable against such temperature perturbation. For a detailed discussion on thermal stability, see Ref. [14].

To conclude this section, the stability of the boundaries of the convective region against radial displacement for a thermally stable steady-state burning plasma is now considered. First, consider the outer edge of the convective region. If there is a perturbation that causes the outer edge to move outward slightly, the increased flux due to convection reduces the pressure gradient and causes the perturbed boundary to move back to its original position. If the perturbation moves the outer edge inward, then the flux at the original boundary is decreased. Consequently, the pressure gradient builds up and triggers the convection to return to its original boundary. A similar argument can be applied to the inner edge of the convective region. Hence, the convective region is stable against radial perturbations on its boundaries.

6. CONCLUSION

We have shown that in a high β stellarator, or a tokamak reactor, convective cells may exist to enhance the transport in regions where the plasma pressure gradient tends to exceed the critical gradient. The mixing-length theory shows that the convective transport is highly efficient and the pressure gradient only needs to exceed the critical gradient slightly in order to provide the necessary driving force for the convective cells. The convective cells in turn transport mass and energy and prevent the pressure gradient from greatly exceeding the critical gradient. Even if the size of the mixing length cannot be accurately calculated, the density and temperature

profiles in the convective region can still be determined if the ratio of the convective energy flux to the convective particle flux is known. The result from the numerical example for the stellarator reactor shows that the location of the convective region depends on the radial profile of the critical pressure gradient determined from MHD stability analysis. Only a qualitative description of the convective cells that may be present in high β reactors has been given in this paper. In order to understand the nonlinear evolution of the convective cells and the mechanisms that lead to the dissolving of a cell, a nonlinear theory with nonideal effects needs to be developed.

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APPENDIX

ESTIMATE OF THE CONVECTIVE CELL VELOCITY USING LINEAR MHD STABILITY ANALYSIS

To simplify the analysis, the "bad" curvature region (the outer region of the torus) is modeled by a slab geometry. Assume a magnetic field pointed along the z -direction and "tied" to the conducting walls at $z = \pm\pi/k_z$, as shown in Fig. 3. The region between $z = \pm\pi/k_z$, which has a length on the order of the connection length, corresponds to the bad curvature region of the torus. If the plasma pressure is increasing in the positive x -direction, then the pressure-driven force due to the bad curvature effect can be modeled by

$$\begin{aligned} p\vec{F} &= - \frac{\rho_i v_{Ti}^2 + \rho_e v_{Te}^2}{2R_{\min}} (1 + \cos k_z z) \hat{e}_x \\ &= - \frac{3}{2} \frac{p}{R_{\min}} (1 + \cos k_z z) \hat{e}_x, \end{aligned}$$

where v_T is the thermal velocity, $R_{\min}$ is the minimum radius of curvature, and the subscripts e and i denote electron and ion species, respectively. Note that the characteristic length in the z -direction is generally much longer than those in the x and y directions. Therefore, gradients in the z -direction are negligible compared to those in the x and y directions.

In order to obtain the initial velocity of the convective cell, the perfect conducting, incompressible MHD equations are linearized by considering an arbitrary small perturbation from the hydrostatic equilibrium with zero fluid velocity. All equilibrium quantities are denoted by a subscript 0 and all perturbed quantities by the subscript 1. The linearized equations are:

$$\frac{\partial p_1}{\partial t} + \vec{v}_1 \cdot \vec{\nabla} p_0 = 0, \quad (\text{since } \frac{\partial p_1}{\partial t} + \vec{v}_1 \cdot \vec{\nabla} p_0 = 0), \quad (A1)$$

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\vec{\nabla} p_1 + \frac{1}{4\pi} (\vec{v} \times \vec{B}_1) \times \vec{B}_0 - \frac{3}{2} \frac{p_1}{R_{\min}} (1 + \cos k_z z) \hat{e}_x, \quad (A2)$$

$$\frac{\partial \vec{B}_1}{\partial t} = \vec{v} \times (\vec{v}_1 \times \vec{B}_0), \quad (A3)$$

$$\vec{v}_1 \cdot \vec{v}_1 = 0, \quad (A4)$$

where the symbol $\perp$ denotes directions that are normal to $\hat{e}_z$. Equation (A4) allows one to choose a stream function ϕ such that $\vec{v}_1 = \nabla \phi \times \hat{e}_z$. To describe convective cells with circular or elliptical flow patterns, the stream function may be represented by

$$\phi = A(t) \sin k_x x \sin k_y y \sin k_z z, \quad (A5)$$

where the function $A(t)$ describes the temporal evolution of the stream function.

To further simplify the calculation, the Boussinesq approximation, i.e., the density gradient is negligible except where it enters the force term, is used here. The perturbed pressure and magnetic field in Eq. (A2) are obtained from Eqs. (A1) and (A3), respectively. Then taking the curl of Eq. (A2) yields

$$\begin{aligned}
& \frac{\partial A}{\partial t} \sin k_x x \sin k_y y \sin k_z z \hat{e}_z \\
& = \left[\frac{3}{2} \frac{k_y^2}{k^2} \frac{1}{\rho_0 R_{\min}} \frac{\partial p_0}{\partial x} (1 + \cos k_z z) - \frac{k_z^2 B_0^2}{4\pi \rho_0} \right] \\
& \quad \times \sin k_x x \sin k_y y \sin k_z z \int_0^t dt' A(t') \hat{e}_z, \quad (A6)
\end{aligned}$$

where $k^2 = k_x^2 + k_y^2$. It is evident from the right-hand side of the equation that the fastest growing modes have the smallest k_x/k_y ratio. Thus, the modes will tend to grow in narrow cells along the x-direction. (The minimum width of the cell is probably determined by plasma viscosity.) The shape of the cells allows one to set $k_y/k \sim 1$ and to approximate the velocity to be entirely along the x-direction in order to estimate the maximum velocity of the convective cell. Then, using Eq. (A6), it can be shown that at $z = 0$ (where the local magnetic field curvature is most unfavorable to mode stability and the driving force p_0^2 has the maximum value)

$$\frac{\partial^2 \xi_x}{\partial t^2} = \left(\frac{3}{\rho_0 R_{\min}} \frac{\partial p_0}{\partial x} - \frac{k_z^2 B_0^2}{4\pi \rho_0} \right) \xi_x, \quad (A7)$$

where the linear displacement ξ_x is defined by

$$\frac{\partial \xi_x}{\partial t} \equiv v_x.$$

If we define the dimensionless parameter

$$\hat{\beta} \equiv \frac{3}{2} \frac{\beta}{k_z^2} \frac{1}{r_{p \min}^R}, \quad (A8)$$

where $r_p^{-1} \equiv |\partial P_0 / \partial x| / P_0$, then the condition for the onset of pressure-driven convective cells is $\hat{\beta} > 1$ according to Eq. (A8). Note that for the case when $k_z^{-1} \sim qR$ (qR is the connection length with q the inverse of the normalized rotational transform t), $\hat{\beta} < 1$ is just the usual stability criterion for ballooning modes. If subscript c denotes the condition at $\hat{\beta} = 1$, then Eq. (A7) can be written as

$$\begin{aligned} \frac{\partial^2 \epsilon_x}{\partial t^2} &= \frac{3}{\rho R_{\min}} \left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial x} \Big|_c \right) \epsilon_x \\ &= \frac{4v_{Ti}^2}{R_{\min} P} \left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial x} \Big|_c \right) \epsilon_x . \end{aligned}$$

Multiplying both sides of the equation by $\partial \epsilon_x / \partial t$ and changing the coordinate system to cylindrical coordinates, the initial velocity of a convective cell can be approximated by

$$\begin{aligned} v_{\text{conv}} &\approx v_r \\ &= \left[\frac{4v_{Ti}^2}{R P} \left(\frac{\partial P}{\partial r} - \frac{\partial P}{\partial r} \Big|_c \right) \right]^{1/2} \epsilon_r , \end{aligned} \quad (\text{A9})$$

where v_r is convective cell velocity in the radial direction, and $R_{\min}$ is approximated by the major radius R . Note that this expression of the velocity is valid until the growth is saturated by nonlinear mechanisms. An attempt to describe the nonlinear behavior of the line-tied interchange modes has been made by Cowley et al. [9].

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TABLE I. REFERENCE STELLARATOR REACTOR PARAMETERS

Toroidal magnetic field (kG)	50
Major radius R(m)	10
Minor radius a(m)	1.6
ϵ_h	0.1

FIGURE CAPTIONS

Fig. 1. Flow diagram for determining the steady-state density and temperature profiles in a plasma column.

Fig. 2. Radial profiles of plasma parameters for the stellarator reactor with machine parameters given in Table I. The particle injection rate is $1.7 \times 10^{15} \text{ cm}^{-3} \cdot \text{sec}^{-1}$ and the power source is $2.8 \times 10^{17} \times \exp[-14.0 (r/a)^2] \text{ keV} \cdot \text{cm}^{-3} \cdot \text{sec}^{-1}$.

- (a) Profile of plasma density.
- (b) Profile of plasma temperature.
- (c) Profile of plasma pressure.
- (d) Profile of plasma pressure gradient. The part of the curve that varies linearly with r corresponds to the convective region.
- (e) Profile of particle flux (per unit length of the plasma column). The solid curve is the total particle flux and the shaded area represents the contribution from the convective particle flux.
- (f) Profile of energy flux (per unit length of the plasma column). The solid curve is the total energy flux and the shaded area represents the contribution from the convective energy flux.

Fig. 3. Plasma confined by magnetic field tied to conducting plates. Plasma pressure is increasing in the positive x -direction and the pressure-driven force due to the bad curvature effect is pointing in the negative x -direction.

Fig. 1

Fig. 2

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Fig. 3

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