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"SENSING" THE ELLIPTICAL TORUS

by

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ABSTRACT

A method is given for testing the location of an arbitrary point $P=(x,y,z)$ relative to the toroidal surface generated by rotation of an ellipse $(y-\bar{y})^2/b^2 + (z-\bar{z})^2/c^2 = 1$, $b, c, \bar{z} > 0$, about the Y axis. This includes the orientation of P with respect to both the outer and inner surfaces of the degenerate torus generated by an ellipse with $c > \bar{z}$.

I. INTRODUCTION

In a previous report,¹ a complete computational method was given for finding all real points of intersection of a line of flight with a toroidal surface of revolution. This naturally depends on the solution of a quartic equation, and the theory required, including the underlying solution of the cubic, was there developed in detail. In the present work, a method is given for location of an arbitrary point relative to such a toroidal surface. The combined reports allow the inclusion of cells bounded by parts of elliptical tori in the geometry of Monte Carlo particle transport.

II. EQUATION OF THE ELLIPTICAL TORUS

The equation of the surface of revolution generated by rotating a curve $f(y', z') = 0$, $z' \geq 0$ of the YZ plane about the Y axis is

$$f(y, (x^2 + z^2)^{1/2}) = 0.$$

For a curve $f(y', z') = 0$, $z' \leq 0$, the equation of the resulting surface is

$$f(y, -(x^2 + z^2)^{1/2}) = 0.$$

Such a rotation of the ellipse

$$f(y', z') \equiv (y' - \bar{y})^2 / b^2 + (z' - \bar{z})^2 / c^2 - 1 = 0; \quad b, c, \bar{z} > 0$$

produces a nondegenerate torus (F) with equation

$$F(x, y, z) \equiv (y - \bar{y})^2 / b^2 + ((x^2 + z^2)^{1/2} - \bar{z})^2 / c^2 = 1; \quad c \leq \bar{z} \quad (1)$$

in the case $c \leq \bar{z}$, since all points (y', z') of the ellipse then have $z' \geq 0$. (See Fig. 1.)

However, if $c > \bar{z}$, a degenerate torus results (Fig. 2), having an outer surface (o) generated by the arc of the ellipse with $z' \geq 0$, and with equation

$$o(x, y, z) \equiv (y - \bar{y})^2 / b^2 + ((x^2 + z^2)^{1/2} - \bar{z})^2 / c^2 = 1; \quad c > \bar{z}, \quad (2)$$

and an inner surface (I) generated by the arc of the same ellipse with $z' \leq 0$, which therefore has equation

$$I(x, y, z) \equiv (y - \bar{y})^2 / b^2 + ((x^2 + z^2)^{1/2} + \bar{z})^2 / c^2 = 1; \quad c > \bar{z}. \quad (3)$$

These surface equations may be written in the form

$$x^2 + z^2 + \rho y^2 - 2\rho \bar{y}y + \beta_0 = 2\bar{z}(x^2 + z^2)^{1/2}; \quad c \leq \bar{z} \quad (F) \quad (4)$$

$$x^2 + z^2 + \rho y^2 - 2\rho \bar{y}y + \beta_0 = \pm 2\bar{z}(x^2 + z^2)^{1/2}; \quad c > \bar{z} \quad \begin{matrix} (o) \\ (I) \end{matrix}, \quad (5)$$

$$\text{where } \rho = c^2 / b^2 \geq 1, \quad \beta_0 = \rho \bar{y}^2 + \bar{z}^2 - c^2. \quad (6)$$

Defining the functions

$$L(x, y, z) \equiv x^2 + z^2 + \rho y^2 - 2\rho \bar{y}y + \beta_0, \quad R(x, y, z) = 2\bar{z}(x^2 + z^2)^{1/2}, \quad (7)$$

Fig. 2. Degenerate case, $c > \bar{z}$.

in either case, we may write (4), (5) as

$$L(x,y,z) = \pm R(x,y,z); \quad c > \bar{z} \quad \begin{matrix} (0) \\ (I) \end{matrix} \quad (9)$$

Squaring, we obtain from both (8) and (9) the equation

for the complete toroidal surface. Solution of this equation with a given line of flight leads to the quartic mentioned in Sec. I. (Since $L^2=R^2$ is equivalent to $(L-R)(L+R)=0$, or $L=\pm R$, it is seen that an extraneous factor,

$$L+R=x^2+z^2+p(y-\bar{y})^2+(\bar{z}^2-c^2)+2\bar{z}(x^2+z^2)^{1/2} \geq 0.$$

has been introduced in the case $c \leq \bar{z}$. This is zero iff $c = \bar{z}$ and $x=0, z=0, y=\bar{y}$, the point of tangency already satisfying $L-R=0$.)

Thus a point $P=(x,y,z)$ is on the complete torus iff $L^2(x,y,z) = R^2(x,y,z)$, as in (10). In the degenerate case $c > \bar{z}$, such a point is on the outer surface (o) iff $L(x,y,z) \geq 0$, and on the inner surface (I) iff $L(x,y,z) \leq 0$. Equality holds in either case iff P is one of the two points on the Y axis.

III. "SENSING" THE TORUS

It is important in Monte Carlo practice to determine the position of an arbitrary point $P=(x,y,z)$ relative to the toroidal surface. The following analysis is based on equations of form

$$(y-\bar{y})^2/(bk)^2 + ((x^2+z^2)^{1/2} - \bar{z})^2/(ck)^2 = 1, \quad (11)$$

$$(y-\bar{y})^2/(bk)^2 + ((x^2+z^2)^{1/2} + \bar{z})^2/(ck)^2 = 1, \quad (12)$$

where $k > 0$ is always understood.

A. Nondegenerate case

Equation of surface (F) is

$$F(x,y,z) \equiv (y-\bar{y})^2/b^2 + ((x^2+z^2)^{1/2} - \bar{z})^2/c^2 = 1; \quad c \leq \bar{z}.$$

We now have the possibilities:

1. $F(x,y,z)=k^2=0$. P is on the circular axis of (F), inside (F).
2. $F(x,y,z)=k^2, 0 < k^2 < 1$. P is on the surface (11), with $ck < c \leq \bar{z}$; this is a nondegenerate torus inside (F).
3. $F(x,y,z)=k^2=1$. P is on the torus (F).
4. $F(x,y,z)=k^2 > 1$. P satisfies equation (11), with $c < ck$. Two cases:
 - a. $c < ck \leq \bar{z}$. P is on a nondegenerate torus outside (F).
 - b. $c \leq \bar{z} < ck$. P is on the outer surface of a degenerate torus outside (F).

B. Degenerate case

Outer surface (o) has equation

$$o(x,y,z) \equiv (y-\bar{y})^2/b^2 + ((x^2+z^2)^{1/2} - \bar{z})^2/c^2 = 1; \quad \bar{z} < c.$$

1. $o(x,y,z)=k^2=0$. P is on the circular axis of (o), inside (o).
2. $o(x,y,z)=k^2$, $0 < k^2 < 1$. P satisfies equation (11), with $ck < c$.

Two cases:

- a. $ck \leq \bar{z} < c$. P is on a nondegenerate torus inside (o).
- b. $\bar{z} < ck < c$. P is on the outer surface of a degenerate torus inside (o).
3. $o(x,y,z)=k^2=1$. P is on outer surface (o).
4. $o(x,y,z)=k^2 > 1$. P satisfies equation (11), with $\bar{z} < c < ck$. P is on outer surface of a degenerate torus outside (o).

C. Degenerate case

Inner surface (I) has equation

$$I(x,y,z) \equiv (y-\bar{y})^2/b^2 + ((x^2+z^2)^{1/2} + \bar{z})^2/c^2 = 1; \bar{z} < c.$$

1. $I(x,y,z)=k^2=0$ is impossible since $\bar{z} > 0$.
2. $I(x,y,z)=k^2$, $0 < k^2 < 1$. P satisfies equation (12) with $ck < c$.

But (12) implies $\bar{z}^2/(ck)^2 \leq 1$ or $\bar{z} \leq ck$. Hence $\bar{z} \leq ck < c$, and we have two cases:

- a. $\bar{z}=ck$. Then P is necessarily the point $P=(0,\bar{y},0)$ inside (I).
- b. $\bar{z} < ck < c$. P is on inner surface of a degenerate torus inside (I).
3. $I(x,y,z)=k^2=1$. P is on (I).
4. $I(x,y,z)=k^2 > 1$. P satisfies (12) with $\bar{z} < c < ck$, and P is on the inner surface of a degenerate torus outside (I).

IV. SUMMARY

Using the definitions of the functions $L(x,y,z)$, $R(x,y,z)$ in (7), and their obvious relations with the functions $F(x,y,z)$, $o(x,y,z)$, $I(x,y,z)$ in (1,2,3), we may summarize these "sensing" criteria in the following more convenient computational form.

A. Nondegenerate case, $c \leq \bar{z}$, surface (F)

- $L < R$, P inside (F)
- $L = R$, P on (F)
- $L > R$, P outside (F).

B. Degenerate case, $c > \bar{z}$, outer surface (o)

$L < R$, P inside (o)

$L = R$, P on (o)

$L > R$, P outside (o).

C. Degenerate case, $c > \bar{z}$, inner surface (I)

$L < -R$, P inside (I)

$L = -R$, P on (I)

$L > -R$, P outside (I).

In the nondegenerate case $c \leq \bar{z}$, the sensing can be done on the radical-free equation (10) of (F), according to the rules

$L^2 < R^2$, P inside (F)

$L^2 = R^2$, P on (F)

$L^2 > R^2$, P outside (F) .

This is true because $L^2 - R^2 \equiv (L-R)(L+R)$, where $L+R \geq 0$, with equality iff $c = \bar{z}$, $x=0$, $z=0$, $y=\bar{y}$, as noted in Sec. II.

However, in the degenerate case, sensing on $L^2 - R^2$ is inconclusive, and the rules B and C above are required. To see this, we observe that, in terms of the open regions labeled 1,2,3 in Fig. 2, the following relations are equivalent:

$$L^2 - R^2 = (L-R)(L+R) < 0 ,$$

$$(L > R \text{ and } L < -R) \text{ or } (L < R \text{ and } L > -R) ,$$

$$(P \text{ in } 3 \text{ and } P \text{ in } 1) \text{ or } (P \text{ in } 1 \text{ or } 2 \text{ and } P \text{ in } 2 \text{ or } 3) ,$$

$$P \text{ in } 2, \text{ inside } o \text{ and outside } I ,$$

and this is decisive. However, one finds similarly, or by default, that $L^2 - R^2 > 0$ iff P is in 1 or in 3, which is inconclusive for either surface o or I.

REFERENCE

1. E. D. Cashwell, C. J. Everett, "Intersection of a Ray with a Surface of Third or Fourth Degree," Los Alamos Scientific Laboratory report LA-4299 (1969).