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IS THE TWO-TERM EXPANSION VALID FOR HIGHLY ANISOTROPIC SYSTEMS? THE
TOWNSEND IONIZATION COEFFICIENT IN STRONG RUNAWAY AS A TEST CASE*

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Abstract

The Townsend ionization coefficient in the strong runaway regime is calculated within the framework of the two-term expansion. Results are compared to the 1-D model. General features of the two models are qualitatively similar, but quantitative differences by factors of ~ 2 are observed.

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A Boltzmann code NUTS¹ has recently been developed to study the evolution of electron-beam-generated plasma channel. The purpose of the code is to study channel physics at all pressures (from tens of microns to atmospheric densities). Hence, the usual simple models associated with the limiting cases at high pressures (local conductivity model) and at the very low pressures (the ion-focused regime (IFR)) are not invoked. Instead, the full Boltzmann Equation is solved with all inertial, nonlocal and electromagnetic terms kept. The theory consists of one single major approximation; namely, the dipole expansion in velocity space. In this treatment, the distribution function is expressed in terms of spherical harmonics:

$$F(\vec{v}) \approx \sum_{\ell m} f_{\ell m}(v) Y_{\ell m}.$$

Only terms with $\ell \leq 1$ are retained:

$$f_{\ell m} = 0 \quad \text{for} \quad \ell > 1.$$

The $\ell = 1, m \neq 0$ terms are also kept in NUTS to allow for current flow in both the longitudinal as well as the transverse direction.

The two-term expansion is rigorously justified when the system is nearly isotropic. This approximation is physically reasonable for describing the bulk of electrons at high pressures. However, one might question its validity at low pressures in the presence of a strong electric field when the distribution functions are highly anisotropic. Even at high pressures, there is generally a small population of energetic electrons (delta rays) whose motion is much more directional than the bulk. In the most extreme case, we may have

$$F(\vec{v}) = f(v_z) \delta(v_x) \delta(v_y).$$

To make a two-term expansion for such a distribution seems arbitrary. In this note, we consider just such a case, and compare the exact solution to the approximate one which results from the two-term expansion. We hope by this example to gain some insight regarding the validity of NUTS in describing phenomena which involve strongly anisotropic distributions.

The example under study² was first considered by R. J. Briggs. He showed that in the strong runaway regime, a Townsend coefficient for ionization can be derived under the following assumptions:

- i. Spatially uniform, time-independent electric field.
- ii. Steady state.
- iii. Frictionless electrons (low pressure).
- iv. Ionizations with secondaries born at rest.

In addition to being a relatively simple problem for which an analytic solution can be obtained, this example is important because it forms the mathematical basis for determining the pressure boundaries of the IFR. The theoretical results seem to be supported by the FX-25 propagation experiments.

The model 1-D Boltzmann Equation for the distribution function $f(v, z)$ has the form

$$v \frac{\partial f}{\partial z} + \frac{eE}{m} \frac{\partial f}{\partial v} = \delta(v) \int_0^{\infty} n_g \sigma(w') v' f(v', z) dv' , \quad (1)$$

where $n_g \sigma(w') v'$ is the ionization rate for an electron with energy $w' = \frac{1}{2} m v'^2 / e$, and $\delta(v)$ is the delta function.

Eq. (1) has a solution that exponentially increases in space

$$f(v, z) = A \Theta(v) e^{-\eta(w-Ez)} , \quad (2)$$

where A is a constant, and $\Theta(v)$ the step function. The parameter η is proportional to the first Townsend ionization coefficient α by

$$\alpha = \eta E , \quad (3)$$

and is determined by the condition

$$\epsilon = \int_0^{\infty} n_g \sigma(w) e^{-\eta w} dw . \quad (4)$$

To facilitate comparisons with the dipole expansion, we first make some observations and definitions regarding the above 1-D calculation. First, we note that Eq. (2) corresponds to an exponentially growing electron density as well as an exponentially growing particle current.

$$N(z) = N(0) e^{\eta E z} \quad (5a)$$

$$J(z) = J(0) e^{\eta E z} . \quad (5b)$$

Secondly, if for the two macroscopic parameters $N(z)$ and $J(z)$, we define corresponding distribution functions in v space $n(v, z)$ and $j(v, z)$ by

$$N(z) = \int_0^{\infty} dv n(v, z) \quad (6a)$$

$$J(z) = \int_0^{\infty} dv j(v, z) , \quad (6b)$$

then, for the 1-D calculation

$$n(v, z) = A \Theta(v) e^{-\eta(w-Ez)} \quad (7a)$$

$$j(v, z) = v A \Theta(v) e^{-\eta(w-Ez)} . \quad (7b)$$

Thirdly, Eq. (2) may be viewed either as the asymptotic solution, valid only at large z , or it could be viewed as the exact solution for all z , provided that the distribution function is properly "prepared" - i.e., the boundary conditions at $z = 0$ are correct. We will take the second point of view to make our comparisons with the dipole treatment unambiguous. We note that this approach implies an η -dependent boundary condition on the macroscopic velocity.

$$v(0) \equiv \frac{J(0)}{N(0)} = \frac{\int_0^\infty dv v e^{-\eta w}}{\int_0^\infty dv e^{-\eta w}} \quad (8)$$

The dipole expansion^{3,4} starts from a distribution function in 3-dimensional velocity space $F(\vec{v}, z)$, satisfying the steady-state 3-D Boltzmann Equation

$$v_z \frac{\partial F}{\partial z} + \frac{eE}{m} \frac{\partial F}{\partial v_z} = \int n_g v' \frac{d^3 \sigma}{dv^3} (v' v) F(v') d^3 v', \quad (9)$$

where the uniform electric field E points in the z -direction. The angular dependence of the distribution function in velocity space is assumed to take on a simple form

$$F(\vec{v}, z) = F_0(v, z) + F_1(v, z) \cos \theta \quad (10)$$

In analogy to the 1-D case, we postulate a simple cross section model

$$\frac{d^3 \sigma}{dv^3} (v' v) \approx \frac{1}{4\pi} \frac{\delta(v)}{v^2} \sigma(v') \quad (11)$$

Making the usual angular decomposition of the operators

$$v_z = v \cos \theta \quad (12)$$

$$\frac{\partial}{\partial v_z} = \cos \theta \frac{\partial}{\partial v} + \frac{\sin^2 \theta}{v} \frac{\partial}{\partial \cos \theta} \quad (13)$$

and performing integration over angles, we obtain two equations which form the analog of Eq. (1) in the dipole treatment

$$\frac{v}{3} \frac{\partial F_1}{\partial z} + \frac{eE}{3mv^2} \frac{\partial}{\partial v} (v^2 F_1) = \frac{\delta(v)}{v^2} \int_0^\infty n_g \sigma(w') v' F_0(v', z) v'^2 dv' \quad (14)$$

$$v \frac{\partial F_0}{\partial z} + \frac{eE}{m} \frac{\partial F_0}{\partial v} = 0 \quad (15)$$

The solution to Eq. (15) is

$$F_0 = A_0 e^{-\eta(w-Ez)} , \quad (16)$$

while Eq. (14) is satisfied by

$$v^2 F_1 = A_1 \theta(v) e^{-\eta(w-Ez)} , \quad (17)$$

with an equation for η given by

$$\frac{eE}{3m} A_1 = A_0 \int_0^\infty n_g \sigma(w) v^3 dv e^{-\eta w} . \quad (18)$$

We first note that the exponential growth in $N(z)$ and $J(z)$ seen in the 1-D model is retrieved in the 2-term expansion since

$$N(z) = 4\pi \int_0^\infty dv v^2 F_0 = N(0) e^{\eta Ez} \quad (19)$$

$$\text{with } N(0) = A_0 4\pi \int_0^\infty v^2 dv e^{-\eta w} \quad (20)$$

$$J(z) = \frac{4\pi}{3} \int_0^\infty dv v^3 F_1 = J(0) e^{\eta Ez} \quad (21)$$

$$\text{with } J(0) = A_1 \frac{4\pi}{3} \int_0^\infty dv v e^{-\eta w} \theta(v) . \quad (22)$$

The current distribution function in v space, defined as in Eq. (6b), has the same functional form as the 1-D result, [Eq. (7b)]

$$j(v,z) = A_1 \frac{4\pi}{3} v \theta(v) e^{-\eta(w-Ez)} , \quad (23)$$

but the density distribution function $n(v,z)$ possesses an additional weighting factor of velocity squared

$$n(v,z) = A_0 4\pi v^2 e^{-\eta(w-Ez)} \quad (24)$$

[cf Eq. (7a)].

It is well known that in describing anisotropic systems with the two-term treatment, the drift velocity is much more accurate than the characteristic energy. The results of our particular example is consistent with this general phenomenon. On the basis of these considerations, we may expect our calculations of the plasma currents to be generally more accurate than the plasma density and temperature.

We now turn to the calculation of the Townsend coefficient. We note that Eq. (18) as it stands, is not closed. An additional relation for A_0 and A_1 is needed. This is obtained by demanding that the initial macroscopic velocity from the ratio of Eq. (22) to Eq. (20) be the same as the 1-D case [Eq. (8)]. The net result is an implicit equation of η as a function of E , given by

$$E = 2 \int_0^\infty n_g \sigma(w) \eta w e^{-\eta w} dw \quad (25)$$

This is the analog of Eq. (4) in the two-term treatment.

In Reference 2, it was shown that in the 1-D case, Eq. (4) could be further reduced if an analytic model for the cross section is employed:

$$\sigma(w) = 4 \sigma_0 \frac{w/w_0}{(1 + w/w_0)^2} \quad (26)$$

A reasonable fit to the nitrogen data was obtained with the parameters $w_0 = 100$ ev, $\sigma_0 = 2.5 \times 10^{-16} \text{ cm}^2$. With a general form of the cross section given by Eq. (26), Eq. (4) reduces to a relation between $\tilde{\eta} = \eta w_0$ and $\tilde{E} = E/4n_g\sigma_0 w_0$ as

$$\tilde{E} = e^{\tilde{\eta}} (1 + \tilde{\eta}) \left[-E_i(-\tilde{\eta}) \right] - 1 \quad [1-D] \quad (27)$$

where E_i is the exponential integral.

A similar calculation can be done in the 2-term treatment, starting with the equation for η [Eq. (25)]. The net result is given by

$$\frac{\tilde{E}}{2} = -\tilde{\eta} e^{\tilde{\eta}} \left[2 + \tilde{\eta} \right] \left[-E_i(-\tilde{\eta}) \right] + (1 + \tilde{\eta}) \quad [2-term] \quad (28)$$

Following Reference 2, we plot on Figure 1 the parameter $n_0 \sigma_0 / \alpha \equiv (4\tilde{\eta}\tilde{E})^{-1}$ as a function of $\tilde{E}$ for the 2 models. The significance of the parameter $n_0 \sigma_0 / \alpha$ is that it represents $\lambda_i / (\lambda_i)_{\min}$, the ratio of the effective ionization mean free path to the minimum mean free path for an electron moving in a straight line.

We note the 2-term expansion falls below the 1-D case at low $\tilde{E}$, but rises steeply as $\tilde{E}$ becomes large. The cross-over point being $\tilde{E} = 1.45$. At the low $\tilde{E}$ end, the value of $\lambda_i / (\lambda_i)_{\min}$ in fact falls below 1. This is physically reasonable since in the 2-term expansion, electrons do not move in a straight line. The sideways dispersion leads to a reduced effective ionization mean free path. At the high $\tilde{E}$ end, Eq. (28) is seen to have a singularity at $\tilde{E} = 2$, where $\tilde{\eta} \rightarrow \infty$. The physical meaning of this singularity is that at large $\tilde{E}$ (> 2), the probability for ionization is always less than 1. This is an artifact resulting from the inaccurate asymptotic behavior of the model cross section [Eq. (26)]. The relative deviations of the 2-term expansion from the 1-D model can be understood in terms of the different weightings associated with the density distribution function $n(v, z)$.

In Figure 2, the Townsend ionization coefficients for N_2 from the 2-models are plotted as a function of E/p . For comparison, we have plotted on the same figure the results from the 1-D and the 2-term calculations by direct numerical evaluation of Eqs. (4) and (25), using real data for cross sections with the proper Born asymptotic limit. This removes the singularities at high E/p associated with the model cross section [Eq. (26).]

The important point to note from the above calculations is that the results from the two-term expansion has qualitatively similar features to the 1-D model, with quantitative differences by factors of two typically. Comparisons of the relevant parameters for the two models are summarized in Table I. We conclude that the two-term expansion gives a reasonable description of this extremely anisotropic example, with quantitative errors of ≤ 2 .

TABLE I 1-D MODEL AND THE TWO-TERM EXPANSION COMPARED

	<u>1-D</u>	<u>TWO-TERM</u>
Number density $N(z)$	$N(0)e^{\eta Ez}$	$N(0)e^{\eta Ez}$
Current density $J(z)$	$J(0)\eta Ez$	$J(0)e^{\eta Ez}$
Particle distribution function $n(v,z)$	$e^{-\eta(w-Ez)}$	$w e^{-\eta(w-Ez)}$
Current distribution function $j(v,z)$	$v\theta(v)e^{-\eta(w-Ez)}$	$v\theta(v)e^{-\eta(w-Ez)}$
Implicit equation for ionization coefficient	$E = \int_0^{\infty} n_0 \sigma(w) e^{-\eta w} dw$	$\frac{E}{2} = \int_0^{\infty} n_0 \sigma(w) \eta w e^{-\eta w} dw$
Ionization coefficient: model cross sections	$\tilde{E} = e^{\tilde{\eta}} (1 + \tilde{\eta}) \left[-E_1(-\tilde{\eta}) \right] - 1$	$\frac{\tilde{E}}{2} = -\tilde{\eta} e^{\tilde{\eta}} \left[2 + \tilde{\eta} \right] \left[-E_1(-\tilde{\eta}) \right] + (1 + \tilde{\eta})$
E/p at $P/\alpha = 1$ (real cross sections)	20 kev/cm-torr	12 kev/cm-torr

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FIGURE CAPTIONS

1. Universal curves for normalized mean free path for ionization as a function of normalized electric field -- 1-D model and two-term expansion compared.
2. Townsend ionization coefficient as a function of E/p for N_2 .

FIG. 1. Universal curves for normalized mean free path for ionization as a function of normalized electric field — 1-D model and two-term expansion compared.

FIG. 2. Townsend ionization coefficient as a function of E/p for N_2 .