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Use of Collineations for
Interpolation on an Irregular Mesh

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Interpolation on an Irregular Mesh**

by

H. Ralph Lewis

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USE OF COLLINEATIONS FOR INTERPOLATION ON AN IRREGULAR MESH

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H. Ralph Lewis

ABSTRACT

A method is described for using collineations for interpolation on an irregular mesh whose elementary cells are quadrilaterals and whose mesh points are ordered in a way topologically equivalent to the ordering of the points of a rectangular mesh. First the irregular mesh is transformed to a rectangular mesh, and then interpolation is performed on the rectangular mesh. Listings of two computer programs that implement the method are given with examples. One of the programs uses bicubic spline interpolation on the rectangular mesh, and the other uses bilinear interpolation.

I. INTRODUCTION

This report describes a method for interpolating on an irregular mesh whose elementary cells are quadrilaterals and whose mesh points are ordered in a way topologically equivalent to the ordering of the points of a rectangular mesh. The function to be interpolated is specified by its values at the points of the mesh. The method starts from the assumption that it is adequate to consider the function to be smooth over each elementary quadrilateral of the mesh but not at the boundaries of the quadrilaterals; only continuity of the function is required at the boundaries. We define a transformation of the space that carries each elementary quadrilateral of the irregular mesh into an elementary rectangle of a rectangular mesh in such a way that the ordering of the points of the rectangular mesh is topologically equivalent to that of the irregular mesh. Given the assumption about the

nature of the function, it is natural to use a transformation that is smooth within each elementary cell but is only required to be continuous at the boundaries. We define the transformation to be a collineation within each elementary cell so that, within each cell, straight lines are mapped into straight lines. In general the collineation will be different for each elementary cell, so that the transformation will, indeed, be only continuous at the boundaries. The interpolation is then made on the rectangular mesh. Two types of interpolation are considered: bicubic spline interpolation and bilinear interpolation.

In Section II we describe the collineations and the interpolation schemes. In Section III we give instructions for using FORTRAN codes for the CDC 6600 computer that implement the method, and three sample problems are described in Section IV.

The computer codes are listed, along with data for the sample problems, in the Appendixes.

II. THE METHOD

A. The Collineations

Let the irregular mesh be defined on a cartesian (x, y) -space such that the x and y coordinates of the mesh points are, respectively, $XS(I, J)$ and $YS(I, J)$, where I and J are positive integers satisfying $1 \leq I \leq \text{NBRY}$ and $1 \leq J \leq \text{NBRX}$. Let $F(I, J)$ be the value at $(x = XS(I, J), y = YS(I, J))$ of the function to be interpolated.

We now transform the (x, y) -space into an (x', y') -space by means of a transformation that carries the points of the irregular mesh into a rectangular array that is ordered in a topologically equivalent fashion. The transformation consists of a set of collineations,¹ one for each elementary quadrilateral of the irregular mesh. That is, we use a transformation of the form

$$\begin{aligned} x' &= \frac{a_1(K)x + b_1(K)y + c_1(K)}{a_0(K)x + b_0(K)y + c_0(K)} \\ y' &= \frac{a_2(K)x + b_2(K)y + c_2(K)}{a_0(K)x + b_0(K)y + c_0(K)} \end{aligned} \quad (1)$$

where the $a_i(K)$, $b_i(K)$, and $c_i(K)$ are constants and K labels the elementary quadrilaterals. Clearly, the transformation is not altered if all the constants associated with a given quadrilateral are multiplied by a common factor. Since a collineation maps straight lines into straight lines, it provides a natural and smooth way of mapping one quadrilateral into another. It can be shown¹ that there always exists a unique collineation that maps one quadrilateral into another, and that the inverse transformation exists and is also a collineation. The transformation is analytic in the interior of each quadrilateral but may be only continuous on the boundaries.

Because the constants of each collineation may all be multiplied by a common factor, we may arbitrarily fix any one of them that is not zero. In the computer codes listed in the Appendixes, we have chosen the $c_0(K)$ to equal unity. If in any particular instance the ratio of $c_0(K)$ to the other constants is, in fact, zero or sufficiently small, it will not be possible in practice to compute the other constants. No provision has been made in the

computer codes for allowing $c_0(K)$ to differ from unity, although it would be simple to make such provision.

The rectangular mesh that we use in the primed space can be chosen at will, and it may be best to choose it differently for different problems. In the computer codes listed in the Appendixes, the rectangular mesh that we use is one whose elementary cells are squares all of the same size. The x' coordinates of the mesh points are the integers running from 1 through NBRX ; the y' coordinates are the integers running from 1 through NBRY .

B. Interpolation

Any method of interpolation can be used on the rectangular mesh in the primed space. As examples, two methods have been used in the computer codes listed in the Appendixes. In the code `SINT`, bicubic spline interpolation² has been used, and bilinear interpolation has been used in the code `BINT`. These two codes are otherwise the same.

With bicubic spline interpolation, the interpolated values of the function and its first and second derivatives are continuous on the rectangular mesh. In some problems this method of interpolation may be unnecessarily refined because the transformation, Eqs. (1), is not generally differentiable at the cell boundaries. As a result, the interpolated values of derivatives are generally not continuous at the cell boundaries on the irregular mesh, in spite of their continuity on the rectangular mesh. Whether bicubic spline interpolation is appropriate must be decided separately for each type of problem.

Bilinear interpolation is accomplished separately in each cell and generally will not provide continuous derivatives even on the rectangular mesh. Such interpolation of a function $u(x, y)$ in the square whose vertices are the points $(0,0)$, $(h, 0)$, (h, h) , and $(0, h)$ is performed with the formula

$$\begin{aligned} u(x,y) &= h^{-2} [u(h,h)xy + u(0,0)(h-x)(h-y) \\ &+ u(h,0)x(h-y) + u(0,h)(h-x)y]. \end{aligned} \quad (2)$$

The boundary conditions on the spline functions that are used in the codes are that all of the derivatives (second and fourth derivatives) that must be specified on the boundary of the square mesh are

set equal to zero.

III. INSTRUCTIONS FOR USING THE COMPUTER CODES

The computer codes that perform interpolation using bicubic spline interpolation on the rectangular mesh are called SINT1 and SINT2, and a test program for using them is called SINT. These codes are listed in Appendix A. The corresponding codes that use bilinear interpolation on the rectangular mesh are called BINT1, BINT2, and BINT and are listed in Appendix B. The instructions for using the two sets of codes are identical except for the change in names (BINT1 instead of SINT1, etc.). The instructions will be given only for SINT, SINT1, and SINT2. The codes are written for the CDC-6600 computer at the Los Alamos Scientific Laboratory and they make explicit use of subprograms available in the Los Alamos Scientific Laboratory program library.

SINT1 and SINT2 are used in the same way as the two two-dimensional spline interpolation codes (E104A, E105A, and E107A) in the Los Alamos Scientific Laboratory program library. First SINT1 is called. (There is only one call to SINT1.) After SINT1 has been called, SINT2 is called once for each interpolation that is desired. SINT1 is a subroutine, and SINT2 is a function subprogram.

The calling sequences for SINT1 and SINT2 are:

```
CALL SINT1(NBRX, X, NBRY, Y, F, FX, FY, FXY, T1,
          T2, T3, MXY, XS, YS,
          A1, B1, C1, A2, B2, C2, AO, BO, CO, MD)
FINT = SINT2(XB, YB, NBRX, X, NBRY, Y, F, FX, FY,
          FXY, K, L,
          A1, B1, C1, A2, B2, C2, AO, BO, CO, MD)
```

1. FINT is the interpolated value of $\partial^{(K+L)} F / \partial x^K \partial y^L$ at $x = XB$, $y = YB$.
2. NBRX and NBRY are defined in Section IIA and are to be supplied by the calling program.
3. XS, YS, and F are arrays defined in Section IIA and are to be supplied by the calling program.
4. MXY is the maximum of NBRX and NBRY. The only other quantity to be supplied by the calling program is MD, defined below in item 7.
5. X and Y are one-dimensional arrays. The dimension of X must be greater than or equal to NBRX; that of Y must be greater than or equal to NBRY.
6. T1, T2, and T3 are one-dimensional arrays whose dimensions must be greater than or equal to MXY.
7. FX, FY, FXY, A1, B1, C1, A2, B2, C2, AO, BO,

and CO are two-dimensional arrays whose dimensions must be the same as those of XS, YS, and F. The first dimension must be greater than or equal to NBRY; the second dimension must be greater than or equal to NBRX. MD equals the first dimension of these arrays.

8. K and L must satisfy $0 \leq K \leq 2$, $0 \leq L \leq 2$. For function interpolation, take $K = L = 0$.

9. The arrays $A1(I, J)$, $B1(I, J)$, ..., $CO(I, J)$ are the coefficients $a_1(K)$, $b_1(K)$, ..., $c_0(K)$, as defined in Eqs. (1), for the collinear transformation in the elementary quadrilateral whose vertices are labeled as shown in the following diagram

Data cards for the test program SINT are as follows, to be read in the order listed.

1. NBRX, NBRY. FORMAT (2I6).
2. (CN(I), I = 1, 6). FORMAT(6E12.6). The elements of CN are constants that may be used for evaluating the function to be interpolated. The function is defined by a function subprogram called FUNC(X, Y, CN). FUNC is used to evaluate the function at $x = X$ and $y = Y$ with the constants CN. CN must be dimensioned CN(6) in FUNC.
3. ((XS(I, J), YS(I, J), J = 1, NBRX), I = 1, NBRY). FORMAT(6E12.3).
4. KF, (XF(I), YF(I), I = 1, KF). FORMAT (I6/(6E12.3)). The points at which interpolation is desired have coordinates $x = XF(I)$, $y = YF(I)$. There are KF such points. $1 \leq KF \leq 100$.

In SINT the dimensions of XF and YF are 100 and that of CN is 6. All other dimension numbers in SINT are 20.

An arbitrary number of test problems can be run with SINT, one after the other, by putting the sets of data cards for the problems one behind the other.

The following information is printed by SINT, in the order listed.

1. NBRX, NBRY, MXY, (CN(I), I = 1, 6). FORMAT (I11, 3I6/1P, 6E20.5).

2. ((XS(I, J), YS(I, J), F(I, J), J = 1, NBRX),
I = 1, NBRY). FORMAT(1H1, 1P, (3E20.8)).
3. XF(K), YF(K), FC, FINT, DIF, DIFFR. FORMAT
(1P, 6E20.8). FC is the actual value of F at
 $x = XF(K)$, $y = YF(K)$; FINT is the interpolated value
of F there. $DIF = FINT - FC$. $DIFFR = DIF/FC$.
4. If a point at which interpolation is desired is
outside of the mesh, then the job will terminate
with the message (XB, YB) OUT OF BOUNDS.
The termination is done in SINT2.

IV. EXAMPLES

For the examples the function to be interpolated is $x^2 + y^2$. This function would be interpolated exactly on a rectangular mesh by bicubic spline functions if the correct boundary conditions were used at the edges of the mesh. The boundary conditions described in SECTION IIB are optimum for arbitrary functions, but not exact for the function $x^2 + y^2$. FUNC for the examples is listed in Appendix C.

The examples are the interpolation of $x^2 + y^2$ on three different meshes. The meshes are shown in Figs. 1, 2, and 3, and data cards for the three cases are listed in Appendixes D1, D2, and D3.

REFERENCES

1. R. N. Sen, A Course of Geometry, Calcutta University, Calcutta, 1953.
2. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, The Theory of Splines and Their Applications, Academic Press, New York, 1967. See also: T. L. Jordan, Smoothing and Multivariable Interpolation with Splines, Los Alamos Scientific Laboratory Report LA-3137, March 1965.

Figure 1. Example 1.

Figure 2. Example 2.

Figure 3. Example 3.

APPENDIX A. LISTINGS OF SINT, SINT1, and SINT2

```

PROGRAM SINT(INPUT,OUTPUT)
C   TO TEST SINT1 AND SINT2.
   DIMENSION X(20),Y(20),F(20,20),FX(20,20),
1  FY(20,20),FXY(20,20),T1(20),T2(20),T3(20),
2  XS(20,20),YS(20,20),A1(20,20),B1(20,20),C1(20,20),
3  A2(20,20),B2(20,20),C2(20,20),AO(20,20),
4  BO(20,20),CO(20,20),CN(6),XF(100),YF(100)
100 READ 1,NBRX,NBRY
   1 FORMAT(2I6)
   MXY=MAX0(NBRY,NBRX)
   READ 2,(CN(I),I=1,6)
   2 FORMAT(6E12.6)
   PRINT 3,NBRX,NBRY,MXY,(CN(I),I=1,6)
   3 FORMAT(1H1,3I6/1P,6E20.5)
   READ 4,((XS(I,J),YS(I,J),J=1,NBRX),I=1,NBRY)
   4 FORMAT(6E12.3)
   DO 10 I=1,NBRY
   DO 11 J=1,NBRX
   F(I,J)=FUNC(XS(I,J),YS(I,J),CN)
11  CONTINUE
10  CONTINUE
   PRINT 12,((XS(I,J),YS(I,J),F(I,J),J=1,NBRX),I=1,NBRY)
12  FORMAT(1H1,1P,(3E20.8))
   CALL SINT1(NBRX,X,NBRY,Y,F,FX,FY,FXY,
1  T1,T2,T3,MXY,XS,YS,A1,B1,C1,A2,B2,C2,
2  AO,BO,CO,20)
   PRINT 13
13  FORMAT(1H1)
   READ 20,KF,(XF(I),YF(I),I=1,KF)
20  FORMAT(I6/(6E12.3))
   DO 21 K=1,KF
   FC=FUNC(XF(K),YF(K),CN)
   FINT=SINT2(XF(K),YF(K),NBRX,X,NBRY,Y,F,FX,FY,
1  FXY,0,0,A1,B1,C1,A2,B2,C2,AO,BO,CO,20)
   DIF=FINT-FC
   DIFFR=DIF/FC
   PRINT 22,XF(K),YF(K),FC,FINT,DIF,DIFFR
22  FORMAT(1P,6E20.8)
21  CONTINUE
   GO TO 100
END

```

```

SUBROUTINE SINT1(NBRX,X,NBRY,Y,F,FX,FY,FXY,
1  T1,T2,T3,MXY,XS,YS,A1,B1,C1,A2,B2,C2,
2  AO,BO,CO,MD)
   DIMENSION X(NBRX),Y(NBRY),F(MD ,NBRX),
1  FX(MD ,NBRX),FY(MD ,NBRX),FXY(MD ,NBRX),
2  T1(MXY),T2(MXY),T3(MXY),LI(4),LJ(4),
3  XS(MD ,NBRX),YS(MD ,NBRX),B(8,1),A(8,8),
4  A1(MD ,NBRX),B1(MD ,NBRX),C1(MD ,NBRX),
5  A2(MD ,NBRX),B2(MD ,NBRX),C2(MD ,NBRX),
6  AO(MD ,NBRX),BO(MD ,NBRX),CO(MD ,NBRX)
C   MXY=MAX0(NBRY,NBRX)
C   FIND SPLINE IN THE PRIMED SPACE.
   DO 1 J=1,NBRX
   1 X(J)=J
   DO 2 I=1,NBRY
   2 Y(I)=I
   DO 3 K=1,6
   3 IBD(K)=1
   DO 4 I=1,NBRY
   FX(I,1)=0.
   4 FX(I,NBRX)=0.
   DO 5 J=1,NBRX
   FY(1,J)=0.

```

```

5  FY(NBRY,J)=0.
   FXY(1,1)=0.
   FXY(1,NBRX)=0.
   FXY(NBRY,1)=0.
   FXY(NBRY,NBRX)=0.
   CALL SPL2D1(NBRX,X,NBRY,Y,F,FX,FY,FX,Y,MD ,
1  IBD,T1,T2,T3)
C  FIND TRANSFORMATION COEFFICIENTS.
   MY=NBRY-1
   MX=NBRX-1
   LI(1)=0
   LI(2)=0
   LI(3)=1
   LI(4)=1
   LJ(1)=0
   LJ(2)=1
   LJ(3)=1
   LJ(4)=0
   DO 100 I=1,MY
   DO 101 J=1,MX
   CO(I,J)=1.
   DO 102 K=1,4
   IO=I+LI(K)
   JO=J+LJ(K)
   XT=XS(IO,JO)
   YT=YS(IO,JO)
   K1=2*K-1
   K2=2*K
   B(K1 ,1)=X(JO)*CO(I,J)
   B(K2,1)=Y(IO)*CO(I,J)
   A(K1 ,1)=XT
   A(K1 ,2)=YT
   A(K1 ,3)=1.
   A(K1 ,7)=-X(JO)*XT
   A(K1 ,8)=-X(JO)*YT
   A(K2,4)=XT
   A(K2,5)=YT
   A(K2,6)=1.
   A(K2,7)=-Y(IO)*XT
   A(K2,8)=-Y(IO)*YT
   A(K1 ,4)=0.
   A(K1 ,5)=0.
   A(K1 ,6)=0.
   A(K2,1)=0.
   A(K2,2)=0.
   A(K2,3)=0.
102 CONTINUE
   CALL LSS(8,1,8,A,B,D,DET)
   A1(I,J)=B(1,1)
   B1(I,J)=B(2,1)
   C1(I,J)=B(3,1)
   A2(I,J)=B(4,1)
   B2(I,J)=B(5,1)
   C2(I,J)=B(6,1)
   A0(I,J)=B(7,1)
   B0(I,J)=B(8,1)
101 CONTINUE
100 CONTINUE
   RETURN
   END

```

```

FUNCTION SINT2(XB,YB,NBRX,X,NBRY,Y,F,FX,FY,
1FX,Y,K,L,A1,B1,C1,A2,B2,C2,A0,B0,C0,MD)
  DIMENSION X(NBRX),Y(NBRY),F(MD ,NBRX),
1FX(MD ,NBRX),FY(MD ,NBRX),FX(Y(MD ,NBRX),
2A1(MD ,NBRX),B1(MD ,NBRX),C1(MD ,NBRX),
3A2(MD ,NBRX),B2(MD ,NBRX),C2(MD ,NBRX),
4A0(MD ,NBRX),B0(MD ,NBRX),C0(MD ,NBRX),
5MSG(5)

```



```

DATA MSG/50H(XB, YB) OUT OF BOUNDS.
C FIND XP AND YP FROM XB AND YB.
C FIRST FIND THE CORRECT TRANSFORMATION.
MY=NBRY-1
MX=NBXR-1
DO 1 I=1,MY
DO 2 J=1,MX
RHO=A0(I,J)*XB+B0(I,J)*YB+C0(I,J)
Z1=A1(I,J)*XB+B1(I,J)*YB+C1(I,J)
XP=Z1/RHO
IF((XP.GT.(X(J)-1.E-10)).AND.(XP.LT.(X(J+1)+1.E-10)))3,2
3 Z2=A2(I,J)*XB+B2(I,J)*YB+C2(I,J)
YP=Z2/RHO
IF((YP.GT.(Y(I)-1.E-10)).AND.(YP.LT.(Y(I+1)+1.E-10)))4,2
2 CONTINUE
1 CONTINUE
INT=0000001
CALL LABRT(4,MSG,INT)
CALL LABRT(1,MSG,INT)
INTERPOLATE.
C 4 SINT2=SPL2D2(XP,YP,NBRX,X,NBRY,Y,F,FX,
1FY,FX,MD ,K,L)
RETURN
END

```

APPENDIX B. LISTINGS OF BINT, BINT1, and BINT2

```

PROGRAM BINT(INPUT,OUTPUT)
C TO TEST BINT1 AND BINT2.
DIMENSION X(20),Y(20),F(20,20),
1
2XS(20,20),YS(20,20),A1(20,20),B1(20,20),C1(20,20),
3A2(20,20),B2(20,20),C2(20,20),A0(20,20),
4B0(20,20),C0(20,20),CN(6),XF(100),YF(100)
100 READ 1,NBRX,NBRY
1 FORMAT(2I6)
MXY=MAX0(NBRY,NBRX)
READ 2,(CN(I),I=1,6)
2 FORMAT(6E12.6)
PRINT 3,NBRX,NBRY,MXY,(CN(I),I=1,6)
3 FORMAT(1H1,3I6/1P,6E20.5)
READ 4,((XS(I,J),YS(I,J),J=1,NBRX),I=1,NBRY)
4 FORMAT(6E12.3)
DO 10 I=1,NBRY
DO 11 J=1,NBRX
F(I,J)=FUNC(XS(I,J),YS(I,J),CN)
11 CONTINUE
10 CONTINUE
PRINT 12,((XS(I,J),YS(I,J),F(I,J),J=1,NBRX),I=1,NBRY)
12 FORMAT(1H1,1P,(3E20.8))
CALL BINT1(NBRX,X,NBRY,Y,F,
1 XS,YS,A1,B1,C1,A2,B2,C2,
2A0,B0,C0,20)
PRINT 13
13 FORMAT(1H1)
READ 20,KF,(XF(I),YF(I),I=1,KF)
20 FORMAT(16/(6E12.3))
DO 21 K=1,KF
FC=FUNC(XF(K),YF(K),CN)
FINT=BINT2(XF(K),YF(K),NBRX,X,NBRY,Y,F,
1 A1,B1,C1,A2,B2,C2,A0,B0,C0,20)
DIF=FINT-FC
DIFFR=DIF/FC
PRINT 22,XF(K),YF(K),FC,FINT,DIF,DIFFR
22 FORMAT(1P,6E20.8)
21 CONTINUE
GO TO 100
END

```

```

SUBROUTINE BINT1(NBRX,X,NBRY,Y,F,
1      XS,YS,A1,B1,C1,A2,B2,C2,
2      AO,BO,CO,MD)
  DIMENSION X(NBRX),Y(NBRY),F(MD ,NBRX),
1
2      LI(4),LJ(4),
3      XS(MD ,NBRX),YS(MD ,NBRX),B(8,1),A(8,8),
4      A1(MD ,NBRX),B1(MD ,NBRX),C1(MD ,NBRX),
5      A2(MD ,NBRX),B2(MD ,NBRX),C2(MD ,NBRX),
6      AO(MD ,NBRX),BO(MD ,NBRX),CO(MD ,NBRX)
  DO 1 J=1,NBRX
1  X(J)=J
  DO 2 I=1,NBRY
2  Y(I)=I
C  FIND TRANSFORMATION COEFFICIENTS.
  MY=NBRY-1
  MX=NBRX-1
  LI(1)=0
  LI(2)=0
  LI(3)=1
  LI(4)=1
  LJ(1)=0
  LJ(2)=1
  LJ(3)=1
  LJ(4)=0
  DO 100 I=1,MY
  DO 101 J=1,MX
  CO(I,J)=1.
  DO 102 K=1,4
  IO=I+LI(K)
  JO=J+LJ(K)
  XT=XS(IO,JO)
  YT=YS(IO,JO)
  K1=2*K-1
  K2=2*K
  B(K1 ,1)=X(JO)*CO(I,J)
  B(K2,1)=Y(IO)*CO(I,J)
  A(K1 ,1)=XT
  A(K1 ,2)=YT
  A(K1 ,3)=1.
  A(K1 ,7)=-X(JO)*XT
  A(K1 ,8)=-X(JO)*YT
  A(K2,4)=XT
  A(K2,5)=YT
  A(K2,6)=1.
  A(K2,7)=-Y(IO)*XT
  A(K2,8)=-Y(IO)*YT
  A(K1 ,4)=0.
  A(K1 ,5)=0.
  A(K1 ,6)=0.
  A(K2,1)=0.
  A(K2,2)=0.
  A(K2,3)=0.
102 CONTINUE
  CALL LSS(8,1,8,A,B,D,DET)
  A1(I,J)=B(1,1)
  B1(I,J)=B(2,1)
  C1(I,J)=B(3,1)
  A2(I,J)=B(4,1)
  B2(I,J)=B(5,1)
  C2(I,J)=B(6,1)
  AO(I,J)=B(7,1)
  BO(I,J)=B(8,1)
101 CONTINUE
100 CONTINUE
  RETURN
  END

```

```

FUNCTION BINT2(XB,YB,NBRX,X,NBRY,Y,F,
1      A1,B1,C1,A2,B2,C2,A0,B0,C0,MD)
  DIMENSION X(NBRX),Y(NBRY),F(MD ,NBRX),
1
2 A1(MD ,NBRX),B1(MD ,NBRX),C1(MD ,NBRX),
3 A2(MD ,NBRX),B2(MD ,NBRX),C2(MD ,NBRX),
4 A0(MD ,NBRX),B0(MD ,NBRX),C0(MD ,NBRX),
5 MSG(5)
  DATA MSG/50H(XB, YB) OUT OF BOUNDS.
C    FIND XP AND YP FROM XB AND YB.
C    FIRST FIND THE CORRECT TRANSFORMATION.
  MY=NBRY-1
  MX=NBRX-1
  DO 1 I=1,MY
  DO 2 J=1,MX
    RHO=A0(I,J)*XB+B0(I,J)*YB+C0(I,J)
    Z1=A1(I,J)*XB+B1(I,J)*YB+C1(I,J)
    XP=Z1/RHO
    IF((XP.GT.(X(J)-1.E-10)).AND.(XP.LT.(X(J+1)+1.E-10)))3,2
3  Z2=A2(I,J)*XB+B2(I,J)*YB+C2(I,J)
    YP=Z2/RHO
    IF((YP.GT.(Y(I)-1.E-10)).AND.(YP.LT.(Y(I+1)+1.E-10)))4,2
4  IO=I
    JO=J
    GO TO 5
2 CONTINUE
1 CONTINUE
  INT=0000001
  CALL LABRT(4,MSG,INT)
  CALL LABRT(1,MSG,INT)
C    INTERPOLATE.
5  XP=XP-X(JO)
  YP=YP-Y(IO)
  BINT2=F(IO+1,JO+1)*XP*YP+F(IO,JO)*(1.-XP)*(1.-YP)
  1+F(IO,JO+1)*XP*(1.-YP)+F(IO+1,JO)*(1.-XP)*YP
  RETURN
  END

```

APPENDIX C. LISTING OF FUNC FOR THE EXAMPLES

```

FUNCTION FUNC(X,Y,CN)
  DIMENSION CN(6)
  FUNC=((X-CN(1))*2+(Y-CN(2))*2+CN(3))*CN(4)
  RETURN
  END

```

APPENDIX D1. DATA FOR THE FIRST EXAMPLE

6	4				
		1.			
0	2	2	2	4	2
6	2	8	2	9999	2
0	4	2	4	4	4
6	4	8	4	9999	4
0	6	2	6	4	6
6	6	8	5	9999	6
0	8	2	8	4	8
6	8	8	8	9999	8
78					
1172	7178	1938	7197	2659	7197
3507	7187	4390	7187	5233	7182
6159	7161	7604	6968	8243	6968
0531	5724	1467	5724	2257	5724
3322	5724	4053	5724	4647	5724
5570	5724	6511	5724	7413	5724
0829	4857	1542	4857	2552	4857
3538	4857	4483	4857	5342	4857
6004	4857	6323	4857	7158	4857
0742	3823	1434	3823	2320	3823
3214	3823	4059	3823	5092	3823
6047	3823	6975	3823	0622	3135
1499	3135	2293	3135	3814	3135
4748	3135	5754	3135	7128	3135
5288	2592	6275	2592	7075	2592
1334	6647	2434	6504	3083	6384
3943	6090	4541	5893	5667	5206
6054	4963	6457	4565	7222	3867
1459	6100	2969	5853	3905	5365
4773	5050	5419	4586	6225	4431
6817	3827	1328	5214	2174	5112
2931	4893	3691	4685	4483	4467
5117	4214	5729	3860	6126	3636
1151	4550	1833	4530	2495	4530
3046	4395	3802	4235	4282	3995
4774	3870	5101	3765	5511	3568

APPENDIX D2. DATA FOR THE SECOND EXAMPLE

6	4	1.				
	1103	7300	2448	7492	4175	7300
	5773	7366	7362	7067	8677	7189
	0244	5734	1536	6236	3587	5245
	4904	5967	5999	5597	7560	5096
	0555	4538	2272	4464	4286	4163
	5742	3880	7016	3215	8017	2579
	0353	2726	1410	2982	2638	2572
	4164	2721	5524	2108	6725	2174
69	1103	7300	2448	7492	4175	7300
	5773	7366	7362	7067	8677	7189
	0244	5734	1536	6236	3587	5245
	4904	5967	5999	5597	7560	5096
	0555	4538	2272	4464	4286	4163
	5742	3880	7016	3215	8017	2579
	0353	2726	1410	2982	2638	2572
	4164	2721	5524	2108	6725	2174
	1172	7178	1938	7197	2659	7197
	3507	7187	4390	7187	5233	7182
	6159	7161	7604	6968	8243	6968
	0531	5724	1467	5724	2257	5724
	3322	5724	4053	5724	4647	5724
	5570	5724	6511	5724	7413	5724
	0829	4857	1542	4857	2552	4857
	3538	4857	4483	4857	5342	4857
	6004	4857	6323	4857	7158	4857
	0742	3823	1434	3823	2320	3823
	3214	3823	4059	3823	5092	3823
	6047	3823	6975	3823	0622	3135
	1499	3135	2293	3135	3814	3135
	4748	3135	5754	3135	7128	3135
	5288	2592	6275	2592	7075	2592

APPENDIX D3. DATA FOR THE THIRD EXAMPLE

6	4	1.				
	1114	7073	3013	6805	4674	6195
	6096	5208	7402	4075	8417	2709
	1020	6042	2212	5935	3625	5651
	4875	5087	6185	4359	7068	3684
	0912	5051	1603	5054	2826	4872
	3960	4616	5242	4073	6254	3540
	0792	4175	1487	4265	2581	4193
	3324	4104	4528	3736	5675	3321
57	1114	7073	3013	6805	4674	6195
	6096	5208	7402	4075	8417	2709
	1020	6042	2212	5935	3625	5651
	4875	5087	6185	4359	7068	3684
	0912	5051	1603	5054	2826	4872
	3960	4616	5242	4073	6254	3540
	0792	4175	1487	4265	2581	4193
	3324	4104	4528	3736	5675	3321
	1334	6647	2434	6504	3083	6384
	3943	6090	4541	5893	5667	5206
	6054	4963	6457	4565	7222	3867
	1459	6100	2969	5853	3905	5365
	4773	5050	5419	4586	6225	4431
	6817	3827	1328	5214	2174	5112
	2931	4893	3691	4685	4483	4467
	5117	4214	5729	3860	6126	3636
	1151	4550	1833	4530	2495	4530
	3046	4395	3802	4235	4282	3995
	4774	3870	5101	3765	5511	3568