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STABILITY LIMIT ON BETA IN A TOKAMAK USING THE COLLISIONLESS ENERGY PRINCIPLE

BY

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Energy Principle

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ABSTRACT

It is shown that use of the collisionless, rather than MHD, energy principle has little effect on the limiting β for stability of a tokamak against ballooning modes.

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Stability Limit on Beta in a Tokamak Using the Collisionless

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It has recently become clear that magnetohydrodynamic (MHD) "ballooning" instabilities impose a fairly severe limit on β , the ratio of plasma pressure p to magnetic pressure $B^2/2$, in a tokamak. Ballooning modes have been identified in numerical minimizations of the MHD energy principle [1, 2], and in time-dependent computations [3]; analytic models have also been developed [4] that give good agreement with these numerical results.

In high temperature plasmas, however, where collisions are not sufficiently frequent to maintain an isotropic perturbed pressure, the MHD energy principle needs to be modified slightly [5, 6]. Moreover, it has been proved [5, 6], quite generally, that the modified energy principle (which we will call the "collisionless" energy principle) always yields more favorable stability criteria than the MHD principle, although the differences are often rather small. In this letter, we will show that the differences are insignificant in the case of ballooning instabilities in tokamaks.

The MHD and collisionless energy principles can both be written in the form

$$W = W_0 + W_1, \quad (1)$$

where W_0 , the same in both cases, is given by

$$W_0 = \int [(\nabla \times \xi_{\perp} \times B)^2 + j \cdot \xi_{\perp} \times \nabla \times (\xi_{\perp} \times B) - \xi_{\perp} \cdot \nabla (\xi_{\perp} \cdot \nabla p)] dV. \quad (2)$$

Here, $\xi_{\perp}$ is a displacement vector that has no component along the unperturbed magnetic field; the remaining notation is standard. The differences between the two energy principles arise in the term W_1 . The MHD principle has

$$W_1 = W_1^{\text{MHD}} = \frac{5}{3} \int p (\nabla \cdot \xi)^2 dV. \quad (3)$$

For closed systems in which the field lines trace out magnetic surfaces, as in tokamaks, the component $\xi_{\parallel}$ can always be chosen so as to make $\nabla \cdot \xi$ vanish. Thus, the compressibility term W_1 in the MHD energy principle has no effect on marginally stable modes.

On the other hand, the collisionless energy principle [5] has

$$W_1 = \frac{15}{4} \int p d\Phi \int_{\alpha=1/B_{\max}}^{\alpha=1/B_{\min}} d\alpha [J(\Phi, \alpha)]^2 / K(\Phi, \alpha) \quad (4)$$

where

$$J(\Phi, \alpha) = \int_{\ell_1(\alpha)}^{\ell_2(\alpha)} \left\{ \left[1 - \frac{3}{2} \alpha B(\ell') \right] \underline{n} \cdot (\underline{n} \cdot \nabla) \xi_{\perp} + \frac{\alpha B(\ell')}{2} \underline{\nabla} \cdot \xi_{\perp} \right\} \frac{d\ell'}{[1 - \alpha B(\ell')]^{1/2}} \quad (5)$$

$$K(\Phi, \alpha) = \int_{\ell_1(\alpha)}^{\ell_2(\alpha)} \frac{d\ell'}{[1 - \alpha B(\ell')]^{1/2}} \quad (6)$$

Here, Φ denotes a magnetic flux tube, and $d\Phi$ an infinitesimal element of flux; $\alpha = v_{\perp}^2 / 2v^2$ represents a particle's pitch angle; $\underline{n}$ is a unit vector along $\underline{B}$; and ℓ is the length along $\underline{B}$, with $\ell_1(\alpha)$ and $\ell_2(\alpha)$ denoting the turning points of trapped-particle orbits. [In a closed system with magnetic surfaces, only the trapped particles will give rise to nonvanishing integrals $J(\Phi, \alpha)$.]

Before evaluating W_1 , it is advantageous to rewrite W_0 in such a way that the order of magnitude of the various terms is more readily apparent. In the first term in W_0 , we can write

$$\begin{aligned} (\nabla \times \underline{\xi} \times \underline{B})_{\parallel} &= [\underline{B} \cdot (\underline{B} \cdot \nabla) \underline{\xi}_{\perp} - B^2 \nabla \cdot \underline{\xi}_{\perp} - \underline{B} \cdot (\underline{\xi}_{\perp} \cdot \nabla) \underline{B}] / B \\ &= -[\underline{\xi}_{\perp} \cdot \nabla (p + B^2/2) + B^2 \nabla \cdot \underline{\xi}_{\perp}] / B \end{aligned} \quad (7)$$

where we have used

$$\nabla \cdot (\underline{p} + B^2/2) = \nabla \cdot \underline{B} B = (\underline{B} \cdot \nabla) B. \quad (8)$$

The part of the second term in W_0 arising from the perpendicular component of $\underline{j}$ can be written

$$\begin{aligned}
 \underline{j}_{\perp} \cdot \underline{\xi}_{\perp} \times (\underline{\nabla} \times \underline{\xi}_{\perp} \times \underline{B}) &= (\underline{B} \times \underline{\nabla} p) \cdot \underline{\xi}_{\perp} \times (\underline{\nabla} \times \underline{\xi}_{\perp} \times \underline{B}) / B^2 \\
 &= -(\underline{\xi}_{\perp} \cdot \underline{\nabla} p) (\underline{\nabla} \times \underline{\xi}_{\perp} \times \underline{B})_{\parallel} / B \\
 &= (\underline{\xi}_{\perp} \cdot \underline{\nabla} p) [\underline{\xi}_{\perp} \cdot \underline{\nabla} (p + B^2) + B^2 \underline{\nabla} \cdot \underline{\xi}_{\perp}] / B^2
 \end{aligned} \tag{9}$$

Integrating by parts over the volume of the system, the third term in W_o can be rewritten

$$-\underline{\xi}_{\perp} \cdot \underline{\nabla} (\underline{\xi}_{\perp} \cdot \underline{\nabla} p) \Rightarrow (\underline{\xi}_{\perp} \cdot \underline{\nabla} p) \underline{\nabla} \cdot \underline{\xi}_{\perp} \tag{10}$$

Squaring the right-hand side of Eq. (7), and adding the right-hand sides of Eqs. (9) and (10), we obtain a collection of terms that can readily be regrouped into the expression

$$[B^2 \underline{\nabla} \cdot \underline{\xi}_{\perp} + \underline{\xi}_{\perp} \cdot \underline{\nabla} (2p + B^2)]^2 / B^2 - (\underline{\xi}_{\perp} \cdot \underline{\nabla} p) [\underline{\xi}_{\perp} \cdot \underline{\nabla} (2p + B^2)] / B^2 \tag{11}$$

Thus, finally, we can rewrite W_o in the form

$$\begin{aligned}
 W_o = \int \{ & (\underline{\nabla} \times \underline{\xi}_{\perp} \times \underline{B})_{\perp}^2 - j_{\parallel} (\underline{\xi}_{\perp} \times \underline{B}) \cdot (\underline{\nabla} \times \underline{\xi}_{\perp} \times \underline{B}) / B \\
 & - (\underline{\xi}_{\perp} \cdot \underline{\nabla} p) [\underline{\xi}_{\perp} \cdot \underline{\nabla} (2p + B^2)] / B^2 \\
 & + [B^2 \underline{\nabla} \cdot \underline{\xi}_{\perp} + \underline{\xi}_{\perp} \cdot \underline{\nabla} (2p + B^2)]^2 / B^2 \} dV.
 \end{aligned} \tag{12}$$

The appropriate ordering in which to analyze ballooning instabilities in a tokamak is $\beta \sim a/R \sim B_p/B_T \sim \epsilon \ll 1$. In this case, the curvature of the magnetic field is approximately that of the vacuum field alone; thus, we may write

$$\nabla (p + B^2/2) = (\underline{B} \cdot \nabla) \underline{B} = -B_T^2 \underline{e}_R / R$$

where $\underline{e}_R$ is a unit vector directed outwardly along the major radius. In our ordering in ϵ , we may choose to regard $\underline{\xi}_\perp$ as of order unity. With this choice, it is easy to see that the first three terms in Eq. (12) for W_0 are all of order ϵ^2 . In order to reduce the fourth term to order ϵ^2 , it is necessary to take $\nabla \cdot \underline{\xi}_\perp \sim \epsilon$. This implies that the lowest order part of $\underline{\xi}_\perp$ must satisfy $\nabla \cdot \underline{\xi}_\perp = 0$. Moreover, the fourth term in Eq. (12) can actually be made to vanish through second order in ϵ , by taking the first order part of $\nabla \cdot \underline{\xi}_\perp$ to satisfy

$$\nabla \cdot \underline{\xi}_\perp = -[\underline{\xi}_\perp \cdot \nabla (2p + B^2)] / B^2 = 2 \underline{\xi}_\perp \cdot \underline{e}_R / R. \quad (13)$$

With this choice of $\nabla \cdot \underline{\xi}_\perp$, W_0 reduces to

$$W_0 = \int \left[(\nabla \times \underline{\xi}_\perp \times \underline{B})_\perp^2 - j_\parallel (\underline{\xi}_\perp \times \underline{B}) \cdot (\nabla \times \underline{\xi}_\perp \times \underline{B}) / B \right. \\ \left. + 2 (\underline{\xi}_\perp \cdot \nabla p) (\underline{\xi}_\perp \cdot \underline{e}_R) / R \right] dV. \quad (14)$$

The next point to be noted is that we can use the lowest-order approximation for $\underline{\xi}_\perp$ in the first term in W_0 ; this follows from expanding

$$\nabla \times (\xi \times B)_\perp = (B \cdot \nabla \xi_\perp - B \nabla \cdot \xi_\perp - \xi_\perp \cdot \nabla B)_\perp = (B \cdot \nabla \xi_\perp - \xi_\perp \cdot \nabla B)_\perp,$$

and observing that $B \cdot \nabla$ and ∇B are both first order quantities. As a lowest-order approximation for $\xi_\perp$, that satisfies $\nabla \cdot \xi_\perp = 0$ to lowest order, we may write

$$\xi_\perp = (B \times \nabla \phi) / B^2 \quad (15)$$

giving

$$(\nabla \times \xi_\perp \times B)_\perp = B \times \nabla (B \cdot \nabla \phi) / B^2. \quad (16)$$

Thus, the first term in W_0 can be written

$$(\nabla \times \xi_\perp \times B)_\perp^2 = [\nabla_\perp (n \cdot \nabla \phi)]^2. \quad (17)$$

In a similar way, the second term in W_0 may be written

$$\begin{aligned} -j_\parallel (\xi \times B) \cdot (\nabla \times \xi \times B) / B &= j_\parallel \xi_\perp \cdot \nabla (n \cdot \nabla \phi) \\ &= j_\parallel \nabla \cdot [\xi_\perp (n \cdot \nabla \phi)]. \end{aligned} \quad (18)$$

Integrating by parts over the volume of the system, the right-hand side of Eq. (18) can be written

$$j_\parallel \nabla \cdot [\xi_\perp (n \cdot \nabla \phi)] \Rightarrow -(n \cdot \nabla \phi) (n \times \nabla \phi \cdot \nabla j_\parallel) / B. \quad (19)$$

We have now reduced W_0 to the form

$$W_0 = \int \left\{ \left[\underline{\nabla}_\perp (\underline{n} \cdot \underline{\nabla} \phi) \right]^2 - (\underline{n} \cdot \underline{\nabla} \phi) (\underline{n} \times \underline{\nabla} \phi \cdot \underline{\nabla} j_\parallel) / B \right. \\ \left. + 2 (\underline{n} \times \underline{\nabla} \phi \cdot \underline{\nabla} p) (\underline{n} \times \underline{\nabla} \phi \cdot \underline{e}_R) / R B^2 \right\} dV. \quad (20)$$

It is easy to see that the expression for W_1 given in Eq. (4) is, formally, of order $\epsilon^{7/2}$. First, the terms $\underline{n} \cdot (\underline{n} \cdot \underline{\nabla}) \underline{\xi}_\perp$ and $\underline{\nabla} \cdot \underline{\xi}_\perp$ appearing in the expression for $J(\Phi, \alpha)$ are both of order ϵ , and $J(\Phi, \alpha)$ occurs to the second power in W_1 . Moreover, W_1 contains a factor β , which introduces a third power of ϵ ; finally, the integration over values of α corresponding to trapped particles gives rise to a factor of order $\epsilon^{1/2}$. Thus, the term W_1 is of order $\epsilon^{3/2}$ smaller than the leading terms in W_0 . It follows that a consistent minimization of W requires us to determine $\underline{\nabla} \cdot \underline{\xi}_\perp$ to first order by means of Eq. (13), and then the calculate the correction term W_1 by substituting this value of $\underline{\nabla} \cdot \underline{\xi}_\perp$ into it. In this way, Eq. (5) becomes

$$J(\Phi, \alpha) = \int_{\ell_1(\alpha)}^{\ell_2(\alpha)} \left[1 - \frac{1}{2} \alpha B(\ell') \right] \frac{\underline{\xi}_\perp \cdot \underline{e}_R}{R} \frac{d\ell'}{[1 - \alpha B(\ell')]^{1/2}} \quad (21)$$

where we have made use of $\underline{n} \cdot (\underline{n} \cdot \underline{\nabla}) \underline{\xi}_\perp = -\underline{\xi}_\perp \cdot (\underline{n} \cdot \underline{\nabla}) \underline{n} = \underline{\xi}_\perp \cdot \underline{e}_R / R$.

Since W_1 is of order $\epsilon^{3/2}$ smaller than the leading terms in W_0 , it may be calculated in a perturbation approximation; that is, we may substitute into W_1 the value of $\underline{\xi}_\perp$ obtained from a minimization of W_0 .

In the present letter, we restrict ourselves to an approximate evaluation of W_1 , for the case of a tokamak configuration with circular cross section and large enough aspect ratio R/a for our expansions to be valid. At the fairly low critical β -values for such configurations, the outward shift in the major radius of the magnetic axis, relative to the plasma boundary, is not very large. Accordingly, we may take the magnetic surfaces to be concentric circles, and we may employ the usual (r, θ, ζ) coordinate system, where θ and ζ are poloidal and toroidal angles, respectively.

Ballooning instabilities have been found to have radially extended structures, that "twist" to follow the field lines [4]. To satisfy the periodicity condition in θ , over a radially extended region, it is necessary that the perturbation amplitudes essentially vanish at $\theta = \pm \pi$, i. e., on the inside of the torus. Specifically, for relatively large values of the toroidal mode number n , the perturbations can be approximated [4] by an expression of the form

$$\phi = \hat{\phi}(\theta) \exp \{ i n [\zeta - q(r) \theta] \}, \quad (22)$$

with

$$\hat{\phi}(\theta) \simeq \hat{\phi}_0 (1 + \cos \theta) / 2. \quad (23)$$

In this model, the strong dependence of ϕ on θ and r is represented by the exponent. The amplitude $\hat{\phi}(\theta)$ represents a weak dependence on θ , following the field line; $\hat{\phi}(\theta)$ will also have a weak dependence on r , but this is unimportant.

Equation (15) gives $\xi_{\perp} = \hat{\xi}_{\perp} \exp \{ \ln [\xi -$
and $\hat{\xi}_{\theta} = - \ln q' \hat{\phi}(\theta) / B$. Thus,

$$\xi_{\perp} \cdot \frac{e}{mR} = \frac{\ln q}{rB} (\cos \theta + \frac{rq'}{q} \theta)$$

We may substitute Eq. (24) into Eq. (21). We may
write $B \approx B_0 (1 - \epsilon \cos \theta)$, where $\epsilon = r/R$, and we may
change the variable α by $k^2 = [1 - \alpha B_0 (1 - \epsilon)] / (2 \epsilon \alpha B_0)$ so
that the trapped particles correspond to values
i.e., αB_0 between $1 - \epsilon$ and $1 + \epsilon$. Writing

$$J(\Phi, \alpha) = \frac{\ln q^2}{rB (2\epsilon)^{1/2}} \int_0^{\theta_k} \frac{[\cos \theta + (rq'/q) \theta]}{[k^2 - \sin^2 \theta]} d\theta$$

where $\theta_k = 2 \arcsin k$. Using Eq. (23) for $\hat{\phi}(\theta)$
in Eq. (25) can all be written in terms of complete
elliptic integrals and obtain

$$J(\Phi, \alpha) = \frac{2 \ln q^2 \hat{\phi}_0}{rB (2\epsilon)^{1/2}} J(k)$$

where

$$J(k) = \frac{1}{3} [(5 - 4k^2) E(k) - 2(1 - k^2) K(k)] \\ + \frac{4}{9} \frac{rq'}{q} [(7 - 5k^2) E(k) - (1 - k^2) K(k)]$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively. In a similar way, Eq. (6) becomes

$$K(\Phi, \alpha) = \frac{4qR}{(2\epsilon)^{1/2}} K(k) . \quad (28)$$

To obtain the final form of W_1 , we simply substitute Eqs. (26) and (28) into Eq. (4), noting that $d\alpha = (2\epsilon/B_0) d(k^2)$ and $d\Phi = B_0 dA/q$, where dA is the element of area of the cross section. (The factor $1/q$ is introduced to compensate for the fact that we have gone around the torus q times in replacing the integrals over l' by integrals over θ .) The final expression for W_1 becomes

$$W_1 = \frac{15}{4} \int \frac{(2\epsilon)^{1/2}}{R} p \left| \hat{\xi}_{r0} \right|^2 I dA , \quad (29)$$

where I , a function of rq'/r is given by

$$I = \int_0^1 \frac{[J(k)]^2}{K(k)} d(k^2) , \quad (30)$$

and where we have written $\left| \hat{\xi}_{r0} \right| = (nq/rB) \left| \hat{\phi}_0 \right|$. The integral I has been evaluated numerically, for a number of values of rq'/q , and the results are given in Table I.

In order to estimate the effect of the term W_1 on the stability limit on β , we evaluate the terms in the expression for W_0 given in Eq.(20), again assuming concentric circular flux surfaces, and again taking

perturbations of the form given in Eqs. (22) and (23). There are three terms in Eq. (20) for W_0 . However, in the limit of large mode number n , the second term, i. e., the term that is responsible for current-driven kink-type instabilities, is unimportant. Using Eqs. (22) and (23), we find

$$\left[\nabla_{\perp} (n \cdot \nabla \phi) \right]^2 = \frac{|\hat{\xi}_{r0}|^2 B_{\theta}^2}{4 r^2} \left(1 + \frac{r^2 q'^2}{q^2} \theta^2 \right) \sin^2 \theta \quad (31)$$

$$\begin{aligned} \frac{\omega}{RB^2} (n \times \nabla \phi \cdot \nabla p)(n \times \nabla \phi \cdot e_R) \\ = \frac{|\hat{\xi}_{r0}|^2 p'}{2R} \left(\cos \theta + \frac{r q'}{q} \theta \sin \theta \right) (1 + \cos \theta)^2, \end{aligned} \quad (32)$$

where $p' = dp/dr$, and where we have again written $|\hat{\xi}_{r0}| = (nq/rB)|\phi_0|$. Substituting these expressions into Eq. (20), and carrying out the indicated integrations over θ , we obtain

$$W_0 = \int \frac{|\hat{\xi}_{r0}|^2 B_{\theta}^2}{8 r^2} \left[1 + \frac{r^2 q'^2}{q^2} \left(\frac{\pi^2}{3} - \frac{1}{2} \right) + \frac{4 p' r^2}{RB_{\theta}^2} \left(1 + \frac{5}{6} \frac{r q'}{q} \right) \right] dV \quad (33)$$

with $dV = 2\pi R dA = (2\pi)^2 R r dr$.

The condition for stability will be that $W_0 + W_1 > 0$ for all possible $|\hat{\xi}_{r0}|$. Using Eq. (29) for W_1 , and Eq. (33) for W_0 , we obtain a local stability condition

$$\frac{4 |p'| r^2}{R B_\theta^2} \left(1 + \frac{5}{6} \frac{r q'}{q} \right) < 1 + \frac{r^2 q'^2}{q^2} \left(\frac{\pi^2}{3} - \frac{1}{2} \right) + \frac{15}{4} \frac{8 p r^2 I}{2 \pi R^2 B_\theta^2} \left(\frac{2r}{R} \right)^{1/2}$$

(34)

taking the usual case where $p' < 0$ and $q' > 0$. From this point, our procedure is as follows. We assume a q -profile of the form $q(r) = q_c + (q_a - q_o) r^2/a^2$, taking a number of values of q_a/q_o in the range $2 < q_a/q_o < 5$. We then calculate the pressure profile that is everywhere marginally stable, by integrating Eq. (34) numerically (replacing $<$ by $=$), treating the last term on the right as a small perturbation. We then calculate $\bar{\beta} = 2 \langle p \rangle / B^2$ for this marginally stable pressure profile. For the range of values of q_a/q_o considered, our numerical results are well approximated by the formula

$$\bar{\beta} < \bar{\beta}_{\text{crit}} = \frac{0.18 a}{R q_a \frac{0.75}{q_o} 1.25} \left[1 + 0.2 \left(\frac{a}{R} \right)^{3/2} \right] \quad (35)$$

Without the correction term of order $(a/R)^{3/2}$, we may note that our result is in surprisingly close agreement with the exact numerical results of Ref. [1]. In Ref. [1], a case with $R/a = 4.6$, $q_o = 1.04$, $q_a = 3.77$ was shown to have $\bar{\beta}_{\text{crit}} = 1.2\%$ for toroidal mode number $n=1$, and $\bar{\beta}_{\text{crit}} = 1.5\%$ for $n=3$; our formula (35) would give $\bar{\beta}_{\text{crit}} = 1.4\%$ in this case. For all realistic aspect ratios, the correction term changes the MHD results very little. Although our analysis has been highly approximate, it seems unlikely that

a more detailed calculation would change this result substantially. We conclude that the use of the collisionless, rather than the MHD, energy principle has no significant effect on the stability limit on β in a tokamak.

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$\frac{rq'}{q}$	0	0.5	1.0	1.5	2.0	2.5
I	0.46	0.86	1.47	2.28	3.29	4.51

TABLE I. Values of integral I.

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