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NEOCLASSICAL TRANSPORT IN STELLARATORS

By

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NEOCLASSICAL TRANSPORT IN STELLARATORS

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ABSTRACT

The stellarator neoclassical transport due to particles trapped in local helical wells is calculated in the low-collisionality regime using a systematic expansion. The behavior of electron transport is found to be the same over a wide range of energies, but the behavior of ion transport for low energy ions is found to be different than that for high energy ions. Furthermore, the electron fluxes do not vary with the change in the radial ambipolar electric field nearly as much as do the ion fluxes. Thus, the particle diffusion is controlled by the electrons. A nonradial ambipolar electric field is induced by ion drift. This electric field enhances the transport by about 15 - 20%. A convenient graphical method that allows one to determine the magnitude of the radial ambipolar field for machines with different parameters is presented. Numerical examples show that electron energy confinement time is comparable to the ion energy confinement time for all the different size stellarators studied. Although the neoclassical losses are large, it is shown that ignition can be achieved in a reasonably sized stellarator reactor. Finally, from the standpoint of reactor economics, the confinement scaling law shows that in order to increase $n\tau$, it is better to increase the aspect ratio than the overall dimensions of the reactor.

I. INTRODUCTION

In recent years, there has been renewed interest in stellarator physics because of encouraging experimental and theoretical results.^{1,2} Experimental results from the Wendelstein VII-A and Heliotron E stellarators²⁻⁴ have shown that the energy confinement time is longer than the confinement time of tokamaks scaled to a comparable size with the same toroidal magnetic field strength. Theoretical studies using three-dimensional computer codes have indicated that stable stellarator operation is possible for plasma β 's between 10 to 20%.^{5,6} Stellarators have another attractive feature of being able to operate without a toroidal current since the rotational transform is provided by externally imposed helical windings. Besides providing steady-state operation, the current-free nature of the stellarator should avoid major disruptions. These advantages make the stellarator an attractive candidate for a future high performance reactor.

On the other hand, stellarators have a major drawback. This arises from the fact that the external helical windings produce local magnetic wells (henceforth, these local magnetic wells are referred to as helical wells) in which trapped particles are badly confined. This is a direct consequence of the lack of toroidal symmetry. The enhanced transport produced by particles trapped in helical wells can be as much as two orders of magnitude larger than the (neoclassical) transport produced by particles on banana orbits in a tokamak. The enhanced transport is strongest in the high-temperature, low-collisionality regime. This is the regime in which a trapped particle can bounce many times in a helical well before it detraps by collisions, and it is also the regime in which reactors are likely to operate. Thus, the nature of neoclassical transport resulting from nonaxisymmetry plays a critical role in determining the confinement in stellarator reactors.

Various authors have analyzed the transport problem in a nonaxisymmetric torus. A brief, but by no means complete, review of the previous studies is given here. Analytical studies carried out by Frieman⁷ and by Connor and Hastie⁸ showed that the diffusion coefficient is inversely proportional to the collision frequency ν , but they did not correctly include the important effect caused by an ambipolar radial electric field on ion transport. Galeev et al.,⁹ using a variational principle, and Galeev and Sagdeev,¹⁰ using an expansion technique, showed that the ambipolar field reduces the ion diffusion coefficient, but they did not investigate the ion behavior over a wide energy range and other effects. Extending Galeev's work, Mynick¹¹ studied the effect on transport of a new nonradial ambipolar electric field. This electric field is generated by ion drift and has a component tangential to magnetic surfaces. Boozer and Kuo-Petravic¹² evaluated the diffusion coefficient using a Monte-Carlo technique. In their work, however, the effect of the self-consistent ambipolar electric field on transport was not discussed in detail. The Monte-Carlo technique used in the ATF report¹³ to estimate the plasma confinement time did not include the diffusion due to a temperature gradient.

From the above review, it can be seen that the physics of transport in stellarators is complex and that it is not clear whether all the important physics have been correctly included. Because of the renewed interest in stellarator physics and the importance of the neoclassical transport in stellarators, it is now appropriate to establish a systematic treatment that can include all the essential physics in stellarator transport. In this paper, the complex stellarator magnetic field is replaced by a simplified model field that has local magnetic wells. In addition, the use of a systematic expansion technique allows the transport calculation to be treated

analytically. From this systematic analysis, previous results are recovered in a natural way, and new transport phenomena are uncovered. Although the calculations given here do not yield a precise confinement time for a real machine geometry, they do indicate that the inclusion of various new effects and the diffusion due to a temperature gradient can increase the estimate of transport by a factor as large as three (e.g., compared to a Monte-Carlo calculation without the inclusion of a temperature gradient). Therefore, it is necessary to treat the physics more precisely.

The results from the analysis in this paper show that electrons with five to six times the thermal energy give the dominant contribution to electron transport and ions with three to four times the thermal energy give the dominant contribution to ion transport. The electron flux is relatively insensitive to variations in the ambipolar electric field and the particle transport is governed by electron diffusion. Thus, a good estimate of the total particle transport can be obtained from electron diffusion without knowing the exact value of the ambipolar field. In contrast, the ion flux depends sensitively on the electric field variations. The nonradial ambipolar field caused by ion drift enhances the transport by about 15 - 20%. Numerical examples show that the electron energy confinement time is approximately equal to that of the ions for machines with different sizes and plasma temperatures. Furthermore, it is shown that the ignition condition can be achieved if the machine size is reasonably large.

In many ways, the calculations and results of the electron and ion diffusion coefficients presented in Secs. IV B and IV C are similar to those of Galeev and Sagdeev.¹⁰ However, our presentation is more detailed and rigorous than theirs. The ion diffusion coefficient given in this paper is smaller than that in Ref. 10 by one order of magnitude after the numerical integration

for the proper energy weighting of the Maxwellian distribution function is carried out correctly.

This paper is organized as follows: The simplified magnetic field geometry is described in Sec. II. A heuristic derivation of the transport coefficients in the low-collisionality regime is presented in Sec. III. The formal analysis starts in Sec. IV. The expansion and solubility conditions for the drift-Boltzmann equation are discussed, and a set of reduced equations for trapped and untrapped particles is obtained in Sec. IV.A. The electron distribution function satisfying the reduced equations is obtained in Sec. IV.B by a second expansion. Electron fluxes are also given in this section. Because of the length of the calculation for ion transport, only the result of the ion calculation is presented in Sec. IV C. The detailed calculation is given in our companion paper.¹⁴ The effect of the nonradial electric field caused by ion drift on transport is examined in Sec. IV. D. In Sec. V a convenient graphical method for determining the self-consistent radial ambipolar electric field is presented, and the scaling laws for confinement are given. Finally, the results are summarized, and their significance for stellarator reactors is discussed in Sec. VI. Some of the detailed calculations which are too long to be included in the main text are covered in the Appendices.

Only neoclassical transport has been addressed in this paper; however, convective transport may be present in a reactor since reactors should operate at the highest possible plasma β in order to give the maximum output power. As a consequence, the β may exceed the critical β limit by a small amount in some regions of the plasma, and this gives rise to pressure-driven convective cells. Thus, the transport in these regions is governed by both neoclassical and convective transport. Such convective transport has been studied by Ho and Kulsrud in Ref. 15.

II. MAGNETIC FIELD GEOMETRY

In order to allow the analysis to be carried out analytically, the magnetic surfaces are assumed to be nested circular tori. The simplified magnetic field strength, which has features in common with a general stellarator field, is approximated by

$$B = B_0 \left[1 - \epsilon_h(r) \cos(l\theta - m\zeta) - \epsilon_t \frac{r}{a} \cos\theta \right] . \quad (1)$$

Here, l is the multipolarity, m is the number of field periods, ϵ_t is the inverse aspect ratio (a/R_0), and ϵ_h is the depth of the helical wells caused by the externally imposed helical windings. The toroidal coordinate system, in terms of minor radius r , poloidal angle θ , and the toroidal angle ζ is shown in Fig. 1.

The first and the third term in the above expression represent the toroidal magnetic field which decreases as the major radius $R = R_0 + r \cos\theta$ increases. The toroidal field gradient results in a grad-B drift of electrons and ions, which is parallel to the major axis. If the toroidal magnetic field points in the direction of increasing ζ , then the electron grad-B drift is toward the upper region ($0 \leq \theta \leq \pi$) of the torus and the ion grad-B drift is toward the lower region ($\pi \leq \theta \leq 2\pi$) of the torus. The second term in B , (1) describes the magnetic field variation due to the helical wells (see Fig. 2). It is this term that destroys the toroidal symmetry. Particles trapped in local helical wells have a grad-B drift component that is normal to the magnetic surface and this produces the major contribution to diffusion in a stellarator.

III. HEURISTIC DERIVATION OF COLLISIONAL DIFFUSION COEFFICIENTS

Qualitative physical arguments are used in this section to obtain the collisional diffusion coefficients in the low-collisionality regime. The electron and ion diffusion coefficients are obtained by using the classical random-walk formula

$$D \sim \Lambda^2 \bar{\nu} F, \quad (2)$$

where Λ is the step-size, $\bar{\nu}$ is the characteristic frequency of taking a step in the random-walk process, and F is the fraction of particles participating in the diffusion process.

Rigorous kinetic theory is applied in Sec. IV to calculate the exact expression for the diffusion coefficients.

A. Electron Diffusion Coefficient

Electrons trapped in local helical wells (henceforth, unless specified, trapped particles mean trapped in helical wells) drift away from their initial magnetic surfaces due to the toroidal grad-B drift. This drift will be either toward an inner or an outer magnetic surface depending on the poloidal location of the trapping helical well. Coulomb collisions with other particles can convert a trapped electron into an untrapped one. This untrapped electron can become trapped again, through further collisions, at any arbitrary poloidal location on the magnetic surface. Thus, electrons random-walk away from their initial magnetic surface with step-size

$$\Lambda \sim \frac{v_B}{v_{eff}}.$$

Here, v_B is the grad-B drift velocity, and v_{eff} is the effective collision frequency, i.e., the frequency for scattering across the trapping region in velocity space (see Fig. 3). Note that v_{eff} is equal to ϵ_h^{-1} times the electron collision frequency v_e (which is the reciprocal of the 90° "deflection time"¹⁶). The characteristic frequency for electrons to take a step is also v_{eff} . Only the trapped electrons, the fraction of which is proportional to $(\epsilon_h)^{1/2}$, participate in this diffusion process. The reason is that the step-size of such a particle is generally large (in the order of centimeters) compared with the Larmor radius. In contrast, untrapped particles stay very close to a magnetic surface and hence make a negligible contribution to diffusion, even when toroidal trapping is included. Thus, by Eq. (2), the diffusion coefficient for electrons is

$$D_e \sim \frac{v_B^2}{v_{eff}} \epsilon_h \quad (3)$$

Because the trapped region in velocity space is small, $mv_{\parallel}^2/2 \approx \epsilon > mv_{\perp}^2/2$ for trapped particles. Here, ϵ is the particle energy. Thus, the magnitude of the grad-B drift velocity for trapped particles can be approximated by

$$v_B \approx \frac{c\epsilon}{eB_0 R_0}$$

Using this approximation and noting that $v_e \propto n\epsilon^{-3/2}$, Eq. (3) shows that $D_e \propto \epsilon^{7/2} n^{-1}$. The large ϵ dependence indicates that the high energy electrons give the dominant contribution to diffusion. Note that although $D_e \propto n^{-1}$, the electron flux is roughly independent of density since the flux is proportional to $D_e(\partial n/\partial r)$. Also note that the helical field gradient also results in a grad-B drift. However, the drift component that is normal to a magnetic surface is zero after bounce averaging [see Eq. (19)].

B. Ion Diffusion Coefficient

In the absence of an ambipolar electric field, the ion diffusion coefficient D_i is obtained by replacing v_e in Eq. (3) by v_i . This value of D_i for a deuteron is approximately sixty times larger than D_e because the large deuteron to electron mass ratio yields a deuteron collision frequency about sixty times smaller than that of the electrons while v_B is the same size. (Likewise, D_i for tritons is approximately seventy times larger than D_e .) Thus, a radial electric field E_r must arise to reduce the ion flux to the same level as the electron flux in order to preserve quasineutrality. This ambipolar electric field reduces the step-size for ion diffusion by changing the ion orbits. The $\vec{E}_r \times \vec{B}$ drift causes particles to rotate poloidally. In general, $|E_r|$ is strong enough to ensure that all the trapped ions, except those with low energy (less than thermal energy), can experience several poloidal rotations before becoming detrapped. Superposition of the grad-B drift upon the poloidal $\vec{E}_r \times \vec{B}$ drift results in an outward shift of the center of the trapped-ion drift orbit from the center of the magnetic surface (see Fig. 4) by a distance

$$\Delta \equiv \frac{v_B}{\Omega_E}.$$

Here,

$$\Omega_E \equiv \frac{c \vec{E}_r \times \vec{B}}{r B^2},$$

is the poloidal rotation frequency of the trapped ions. The shift Δ is the random-walk step-size for ions.

Although the $\vec{E}_r \times \vec{B}$ drift velocity is the same for electrons, the radial electric field has a smaller effect on electron transport than on ion transport. The reason is that v_e is large compared to Ω_E and the trapped electrons are detrapped before they rotate through an appreciable poloidal angle.

To zeroth order in ϵ_h , the grad-B drift velocity of trapped ions is independent of their pitch angle. Consequently, the shift Δ is the same for all trapped ions, and they do not experience any random-walk until they leave the trapped region. Therefore, the dominant contribution to ion diffusion arises from the barely trapped ions. The characteristic time for a barely trapped ion to detrap through collisions is Ω_E^{-1} . The fraction of ions in the barely trapped region in velocity space that can become detrapped in the time Ω_E^{-1} is $\Omega(v_i/\Omega_E)^{1/2}$. Thus, from Eq. (2), the diffusion coefficient for ions is¹⁷

$$D_i(\epsilon) \sim \frac{v_B^2}{\Omega_E} \left(\frac{v_i}{\Omega_E} \right)^{1/2}. \quad (4)$$

The $\Omega_E^{-3/2}$ dependence indicates that the effect of the electric field in changing the ion orbits is very efficient in reducing the ion diffusion. In addition, ion mobility in E_r reduces the ion flux somewhat further. (See the expressions for the ion particle and energy fluxes given in Sec. IV. C.)

If the helical well depth ϵ_h varies with minor radius, then ions can collisionlessly detrap out of and entrap into helical wells. This effect is discussed in Ref. 14.

IV. FORMAL ANALYSIS OF COLLISIONAL TRANSPORT

A. Expansion and the Solubility Conditions for the Drift-Boltzmann Equation

Plasmas in some of the recent experimental stellarators and in stellarator reactors (see Table I) are in the long-mean-free path regime; furthermore, the characteristic length and plasma confinement times in transport studies are long compared to particle gyroradii and gyroperiods, respectively. Therefore, it is appropriate to use the drift-Boltzmann equation^{18,19} to study collisional diffusion. If the magnetic field is time independent, the drift-Boltzmann equation has the form

$$\frac{\partial f_t}{\partial t} + (\vec{v}_I + \vec{v}_D) \cdot \vec{\nabla} f_t + e \frac{\partial \Phi}{\partial t} \frac{\partial f_t}{\partial \epsilon} = C(f_t, f) \quad , \quad (5)$$

where the guiding-center distribution function of species t is

$$f_t = f_t(\epsilon, \mu, \vec{r}, t) \quad ,$$

the particle energy is

$$\epsilon = \frac{m_t v^2}{2} + e_t \Phi \quad ,$$

the magnetic moment is

$$\mu = \frac{m_t v_\perp^2}{2B} \quad ,$$

and the position vector $\vec{r}$ is in (r, θ, ζ) coordinates. Each of the partial derivatives in Eq. (5) is performed by holding five of the six independent variables $\epsilon, \mu, \vec{r}, t$ constant.

The drift velocity $\vec{v}_D$ is the part of the guiding-center velocity that is perpendicular to the line of force:

$$\vec{v}_D = \frac{m_e c}{e B} \left(\frac{v_1^2}{2} + v_1^2 \right) \frac{\vec{B} \times \vec{\nabla} B}{B^2} + c \frac{\vec{B} \times \vec{\nabla} \phi}{B^2} \quad (6)$$

Inserting the expression for the magnetic field, Eq. (1), into the above equation, the radial and poloidal components of $\vec{v}_D$ are

$$\begin{aligned} \vec{v}_D \cdot \hat{e}_r &= \frac{(\mu + m v_1^2/B) B_0}{m_e \omega_t R_0} \left[\sin \theta + \frac{\ell \epsilon_h}{(r/R_0)} \sin(\ell \theta - m \zeta) \right] + \frac{c}{B} \frac{1}{r} \frac{\partial \phi}{\partial \theta} , \\ \vec{v}_D \cdot \hat{e}_\theta &= \frac{(\mu + m v_1^2/B) B_0}{m_e \omega_t R_0} \left[\cos \theta + \frac{\ell \epsilon_h}{(r/R_0)} \cos(\ell \theta - m \zeta) \right] + \frac{c}{B} \frac{\partial \phi}{\partial r} , \end{aligned} \quad (7)$$

where $\omega_t \equiv e B / m_e c$. In Eq. (7) the $\sin \theta$ and $\cos \theta$ terms are the part of the drift velocity resulting from the gradient and curvature of the toroidal magnetic field, the $\ell \epsilon_h$ terms are the part of the drift velocity due to the helical field variation, and the $\vec{\nabla} \phi$ terms are the part due to the $\vec{E} \times \vec{B}$ drift velocity (where E is an ambipolar electric field).

In the low-collisionality limit, the collision frequency ν and the effective collision frequency ν_{eff} are small compared to the respective trapped and untrapped particle bounce frequency ω_b . For untrapped particles, this limit can be expressed as

$$\nu \ll \omega_b \text{ (untrapped)} \sim \frac{v_1}{qR} \quad (8a)$$

where qR is the connection length with q the inverse of the normalized rotational transform J . The corresponding condition for helically trapped particles is

$$v_{\text{eff}} \ll \omega_b \text{ (trapped)} \sim \frac{(\epsilon_h)^{1/2} v_i}{L}, \quad (8b)$$

where L is the distance between two helical mirror points. As an example, using the parameters for the reference stellarator (see Table I) with $m = 6$, it can be shown that for trapped thermal electrons (ions) $v_{\text{eff}} \approx 8.5 \times 10^4 \text{ sec}^{-1}$ ($1.4 \times 10^3 \text{ sec}^{-1}$) and $\omega_b = 10^6 \text{ sec}^{-1}$ (10^4 sec^{-1}). Further, in one bounce period, a particle only drifts a tiny distance away from a magnetic surface, i.e.,

$$\frac{v_D}{a} \ll \omega_b. \quad (9)$$

For both electrons and ions, $v_B/a \approx 10^2 \text{ sec}^{-1}$ for the reference stellarators. Note that inequalities (8) and (9) are also valid in some of the recent experimental stellarators with parameters given in Table I.

Inequalities (8) and (9) suggest solving Eq. (5) by expanding f_t in terms of $1/v_i$:

$$f_t = F_{t0} + F_{t1} + F_{t2} + \dots \quad (10)$$

Equation (5) to lowest order is

$$\vec{v}_i \cdot \vec{\nabla} F_{t0} = 0. \quad (11)$$

This implies that F_{t0} is constant along a line of force. For the untrapped particle distribution function $f_t^{U.T.}$, Eq. (11) implies

$$F_{t0}^{U,T} = F_{t0}^{U,T}(\epsilon, \mu, r, t) \quad (12)$$

The reason that $F_{t0}^{U,T}$ is independent of θ is as follows. On any irrational surface, a line of force can come arbitrarily close to any point. Therefore, because $F_{t0}^{U,T}$ is constant along a line of force, it must also be constant everywhere on a surface. Constancy on a rational surface follows from the fact that there is always an irrational surface lying arbitrarily close to a rational surface. Therefore, by continuity, the distribution function can also be assumed to be independent of poloidal or toroidal angle on any rational surface. Possible island structure of surfaces is not considered in this study.

Now consider the trapped particle distribution function f_t^T . If the rotational transform per helical period is small, a line of force between the helical mirror points is localized in poloidal angle. Thus, F_{t0}^T depends on θ but not ζ , i.e.,

$$F_{t0}^T = F_{t0}^T(\epsilon, \mu, r, \theta, t) \quad (13)$$

and unlike $F_{t0}^{U,T}$, F_{t0}^T can have different values at different poloidal locations. Some stellarators, e.g., Helicac, do not have small rotational transforms. Therefore, the line between the mirror points may have a finite poloidal excursion. However, including this effect in the transport calculation does not give a result which differs qualitatively from the result for the case when the line is localized in poloidal angle. Hence, for simplicity, any line between two mirror points is assumed to be fixed in poloidal angle in this study.

The expansion in $1/v_{\parallel}$ breaks down in a thin transition boundary layer of thickness $(v/\omega_p)^{1/2}$ between the trapped and untrapped regions in velocity space. This layer produces discontinuities in velocity-space flux between trapped and untrapped regions at all poloidal angles since particles leaving the trapped region at some θ can flow along B to a different θ before becoming essentially untrapped. Hence, only the total flux (i.e., the flux summed over θ) is continuous between trapped and untrapped regions [see Eq. (22)]. For a detailed treatment of the distribution function in the boundary layer using the Wiener-Hopf technique, see Hinton and Rosenbluth²¹ and Shaing and Callen.²²

Equation (5) to first order is

$$\frac{\partial F_{t0}}{\partial t} + \vec{v}_{\parallel} \cdot \vec{\nabla}_F F_{t1} + \vec{v}_D \cdot \vec{\nabla}_F F_{t0} + e \frac{\partial \phi}{\partial t} \frac{\partial F_{t0}}{\partial \varepsilon} = C(F_{t0}, F_0) \quad (14)$$

Solubility conditions on F_{t0} are obtained differently for untrapped particles than for the trapped particles. For untrapped particles, the above equation can be expressed as

$$\vec{B} \cdot \vec{\nabla}_F F_{t1}^{U.T.} = S \quad (15)$$

where

$$S \equiv -\frac{B}{v_{\parallel}} \left[\frac{\partial F_{t0}^{U.T.}}{\partial t} + \vec{v}_D \cdot \vec{\nabla}_F F_{t0}^{U.T.} + e \frac{\partial \phi}{\partial t} \frac{\partial F_{t0}^{U.T.}}{\partial \varepsilon} - C(F_{t0}^{U.T.}, F_0) \right]$$

Equation (15) is a "magnetic differential equation."²³ Single-valued $F_{t1}^{U.T.}$ requires a solubility condition on $F_{t0}^{U.T.}$ obtained from the integration over the shell volume between two neighboring magnetic surfaces (constant pressure P surface) $p = P$ and $p = P + dP$, to vanish, i.e.,

$$\int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \left[\frac{\partial F_{t0}^{U,T.}}{\partial t} + \vec{v}_D \cdot \vec{\nabla} F_{t0}^{U,T.} + e \frac{\partial \phi}{\partial t} \frac{\partial F_{t0}^{U,T.}}{\partial \epsilon} - C(F_{t0}^{U,T.}, F_0) \right] = 0 \quad (16)$$

Since $d\tau = -(dp/|\nabla p|)dS$ where dS is an area element on a magnetic surface, the solubility condition on $F_{t0}^{U,T.}$ can be expressed as

$$\int \frac{dS}{|\nabla p|} \frac{B}{v_{\parallel}} \left[\frac{\partial F_{t0}^{U,T.}}{\partial t} + e \frac{\partial \phi}{\partial t} \frac{\partial F_{t0}^{U,T.}}{\partial \epsilon} - C(F_{t0}^{U,T.}, F_0) \right] = 0 \quad (17)$$

Note that after surface integration, the $\vec{v}_D \cdot \vec{\nabla} F_{t0}^{U,T.}$ term either vanishes or is negligibly small. For particles trapped in toroidal wells, however, this term may not vanish after bounce averaging over a banana orbit since the mirror points in a toroidal well may not have poloidal symmetry due to the helical variation. Nevertheless, this effect only makes a small contribution to diffusion and shall be ignored.

For helically trapped particles, single valuedness of F_{t1}^T requires the $\vec{v}_{\parallel} \cdot \vec{\nabla} F_{t1}^T$ term in Eq. (14) to vanish after applying the operator $\oint dl/v_{\parallel}$ between two mirror points. Defining a trapped particle bounce-average operator $\langle F \rangle \equiv (\oint dl/v_{\parallel})^{-1} \oint dl F/v_{\parallel}$, and applying this operator to Eq. (14) gives the solubility condition on F_{t0}^T :

$$\frac{\partial F_{t0}^T}{\partial t} + \langle \vec{v}_D \cdot \vec{\nabla} F_{t0}^T \rangle + e \frac{\partial \phi}{\partial t} \frac{\partial F_{t0}^T}{\partial \epsilon} = \langle C(F_{t0}^T, F_0) \rangle \quad (18)$$

For helical wells localized in poloidal angle, the bounce-averaged drift velocity $\langle \vec{v}_D \rangle$ can be obtained in closed form if the parallel velocity is approximated by

$$v_{\parallel} \approx \pm \left[\frac{2}{m} (\epsilon - \mu B_0 [1 - \epsilon_h \cos(\ell\theta - m\zeta)]) \right]^{1/2} .$$

The result is¹⁰

$$\langle \vec{v}_D \rangle = v_B (\sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta) + r (\Omega_E + \Omega_H) \hat{e}_\theta, \quad (19)$$

where the expressions for v_B and Ω_E have already been given in Sec. III (v_B has a positive sign for electrons and a negative sign for ions), and

$$\Omega_H = \frac{v_B \frac{1}{2} \epsilon_h}{(r^2/R)} \left(\frac{2E(\kappa)}{K(\kappa)} - 1 \right)$$

is the bounce-averaged poloidal rotational frequency. $K(\kappa)$ and $E(\kappa)$ are the elliptical integrals of the first and second kind, and their argument

$$\kappa \equiv \left\{ \left[\frac{1}{2\mu B_0 \epsilon_h} \right] [c - \mu B_0 (1 - \epsilon_h)] \right\}^{1/2},$$

is essentially $v_i / \sqrt{2\epsilon_h} v$ at the local magnetic field minimum. For particles trapped at different depths in the helical well, the value of κ ranges from 0 to 1.

This section will be closed by formally establishing the boundary conditions for the distribution function just outside the boundary layer between the trapped and untrapped regions.

Because the size of the boundary layer region in velocity space, $\Delta \xi$ is small, f_c varies rapidly in pitch angle in this region. Thus, to the lowest order in small $\Delta \xi$, the dominant contribution to the Fokker-Planck collision operator in the boundary layer region is the pitch-angle scattering operator, i.e.,

$$\sum_j C_{tj}(f_t, f_j) \approx \frac{\sum_j v_{tj}}{4} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial f_t}{\partial \xi} \quad (20)$$

where $\xi = v_{\parallel}/v$ is the pitch-angle variable. The condition for trapping is $|\xi| < |\xi_c| \equiv (2\varepsilon_h)^{1/2}$. The summation j is taken over all kinds of field particles (e.g., electrons, deuterons, and tritons). The reason that the coefficient is $1/4$ instead of the usual $1/2$ is because only the change in $v_{\parallel}^2$ in the pitch-angle direction, rather than the change in the total v^2 , is of concern to the trapping and detrapping processes.

Using the approximation (20), the drift-Boltzmann equation [Eq. (5)] is integrated with respect to ξ across the boundary layer and then integrated over a thin shell between the two adjacent magnetic surfaces in order to eliminate the $\vec{v}_{\parallel} \cdot \vec{\nabla} f$ term. The result is

$$\begin{aligned} \int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \left[\frac{\sum_j v_{tj}}{4} \left(\frac{\partial f_t}{\partial \xi} \right)_{\xi_c^+} - \frac{\partial f_t}{\partial \xi} \right]_{\xi_c^-} \\ = \int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \left(\frac{\partial f_t}{\partial t} + \vec{v}_D \cdot \vec{\nabla} f_t + e \frac{\partial \Phi}{\partial t} \frac{\partial f_t}{\partial \varepsilon} \right) \Delta \xi \quad (21) \end{aligned}$$

Note that $\Delta \xi = 0(v/\omega_b)^{1/2}$. Thus, the ratio of the dominant term (the second term) on the right-hand side to the terms on the left-hand side is of order $(v_D/av)(v/\omega_b)^{1/2}$, and is much less than unity. Hence, in the asymptotic limit when $(v_D/av)(v/\omega_b)^{1/2}$ approaches zero, the right-hand side vanishes and the boundary condition for the distribution function is

$$\int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \frac{\partial f_t}{\partial \xi} \bigg|_{\xi_c^+} = \int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \frac{\partial f_t}{\partial \xi} \bigg|_{\xi_c^-} \quad (22)$$

It can also be shown that

$$\int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \ln[f_t^{U,T}(\xi_c^+)] \frac{\partial f_t}{\partial \xi} \bigg|_{\xi_c^+} = \int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \ln[f_t^{T*}(\xi_c^-)] \frac{\partial f_t}{\partial \xi} \bigg|_{\xi_c^-} . \quad (23)$$

Equation (23) will be used in the following section when the Boltzmann H-theorem is applied to show that the distribution function is a Maxwellian.

The continuity condition is

$$f_t^{U,T}(\xi_c^+) = f_t^{T*}(\xi_c^-) . \quad (24)$$

The reason for this condition is because, otherwise, there will be no other term in the drift-Boltzmann equation to balance the collision operator term, and the discontinuity will be smoothed out by collisions immediately.

B. Electron Transport

The step size, i.e., the distance travelled by an electron during a collisional detrapping time due to the grad-B drift, is small compared to the minor radius. Thus

$$\frac{v_B}{av_{\text{eff}}} \ll 1 . \quad (25)$$

For example, using the reference stellarator parameters given in Table I, it can be shown that $v_B/av_{\text{eff}} = 0(10^{-3})$ for thermal electrons. The radial ambipolar electric field can be obtained from the approximation $(e/T)(\partial\phi/\partial r) \approx (1/n)(\partial n/\partial r)$. Then $\Omega_E = 0(10^3 \text{ sec}^{-1})$, and hence

$$\frac{\Omega_E}{v_{\text{eff}}} \ll 1 . \quad (26)$$

This means that electrons trapped in helical wells rotate a negligible amount in the poloidal direction before detrapping, so that the poloidal rotation is not important in the study of electron transport. A similar remark applies to Ω_H . Note that inequalities (25) and (26) are also valid in some of the recent experimental stellarators with parameters given in Table I.

Inequality (25) suggests a perturbation analysis to solve Eqs. (17) and (18). Expanding F_{e0} in powers of $v_B/a v_{eff}$ gives

$$F_{e0} = f_{e0} + f_{e1} + f_{e2} + \dots \quad (27)$$

The Ω_E and Ω_H terms will be treated as second order in this expansion. Since collisions are dominant [inequalities (25) and (26)] and Eqs. (16) and Eq. (18) result in no particle flow out of the magnetic surface in the lowest order in expansion Eq. (27), f_{e0} must be a Maxwellian by the Boltzmann H-theorem. This is formally proved by the following argument.

For untrapped electrons, Eq. (17) to lowest order is

$$\int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \sum_j C_{ej} (f_{e0}^{U.T.}, f_{j0}) = 0 \quad (28)$$

where the summation j is over electrons and ions. The ions are assumed to have no mean velocity but otherwise their distribution is arbitrary. Since $f_{e0}^{U.T.}$ is a function of only r , ϵ , and μ , the factor $\ln f_{e0}^{U.T.}$ can be inserted inside the integral in Eq. (28) so that

$$\int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \ln f_{e0}^{U.T.} \sum_j C_{ej} (f_{e0}^{U.T.}, f_{j0}) = 0 \quad (29)$$

For trapped electrons, Eq. (18) to lowest order is

$$\oint \frac{dl}{v_{||}} \sum_j C_{ej}(f_{e0}^T, f_{j0}) = 0 \quad (30)$$

Since f_{e0}^T is independent of l , the factor $\ln f_{e0}^T$ can be inserted inside the line integral and Eq. (30) becomes

$$\oint \frac{dl}{v_{||}} \ln f_{e0}^T \sum_j C_{ej}(f_{e0}^T, f_{j0}) = 0 \quad (31)$$

Integrate Eqs. (29) and (31) with respect to ϵ and μ over the respective untrapped and trapped regions in phase space. Then convert the line integral condition for the trapped particles into a shell integral condition. [This is done by the following procedure. First, conservation of magnetic flux allows a line integral condition to be converted into a tube integral condition. (The tube is defined by the line in question, a neighboring line on the same magnetic surface, and a neighboring magnetic surface.) Then, summing over all the tubes between the two neighboring surfaces gives the desired shell integral condition.] Adding these two shell integrals for the untrapped and trapped electrons and then converting the combined integral into an ordinary velocity-space integral making use of the Jacobian relation

$$\begin{aligned} \int d^3v &= \int \frac{4\pi B d\mu d\epsilon}{m_e^2 |v_{||}|} \quad , \\ &= \int \frac{4\sqrt{2}\pi}{m_e^{3/2}} d\epsilon \sqrt{\epsilon + e\Phi} d\epsilon \quad , \end{aligned} \quad (32)$$

gives

$$\int_{\text{shell}} d\tau \int d^3v \ln f_{e0} [C_{ee}(f_{e0}, f_{e0}) + \sum_i C_{ei}(f_{e0}, f_{i0})] = 0 \quad (33)$$

The summation 1 is taken over the field ions. Note that f_{e0} satisfies boundary conditions given by Eqs. (22) and (24). Also note that, from Eq. (23), the boundary layer region does not give a contribution to Eq. (33).

From the Boltzmann H-theorem

$$\int d^3v \ln f_{e0} C_{ee}(f_{e0}, f_{e0}) \leq 0 ,$$

where equality holds only if f_{e0} is a Maxwellian (f_{eM}). Using the smallness of the mass ratio m_e/m_i to approximate the electron-ion Fokker-Planck collision operator, Hinton and Hazeltine²⁴ showed that

$$\begin{aligned} & \int d^3v \ln f_{e0} C_{ei}(f_{e0}, f_{ei}) \\ &= - \frac{2\pi e^4 n \ln \Lambda}{m_e^2} \int d^3v \frac{1}{f_{e0}} \frac{\partial f_{e0}}{\partial \vec{v}} \cdot \vec{U}(\vec{v}) \cdot \frac{\partial f_{e0}}{\partial \vec{v}} \\ &\leq 0 , \end{aligned}$$

where $\vec{U}(\vec{v}) = (\vec{I} - \vec{v}\vec{v})/v^3$. Equality holds only if f_{e0} is isotropic. Therefore, f_{e0} in Eq. (33) must be a Maxwellian. The Maxwellian must be independent of θ since $f_{e0}^{U \cdot T} = f_{eM}$ is. Hence,

$$f_{e0} = f_{eM}$$

$$= n_e \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \exp\left(-\frac{\varepsilon + e\phi}{T_e}\right) , \quad (34)$$

where n_e , T_e , and ϕ are functions of r and t . To this order, the dependence of n_e and T_e on r is arbitrary; this dependence is determined by annihilation conditions in second order.

Note that in the above proof, it is not necessary to assume that electrons and ions have equal temperatures. The reason for this is that the energy transfer time between the two species is slow compared to the transport time scale, because of the small electron to ion mass ratio.

For untrapped electrons, the first-order equation is

$$\int_{\text{shell}} d\tau \frac{B}{v_{\parallel}} \sum_j C_{ej}(f_{e1}^{U.T.}) = 0 \quad , \quad (35)$$

where the linearized collision operator is

$$C_{ej}(f_{e1}) = C_{ej}(f_{e1}, f_{j0}) + C_{ej}(f_{e0}, f_{j1}) \quad .$$

For trapped electrons, the first-order equation is

$$\langle \sum_j C_{ej}(f_{e1}^{T.}) \rangle = v_B \sin\theta \left. \frac{\partial f_{eM}}{\partial r} \right|_e \quad . \quad (36)$$

In order to solve for $f_{e1}^{T.}$, let

$$f_{e1}^{T.} = \overline{f_{e1}^{T.}} + \mathcal{F}_e(\xi) \sin\theta \quad , \quad (37)$$

with the boundary condition $\mathcal{F}_e(\xi_c) = 0$. Thus, Eq. (36) can be separated into two equations

$$\langle \sum_j C_{ej}(\overline{f_{e1}^{T.}}) \rangle = v_B \left. \frac{\partial f_{eM}}{\partial r} \right|_e \quad , \quad (38a)$$

and

$$\langle \sum_j C_{ej} \overline{(f_{e1}^T)} \rangle = 0 \quad (38b)$$

Using exactly the same H-theorem procedure employed in proving that the lowest order distribution function f_{e0} is a Maxwellian, Eqs. (35) and (38) can be used to demonstrate that $f_{e1}^{U'}$ and $\overline{f_{e1}^T}$ have the form

$$f_{e1} = (\alpha + \gamma v^2) f_{eM} \quad (39)$$

where α and γ are constants. This is a distorted Maxwellian distribution function and can be absorbed into the lowest order solution. Thus, Eq. (39) becomes

$$f_{e1} = 0 \quad (40)$$

Now consider $\mathcal{F}_e(\xi)$. If the size of the trapping region is small (i.e., $|\xi_0| \ll 1$), the dominant contribution to the Fokker-Planck collision operator in this region is the pitch-angle scattering operator. The bounce-averaged pitch-angle scattering operator is evaluated in Appendix A, and the result is

$$\langle \sum_j C_{ej} \mathcal{F}_e \rangle \sim \frac{(v_{ee} + \frac{1}{2} v_{ei})_1}{4} \frac{\partial}{\partial \xi_0} \text{JD} \frac{\partial e}{\partial \xi_0} \quad (41)$$

where

$$\xi_0 \equiv \left(\frac{E - \mu_B + e\Phi}{e} \right)^{1/2} = \frac{v_{\parallel}}{v} \quad \text{at the center of a magnetic well.}$$

$$D \equiv (\kappa^2)^{-1} \left[\frac{E(\kappa)}{K(\kappa)} + (\kappa^2 - 1) \right] ,$$

$$\kappa \equiv \frac{\xi_0}{(2\epsilon_h)^{1/2}} , \quad 0 < \kappa \leq 1$$

and

$$J \equiv \frac{2KK(\kappa)R}{m} .$$

Substitution of Eq. (41) into the left-hand side of Eq. (38a) yields

$$\frac{(v_{ee} + \frac{1}{2} v_{ei})}{4} \frac{\partial}{\partial \xi_0} JD \frac{\partial \mathcal{F}_e}{\partial \xi_0} = v_B \frac{\partial f_{eM}}{\partial r} \Big|_{\epsilon} . \quad (42)$$

The second boundary condition for $\mathcal{F}_e$ just outside the thin boundary layer must be established before solving for $\mathcal{F}_e$. Using the relation $d\xi_0/d\xi = \xi/\xi_0$ as indicated in Appendix A and assuming that the magnetic field is independent of the poloidal angle, Eq. (22) can be used to demonstrate that

$$\oint d\theta \left(\frac{\partial \mathcal{F}_e}{\partial \xi_0} + \overline{\frac{\partial f_{e1}^T}{\partial \xi_0}} \right) \Big|_{\xi_0 = \xi_c^-} = \oint d\theta \frac{\partial f_{e1}^{U,T}}{\partial \xi_0} \Big|_{\xi_0 = \xi_c^+} .$$

Note that $\overline{f_{e1}^T} = f_{e1}^{U,T} = 0$. Thus, the boundary conditions are

$$\oint d\theta \frac{\partial \mathcal{F}_e}{\partial \xi_0} \Big|_{\xi_0 = \xi_c} = 0 , \quad (43a)$$

and, from before,

$$\mathcal{F}_e \Big|_{\xi_0 = \xi_c} = 0 . \quad (43b)$$

Integrating Eq. (42) with respect to ξ_0 gives

$$\frac{\partial \mathcal{F}_e}{\partial \xi_0} = \frac{\partial \mathcal{F}_e}{\partial \xi_0} \Big|_{\xi_0=0} + \left(\frac{4}{v_{ee} + \frac{1}{2} v_{ei}} \right) v_B \frac{1}{\partial D} \left(\int_0^{\xi_0} J' d\xi_0' \right) \frac{\partial f_{eM}}{\partial r} \Big|_e .$$

Note that $\partial \mathcal{F}_e / \partial \xi_0 \big|_{\xi_0=0}$ is zero since e is even in ξ_0 . Integrating the above equation once again in ξ_0 and using the boundary condition (Eq. 43b) gives

$$\mathcal{F}_e = - \left(\frac{4}{v_{ee} + \frac{1}{2} v_{ei}} \right) v_B H(\xi_0, r) \frac{\partial f_{eM}}{\partial r} \Big|_e , \quad (44)$$

where

$$H(\xi_0, r) \equiv \int_{\xi_0}^{\xi_c} \int_0^{\xi_0'} \frac{J''}{J'D'} d\xi_0'' d\xi_0' .$$

Note that ξ_c is, in general, a function of r . In Appendix B it is shown that

$$H(\xi_0, r) = \epsilon_h(r) (1 - \kappa^2) .$$

Equation (17) to second order is

$$\int \frac{dS}{|VP|} \frac{B}{v_{\parallel}} \left(\frac{\partial f_{eM}}{\partial t} - e \frac{\partial \Phi}{\partial t} \frac{\partial f_{eM}}{\partial \epsilon} \right) = \int \frac{dS}{|VP|} \frac{B}{v_{\parallel}} \sum_j C_{ej}(f_{e2}^{U.T.}) , \quad (45)$$

where the linearized collision operator

$$C_{ej}(f_{e2}) = C_{ej}(f_{e2}, f_{j0}) + C_{ej}(f_{e1}, f_{j1}) + C_{ej}(f_{eM}, f_{j2}) .$$

Equation (18) to second order is

$$\begin{aligned}
& \oint \frac{dt}{v_i} \left[\frac{\partial f_{eM}}{\partial t} - e \frac{\partial \phi}{\partial t} \frac{\partial f_{eM}}{\partial \epsilon} + v_B \sin^2 \theta \frac{\partial \mathcal{F}_e}{\partial r} \right. \\
& \quad \left. + (v_B \cos \theta + r \Omega_E + r \Omega_H) \frac{\cos \theta}{r} \mathcal{F}_e \right] \\
& = \oint \frac{dt}{v_i} \sum_j c_{ej} (f_{e2}^T) \quad . \quad (46)
\end{aligned}$$

Equations (45) and (46) together represent an inhomogeneous symmetric integral equation for f_{e2} . The solubility conditions on the temporal evolution are that the inhomogeneous terms be orthogonal to 1 and v^2 (i.e., the particle and energy conservation properties of the collision operator). These two conditions yield constraints on f_{eM} and give the temporal evolution of its parameters n_e and T_e . That is, it yields the desired results for particle and energy transport.

To determine the first solubility condition, first integrate Eqs. (45) and (46) with respect to ϵ and μ over the respective untrapped and trapped regions. Then, the result for the trapped particles is summed over flux tubes to convert it into a shell integral. Finally, adding these two shell integrals gives

$$\begin{aligned}
& \int \frac{dS}{|VP|} \int \frac{4\pi B d\mu d\epsilon}{m_e^2 |v_i|} \left[\frac{\partial f_{eM}}{\partial t} - e \frac{\partial \phi}{\partial t} \frac{\partial f_{eM}}{\partial \epsilon} + v_B \sin^2 \theta \frac{\partial \mathcal{F}_e}{\partial r} \right. \\
& \quad \left. + (v_B \cos \theta + r \Omega_E + r \Omega_H) \frac{\cos \theta}{r} \mathcal{F}_e \right] = 0 \quad . \quad (47)
\end{aligned}$$

The energy integration in this equation, strictly speaking, is from $-e\phi$ to an upper limit. Above this limit, the expansion (27) breaks down since the high energy particles deplete at a very rapid rate. Thus, the lowest order

distribution function f_{e0} is no longer a Maxwellian at high energies. However, this upper limit is generally larger than the energy of the electrons giving the dominant contribution to particle and energy fluxes [see Figs. 5(a) and 5(b)]. Therefore, the energy integration can be performed from $-e\phi$ to infinity for all practical purposes.

It will now be shown that Eq. (47) is equivalent to the particle conservation law. The factor $|VP|$ can be removed from the integral if it is assumed to be only a function of the minor radius. Dividing Eq. (47) by the total surface area of the torus, which is approximated by $(2\pi)^2 r_{R0}$, and noting that in the ξ, ν coordinates $\partial v_{||}/\partial t = e(\partial\phi/\partial t)(\partial v_{||}/\partial \xi)$, we see that the first two terms on the left-hand side of Eq. (47) can be written as

$$\begin{aligned} & \frac{1}{(2\pi)^2 r_{R0}} \int dS \int \frac{4\pi B d\nu d\xi}{m_e^2 |v_{||}|} \left(-\frac{\partial f_{eM}}{\partial t} - e \frac{\partial \phi}{\partial t} \frac{\partial f_{eM}}{\partial \xi} \right) \\ &= \frac{1}{(2\pi)^2 r_{R0}} \int dS \int \frac{4\pi B d\nu d\xi}{m_e^2 |v_{||}|} \left(-\frac{\partial f_{eM}}{\partial t} - m_e v_{||} \frac{\partial v_{||}}{\partial t} \frac{\partial f_{eM}}{\partial \xi} \right) . \end{aligned}$$

Then, using integration by parts for the energy variable gives

$$\frac{1}{(2\pi)^2 r_{R0}} \int dS \frac{\partial}{\partial t} \int \frac{4\pi B d\nu d\xi}{m_e^2 |v_{||}|} f_{eM} = \frac{\partial \langle n_e \rangle}{\partial t} , \quad (48)$$

where $\langle n_e \rangle$ is the surface-averaged density. In evaluating the remaining terms on the left-hand side of Eq. (47), note that the Ω_E and Ω_H terms vanish after θ integration. The v_B terms can be combined and expressed as

$$\begin{aligned}
& \frac{1}{(2\pi)^2 r R_0} \int r d\theta R_0 d\zeta \int \frac{4\pi B du dv}{m_e^2 |v_\parallel|} v_B (\sin^2 \theta \frac{\partial \mathcal{F}_e}{\partial r} + \cos^2 \theta \frac{\mathcal{F}_e}{r}) \\
& = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\pi}{(2\pi)^2 R_0} \int R_0 d\zeta \int d^3 v v_B \mathcal{F}_e , \\
& = \frac{1}{r} \frac{\partial}{\partial r} r \Gamma_e , \tag{49}
\end{aligned}$$

which defines Γ_e , the electron particle flux per unit area. Substituting Eqs. (48) and (49) into Eq. (47) gives

$$\frac{\partial \langle n_e \rangle}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} r \Gamma_e = 0 . \tag{50}$$

This is the particle conservation equation for electrons. To obtain the second solubility condition for the temporal evolution, Eqs. (45) and (46) are first multiplied by $m_e v^2/2$, and then a similar procedure to that employed in obtaining the particle conservation equation is used. The result is the energy conservation equation which has the form

$$\frac{\partial}{\partial t} \frac{3}{2} \langle n_e T_e \rangle + \frac{1}{r} \frac{\partial}{\partial r} r Q_e - e \Gamma_e \frac{\partial \Phi}{\partial r} = 0 , \tag{51}$$

where $(3/2) \langle n_e T_e \rangle$ is the surface-averaged energy density, and the electron energy flux per unit area is

$$Q_e = \frac{\pi}{(2\pi)^2 R_0} \int R_0 d\zeta \int d^3 v \frac{m_e^2 v^2}{2} v_B \mathcal{F}_e .$$

Note that Eq. (51) is obtained under the assumption that $T_e = T_i$, and depends on there being no mean parallel ion velocity; otherwise, additional terms will

appear.²⁵ If the temporal evolution of density and energy in Eqs. (50) and (51) are replaced by steady-state particle and energy sources, respectively, then the density and temperature profiles can be determined from these two equations.

The electron particle and energy fluxes will now be evaluated. Substituting Eq. (44) into the expression for the particle flux gives

$$\Gamma_e = - \frac{m\pi}{(2\pi)^2 R_0} \int_{-\pi/m}^{\pi/m} R_0 d\zeta \frac{4\sqrt{2\pi}}{m_e^{3/2}} \int_{-e\phi}^{\infty} d\varepsilon \sqrt{\varepsilon + e\phi} \\ \times \frac{4v_B^2}{(v_{ee} + \sum_i v_{ei})} \left[\int_0^1 H(\xi_0, r) d\xi \right] \frac{\partial f_{eM}}{\partial r} \Big|_e ,$$

where the ζ integration is over one helical period. Note that the integral

$$\int_{-\pi/m}^{\pi/m} R d\zeta \int_0^1 H d\xi = \int_0^{\xi_c} J H d\xi_0 ,$$

since

$$J = \int_{-\pi/m}^{\pi/m} R d\zeta \frac{d\xi}{d\xi_0}$$

(see Appendix A) and the function $H(\xi_0, r)$ is nonzero only between $0 \leq \xi_0 \leq \xi_c$. Using the result from Appendix B, we find

$$\int_0^{\xi_c} J H d\xi_0 = \frac{4}{9} \frac{[2\varepsilon_h(r)]^{3/2} R}{m} .$$

Therefore, the electron particle flux per unit area can now be expressed as

$$\Gamma_e = - \frac{4}{9\pi} [2\varepsilon_h(r)]^{3/2} \frac{4\sqrt{2\pi}}{m_e^{3/2}} \int_{-e\phi}^{\infty} d\varepsilon \sqrt{\varepsilon + e\phi} \frac{v_B^2}{(v_{ee} + \sum_i v_{ei})} \frac{\partial f_{eM}}{\partial r} \Big|_e . \quad (52)$$

From this equation, the electron diffusion coefficient at a given energy is found to be proportional to $\left[v_B^2 / (v_{ee} + \sum_i v_{ei}) \right] [\epsilon_h(r)]^{3/2}$. Thus, the rigorous calculation verifies the result of the heuristic argument.

To carry out the energy integration in Eq. (52), note that the reciprocal of the Spitzer 90° "deflection time" for electrons scattered by background electrons and ions is

$$v_{ee}(v) + \sum_i v_{ei}(v) = A_D l_e^3 \frac{\{ [\Phi(\sqrt{x}) - G(\sqrt{x})] + \sum_i [\Phi(\sqrt{(m_i/m_e)x}) - G(\sqrt{(m_i/m_e)x})] \}}{x^{3/2}}.$$

Here,

$$A_D = \frac{8\pi e^4 Z^2 n_j^2 \ln \Lambda}{m_e^2},$$

with the subscript j denoting the quantity for the field particles,

$$l_e = \left(\frac{m_e}{2T_e} \right)^{1/2},$$

the Chandrasekhar function

$$G(\sqrt{x}) = \frac{\Phi(\sqrt{x}) - \sqrt{x} \Phi'(\sqrt{x})}{2x},$$

the usual error function

$$\Phi(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-y^2} dy,$$

and

$$x \equiv \frac{\varepsilon + e\phi}{T}.$$

Since $(m_i/m_e)^{1/2}$ is much greater than unity, the asymptotic limit ($v \gg v_i$) of the reciprocal of the 90° "deflection time" has the form

$$v_{ee}(v) + \frac{1}{2} v_{ei}(v) \sim A_D l_e^3 \frac{[\phi(\sqrt{x}) - G(\sqrt{x}) + 1]}{x^{3/2}}. \quad (53)$$

Using this expression, Eq. (52) can be written as

$$\begin{aligned} \Gamma_e = & -\frac{4}{9\pi} \left(\frac{cT}{eB_0 R_0} \right)^2 \frac{[2\varepsilon_h(r)]^{3/2}}{A_D l_e^3} n_e \left\{ \int_0^\infty dx \sqrt{x} \frac{x^{3.5} e^{-x}}{[\phi(\sqrt{x}) - G(\sqrt{x}) + 1]} \right. \\ & \times \left[\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial \phi}{\partial r} + \left(x - \frac{3}{2}\right) \frac{1}{T} \frac{\partial T}{\partial r} \right] \Bigg\} \div \int_0^\infty dx \sqrt{x} e^{-x}. \quad (54) \end{aligned}$$

The electron energy flux per unit area can be obtained in an entirely analogous manner and is

$$\begin{aligned} Q_e = & -\frac{4}{9\pi} \left(\frac{cT}{eB_0 R_0} \right)^2 \frac{[2\varepsilon_h(r)]^{3/2}}{A_D l_e^3} n_e T \left[\int_0^\infty dx \sqrt{x} \frac{x^{4.5} e^{-x}}{[\phi(\sqrt{x}) - G(\sqrt{x}) + 1]} \right. \\ & \times \left[\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial \phi}{\partial r} + \left(x - \frac{3}{2}\right) \frac{1}{T} \frac{\partial T}{\partial r} \right] \Bigg] \div \int_0^\infty dx \sqrt{x} e^{-x}. \quad (55) \end{aligned}$$

To exhibit the contribution to particle and energy diffusion of electrons with energy between ε and $\varepsilon + C\varepsilon$ (these contributions are equal to $d\Gamma_e/d\varepsilon$ and $dQ_e/d\varepsilon$, respectively), the integrands in Eqs. (54) and (55) are plotted versus energy. Assuming $(1/n)(\partial n/\partial r) = (e/T)(\partial \phi/\partial r) = (1/T)(\partial T/\partial r)$, the integrands become $[x^4 e^{-x}/(\phi(\sqrt{x}) - G(\sqrt{x}) + 1)](x + 1/2)$ and $[x^5 e^{-x}/(\phi(\sqrt{x}) - G(\sqrt{x}) + 1)](x + 1/2)$ which are plotted versus x in Figs. 5(a) and 5(b), respectively. The figures show that electrons with approximately five and six times the

thermal energy give the dominant contributions to particle and energy diffusion, respectively.

The integral in the denominators of Eqs. (54) and (55) has a value of $\Gamma(3/2)$, while the integrals in the numerators are evaluated numerically. The results are:

$$\Gamma_e = -\frac{4}{9\pi} \left(\frac{cT}{eB_0 R_0} \right)^2 \frac{[2\epsilon_h(r)]^{3/2}}{A_D l_e^3} n_e \left[14.4 \left(\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial \phi}{\partial r} - \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial r} \right) + 71.2 \frac{1}{T} \frac{\partial T}{\partial r} \right] \quad (56a)$$

$$Q_e = -\frac{4}{9\pi} \left(\frac{cT}{eB_0 R_0} \right)^2 \frac{[2\epsilon_h(r)]^{3/2}}{A_D l_e^3} n_e T \left[71.2 \left(\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial \phi}{\partial r} - \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial r} \right) + 423.4 \frac{1}{T} \frac{\partial T}{\partial r} \right] \quad (56b)$$

Substituting the value of $A_D l_e^3$ into Eqs. (56a) and (56b) gives

$$\Gamma_e = - (2\epsilon_h(r))^{3/2} \frac{T_4^{7/2}}{B_4^2 R_m^2} \left[14.4 \left(\frac{1}{n_{14}} \frac{\partial n_{14}}{\partial r_m} - \frac{e}{T_4} \frac{\partial \phi}{\partial r_m} - \frac{3}{2} \frac{1}{T_4} \frac{\partial T_4}{\partial r_m} \right) + 71.2 \frac{1}{T_4} \frac{\partial T_4}{\partial r_m} \right] (10^{19} \text{ cm}^{-2} \cdot \text{sec}^{-1}) \quad (57a)$$

$$Q_e = - (2\epsilon_h(r))^{3/2} \frac{T_4^{9/2}}{B_4^2 R_m^2} \left[71.2 \left(\frac{1}{n_{14}} \frac{\partial n_{14}}{\partial r_m} - \frac{e}{T_4} \frac{\partial \phi}{\partial r_m} - \frac{3}{2} \frac{1}{T_4} \frac{\partial T_4}{\partial r_m} \right) + 423.4 \frac{1}{T_4} \frac{\partial T_4}{\partial r_m} \right] (10^{20} \text{ keV} \cdot \text{cm}^{-2} \cdot \text{sec}^{-1}) \quad (57b)$$

where r_m and R_m are in meters, B_4 is in units of 10 kG, n_{14} is in 10^{14} cm^{-3} , and T_4 and $e\phi$ are in 10 keV. Equations (57a) and (57b) show that diffusion due to the temperature gradient (the thermal diffusion) is greater than that

due to the combined contribution of density and electric potential gradients by factors of approximately three and four, respectively. These equations also show that Q_e/T is larger than Γ_e by a factor of approximately six. The large numerical coefficients in front of the gradients in these two equations result from the energy weighting of the high energy dependent diffusion coefficient. In Sec. V, the exact value of Φ is obtained by matching the electron and the ion particle fluxes.

Finally, Eqs. (56a) and (56b) can be written in a form displaying Onsager symmetry. Note that $Q_e - (5/2)T\Gamma_e$ is the radial heat flux q_e (i.e., that portion of energy flux not transported by mass motion). If "forces" A_n are defined as (Ref. 24):

$$A_1 = \frac{1}{P} \frac{\partial P}{\partial r} + \frac{e}{T} \frac{\partial \Phi}{\partial r} ,$$

$$A_2 = \frac{1}{T} \frac{\partial T}{\partial r} ,$$

then Eqs. (56a) and (56b) can be cast into the form

$$\begin{pmatrix} \Gamma_e \\ \frac{q_e}{T} \end{pmatrix} = - \frac{4}{9\pi} \left(\frac{cT}{eBR} \right)^2 \frac{(2\varepsilon_h(r))^{3/2}}{A_D t_e^3} n_e \begin{pmatrix} 14.4 & 35.2 \\ 35.2 & 245.4 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} . \quad (58)$$

The symmetry of this matrix is equivalent to Onsager symmetry.

C. Ion Transport

The detailed kinetic calculation for ion transport is given in our companion paper.¹⁴ The results are outlined in this section.

Since the ion collision frequency is much lower than that of the electrons, the $\vec{E}_\perp \times \vec{B}$ drift can take a trapped ion through several poloidal

rotations during a collisional detrapping time. It can be shown that in the reference stellarator,

$$\frac{v_B}{av_{\text{eff}}} \approx \frac{v_B}{a\Omega_E} = 0(10^{-1}) \quad (59)$$

Except for low energy ions, i.e., $\epsilon < T$, this result is valid for ions in all the machines with parameters given in Table I. Thus, Eqs. (17) and (18) can be solved by expanding F_{i0} in powers of v_B/av_{eff} or $v_B/a\Omega_E$:

$$F_{i0} = f_{i0} + f_{i1} + f_{i2} + \dots \quad (60)$$

As before, f_{i0} is a Maxwellian. For the case without the effect of collisionless detrapping and entrapping, the first-order distribution functions for the untrapped and trapped particles are:

$$f_{i1}^{U.T.} = 0$$

and

$$f_{i1}^T = \frac{v_B}{\Omega_E} \exp(k(\xi_0 - \xi_c)) \left[\sin k(\xi_0 - \xi_c) \right] \frac{\partial f_{iM}}{\partial r} \Big|_{\epsilon} \sin\theta \\ + \frac{v_B}{\Omega_E} [1 - \exp(k(\xi_0 - \xi_c))] \left[\cos k(\xi_0 - \xi_c) \right] \frac{\partial f_{iM}}{\partial r} \Big|_{\epsilon} \cos\theta \quad (61)$$

The f_{i1}^T shown here is the approximate result for the case in which $k \equiv \sqrt{4.93\Omega_E/T_j} i_j$ is large. This is the "rotational limit" that is valid for most of the ions. The $\cos\theta$ term comprises the bulk of the trapped particle distribution function. However, in the second order, the $\cos\theta$ terms become

zero after surface integration and thus only the terms involving $\sin\theta$ contribute to diffusion. Nevertheless, the $\cos\theta$ term in f_{i1}^T gives rise to a nonradial ambipolar electric field which causes some enhancement of electron and ion diffusion. This effect is discussed in the next section. The low energy ions are "nonrotational." Thus, k is small and the approximate first-order solution is

$$f_{i1}^T = \frac{4}{\sum_j v_{ij}} v_B \epsilon_h(r) \left(1 - \frac{\epsilon_0^2}{2\epsilon_h}\right) \frac{\partial f_{iM}}{\partial r} \Big|_e \sin\theta. \quad (62)$$

This has exactly the same functional form as the first-order electron distribution function. Thus, the behavior of low energy ion transport is identical to that of the electrons. Using Eqs. (61) and (62), the ion particle and energy fluxes can be shown to have the form

$$\begin{aligned} \Gamma_i = & -\frac{4}{9\pi} (2\epsilon_h(r))^{3/2} \frac{4\sqrt{2\pi}}{m_i^{3/2}} \int_{\epsilon_T}^{\epsilon_0} d\epsilon \sqrt{\epsilon - e\phi} \frac{v_B^2}{\sum_j v_{ij}} \frac{\partial f_{iM}}{\partial r} \Big|_e \\ & - \frac{0.44}{4\sqrt{2}} \frac{4\sqrt{2\pi}}{m_i^{3/2}} \int_{\epsilon_T}^{\infty} d\epsilon \sqrt{\epsilon - e\phi} \frac{v_B^2}{\Omega_E} \left(\frac{\sum_j v_{ij}}{\Omega_E}\right)^{1/2} \frac{\partial f_{iM}}{\partial r} \Big|_e \end{aligned} \quad (63a)$$

$$\begin{aligned} Q_i = & -\frac{4}{9\pi} (2\epsilon_h(r))^{3/2} \frac{4\sqrt{2\pi}}{m_i^{3/2}} \int_{\epsilon_T}^{\epsilon_0} d\epsilon \sqrt{\epsilon - e\phi} \left(\frac{m_i v^2}{2}\right) \frac{v_B^2}{\sum_j v_{ij}} \frac{\partial f_{iM}}{\partial r} \Big|_e \\ & - \frac{0.44}{4\sqrt{2}} \frac{4\sqrt{2\pi}}{m_i^{3/2}} \int_{\epsilon_T}^{\infty} d\epsilon \sqrt{\epsilon - e\phi} \left(\frac{m_i v^2}{2}\right) \frac{v_B^2}{\Omega_E} \left(\frac{\sum_j v_{ij}}{\Omega_E}\right)^{1/2} \frac{\partial f_{iM}}{\partial r} \Big|_e, \end{aligned} \quad (63b)$$

where ϵ_T is the "transition energy" above which the rotational approximation holds and below which the nonrotational approximation holds.

The first term on the right-hand side of Eq. (63) is generally small compared to the second term, and thus the ion fluxes can be approximated by integrating the second term from zero to infinity. For deuterons, the

particle flux is

$$\begin{aligned} \Gamma_D \approx & -\frac{0.44}{4\sqrt{2}} \left[\left(\frac{cT}{eB_0 R_0} \right)^2 / \Omega_E \right] \left(\frac{A_D^3}{\Omega_E} \right)^{1/2} n_D \left[1.63 \left(\frac{1}{n_D} \frac{\partial n_D}{\partial r} \right) \right. \\ & \left. + \frac{e}{T} \frac{\partial \Phi}{\partial r} - \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial r} \right] + 4.63 \frac{1}{T} \frac{\partial T}{\partial r} \quad (64) \end{aligned}$$

The energy flux is

$$\begin{aligned} Q_D \approx & -\frac{0.44}{4\sqrt{2}} \left[\left(\frac{cT}{eB_0 R_0} \right)^2 / \Omega_E \right] \left(\frac{A_D^3}{\Omega_E} \right)^{1/2} n_D T \left[4.63 \left(\frac{1}{n_D} \frac{\partial n_D}{\partial r} \right) \right. \\ & \left. + \frac{e}{T} \frac{\partial \Phi}{\partial r} - \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial r} \right] + 17.68 \frac{1}{T} \frac{\partial T}{\partial r} \quad (65) \end{aligned}$$

From Eq. (64), the ion diffusion coefficient is $0.13 [cT/(eBR)]^2 / \Omega_E (A_D^3 / \Omega_E)^{1/2}$, which is one order of magnitude smaller than the ion diffusion coefficient given in Ref. 10. The reason is because the numerical integration for the proper energy weighting of the Maxwellian distribution function is carried out correctly here. In Ref. 14, it is shown that ion particle and energy diffusion is dominated by particles having three to four times the thermal energy. Note that Eqs. (64) and (65) can be cast into a matrix form that possesses the Onsager symmetry as is the case for electrons. Also note that from Eqs. (64) and (65), the ratio of Q_D/T to Γ_D is roughly five and from Sec. IV B, the ratio of Q_e/T to Γ_e is roughly six. Since $\Gamma_e = \Gamma_i$, the ion energy confinement time should be comparable to the electron energy confinement time. This is confirmed by numerical examples in Sec. V.

D. Enhanced Transport Induced by the Nonradial Electric Field Caused by Ion Drift

In the $k \gg 1$ rotational limit, the first-order trapped ion distribution function is essentially the $\cos\theta$ dependent term over the bulk of the trapped region. This term has a maximum at $\theta = 0$ and a minimum at $\theta = \pi$. The reason for this behavior is that the poloidal $\vec{E}_r \times \vec{B}$ rotation and the shift of the drift orbits cause an excess of ions on the outer region and a deficiency of ions on the inner region of the torus. This first-order distribution function creates an imbalance of charge that gives rise to an ambipolar electric field E_p which has a θ component. This electric field has the effect of enhancing electron and ion diffusion. The effect on transport caused by this nonradial electric field was first considered by Mynick.¹¹ However, our approach and conclusions given in this section differ considerably from his. This section presents the calculation for the $\vec{E}_p \times \vec{B}$ drift induced by the nonradial ambipolar field. Then, the corrected first-order electron distribution function is obtained by including the $\vec{E}_p \times \vec{B}$ drift. Finally, using the corrected first-order distribution function, the enhanced particle and energy fluxes are obtained.

To obtain the ambipolar potential ϕ_p induced by the perturbed ion density, the quasineutrality condition is applied. This condition requires that

$$\int d^3v (f_{iM} + f_i^1) = \int d^3v f_{eM} \quad . \quad (66)$$

Here $f_{iM} = n_i(r) \exp(-mv^2/2 - e\phi_p/T)$ and $f_{eM} = n_e(r) \exp(-mv^2/2 + e\phi_p/T)$, where ϕ_p is the potential, necessary for charge neutrality, which produces E_p .

After expanding the factor $\exp(-e\phi_p/T)$ and using the quasineutrality approximation $n_e \approx n_i$, Eq. (66) becomes

$$\phi_p = \frac{T}{2en} \int d^3v f_i^1 \quad (67)$$

In the rotational limit, f_i^1 has the form

$$\lim_{k \rightarrow 0} f_i^1 = \frac{v_B}{\Omega_E} \frac{\partial f_{iM}}{\partial r} \Big|_e \cos\theta \quad .$$

Thus,

$$\phi_p \approx \frac{T}{2en} \cos\theta \int_{\text{trapped}} d^3v \frac{v_B}{\Omega_E} \frac{\partial f_{iM}}{\partial r} \Big|_e \quad .$$

The component of the $\vec{E}_p \times \vec{B}$ drift normal to a magnetic surface is

$$\begin{aligned} \vec{v}_{E \times B} \cdot \hat{e}_r &= -\frac{c\partial\phi}{rB\partial\theta} \\ &= \frac{cT}{2reBn} \sin\theta \int_{\text{trapped}} d^3v \frac{v_B}{\Omega_E} \frac{\partial f_{iM}}{\partial r} \Big|_e \quad (68) \end{aligned}$$

Note that $\Omega_E = (c/rB)(\partial\phi/\partial r)$. The ambipolar radial electric field $-\partial\phi/\partial r$ can be approximated by $C(T/e)(1/n)(\partial n/\partial r)$ where the coefficient C , of order unity, is determined in Sec. V by requiring the electron and ion particle fluxes to be ambipolar. Thus, Eq. (68) can be expressed as

$$\begin{aligned} \vec{v}_{E \times B} \cdot \hat{e}_r &= -\frac{3}{4} \frac{cT}{eB_0 r} \frac{(2e_h(r))^{1/2}}{C(1/n)(\partial n/\partial r)} \sin\theta \\ &\times \left(\frac{1}{n_i} \frac{\partial n_i}{\partial r} + \frac{e}{T} \frac{\partial\phi}{\partial r} + \frac{1}{T} \frac{\partial T}{\partial r} \right) \quad , \quad (69) \end{aligned}$$

which is of $\frac{0(cT\sqrt{2\epsilon_h}/eBR)}{}$. For $\epsilon_h = 0.1$, $v_{E \times B}/v_B = 0.5$ at $\epsilon = T$. The effect of this $\vec{E}_p \times \vec{B}$ drift on electron transport will be discussed now, and the effect on ion transport will be discussed later in this section.

Note that the radial component of $\vec{v}_{E \times B}$ and $\vec{v}_B$ are proportional to $\sin\theta$, and that their directions are parallel for electrons. Therefore, the modified electron diffusion coefficient is

$$D_e(\epsilon) \sim \frac{1}{v} \left(\frac{cT}{eBR} \right)^2 \left[\frac{\epsilon}{T} + \sqrt{2\epsilon_h} \right]^2. \quad (70)$$

For example, if $\epsilon_h = 0.1$, then the diffusion coefficient is enhanced by about 50% for thermal electrons. However, as we have found in Sec. IV B, electron particle and energy diffusion is dominated by particles having five or six times the thermal energy. Thus, when the proper energy weighing is considered, the electron particle and energy fluxes are only enhanced by about 20 and 15%, respectively. This is shown in the following calculation.

Since $\vec{v}_{E \times B}$ and $\vec{v}_B$ are the same order of magnitude, the first-order electron distribution, given by Eq. (44), must be modified. Including the $\vec{E}_p \times \vec{B}$ drift into Eq. (42) yields

$$\begin{aligned} & \frac{(v_{ee} + \frac{1}{2}v_{ei})}{4} \frac{1}{J} \frac{\partial}{\partial \xi_0} JD \frac{\partial e}{\partial \xi_0} \\ &= [v_B \sin\theta + \frac{c}{B} (-\frac{\partial \phi}{r \partial \theta})] \frac{\partial f_{eM}}{\partial r} \Big|_{\epsilon} \end{aligned} \quad (71)$$

Using the exact procedure as was employed in solving Eq. (42), Eq. (71) gives

$$F_e = - \frac{4}{(v_{ee} + \frac{1}{2}v_{ei})} [v_B \sin\theta + \frac{c}{B} (-\frac{\partial \phi}{r \partial \theta})] H(\xi_0, r) \frac{\partial f_{eM}}{\partial r} \quad (72)$$

Following Eq. (49), it can be shown that the modified electron particle flux per unit area is

$$\begin{aligned}
 \Gamma_e &= \frac{m}{(2\pi)^2 R} \int_0^{2\pi} d\theta \int_{-\pi/m}^{\pi/m} R d\zeta \int d^3v [v_B \sin\theta + \frac{c}{B} (-\frac{\partial\phi}{r\partial\theta})] e \\
 &= -\frac{4}{9\pi} \left(\frac{cT}{eB_0 R_0}\right)^2 \frac{(2\varepsilon_h(r))^{3/2}}{A_D l^3} n_e \left\{ \int_0^\infty dx \sqrt{x} \frac{x^{3.5} e^{-x}}{[(\phi(\sqrt{x}) - G(\sqrt{x})) + 1]} \right. \\
 &\quad \times \left[x - \frac{3}{4} \frac{\sqrt{2\varepsilon_h(r)}}{C(1/n_e)(\partial n_e/\partial r)} \left(\frac{1}{n_i} \frac{\partial n_i}{\partial r} + \frac{e}{T} \frac{\partial\phi}{\partial r} + \frac{1}{T} \frac{\partial T}{\partial r} \right) \right]^2 \\
 &\quad \times \left(\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial\phi}{\partial r} + \left(x - \frac{3}{2} \right) \frac{1}{T} \frac{\partial T}{\partial r} \right) \Big\} \\
 &\quad \div \int_0^\infty dx \sqrt{x} e^{-x} , \\
 &= -\frac{4}{9\pi} \left(\frac{cT}{eB_0 R_0}\right)^2 \frac{(2\varepsilon_h(r))^{3/2}}{A_D l^3} n_e \\
 &\quad \times \left\{ [14.4 \left(\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial\phi}{\partial r} + \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial r} \right) + 71.2 \frac{1}{T} \frac{\partial T}{\partial r}] \right. \\
 &\quad \left. - \frac{3}{2} \frac{\sqrt{2\varepsilon_h(r)}}{C(1/n_e)(\partial n_e/\partial r)} \left(\frac{1}{n_i} \frac{\partial n_i}{\partial r} + \frac{e}{T} \frac{\partial\phi}{\partial r} + \frac{1}{T} \frac{\partial T}{\partial r} \right) \right. \\
 &\quad \left. \times [3.8 \left(\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial\phi}{\partial r} - \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial r} \right) + 14.4 \frac{1}{T} \frac{\partial T}{\partial r}] \right\} . \tag{73}
 \end{aligned}$$

The second term on the right-hand side in the last equation is the enhancement for the electron particle flux.

Similarly, the modified energy flux can be obtained and has the form

$$\begin{aligned}
Q_e = & -\frac{4}{9\pi} \left(\frac{cT}{eB_0 R_0} \right)^2 \frac{(2\varepsilon_h(r))^{3/2}}{A_D^3} n_e T \\
& \times \left\{ [71.2 \left(\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial \Phi}{\partial r} + \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial r} \right) + 423.4 \frac{1}{T} \frac{\partial T}{\partial r}] \right. \\
& - \frac{3}{2} \frac{\sqrt{2\varepsilon_h(r)}}{C(1/n_e)(\partial n_e/\partial r)} \left(\frac{1}{n_e} \frac{\partial n_e}{\partial r} + \frac{e}{T} \frac{\partial \Phi}{\partial r} + \frac{1}{T} \frac{\partial T}{\partial r} \right) \\
& \left. \times [14.4 \left(\frac{1}{n_e} \frac{\partial n_e}{\partial r} - \frac{e}{T} \frac{\partial \Phi}{\partial r} + \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial r} \right) + 71.2 \frac{1}{T} \frac{\partial T}{\partial r}] \right\} \quad (74)
\end{aligned}$$

where the $(3/2)[\sqrt{2\varepsilon_h}/C(1/n)(\partial n/\partial r)]$ term is the enhancement factor for the electron energy flux. The term involving $(v_{E_p \times B})^2$ has not been included in Eqs. (73) and (74) because this term would be smaller than any other term in these two equations. Equations (73) and (74) indicate that the enhanced electron particle transport due to the electric field caused by ion drift is about 20% of the ordinary electron particle transport, as opposed to about 15% for energy transport. Note that Eqs. (73) and (74) can be cast into a matrix that possesses Onsager symmetry.

The effect of the $\vec{E}_p \times \vec{B}$ drift on ion transport will now be discussed. Since the $\vec{E}_p \times \vec{B}$ and grad-B drifts are in opposite direction for ions, the modified diffusion coefficient for ions is

$$D_i(\varepsilon) \sim \frac{1}{\Omega_E} \left(\frac{cT}{eBR} \right)^2 \left(\frac{e}{T} - \sqrt{2\varepsilon_h} \right)^2 \left(\frac{v_i}{\Omega_E} \right)^{1/2} \quad (75)$$

Hence, it may be thought that the nonradial electric field reduces ion transport. However, this is not the case. In fact, the ion particle transport must be increased by the same amount as the enhanced electron loss in order to preserve charge neutrality. Since ion diffusion is very sensitive to the radial electric field, while electron diffusion is not (see Sec. V), a

slight decrease in radial electric field increases the ion transport to restore the charge imbalance.

The electron drift also creates a nonradial ambipolar electric field. However, the $\vec{E}_p \times \vec{B}$ drift caused by this nonradial electric field is smaller than the grad-B drift velocity by a factor of v_B/av_{eff} . Therefore, the corrections to the first-order electron and ion distribution functions are negligible, and this nonradial electric field has no important effect on random-walk type diffusion.

V. RADIAL AMBIPOLAR ELECTRIC FIELD AND SCALING LAWS FOR CONFINEMENT

The expressions for the electron and ion particle fluxes given in Secs. IVB and IVC depend on the magnitude of the ambipolar electrostatic potential $\phi(r)$. The first part of this section discusses the behavior of the electron and ion fluxes as the magnitude of the radial electric field $E_r = -\partial\phi(r)/\partial r$ varies. The second part of this section presents a convenient method of determining the self-consistent radial ambipolar potential and discusses the scaling laws for confinement.

In this section, it is assumed for simplicity that the radial electric field is related to the density gradient by a constant C , i.e.,

$$\frac{e}{T} \frac{\partial \phi}{\partial r} = -C \frac{1}{n} \frac{\partial n}{\partial r} \quad . \quad (76)$$

Using this condition, the electron and ion fluxes are plotted versus C for two different central densities in Fig. 6 for the reference stellarator at $r=a/2$. Note that the electron flux is independent of central density (see Sec. III).

In this plot, both the density and temperature profiles are assumed to be

parabolic, i.e., $n(r) = n_0[1 - (r/a)^2]$ and $T(r) = T_0[1 - (r/a)^2]$. (The reason for choosing parabolic profiles is given later in this section.)

Figure 6 shows that the ambipolar condition $\Gamma_e = \Gamma_i$ is satisfied at $C = 0.7$ for a peak density $n_0 = 2 \times 10^{14} \text{ cm}^{-3}$. As n_0 decreases to 10^{14} cm^{-3} , the ambipolar condition is satisfied at three different values of C . This behavior of multiple roots for the ambipolar field has been discussed for stellarators by Mynick and Hitchon,²⁶ for tandem mirrors by Mirin *et al.*,²⁷ and for Elmo Bumpy Torus by Jaeger *et al.*²⁸ Of the three roots shown in Fig. 7, the one at C_2 is unstable. A slight decrease in the positive charge moves C_2 to the stable equilibrium C_3 with the direction of the radial electric field pointing toward the minor axis. On the other hand, a slight increase in the positive charge moves C_2 to the stable equilibrium C_1 with the radial electric field pointing away from the minor axis. Although the electron and ion particle fluxes are very small at the stable equilibrium C_1 , as indicated in Fig. 6, the energy fluxes are not necessarily small. This is because the energy of the particles that gives the dominant contribution to energy diffusion is higher than the energy of the particles that gives the dominant contribution to particle diffusion. In the following calculations, the electric field that gives the stable equilibrium C_3 is taken. For low density operation, this electric field is likely to be the equilibrium field that the plasma will adjust to naturally, and for high density operation, this electric field is the only equilibrium field that can satisfy the ambipolar condition.

It is important to note that the ion flux is very sensitive to radial electric field variations near $C = 0$. The reason is that a small increase in the magnitude of the electric field causes the ions to execute full poloidal rotations. This drastically reduces the ion particle flux. In contrast,

electrons are relatively insensitive to a change in radial electric field because the electron effective collision frequency is less than the $\vec{E}_r \times \vec{B}$ poloidal rotation frequency. Therefore, if a rough estimate of the magnitude of the ambipolar field is given (e.g., $\vec{E}_r = 0$), the electron particle diffusion is well approximated and should provide a good estimate for the total particle confinement time.

Once the behavior of particle fluxes as a function of the radial electric field is determined, the problem is reduced to the determination of the magnitude of this radial electric field. In principle, Fig. 6 can be used to obtain the magnitude of the self-consistent radial electric field. This method requires, however, a different set of curves for each set of machine parameters. To obviate this, we propose an approximate method which can be used for all machine parameters. The approximation is the same approximation that is used to obtain Eq. (63), i.e., below ϵ_T , the nonrotational limit for ion transport is employed and above ϵ_T , the rotational limit is employed. Thus, matching the two limits gives

$$\Omega_E = \left(\frac{0.44 \times 9\sqrt{\pi}}{16\sqrt{2}} \right)^{2/3} \frac{A_D^3}{2\epsilon_h(r)} \frac{[\Phi(\sqrt{\epsilon_T/T}) - G(\sqrt{\epsilon_T/T})]}{(\epsilon_T/T)^{3/2}} \quad (77)$$

Since $\Omega_E = (c/rB)(\partial\Phi/\partial r) \approx C(cT/reBn)(\partial n/\partial r)$, Eq. (77) can be rewritten as

$$|C| = M \frac{[\Phi(\sqrt{\epsilon_T/T}) - G(\sqrt{\epsilon_T/T})]}{(\epsilon_T/T)^{3/2}} \quad (78)$$

and the machine dependent parameter is

$$M = 0.27 \times 10^{-2} \frac{a_m^2 n_{14}^2(r)}{\epsilon_h(r) n_{014} T_4^{5/2}(r)} \quad (79)$$

Here, n_{014} is the central plasma density in 10^{14} cm^{-3} .

The second equation that related C to ϵ_T/T is obtained from the ambipolar condition $\Gamma_e = \Gamma_i$. This equation is derived in Appendix C and has the form

$$C = \frac{-D(\epsilon_T/T)/2 + E(\epsilon_T/T) - 0.93}{D(\epsilon_T/T) + 0.21} \quad (80)$$

where

$$\begin{aligned} D(\epsilon_T/T) = & \left(\frac{\epsilon_T}{T}\right)^5 \int_0^1 dx \frac{x^4 \exp(-(\epsilon_T/T)x)}{[\Phi(\sqrt{(\epsilon_T/T)x}) - G(\sqrt{(\epsilon_T/T)x})]} \\ & + \frac{1}{[\Phi(\sqrt{\epsilon_T/T}) - G(\sqrt{\epsilon_T/T})]} \left(\frac{\epsilon_T}{T}\right)^5 \int_1^\infty dx [\Phi(\sqrt{(\epsilon_T/T)x}) \\ & - G(\sqrt{(\epsilon_T/T)x})]^{1/2} x^{7/4} \exp(-(\epsilon_T/T)x) \end{aligned}$$

and

$$\begin{aligned} E(\epsilon_T/T) = & \left(\frac{\epsilon_T}{T}\right)^6 \int_0^1 dx \frac{x^5 \exp(-(\epsilon_T/T)x)}{[\Phi(\sqrt{(\epsilon_T/T)x}) - G(\sqrt{(\epsilon_T/T)x})]} \\ & + \frac{1}{[\Phi(\sqrt{\epsilon_T/T}) - G(\sqrt{\epsilon_T/T})]} \left(\frac{\epsilon_T}{T}\right)^6 \int_1^\infty dx [\Phi(\sqrt{(\epsilon_T/T)x}) \\ & - G(\sqrt{(\epsilon_T/T)x})]^{1/2} x^{11/4} \exp(-(\epsilon_T/T)x) \end{aligned}$$

with $x = (\epsilon/T)/(\epsilon_T/T)$. Note that Eq. (80) involves only C and ϵ_T/T . With two equations for two unknowns, C and ϵ_T can be determined from Eqs. (78) and (80) if the machine parameters are given.

Equating the right-hand sides of Eqs. (78) and (80) gives

$$M = \frac{|(-D(\epsilon_T/T)/2 + E(\epsilon_T/T) - 0.93)(\epsilon_T/T)^{3/2}|}{|(\phi(\sqrt{\epsilon_T/T}) - G(\sqrt{\epsilon_T/T}))(D(\epsilon_T/T) + 0.21)|} \quad (81)$$

The parameter M is plotted versus ϵ_T/T in Fig. 7(a) using Eq. (81). The parameter C is plotted versus ϵ_T/T in Fig. 7(b) using Eq. (80). For any given set of machine parameters, M can be determined from Eq. (79). Once M is known, ϵ_T/T can be found from Fig. 7(a) and then C from Fig. 7(b). Knowing the normalized strength of the ambipolar field C , the neoclassical particle and energy fluxes can then be determined. Figure 7(a) shows that when the value of M is sufficiently small (e.g., if density is low), the ambipolar condition can be satisfied at three values of C as mentioned in the beginning of this section.

The neoclassical fluxes are most easily specified by stating the confinement time τ , i.e., the time during which the particle and energy must be replaced to maintain a steady state. For a reactor, however, the evaluation of τ is complicated by the fact that the reactor will most likely operate near marginal stability set by some limiting modes (e.g., the ballooning instabilities). The profile of the critical pressure gradient for ballooning instabilities is obtained from MHD stability analysis. In some regions of the plasma, the plasma pressure gradient will be subcritical and the entire flux will be carried by neoclassical transport. In other regions, the plasma will be marginally unstable, i.e., the plasma pressure gradient exceeds the critical gradient by a small amount, and the flux will be carried partially by turbulent convection and partially by neoclassical transport (see Ref. 15 for a discussion on convective equilibrium and transport). Therefore, if the particle confinement time at any minor radius is approximated by

$$\tau_p = \frac{\int_0^r dr \, 2\pi r n_e(r)}{2\pi r \Gamma_e(r)}, \quad (82)$$

where Γ_e is the neoclassical particle flux, then τ_p will be greater than or equal to the confinement time (because Γ_e is less than or equal to the actual flux). Thus, a reasonable estimate of the confinement time could be obtained by minimizing Eq. (82) with respect to minor radius. If $n(r)$ and $T(r)$ are parabolic, as assumed before (a good approximation if the plasma is marginally unstable over a good part of the radius),¹⁵ then Eq. (82) has its minimum value near $r = a/2$. The same argument can be used to estimate the energy confinement time τ_e . Hence, in quoting stellarator confinement times, the profiles are assumed to be parabolic and r is evaluated at $r = a/2$. (For smaller experimental stellarators, this prescription should still give an adequate estimate of the effective confinement time.) Inserting the expression for Γ_e [given by Eq. (57a)] into Eq. (82) and subsequently simplifying the equation by using the parabolic profile assumption, Eq. (82) becomes

$$\tau_p = 0.31 \times 10^{-4} \frac{n_{014} B_4^2 R_m^2 a_m^2}{[4.4 + C] (2\epsilon_h(a/2))^{3/2} T_{04}^{7/2}} \quad (83)$$

where T_{04} is the central plasma temperature in 10 keV.

The energy confinement time is

$$\tau_e = \frac{\int_0^r dr \, 2\pi r (3/2) n(r) T(r)}{2\pi r Q(r)}, \quad (84)$$

where $Q(r)$ is the neoclassical energy flux. Inserting Eq. (56b) into Eq. (84) gives the electron energy confinement time,

$$\tau_{\epsilon}^e = 0.1 \times 10^{-4} \frac{n_{014} B_4^2 R_m^2 a_m^2}{(5.4 + C)(2\epsilon_h(a/2)^{3/2} T^{7/2})^{0.4}} \quad (85)$$

Using Eqs. (84), (85), and the equation for ion energy flux [Eq. (63b)], it can be shown that the ion energy confinement time is

$$\tau_{\epsilon}^i \sim R(\epsilon_T/T) \tau_{\epsilon}^e, \quad (86)$$

where the ratio of the ion to electron energy confinement times is

$$R(\epsilon_T/T) = \frac{5.4 + C}{[(-C - 0.5)E(\epsilon_T/T) + F(\epsilon_T/T)]} \quad (87)$$

with

$$\begin{aligned} F(\epsilon_T/T) = & \left(\frac{\epsilon_T}{T}\right)^7 \int_0^1 dx \frac{x^6 \exp(-(\epsilon_T/T)x)}{(\Phi(\sqrt{(\epsilon_T/T)x}) - G(\sqrt{(\epsilon_T/T)x}))} \\ & + \frac{1}{(\Phi(\sqrt{\epsilon_T/T}) - G(\sqrt{\epsilon_T/T}))^{3/2}} \left(\frac{\epsilon_T}{T}\right)^7 \int_1^{\infty} dx [(\Phi(\sqrt{(\epsilon_T/T)x}) - G(\sqrt{(\epsilon_T/T)x}))]^{1/2} \\ & \times x^{15/4} \exp(-(\epsilon_T/T)x). \end{aligned}$$

The function R is plotted versus ϵ_T/T in Fig. 8. Note that, although the four sets of machine parameters listed in Table I are different, Eqs. (83)-(86) are still applicable since the expansion parameters used in obtaining the neoclassical fluxes are valid for all these machines, as mentioned in Secs. IVB and IVC.

Using Eqs. (83)-(86), the confinement times estimated for the four machines are given in Table II. The results show that in all the machines,

the electron energy confinement time is comparable to the ion energy confinement time, but is only one third of the particle confinement time. Note that ϵ_T/T is less than unity for all cases. For the reactor, $n\tau_E^i \approx 1.7 \times 10^{14} \text{ cm}^{-3}\cdot\text{sec}$, which roughly satisfies the Lawson criterion for ignition. Equation (86) shows that the parameters $n\tau_E^i$ can be increased by increasing the major radius, the minor radius, the magnetic field, the plasma density, or by decreasing the plasma temperature. Finally, Eqs. (85) and (86) show that for a fixed minor radius, $n\tau_E$ is proportional to the aspect ratio squared or the volume squared. For fixed aspect ratio, $n\tau_E$ is proportional to $(\text{volume})^{4/3}$. Thus, if cost scales with volume, it is cheaper to increase the aspect ratio than the overall dimensions.

VI. CONCLUSION

Stellarator neoclassical transport in the low-collisional regime is calculated by using the drift-Boltzmann equation. By expanding the drift-Boltzmann equation and then applying the solubility conditions, a set of reduced equations for trapped and untrapped particles is obtained in Sec. IV A. The electron distribution function satisfying the reduced equations is obtained in Sec. IV B by a second expansion. The electron fluxes are then calculated. It is found that electrons with five to six times the thermal energy give the dominant contribution to electron diffusion. The expressions for ion fluxes are given in Sec. IV C. Ions with three to four times the thermal energy give the dominant contribution to diffusion. Diffusion due to temperature gradient is larger than that due to density and electric potential gradients. The nonradial ambipolar electric field induced by ion drift enhances the particle transport by about 20% and energy transport by about 15%. A convenient graphical method for determining the self-consistent radial

ambipolar field for machines with different parameters is presented in Figs. 8 and 9. Numerical calculations show that electron energy confinement time is always comparable to that of the ions regardless of machine parameters. Although the neoclassical losses are large, it is shown that ignition can be achieved in a reasonable size reactor. Confinement scaling law shows that in order to increase nt , it is cheaper to increase the aspect ratio than the overall dimensions of the reactor.

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Appendix A. BOUNCE AVERAGE OF THE PITCH-ANGLE SCATTERING OPERATOR

The bounce-average operator in Eq. (41) is

$$\left\langle \frac{v}{4} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial f}{\partial \xi} \right\rangle = \frac{(v/4) \oint (dl/v_{\parallel}) (\partial/\partial \xi) (1 - \xi^2) (\partial f/\partial \xi)}{\oint (dl/v_{\parallel})} \quad (A1)$$

Let the value of ξ at the center of a magnetic well be ξ_0 . Using the first adiabatic invariant, it can be shown that at an arbitrary fixed position in a magnetic well

$$\frac{d\xi}{d\xi_0} = \frac{\xi_0}{\xi} \quad , \quad (A2)$$

where

$$\xi_0 \equiv (2\epsilon_h)^{1/2} \kappa,$$

with $1 \leq \kappa \leq 0$. Using Eq. (A2) and the fact that $\xi_0^2 \ll 1$ for trapped particles, Eq. (A1) can be written as

$$\left\langle \frac{v}{4} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial f}{\partial \xi} \right\rangle \approx \frac{(v/4) (\partial/\partial \xi_0) (\xi_0)^{-2} [\oint dl (\xi_0/\xi) (\partial dl \xi / \partial dl / \xi) (\partial f / \partial \xi_0)]}{\oint dl (\xi_0/\xi)} \quad (A3)$$

The integrals in the numerator and in the denominator are evaluated by keeping only the helical variations in the magnetic field strength [i.e., $B = B_0[(1 - \epsilon_h) \cos(l\theta - m\zeta)]$]. The results are:

$$\oint dl \xi = \oint dl [\epsilon_h (\cos hl - \cos h l_{\max})]^{1/2}$$

$$= \frac{\sqrt{\epsilon_h} R}{m} [\sqrt{8} E(\kappa) + \sqrt{8} (\kappa^2 - 1) K(\kappa)] ,$$

$$\oint \frac{dl}{\xi} = \oint \frac{dl}{[\epsilon_h (\cos hl - \cos hl_{\max})]^{1/2}}$$

$$= \frac{R}{\sqrt{\epsilon_h m}} [\sqrt{2} K(\kappa)] ,$$

where $hl = l\theta - m\xi$ and $K(\kappa)$ [$E(\kappa)$] is the complete elliptical integral of the first [second] kind. The bounce-averaged pitch-angle scattering operator becomes

$$\left\langle \frac{\nu}{4} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial f}{\partial \xi} \right\rangle = \frac{\nu}{4} \frac{1}{j} \frac{\partial}{\partial \xi_0} J D \frac{\partial f}{\partial \xi_0} , \quad (A4)$$

where

$$D \equiv (\kappa^2)^{-1} \left[\frac{E(\kappa)}{K(\kappa)} + (\kappa^2 - 1) \right] ,$$

$$J \equiv \oint dl \frac{\xi_0}{\xi} = \frac{2\kappa K(\kappa) R}{m} .$$

Appendix B. EVALUATION OF THE INTEGRAL $H(\xi_0, r)$ AND THE INTEGRAL $\int_0^{\xi_c} J H d\xi_0$

The integral $H(\xi_0, r)$ that appeared in Eq. (44) has the form

$$H(\xi_0, r) \equiv \int_{\xi_0}^{\xi_c(r)} \int_0^{\xi_0'} \frac{J''}{J' D'} d\xi_0'' d\xi_0' . \quad (B1)$$

Note that ξ_c is generally a function of r . Since $\xi_0 \equiv (2\varepsilon_h)^{1/2}\kappa$, the inner integral can be written as

$$\begin{aligned} \int_0^{\xi_0} J'' d\xi_0'' &= \frac{2\xi_c(r)R}{m} \int_0^{\kappa'} \kappa'' K(\kappa'') d\kappa'' , \\ &= \frac{2\xi_c(r)R}{m} [E(\kappa') + (\kappa'^2 - 1)K(\kappa')] . \end{aligned}$$

Thus,

$$\begin{aligned} H(\xi_0, r) &= \xi_c^2(r) \int_{\kappa}^1 \kappa' d\kappa' , \\ &= \varepsilon_h(r) (1 - \kappa^2) . \end{aligned} \quad (B2)$$

Using this result, the integral $\int_0^{\xi_c(r)} J H d\xi_0$ becomes

$$\begin{aligned} \int_0^{\xi_c(r)} J H d\xi_0 &= \frac{[2\varepsilon_h(r)]^{3/2} R}{m} \int_0^1 \kappa (1 - \kappa^2) K(\kappa) d\kappa , \\ &= \frac{4}{9} \frac{[2\varepsilon_h(r)]^{3/2} R}{m} . \end{aligned} \quad (B3)$$

Appendix C. DERIVATION OF EQ. (80)

Equation (80) is obtained from the ambipolar condition, $\Gamma_e = \Gamma_i$. An expression for this condition can be obtained by equating the right-hand sides of Eqs. (54) and (63). Then, assuming $(1/n)(\partial n/\partial r) = (1/T)(\partial T/\partial r)$ and using Eq. (76) for E_r and Eq. (77) for Ω_E , the ambipolar condition becomes

$$\begin{aligned}
& \frac{A_D l_D^3}{A_D l_e^3} \left[\int_0^\infty dx \frac{x^4 e^{-x}}{(\Phi(\sqrt{x}) - G(\sqrt{x}) + 1)} \left(C - \frac{1}{2} + x \right) \right] \\
&= \int_0^{x_T} dx \frac{x^4 e^{-x}}{(\Phi(\sqrt{x}) - G(\sqrt{x}))} \left(-C - \frac{1}{2} + x \right) \\
&+ \frac{x_T^{9/4}}{(\Phi(\sqrt{x_T}) - G(\sqrt{x_T}))^{3/2}} \int_{x_T}^\infty dx (\Phi(\sqrt{x}) - G(\sqrt{x}))^{1/2} x^{7/4} e^{-x} \left(-C - \frac{1}{2} + x \right),
\end{aligned} \tag{C1}$$

where $x = \varepsilon/T$. Note that $(A_D l_D^3)/(A_D l_e^3) = 1.64 \times 10^{-2}$. Upon performing the integration on the left-hand side of Eq. (C1), this equation becomes

$$\begin{aligned}
0.21C + 0.93 &= \int_0^{x_T} dx \frac{x^4 e^{-x}}{(\Phi(\sqrt{x}) - G(\sqrt{x}))} \left(-C - \frac{1}{2} + x \right) \\
&+ \frac{x_T^{9/4}}{(\Phi(\sqrt{x_T}) - G(\sqrt{x_T}))^{3/2}} \int_{x_T}^\infty dx (\Phi(\sqrt{x}) - G(\sqrt{x}))^{1/2} x^{7/4} e^{-x} \left(-C - \frac{1}{2} + x \right).
\end{aligned} \tag{C2}$$

After rearranging terms, this equation can be written as

$$C = \frac{-D(x_T)/2 + E(x_T) - 0.93}{D(x_T) + 0.21}. \tag{C3}$$

This is Eq. (80) with $D(x_T)$ and $E(x_T)$ given in Sec. V.

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TABLE I. STELLARATOR PARAMETERS

Physical quantity	Reactor	Reference stellarator	ATF ^a	Small ^b Heliac
Peak plasma density (cm^{-3})	6×10^{14}	2×10^{14}	10^{14}	10^{14}
Peak plasma temperature (keV)	10	10	1	1
Toroidal magnetic field (kG)	60	50	18	10
Peak β	13.4	6.4	2.5	8
R(m)	10	10	2.1	1.2
a(m)	1.0	1.0	0.3	0.25
$\epsilon_h(r=a/2)$	0.1	0.1	0.2	0.1
Deuteron thermal velocity ($\text{cm} \cdot \text{sec}^{-1}$)	1.1×10^8	1.1×10^8	3.4×10^7	3.4×10^7
Deuteron 90° deflection time (sec)	2.3×10^{-3}	7.0×10^{-3}	0.44×10^{-3}	0.44×10^{-3}

^a Reference 12.^b Reference 18.

TABLE II. Confinement Properties in Stellarators^a

Confinement Parameters	Reactor	Reference Stellarator	ATF	Small Helias
Particle confinement time τ_p (sec)	1.25	0.35	0.266	0.019
Electron energy confinement time τ_e^e (sec)	0.35	0.10	0.072	0.005
Ion energy confinement time τ_e^i (sec)	0.28	0.13	0.079	0.006
R	0.8	1.3	1.2	1.25
N	1.65	0.30	0.69	0.60
C	1.6	0.5	0.9	0.8
ϵ_T/T	0.71	0.38	0.47	0.44

^a In this table, all results are obtained with parabolic density and temperature profiles at $r = a/2$.

FIGURE CAPTIONS

- Fig. 1 Toroidal coordinate system (r, θ, ζ) .
- Fig. 2 The variation in B , the magnitude of the magnetic field, along a field line.
- Fig. 3 Trapped and untrapped regions in velocity space, in terms of the parallel ($v_{\parallel}$) and perpendicular ($v_{\perp}$) velocity. The trapped region is shaded. The pitch angle θ of the gyro-orbit is defined by $\cos\theta = v_{\parallel}/v$.
- Fig. 4 Projection of a typical collisionless ion orbit on a poloidal cross section. Point O is the center of the magnetic surface and $\Delta \equiv v_B/\Omega_E$.
- Fig. 5 (a). The relative electron particle flux per unit energy, $[x^4 e^{-x}/(\Phi(\sqrt{x}) - G(\sqrt{x}) + 1)](x + 1/2)$, as a function of the normalized energy $x = \varepsilon/T$.
- (b). The relative electron energy flux per unit energy, $[x^5 e^{-x}/(\Phi(\sqrt{x}) - G(\sqrt{x}) + 1)](x + 1/2)$, as a function of x .
- Fig. 6 Plot of the electron and ion particle fluxes Γ versus the normalized ambipolar electric field strength C for the reference stellarator at $r/a = 0.5$.
- Fig. 7 (a). Plot of the machine dependent parameter M [given by Eq. (81)] versus ε_T/T .
- (b). Plot of the normalized ambipolar electric field C [given by Eq. (80)] versus ε_T/T .
- Fig. 8 Plot of the ratio of the ion energy to electron energy confinement time, R [given by Eq. (87)], versus ε_T/T .

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Fig. 1

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Fig. 2

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Fig. 3

Fig. 4

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Fig. 5

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Fig. 6

Fig. 7

Fig. 8

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