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An Analytical Model for a Vertical Buoyant Jet

D. W. Lee

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DEPARTMENT OF ENERGY**

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Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22161
NTIS price codes—Printed Copy: A04; Microfiche A01

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Contract No. W-7405-eng-26

Energy Division

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D. W. Lee

Date Published - October 1980

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AN ANALYTICAL MODEL FOR A VERTICAL BUOYANT JET

D. W. Lee

ABSTRACT

An analytical model for a round and two-dimensional turbulent buoyant jet which is discharged vertically into a stagnant ambient is developed. The buoyant jet is considered to have three separate zone models which are matched to form a complete solution. The velocity field is determined for the entire jet and plume regions by the use of an eddy viscosity which varies along the axis of the jet. The centerline decay of buoyancy is determined throughout and the results are compared to existing numerical codes. The model is applied to the disposal of carbon dioxide enriched seawater. The results can be used to provide design information for minimizing or maximizing the dilution of a discharge by the receiving environment.

1. INTRODUCTION

Turbulent buoyant jets generated by discharges from submerged outfalls, smokestacks, and cooling towers have become an important concern in society's effort to protect the environment while optimizing the performance of industrial processes. Historically, the investigation of the physical behavior of buoyant jets has focused on predicting the characteristics of the plume in the receiving environment using either numerical models or physical models for a specified design (Dunn, 1975; Davis, 1978). While predictive efforts have successfully analyzed an impressive amount of data describing the spatial and temporal characteristics of the receiving environment, they have not been successful in providing a parametric description of a buoyant jet's behavior for the analysis and optimization of a design.

Buoyant jets, also referred to as forced plumes, often are described as having two zones of flow identified as a zone of flow establishment and a zone of established flow. Within the zone of flow establishment, the profiles of velocity, concentration, and temperature develop from a "top hat" shape at the discharge to a "bell-shaped" distribution. Within the zone of established flow, the buoyant jet develops from a jet dominated by the initial momentum from the discharge to a plume dominated by the jet's buoyancy.

The zone of flow establishment has been studied experimentally and numerically (Albertson, 1950; Hirst, 1971b). In a classical experimental study, Albertson et al. (1950) determined the length of the zone of flow establishment for round and two-dimensional pure momentum jets discharged vertically into a stagnant environment. No experimental data is available that quantifies the effect that buoyancy has on the size of the zone of flow establishment. Hirst (1971b) developed an integral numerical model of the zone of flow establishment for a round buoyant jet discharged into several different types of receiving environments. The results demonstrated that for stagnant environments, buoyancy reduced the size of the zone of flow establishment significantly for densimetric Froude numbers less than 30 and approached Albertson's results for large densimetric Froude numbers. Theoretical studies of the zone of flow

establishment have been limited providing only a general characterization of the conditions at the end of the zone (Fan, 1967).

The zone of established flow has been studied extensively by the use of experiments, numerical models, and physical models. Numerical models have been developed using the concept of entrainment introduced by Morton, Taylor, and Turner (1956) and Priestley, and Ball (1955) for closure of the solution. A review of the current status of modeling the zone of established flow has been prepared by Davis and Shirazi (1978). Experiments that have measured transport properties in the zone of established flow are extensive; however, most measurements have been shown to correlate closely with the early work performed by Rouse, Yih, and Humphreys (1952). Physical models of discharges have also been extensively studied for the assessment of the physical impacts of power plant discharges. The results of these efforts have led to many of the innovations in the design of submerged discharges. In spite of the extensive effort to determine the behavior of the zone of established flow, theoretical investigations of buoyant jets have been of limited success. However, theoretical studies of momentum jets and buoyant plumes have been successfully performed (Görtler, 1942; Yih, 1977).

The purpose of this investigation is to utilize the past investigations of the zone of flow establishment, momentum jets, and buoyant plumes and derive a theoretical model for a vertical buoyant jet that can be used for the design and analysis of submerged and atmospheric discharges. The works of Albertson et al. (1950), Fan (1967), Görtler (1942), and Yih (1977) are used to determine a three-zone, matched solution of the concentration and velocity profiles in a buoyant jet. Both vertical, round buoyant jets and vertical, two-dimensional, buoyant jets are theoretically modeled. The results are compared with existing numerical models and experimental data. Finally, the theory is applied to the analysis of a submerged, dense, buoyant jet generated by a downwardly directed discharge with the specified performance criteria of minimum dilution.

For the round buoyant jet and the two-dimensional buoyant jet the previous work that is utilized is presented to make the theoretical

development clear and to avoid the need for extensive cross-referencing. From the complete model, the variation of the eddy-viscosity within the buoyant jet is determined providing formulations that can be applied to existing numerical codes.

2. VERTICAL ROUND BUOYANT JET

2.1 Model Description and Assumptions

The vertical, round, buoyant jet is modeled by considering the flow to have three distinct zones that are matched at zone boundaries determined by the discharge characteristics. The three zones are illustrated in Fig. 1 and are identified as:

1. Zone of flow establishment
2. Jet zone
3. Plume zone.

The zone of flow establishment begins at the discharge port where the velocity and density profiles are "top-hat" in shape and ends at the downstream location where the velocity profile begins to assume a "bell-shaped" distribution. Within the zone of flow establishment, the momentum flux (J) and the specific-weight flux (G) are conserved. This region is characterized as having velocity and density profiles which are not similar. Therefore, similarity solutions are not applicable. The diffusion of the density profile begins before the velocity profile begins to assume a "bell-shaped" distribution. The length of the zone of flow establishment is determined from the experiments of Albertson et al. (1950) and the centerline characteristics of the density profile are determined by the approximate analysis of Fan (1967). By definition the centerline value of the velocity is the discharge velocity.

The jet zone begins at the end of the zone of flow establishment. The jet zone is the region of the flow where the initial discharge momentum of the buoyant jet is the dominant transport mechanism. At the end of the jet zone, the momentum is generated by the jet's buoyancy. Within the jet zone, the momentum flux and the specific-weight flux are

Fig. 1. Schematic of a buoyant jet.

conserved but the buoyancy forces are not considered in the solution. The jet zone profiles are determined using Görtler's theory (1942) because of its capacity to accurately predict the centerline behavior of jets. The location of the virtual source for the jet is determined from matching the jet-zone solution for the centerline velocity with the centerline velocity at the end of the zone of flow establishment.

The plume zone, which begins at the end of the jet zone, is the region of the flow where the buoyancy of the discharge is the significant mechanism of transport. Within the plume zone, the momentum flux and the specific-weight flux are conserved and the buoyancy forces are incor-

porated into the solution by the use of Yih's theory (1977). Yih's theory uses the same assumptions as Görtler's jet theory, and consequently the plume zone can be matched easily to the jet zone by requiring the centerline values of velocity and specific-weight to be equal at the matching location. This matching procedure locates the virtual source for the plume.

Several assumptions about the nature of the receiving environment and the buoyant jet need to be made before a theoretical model can be formulated. The assumptions limit the generality of the model, but they provide for a closed-form solution that can be used for design and analysis. The following assumptions are made and applied to all zones of the buoyant jet.

1. The fluid is incompressible.
2. The ambient fluid has a constant density.
3. Variations of fluid density are small throughout the flow field compared to the ambient density. Hence, the Boussinesq approximation is valid.
4. The flow field is steady-state.
5. The buoyant jet is turbulent and the mixing can be described by an eddy-viscosity concept.
6. Pressure is hydrostatic throughout the flow field.
7. Similarity profiles for velocity and specific-weight exist.
8. The eddy viscosity is constant for any cross section of the buoyant jet.
9. The Schmidt number is constant throughout the flow field.
10. The receiving fluid is initially at rest.
11. Longitudinal turbulent transport is small compared to longitudinal convective transport.

These assumptions are the same as those commonly used in numerical models except for the assumption of a constant eddy viscosity in the radial direction. This assumption is pivotal to Görtler's theory and has been confirmed by the measurements of Laufer (1953) to be valid in the core of turbulent flow in a pipe. Therefore, the simpler theory that results from this assumption can be applied to the core of a buoyant jet

with the understanding that near the edges of the buoyant jet the predictions will be less accurate.

In the next section the theories to be applied in each of the three zones are presented in a consistent notation to clarify the model formulation that follows. Once formulated, the complete model is compared to existing numerical models and experimental data.

2.2 Zone Models

2.2.1 Zone of Flow Establishment

Albertson, Da1, Jensen, and Rouse (1950) have conducted experimental studies within the zone of flow establishment of a momentum jet which determined the length of the zone of flow establishment to be 6.2 discharge diameters. Numerical modeling by Hirst (1971b) examined the zone of flow establishment for a round buoyant jet and compared his predictions with the graphical methods of Abraham (1963). For vertical jets with a large densimetric Froude number, the results approached Albertson's measurements, and as the densimetric Froude number decreased the effect of buoyancy reduced the length of the zone of flow establishment. For a large class of buoyant jets, the densimetric Froude number is large, such that Albertson's measurements are valid. Furthermore, no experimental data exists for the zone of flow establishment of a buoyant jet to verify the numerical predictions. Since the error associated with using Albertson's fixed value for the length of the zone of flow establishment for any densimetric Froude number is small, the results of Albertson are used with the understanding that the length of the zone of flow establishment may be smaller for small densimetric Froude numbers.

In addition to an expression for the length of the zone of flow establishment, an understanding of the centerline value of the specific weight at the end of the zone in terms of the specific weight at the discharge port is needed. Since the flow within the zone of flow establishment is not self-similar, only an approximate analysis is possible. Fan (1967) analyzed this problem by considering the specific-weight flux defined as

$$G = - \int_A w \Delta \gamma \, dA ,$$

where A is the cross-sectional area of the buoyant jet, w is the velocity in the vertical direction, and $\Delta\gamma$ is the difference in the specific weight (γ) in the plume and the specific weight of the receiving environment (γ_o). The profile of w and $\Delta\gamma$ is considered to be "top-hat" at the discharge port. Hence, at the discharge port,

$$G_o = - \frac{\pi}{4} D_o^2 U_o \Delta\gamma_o ,$$

where D_o is the discharge port diameter, U_o is the discharge velocity, $\Delta\gamma_o$ is the initial excess of specific-weight. At the end of the zone of flow establishment, Fan approximated the velocity and specific-weight profiles as having a Gaussian distribution. Hence, at the end of the zone,

$$G_1 = - \frac{S^2}{1+S^2} b^2 \pi U_o \Delta\gamma_1 ,$$

where S is the turbulent Schmidt number and b is the characteristic width of the buoyant jet. Additionally, the momentum flux (J) defined as

$$J = \int_A \rho w^2 dA ,$$

where ρ is the density of the receiving environment, can be calculated at the beginning and end of the zone of flow establishment to be

$$J_o = \frac{\pi}{4} \rho D_o^2 U_o^2 ,$$

$$J_1 = \frac{\pi}{2} \rho b^2 U_o^2 .$$

The momentum flux and the specific-weight flux are conserved within the zone of flow establishment. Equating the fluxes and simplifying the resulting expression for the centerline value of the difference in specific weights gives

$$\Delta\gamma_1 = \frac{1+S^2}{2S^2} \Delta\gamma_o .$$

Using $S = 1.10$ (a value determined from the plume zone described in Sect. 2.2.3) gives

$$\Delta\gamma_1 = 0.91 \Delta\gamma_0 .$$

Although this result is only approximate because the profiles of velocity and specific weight are not exactly Gaussian, the approximate expression for the centerline specific weight provides a means for evaluating the dilution in the zone of flow establishment.

In summary, the model to be applied to the round buoyant jet defines the length of the zone of flow establishment (z_1) as

$$z_1 = 6.2 D_0$$

and the "bell-shaped" profiles of velocity and specific weight to have the maximum values on the centerlines,

$$w_1 = U_0$$

$$\Delta\gamma_1 = 0.91 \Delta\gamma_0 .$$

2.2.2 Jet Zone

The theory of round turbulent jets developed by Schlichting (1968), which was based on the work of Görtler, serves as the basis of the buoyant jet model. By using an eddy viscosity that is constant for the cross-section, a solution to the equation of motion can be obtained which is very accurate near the core of the jet, where the eddy viscosity is constant, but overpredicts the velocities near the edges of the jet, where the eddy viscosity varies drastically. This simpler theory is advantageous for its ease of application and its inherent superiority to Tollmein's theory (1926) which predicts a cusp in the velocity profile at the jet's centerline.

Using the assumptions in Sect. 2.1, the jet model is generated by a point source located at the origin. The equation of motion for a pure momentum jet in cylindrical coordinates is

$$w \frac{\partial w}{\partial z} + v \frac{\partial w}{\partial r} = \frac{\epsilon}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) ,$$

where r, z are the radial and vertical coordinates with v, w the velocities in the r, z directions, respectively, and ϵ is the eddy viscosity which can be a function of z only. The equation of continuity is

$$\frac{\partial}{\partial z} (rw) + \frac{\partial}{\partial r} (rv) = 0 .$$

A solution is permitted by the use of the Stoke's stream function defined as

$$w = \frac{1}{r} \Psi_r , \quad v = - \frac{1}{r} \Psi_z$$

The boundary conditions for the jet are:

$$\frac{\partial w}{\partial r} = 0 = v , \text{ at } r = 0 ,$$

$$\Psi \text{ is finite, at } r = \infty .$$

A dimensional analysis of the turbulent eddy viscosity gives

$$\epsilon = \lambda \left(\frac{J}{\rho} \right)^{1/2} ,$$

where λ is an experimental constant. A similarity solution exists if the following transformation is used,

$$\Psi = - z f(\eta) + z \eta f'(\eta) ,$$

where

$$\eta = \frac{r}{z} .$$

The velocity components become as a result

$$w = \frac{1}{z} f'' ,$$

$$v = \frac{1}{r} f - \frac{1}{z} f' + \frac{\eta}{z} f'' ,$$

where the prime denotes differentiation with respect to η . Substituting the velocities in the equation of motion gives the governing equation,

$$\left(f''f''\right)' + \varepsilon \left(f'''' + \frac{1}{\eta} f'''\right) - \frac{1}{\eta} f''' f = 0 .$$

The boundary conditions become

$$f'''(0) = 0 ,$$

$$f(0) = f'(0) = 0 ,$$

$$\eta f'(\eta) - f(\eta) \text{ finite as } \eta \rightarrow \infty .$$

Using a solution of the form

$$f = B\eta \tan^{-1} A\eta ,$$

where A, B are constants gives

$$B = 4\lambda \left(\frac{J}{\rho}\right)^{1/2} ,$$

$$\psi = 4\lambda \left(\frac{J}{\rho}\right)^{1/2} z\eta^2 A \left(1 + A\eta^2\right)^{-1} ,$$

$$w = \frac{1}{z} 8\lambda \left(\frac{J}{\rho}\right)^{1/2} A \left(1 + A\eta^2\right)^{-2} ,$$

$$v = \frac{1}{z} 4\lambda \left(\frac{J}{\rho}\right)^{1/2} A\eta \left(1 - A\eta^2\right) \left(1 + A\eta^2\right)^{-2} .$$

The values of λ, A are determined from experiments conducted by Reichardt (1942) to be

$$A = 57.6 ,$$

$$\lambda = 0.016 .$$

A similarity solution for the diffusion of the specific-weight profile within the jet zone is not known, but a characterization of the center-line decay of $\Delta\gamma$ within the jet can be identified. The dimensional

quantities of interest in the jet zone are $\Delta\gamma$, G , J , ρ , n , z . Applying Buckingham's pi-theorem gives as a relation

$$-\Delta\gamma = \frac{G}{z} \left(\frac{\rho}{J} \right)^{1/2} \theta(\xi, \eta),$$

where

$$\xi = \left(\frac{G}{\rho} \right)^2 \left(\frac{\rho}{J} \right)^3 \frac{1}{z},$$

subject to the conditions

$$\frac{\partial}{\partial \eta} \theta(\xi, 0) = 0,$$

$$\theta(\xi, \infty) = 0.$$

By expanding the function $\theta(\xi, \eta)$ as a Taylor series about $(\xi_0, 0)$ to the first order, and then applying the boundary condition gives

$$\theta(\xi, \eta) = \theta(\xi_0, 0) + (\xi - \xi_0) \theta_{\xi}(\xi_0, 0) + \dots$$

When the problem of interest is such that $(\xi - \xi_0)$ is small, sufficient accuracy can be preserved by neglecting the second term, such as problems where the length of the jet zone is small or the momentum flux is large as compared to the specific-weight flux.

Consequently, the centerline behavior of the change in specific weight ($\Delta\gamma_{cl}$) becomes

$$-\Delta\gamma_{cl} = \frac{1}{z} G \left(\frac{\rho}{J} \right)^{1/2} \theta(\xi_0, 0).$$

With this formulation, the centerline value of $\Delta\gamma$ needs to be defined at one location only in order to be determinate throughout the jet zone. Although the solution for $\theta(\xi, \eta)$ is not apparent, the distribution can be understood as a "bell-shaped" form which is nearly Gaussian based on experimental investigations (Reichardt, 1942).

In summary, the vertical velocity and centerline decay of the specific weight within the jet zone have been determined as

$$w = \frac{7.41}{z} \left(\frac{J}{\rho} \right)^{1/2} \left(1 + 57.6 \eta^2 \right)^{-2} ,$$

$$-\Delta\gamma_{cl} = \frac{G}{z} \left(\frac{\rho}{J} \right)^{1/2} \theta(\xi_0, 0) .$$

2.2.3 Plume Zone

The theory of round turbulent buoyant plumes was developed by Yih (1977) based on the theory used by Görtler for jets. In the theory, an eddy viscosity that is constant at each cross-section is assumed. The buoyancy force of the plume is expressed as a deficiency in specific weight. A similarity solution to the equation of motion and diffusion of specific weight is determined that can be matched to the jet zone for a complete buoyant-jet solution.

The plume model is generated by a point source located at the origin with the z -axis measured vertically and with the r -axis measured radially. Let w, v denote the velocity components in the z, r directions, respectively. The equation of motion in the z direction using the assumptions in 2.1 is

$$v \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = - \frac{\Delta\gamma}{\gamma} g + \frac{\epsilon}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) ,$$

where γ is the specific weight of the receiving environment, ϵ is the eddy viscosity, g the gravitational acceleration, and $\Delta\gamma$ the difference between the specific weight of the plume and the receiving environment. The equation of continuity is

$$\frac{\partial}{\partial r} (rv) + \frac{\partial}{\partial z} (rw) = 0 .$$

The advective-diffusion equation is

$$w \frac{\partial}{\partial z} \Delta\gamma + v \frac{\partial}{\partial r} \Delta\gamma = \frac{\epsilon}{Sr} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \Delta\gamma \right) ,$$

where S is the turbulent Schmidt number. A solution is permitted by the use of the Stoke's stream function defined as

$$w = \frac{1}{r} \psi_r, \quad v = -\frac{1}{r} \psi_z.$$

The boundary conditions are at $r = 0$,

$$\frac{\partial w}{\partial r} = 0,$$

$$v = 0,$$

$$\frac{\partial}{\partial r} \Delta \gamma = 0,$$

and at $r = \infty$,

$$\Delta \gamma = 0,$$

ψ is finite.

The specific-weight flux is

$$G = -2\pi \int_0^\infty r w \Delta \gamma dr.$$

Using dimensional analysis,

$$\varepsilon = \lambda \left(G z^2 / \rho \right)^{1/3}.$$

where λ is an experimental constant and ρ is the density of the receiving environment. A similarity solution can be achieved by using the following transformations,

$$\begin{aligned} \psi &= 3\lambda \left(G z^5 / \rho \right)^{1/3} f(\eta), \\ -\Delta \gamma &= 3\lambda^2 \left(\rho G^2 / z^5 \right)^{1/3} \theta(\eta), \end{aligned}$$

where $\eta = x/z$. Using these transformations the velocity components are

$$w = \frac{3\lambda}{\eta} \left(G/\rho z \right)^{1/3} f' ,$$

$$v = \lambda \left(G/\rho z \right)^{1/3} \left[3 f' - \frac{5}{\eta} f \right] .$$

Substituting these expressions into the equation of motion and the advective-diffusion equation gives

$$\left(1 - 5 f \right) \left(\frac{f'}{\eta} \right)' - \frac{(f')^2}{\eta} = f'''' + \eta \theta ,$$

$$- 5S \left(f \theta \right)' = \left(\eta \theta' \right)' ,$$

subject to the boundary conditions,

$$f(0) = f'(0) = \theta'(0) = 0 ,$$

$$f(\infty) \text{ is finite, } \theta(\infty) = 0 .$$

Select as a form for the solution,

$$f = B \left[1 - \left(1 + A\eta^2 \right)^{-1} \right] ,$$

$$\theta = \frac{C}{\left(1 + A\eta^2 \right)^m} ,$$

where $m = 5SB/2$, and A, B, C are constants. Substituting the solution forms into the governing equations and applying the boundary conditions gives two solutions, one of which is of interest, namely,

$$m = 3 ,$$

$$B = 12/11 ,$$

$$S = 1.10 ,$$

$$C = \frac{1536}{121} A^2 .$$

To determine the constants, first substitute the variable transformations into the specific-weight flux to get

$$1 = 18\pi\lambda^3 \int_0^\infty f' \theta d\eta .$$

Substituting the solutions for f , θ gives

$$C\lambda^3 = \frac{11}{54\pi} ,$$

$$A^2\lambda^3 = \frac{1331}{82944\pi} .$$

Using the experimental results of Yih (1951, 1956) the values of A , λ , B can be determined such that

$$A = 38.58 ,$$

$$\lambda = 0.0151 ,$$

$$6\lambda AB = 3.8 .$$

The final form for the vertical velocity and difference in specific weight can be summarized as

$$w = \left(\frac{G}{\rho z} \right)^{1/3} \frac{3.80}{(1 + 38.6\eta^2)^2} ,$$

$$-\Delta\gamma = \left(\frac{\rho G^2}{z^5} \right)^{1/3} \frac{12.9}{(1 + 38.6\eta^2)^3} .$$

2.3 Model Formulation

In this section the model for the round buoyant jet is formulated by matching the separate zone models described in Sect. 2.2. In the matching procedure the location for the transition from one zone model to the next is identified and the virtual source for the models of the jet and plume zone are determined. The complete theory is then compared to existing numerical codes and experimental data to illustrate the model's validity.

As a result of the complete analytical model, important information is revealed describing the parametric behavior of the vertical, round, turbulent, buoyant jet.

2.3.1 Zone Matching

2.3.1.1 Zone of flow establishment and jet zone. The transition from the zone of flow establishment to the jet zone occurs when the velocity profile becomes "bell-shaped" and the centerline value of the distribution is the discharge velocity. Referring to the discussion in Sect. 2.2.1, the length of the zone of flow establishment has been determined to be 6.2 discharge diameters. Hence the matching of the two zones occurs at

$$z_1 = 6.2 D_o .$$

The velocity profile at the matching location is expressed as

$$w(\eta) = U_o h(\eta) ,$$

where $h(\eta)$ is an unknown function with the property

$$h(0) = 1 .$$

At the matching location, the centerline value for the velocity of the jet zone must be equal to U_o , hence

$$U_o = \frac{7.41}{z_1 - z_j} \left(\frac{J}{\rho} \right)^{1/2} ,$$

where z_j is the virtual source for the jet. Solving for z_j gives

$$z_j = -0.4 D_o .$$

From the analysis of the zone of flow establishment recall that the centerline value of $\Delta\gamma$ is

$$\Delta\gamma_1 = 0.91 \Delta\gamma_0 .$$

For the jet zone model the centerline value of $\Delta\gamma$ at the matching location is

$$-\Delta\gamma_1 = \frac{G}{z - z_j} \left(\frac{\rho}{J} \right)^{1/2} \theta(\xi_0, 0) .$$

Equating these two expressions and solving for $\theta(\xi_0, 0)$ gives

$$\theta(\xi_0, 0) = 6.78 .$$

In summary, within the jet region defined by $6.2D_0 \leq z \leq z_2$ the vertical velocity and difference in specific weight are

$$w = \frac{7.41}{z + 0.4D_0} \left(\frac{J}{\rho} \right)^{1/2} \left(1 + 57.6\eta^2 \right)^{-2} ,$$

$$-\Delta\gamma_{c1} = \frac{6.78}{z + 0.4D_0} G \left(\frac{\rho}{J} \right)^{1/2} ,$$

where

$$\eta = \frac{r}{z + 0.4D_0} .$$

2.3.1.2 Jet zone to plume zone. The transition from the jet zone to the plume zone is not as apparent as the previous transition because the length of the jet zone is unknown and is a variable depending on the discharge characteristics. The procedure for matching is the same as for the previous matching, namely, requiring continuity of the centerline velocity and specific weight. By imposing these two conditions, the length of the jet zone and the virtual source for the plume model is determined.

Let the matching location be identified as z_2 and the virtual source for the plume be identified as z_p . Matching the jet and plume vertical velocities on the centerline gives

$$1.95 \left(\frac{J}{\rho} \right)^{1/2} \left(\frac{\rho}{G} \right)^{1/3} = \frac{z_2 - z_j}{(z_2 - z_p)^{1/3}},$$

where z_j has been determined to be $-0.4D_0$. Matching the jet and plume centerline values of the difference in specific weight at z_2 gives

$$0.53 \left(\frac{G}{\rho} \right)^{1/3} \left(\frac{\rho}{J} \right)^{1/2} = \frac{z_2 - z_j}{(z_2 - z_p)^{5/3}}.$$

Solving the two conditions for the unknowns gives

$$z_2 = -0.4D_0 + 2.70 \left(\frac{J}{\rho} \right)^{3/4} \left(\frac{\rho}{G} \right)^{1/2},$$

$$z_p = -0.4D_0 + 0.031 \left(\frac{J}{\rho} \right)^{3/4} \left(\frac{\rho}{G} \right)^{1/2}.$$

In summary, the vertical velocity and the difference in specific weight are expressed as

$$w = \left[\frac{G}{\rho(z - z_p)} \right]^{1/3} \frac{3.80}{(1 + 38.6\eta^2)^2},$$

$$-\Delta\gamma = \left[\frac{\rho G^2}{(z - z_p)^5} \right]^{1/3} \frac{12.9}{(1 + 38.6\eta^2)^3},$$

for the plume zone $z_2 \leq z$. With the matching of the three zone models to each other, the complete analytical description of a buoyant jet can

be applied to the analysis and design of vertical discharges into stagnant receiving environments.

2.3.2 Results and Discussion

The complete round buoyant jet model can now be used to provide analytical expressions for the performance of a vertical discharge in a stagnant receiving environment for comparison with existing numerical codes and experimental data. For these purposes, the largest data base exists for thermal discharges in an aqueous system. The analytical model that has been developed, however, is applicable to gaseous discharges into the atmosphere or density enriched discharges into an aqueous environment. The behavior of thermal buoyant jets is generally described by determining the centerline decay of velocity, the centerline dilution, and the jet width. The behavior of thermal buoyant jets can be described in these terms by relating the temperature to the density with the use of the equation of state for the fluid.

To compare the results of the theory with existing numerical codes and experimental data, a reference design is needed. Table 1 is a listing of the reference design that is similar to a thermal diffuser that might be utilized at a power plant in a brackish water environment. Also shown on Table 1 are the fluxes (J, G, Q) calculated for the reference design.

Table 1. Round buoyant jet reference design

Discharge diameter, m	0.5
Discharge velocity, m/s	5.0
Discharge density, kg/m ³	1115
Ambient salinity, ppt	36
Ambient density, kg/m ³	1020
Ambient temperature, °C	20
Densimetric Froude No.	32.3
Momentum flux (J), nt	5007
Specific weight flux (G), nt/s	48.11
Mass flux (Q), kg/s	1001

The centerline velocity decay predicted by the theory was compared with the numerical code of Hirst (1971a). The comparison with existing experimental data can only be made with the work of Reichart (1942) and Yih (1951) for the separate zones because centerline velocity data for buoyant jets has not been collected. Naturally, the theory matches these experimental data because the experimental results were used to obtain the constants in the theory. The results from the comparison with Hirst showed identical predictions. Figure 2 shows the centerline decay of the temperature (the inverse of this curve represents the buoyant jet dilution) as compared to the numerical codes of Hirst (1971a), Koh and Fan (1970). The separate zones included in the theory are identified to emphasize the effect of each zone on the centerline behavior. The slightly higher concentrations predicted by the theory can be attributed to the simplified series approximation for $\theta(\xi, \eta)$.

Fig. 2. Theory compared to numerical codes.

Figure 3 compares the plume width with the widths predicted by Koh and Fan (1970) and Hirst (1971a). The jet width of a buoyant jet has been identified by a variety of definitions which is understandable because the velocity profile of the jet extends to infinity. Consequently, the definition for the jet width that is used is somewhat arbitrary. Hirst specifies the jet half-width as the radial location where the vertical velocity is one-half of the centerline velocity. Koh and Fan specify the jet half-width as the radial location determined by

$$b = \frac{Q}{\sqrt{\pi J}}$$

Finally, a third specification for the jet half-width by Jirka et al. (1976) defines the jet half-width to be the radial location where the vertical velocity is equal to the centerline velocity divided by e or the "e-folding" width. The theory has been used to determine the jet half-width according to Jirka's definition in Figure 3.

Fig. 3. Buoyant jet half-width: comparison of theory with numerical codes.

The theory, numerical codes, and the experimental data display a satisfying level of agreement. An additional result is obtained from comparing the location of the transition from the jet zone to the plume zone with Turner's experiments (1966). In these experiments a salt enriched jet was injected vertically into a stagnant pool, and the maximum height of the rise was measured. Using dimensional reasoning the maximum height of rise was determined as

$$z_m = C \left(\frac{J}{\rho} \right)^{3/4} \left(\frac{G}{\rho} \right)^{-1/2},$$

where C was an experimental constant reported from a line drawn to fit the experimental data to be 1.85. For the experiments, nozzle diameters of 1.4 cm, 0.96 cm, and 0.65 cm were used. The comparison of the data with the Turner formulation and the predictions for z_2 are shown in Fig. 4. The similarity of the theoretical prediction with the data is

Fig. 4. Measured mean heights of the tops of salt jets projected upwards compared with the form suggested by the dimensional arguments. The symbols refer to different nozzle diameters: ●, 1.40 cm; ○, 0.96 cm; +, 0.65 cm.

striking, and the calculation of Turner for $C = 1.85$ is apparently in error. The result gives support to the concept of separating the zone of established flow into a jet zone and a plume zone.

With the theory comparing favorably with existing experimental results and numerical codes, confidence can be attached to the theory for use in diffuser and stack design.

3. VERTICAL TWO-DIMENSIONAL BUOYANT JET

3.1 Model Description and Assumptions

The vertical, two-dimensional, buoyant jet is modeled by considering the flow to have three distinct zones that are matched at zone boundaries determined by the discharge characteristics. The zones and their characteristics are defined in the same manner as for the round buoyant jet. The zone of flow establishment is determined from the experiments of Albertson et al. (1950) and the analysis of Fan (1967) for two-dimensional buoyant jets. The jet zone is determined using Görtler's theoretical solution (1942) which utilizes the same conceptual development as for round jets. The plume zone is determined using Yih's theory (1977) which also determined the two-dimensional plume solution.

The locations of the virtual sources for the jet and the plume are determined, and the matching locations are explicitly determined. The assumptions for the two-dimensional buoyant jet are the same as for the round buoyant jet except for symmetry in the lateral direction instead of the radial direction. Consequently, the two-dimensional buoyant jet model is identical in concept to the round buoyant jet but differs in detail. The notation used for the two-dimensional jet is similar but necessarily the two-dimensional parameters differ in definition.

In the next section the theories to be applied in each of the three zones are presented in a consistent notation to clarify the model formulation that follows. Once formulated the complete model is compared to existing numerical models and data.

3.2 Zone Models

3.2.1 Zone of Flow Establishment

In the experimental investigation of submerged jets by Albertson et al. (1950) two-dimensional jets were shown to have a zone of flow establishment with a length equal to 5.2 discharge widths. Although detailed investigations of the zone of flow establishment for two-dimensional buoyant jets are limited, the behavior of two-dimensional buoyant jets should have characteristics similar to round buoyant jets. Namely, the length of the zone could be expected to decrease as the densimetric Froude number decreases. Since definitive work to describe this effect is lacking and since the degree of error that would be introduced by using Albertson's results is small, these results are used with the understanding that the length of the zone of flow establishment may be reduced for small densimetric Froude numbers.

The centerline value of the specific weight at the end of the zone of flow establishment in terms of the specific weight at the discharge port is required. Since the flow within the zone of flow establishment is not self-similar, only an approximate analysis is possible. Fan (1967) analyzed the two-dimensional buoyant jet by using the same technique as the round buoyant jet. By equating the specific-weight flux and momentum flux at the discharge port with the respective fluxes at the end of the zone of flow establishment, the centerline difference in specific weight can be expressed as

$$\Delta\gamma_1 = \sqrt{\frac{1 + S^2}{2\pi S^2}} \Delta\gamma_0 ,$$

where S is the turbulent Schmidt number, $\Delta\gamma_0$ is the difference in specific weight (γ) of the discharge and the specific weight of the receiving environment, and $\Delta\gamma_1$ is the difference of the centerline specific weight at the end of the zone of flow establishment and the specific weight of the receiving environment. Using $S = 2/3$ (a value determined from the plume zone described in Sect. 3.2.3) gives

$$\Delta\gamma_1 = 0.72 \Delta\gamma_0 .$$

In summary, the model to be applied to the two-dimensional buoyant jet defines the length of the zone of flow establishment (x_1) as

$$x_1 = 5.2 W ,$$

where W is the discharge width. The "bell-shaped" profiles of velocity and specific weight have initial maximum values on the centerlines of $U_0, \Delta\gamma_0$, respectively. At the end of the zone of flow establishment the centerline values are

$$u_1 = U_0 ,$$

$$\Delta\gamma_1 = 0.72\Delta\gamma_0 .$$

3.2.2. Jet Zone

The theory of two-dimensional jets developed by Görtler (1942) serves as the basis for the buoyant jet model. This theory is conceptually identical to the theory applied to the round buoyant jet. The theory is based on the assumption that the eddy viscosity is constant for each cross-section. As a result, an exact solution to the Navier-Stokes equation is obtained which is very accurate near the core of the jet, but overpredicts the velocity near the edges of the jet where the eddy viscosity varies.

Using the assumptions identified in Sect. 2.1, the jet model is generated by a line source located at the origin along the z -axis. The equation of motion for a pure momentum jet in Cartesian coordinates (x, y) with respective velocity components u, v is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \epsilon \frac{\partial^2 u}{\partial y^2} ,$$

where ϵ is the turbulent eddy viscosity which is the function of x . The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .$$

A solution is permitted by the use of the Stoke's stream function defined as

$$u = \psi_y , \quad v = -\psi_x .$$

The boundary conditions for the jet are:

$$\frac{\partial u}{\partial y} = 0 = v , \quad \text{at } x = 0 ,$$

$$\psi \text{ is finite, at } x = \infty .$$

A dimensional analysis of the turbulent eddy viscosity gives

$$\epsilon = \lambda \left(\frac{J}{\rho} \right)^{1/2} \sqrt{x} ,$$

where J is the momentum flux per unit length defined by

$$J = \rho \int_{-\infty}^{\infty} u^2 dy ,$$

with ρ being the density of the receiving environment and λ an experimental constant. A similarity solution exists if the following transformation is made,

$$\psi = \sqrt{x} \quad F(\eta) ,$$

where

$$\eta = \frac{y}{x} .$$

The velocity is now expressed in its components as

$$u = \frac{1}{\sqrt{x}} F' ,$$

$$v = \frac{1}{\sqrt{x}} \left\{ \eta F' - \frac{1}{2} F \right\} .$$

Substituting these forms and the form of ϵ into the equation of motion gives as a governing equation

$$-\frac{1}{4\lambda} \left(\frac{\rho}{J} \right)^{1/2} (F^2)'' = F''''.$$

The boundary conditions become

$$\begin{aligned} F(0) &= 0, \\ F''(0) &= 0, \\ F(\infty) &\text{ is finite.} \end{aligned}$$

Using a solution of the form

$$F = C \tanh \sigma \eta,$$

where C, σ are constants gives as a result

$$\sigma = \frac{C}{4\lambda} \left(\frac{\rho}{J} \right)^{1/2}.$$

To determine C , note that at the end of the zone of flow establishment ($x = x_1$) the centerline velocity is

$$u_1 = \frac{1}{\sqrt{x_1}} F'(0) = U_0,$$

where U_0 is the discharge velocity. Hence,

$$U_0 = \frac{C\sigma}{\sqrt{x_1}}.$$

Solving for C gives

$$C^2 = 4\lambda \left(\frac{J}{\rho} \right)^{1/2} U_0 \sqrt{x_1}.$$

Summarizing the solution,

$$u = U_o \left(\frac{x_1}{x} \right)^{1/2} \text{sech}^2 \sigma \eta ,$$

$$v = U_o \left(\frac{x_1}{x} \right)^{1/2} \left(\eta \text{sech}^2 \sigma \eta - \frac{1}{2\sigma} \tanh \sigma \eta \right) ,$$

where

$$\sigma = \frac{1}{2} \left(\frac{U_o}{\lambda} \right)^{1/2} \left(\frac{\rho x_1}{J} \right)^{1/4} .$$

Reichart's experimental study (1942) of two-dimensional jets determined $\sigma = 7.57$. Using Albertson's (1950) determination for x_1 (Sect. 3.2.1) of 5.2 W gives a value of $\lambda = 0.0097$.

A similarity solution for the diffusion of the specific-weight profile within the jet zone is not known, but a characterization of the centerline decay of $\Delta\gamma$ within the jet can be identified. The dimensional quantities of interest in the jet zone are $\Delta\gamma$, G , J , ρ , y , x . Applying Buckingham's pi-theorem gives as a relation

$$\Delta\gamma = \left(\frac{\rho G^2}{Jx} \right)^{1/2} \theta(\xi, \eta) ,$$

where

$$\xi = \left(\frac{\rho G}{Jx} \right)^{2/3} ,$$

subject to the conditions,

$$\frac{\partial}{\partial \eta} \theta(\xi, 0) = 0 ,$$

$$\theta(\xi, \infty) = 0 .$$

Expanding the functions $\theta(\xi, \eta)$ as a Taylor series about $(\xi_0, 0)$ to the first order, and then applying the boundary conditions, gives

$$\theta(\xi, \eta) = \theta(\xi_0, 0) + (\xi - \xi_0) \theta_{\xi}(\xi_0, 0) + \dots$$

When the problem of interest is such that $(\xi - \xi_0)$ is small, sufficient accuracy can be preserved by neglecting the second term, such as problems where the length of the jet zone is small or the momentum flux is large as compared to the specific-weight flux. Consequently, the centerline decay of specific weight becomes

$$-\Delta\gamma_{cl} = \left(\frac{\rho G^2}{Jx} \right)^{1/2} \theta(\xi_0, 0) .$$

Thus, by knowing the centerline value of $\Delta\gamma$ at the end of the zone of flow establishment, the centerline decay of $\Delta\gamma$ can be determined throughout the jet zone. Although the solution for $\theta(\xi, \eta)$ is not apparent, the distribution can be understood as "bell shaped" from experimental investigations (Reichardt, 1942).

3.2.3 Plume Zone

The theory of two-dimensional turbulent buoyant plumes was developed by Yih (1977) based on Görtler's theory for jets. A similarity solution is determined which utilizes the assumption that the eddy viscosity is constant for each cross-section of the plume.

The plume model is generated by a line source located at the origin with a Cartesian coordinate system (x, y) and with velocity components (u, v) , respectively. The x -axis is vertically oriented and the line source lies along the z -axis. The equation of motion is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \epsilon \frac{\partial^2 u}{\partial y^2} - g \frac{\Delta\gamma}{\gamma} ,$$

where ϵ is the turbulent eddy viscosity, $\Delta\gamma$ the difference in the specific weight of the plume and the receiving environment and g is the acceleration due to gravity. The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .$$

The advective-diffusion equation is

$$u \frac{\partial}{\partial x} \Delta \gamma + v \frac{\partial}{\partial y} \Delta \gamma = \frac{\epsilon}{S} \frac{\partial^2}{\partial y^2} \Delta \gamma ,$$

where S is the turbulent Schmidt number. A solution is permitted by the use of the Stoke's stream function defined as

$$u = \psi_y , \quad v = - \psi_x .$$

The boundary conditions at $y = 0$ are

$$v = 0 ,$$

$$\frac{\partial u}{\partial y} = 0 ,$$

$$\frac{\partial}{\partial y} \Delta \gamma = 0 ,$$

and at $y = \pm\infty$ are

$$\psi \text{ is finite,}$$

$$\Delta \gamma = 0 .$$

The specific-weight flux (G) of the line-source per unit length is

$$G = - \int_{-\infty}^{\infty} u \Delta \gamma \, dy .$$

Using dimensional analysis, the eddy viscosity can be represented as

$$\epsilon = \lambda x \left(\frac{G}{\rho} \right)^{1/3} .$$

A similarity solution is determined by making the transformations

$$\psi = \left(\frac{G}{\rho} \right)^{1/3} x f(\eta) ,$$

$$- \Delta \gamma = \frac{\rho G^2}{x^3} \theta(\eta) .$$

Using these transformations the velocity components become

$$u = \left(\frac{G}{\rho}\right)^{1/3} f'(\eta),$$

$$v = -\left(\frac{G}{\rho}\right)^{1/3} \left(f - \eta f'\right).$$

Substituting into the equation of motion and the advective-diffusion equation gives

$$-ff'' = \lambda f'''' + \theta,$$

$$-S(\theta f)' = \lambda \theta'',$$

subject to the boundary conditions,

$$f(0) = f''(0) = \theta'(0) = 0,$$

$$\theta(\pm\infty) = 0,$$

$$f(\pm\infty) \text{ is finite.}$$

Select as a form for the solution

$$f = A \tanh B\eta,$$

$$\theta = C \operatorname{sech}^m B\eta,$$

where

$$m = \frac{AS}{B\lambda}.$$

Substituting the forms of solution into the governing equations and applying the boundary conditions gives two solutions, one of which is of interest, namely,

$$S = 2/3,$$

$$A = 3B\lambda,$$

$$C = 6B^4\lambda^2.$$

To determine the constants, use the definition of G to give

$$\begin{aligned} AC &= 3/4, \\ 24B^5\lambda^3 &= 1, \end{aligned}$$

which allows an expression for B in terms of λ . In summary

$$\begin{aligned} u &= \left(\frac{G}{\rho}\right)^{1/3} AB \operatorname{sech}^2 B\eta, \\ -\Delta\gamma &= \left(\frac{\rho G^2}{x^3}\right)^{1/3} C \operatorname{sech}^2 B\eta. \end{aligned}$$

Using the experimental data of Rouse et al. (1952) gives as a final form

$$\begin{aligned} u &= 1.99 \left(\frac{G}{\rho}\right)^{1/3} \operatorname{sech}^2 7.05\eta, \\ -\Delta\gamma &= 2.66 \left(\frac{\rho G^2}{x^3}\right)^{1/3} \operatorname{sech}^2 7.05\eta. \end{aligned}$$

3.3 Model Formulation

In this section the model for the two-dimensional buoyant jet is formulated by matching the separate zone models described in Sect. 3.2. As a consequence of the matching procedure, the location for the transition from one zone model to the next is determined and the virtual source for the jet and plume models is identified. The complete theory is then compared to existing numerical codes and experimental data to illustrate the model's validity. As a result of the complete model, a description of the parametric behavior of the vertical, two-dimensional, turbulent, buoyant jet is obtained.

3.3.1 Zone Matching

3.3.1.1 Zone of flow establishment and jet zone. The transition from the zone of flow establishment to the jet zone occurs when the velocity

profile becomes "bell shaped" and the centerline value of the distribution is equal to the discharge velocity. In the development of the model for the jet zone, the effect of the zone of flow establishment is included. Consequently, the virtual source for the jet is the origin of the discharge, unlike the round buoyant jet, and the length of the zone of flow establishment is $5.2 W$.

To determine the centerline decay of $\Delta\gamma$ in the jet zone, one constant needs to be determined. At the end of the zone of flow establishment, the centerline $\Delta\gamma$ is

$$\Delta\gamma_1 = 0.72 \Delta\gamma_0 .$$

For the jet zone model the centerline $\Delta\gamma$ at the matching location is

$$- \Delta\gamma_1 = \left(\frac{\rho G^2}{J x_1} \right) \theta(\xi_0, 0) .$$

Equating these two expressions at $x_1 = 5.2 W$ and solving gives

$$\theta(\xi_0, 0) = 1.64 .$$

In summary, within the jet zone defined by $5.2 W \leq x \leq x_2$ the vertical velocity and centerline difference in specific weight are

$$u = 2.28 \left(\frac{J}{\rho x} \right)^{1/2} \text{sech}^2 7.67\eta ,$$

$$- \Delta\gamma_{cl} = 1.64 \left(\frac{\rho G^2}{J x} \right)^{1/2} .$$

3.3.1.2 Jet zone and plume zone. The transition from the jet zone to the plume zone is not as apparent as the previous transition, because the length of the jet zone is unknown, and the location of the virtual source for the plume is not specified. Both of these unknowns are dependent on the discharge characteristics. The method used for matching is the same as the method applied to the round buoyant jet.

Let the matching location be defined as x_2 and the virtual source for the plume be identified as x_p . Matching the vertical velocities of the jet and the plume on the centerline at x_2 gives

$$x_2 = 1.31 \left(\frac{J^3}{\rho G^2} \right)^{1/3}.$$

Matching the centerline values of the difference in specific weight at x_2 gives

$$x_p = -0.55 \left(\frac{J^3}{\rho G^2} \right)^{1/3}.$$

In summary, the vertical velocity and the difference in specific weight are expressed as

$$u = 1.99 \left(\frac{G}{\rho} \right)^{1/3} \text{sech}^2 7.05\eta ,$$

$$-\Delta\gamma = 2.66 \left(\frac{\rho G^2}{[x - x_p]} \right)^{1/3} \text{sech}^2 7.05\eta .$$

for the plume zone $x_2 \leq x$. With the matching of the three zone models to each other the complete analytical description of a two-dimensional buoyant jet can be applied to the analysis and design of vertical discharges into stagnant receiving environments.

3.3.2 Results and Discussion

The complete two-dimensional buoyant jet model can now be used to provide analytical expressions for the performance of a vertical discharge in a stagnant receiving environment for comparison with existing numerical codes and experimental data. For these purposes the largest data base exists for thermal discharges in an aqueous system. The analytical model that has been developed, however, is applicable to gaseous discharges into the atmosphere or density enriched discharges into an aqueous environment. The behavior of thermal buoyant jets is generally described by determining the centerline decay of velocity, the centerline dilution, and the jet width. The behavior of thermal buoyant jets can be described in these

terms by relating the temperature to the density by the use of the equation of state for the fluid.

Numerical codes and experimental data for two-dimensional buoyant jets are not as extensive as for round buoyant jets. Consequently, the results are somewhat more difficult to compare. The numerical code of Koh and Fan that is used for comparison was prepared for a multiple port diffuser composed of a series of round ports. Once the round jets interact, the solution is computed as if the discharge were a two-dimensional diffuser. For comparison with the theory presented, the spacing between ports was selected so that jet interaction occurred at the end of the zone of flow establishment.

To compare the results of the theory with existing numerical codes, a reference design is needed. Table 2 is a listing of the reference design that is similar to a multiport thermal diffuser after the multiple jets interact. The ambient conditions are similar to a brackish water environment. The fluxes (J, G, Q) have been calculated for the reference design behaving as a two-dimensional buoyant jet. Figure 5 shows the centerline decay of the temperature (the inverse of this curve represents the centerline buoyant jet dilution) as compared to the numerical code of Koh and Fan (1970). The separate zones of the theory have been identified to emphasize the effect of each zone on the centerline behavior. Figure 6 compares the plume width with the widths predicted by Koh and

Table 2. Two-dimensional buoyant jet reference design

Discharge width, m	0.13
Discharge velocity, m/s	1.82
Discharge density, kg/m ³	1015
Salinity, ppt	30
Ambient density, kg/m ³	1020
Ambient temperature, °C	23.6
Densimetric Froude No. (Fr)	23
Momentum Flux (J), nt/m	293
Specific-weight flux (G), nt/s-m	7.73
Mass flux (Q), kg/m-s	161

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Fig. 5. Comparison of theory with numerical code: two-dimensional buoyant jet.

Fig. 6. Plume half-width: comparison to computer model two-dimensional buoyant jet.

Fan (1970). The plume width determined by the theory has been identified as the location where the vertical velocity is equal to the centerline velocity divided by e or the "e-folding" width. The theory and numerical codes display a satisfying level of agreement. With this favorable agreement, confidence can be attached to the theory for use in diffuser and stack design.

4. APPLICATION TO THE DISPOSAL OF CO₂ ENRICHED EFFLUENT IN SEAWATER

4.1 Project Description and Design Criteria

For the past twenty years of monitoring, the concentrations of CO₂ in the atmosphere have been increasing at about 4% per year (Baes, Goeller, Olson, and Rotty, 1977). This increase has been attributed to the increased combustion of fossil fuels. Should the level of growth in the concentration of atmospheric CO₂ continue, unacceptable changes in the global ecology are predicted. A means to allow for the combustion of fossil fuels while reducing the concentration of atmospheric CO₂ is the capture and disposal of CO₂ prior to its introduction into the atmosphere. One proposed technology that is presently being evaluated would collect CO₂ from stationary sources such as power plants and dispose of the CO₂ in the ocean (Baes, Beall, and Marland, 1979). Once CO₂ has been collected and transported to the ocean, the gas would be dissolved in seawater, creating a density-enriched effluent. The resulting effluent plume would have a natural tendency to sink. If the diffuser could be designed to promote the sinking of the effluent for maximum retention within the ocean, then the dissolved CO₂ could react with basic minerals in the ocean and thereby reduce the introduction of CO₂ into the atmosphere. The theory developed in the previous sections can be applied to this problem to provide necessary insight into the behavior of the effluent plume and to give guidance in the design of the discharge structure to enhance performance.

The design of a proposed facility for dissolving CO₂ in seawater is not at hand, but several performance features of such a facility have been determined that provide criteria for design and evaluation. The

increase in density from a gas dissolver can range to a maximum of 6 kg/m^3 . A facility dedicated to the disposal of CO_2 from a 1000 MW(e) coal plant would be required to dissolve 210 kg/sec of CO_2 . If the CO_2 were dissolved in seawater at a concentration of 0.5 moles/kg, 10,000 kg/sec of density-enriched seawater would be discharged. This rate of discharge can be considered as a lowerbound with an upperbound of 380,000 kg/sec being the typical rate of discharge from a 100 MW(e) OTEC plant. For the density increase of the discharge to be enhanced, the depth of discharge should be enhanced. Additionally, the location of the discharge below the thermocline (typically 1000 m) reduces the threat of the facility on aquatic life and increases the potential of the plume reaching the ocean floor. Using these design criteria and the theory that has been developed, the discharge can be designed to enhance the performance of an effluent plume by minimizing dilution and, thus, to increase the capacity of the ocean to retain CO_2 .

4.2 Discharge Design and Performance

The temporal and spatial behavior of an effluent plume in the ocean is a complex process which would require a complete description of the behavior of ocean circulation in the area of interest. A complete description of ocean circulation at any location is beyond the current state of knowledge so that some assumed behavior is required as a basis for analysis. For the disposal of CO_2 to be confined to the smallest area and for the oceans retention time to be maximized, the absence of any ambient currents is most desirable. Additionally, by making the assumption of a stagnant environment, the analytical solution of the behavior of a buoyant jet is possible. The present conceptualization of a facility for the disposal of CO_2 suggests that the discharge would occur below the thermocline where the density gradient is small. Consequently, by assuming the ambient density to be constant with depth, the results will approximate the natural conditions and will provide a more complete description of the buoyant jet's behavior. Using these assumptions, a model for the discharge of CO_2 can be constructed for discharge into an infinitely deep receiving environment.

The design criteria of minimum dilution suggests the densimetric Froude number to be as small as practicable. By maintaining a small Froude number, the jet zone is reduced in size where the eddy viscosity is largest, resulting in the dilution being reduced by minimizing the plume area exposed to entrainment. Except at large distances from the discharge port, a single round buoyant jet best accomplishes this objective. For multiple port discharges, the merging of adjacent round plumes at large distances from the discharge port would result in the best means to minimize dilution.

Using the theory developed in Sect. 2, the sensitivity of the center-line dilution to changes in the discharge density was examined. Dilution of the discharge is increased with increasing discharge density. However, the dilution of the discharge is much more sensitive to comparable changes in the discharge rate. Consequently, to minimize dilution, the maximum discharge density with the minimum discharge rate is most desirable.

Based on the sensitivity study, an illustrative design is presented. For discharge conditions a rate of 50,000 kg/sec with a density increase of 6 kg/m³ was selected arbitrarily as typical conditions.

With a discharge velocity arbitrarily selected to be 2.0 m/s, continuity gives the discharge diameter as 5.55 meters. The densimetric Froude number can be calculated as 3.89 which is a low value as compared to typical thermal diffusers designed for maximum dilution. The arbitrarily selected variables are those which could be adjusted to improve facility performance. The design parameters are summarized in Table 3.

Table 3. CO₂ disposal-buoyant jet illustrative design

Discharge diameter, m	5.55
Discharge velocity, m/s	2.0
Discharge density, kg/m ³	1033
Ambient density, kg/m ³	1027
Densimetric Froude No. (Fr)	3.89
Momentum Flux (J), nt	99,380
Specific-weight flux (G), nt/s	2845
Mass flux (Q), kg/s	50,000

Using the theory developed in Sect. 2, the velocity profiles and concentration profiles are presented in Figs. 7 and 8 respectively. The plume width, centerline velocity, and the centerline difference in specific weight are presented in Fig. 9. The physical characteristics of the discharge revealed by the theory are summarized on Table 4. The

Fig. 7. Vertical component of velocity-isovels: illustrative example.

Fig. 8. Dissolved CO_2 isopleths: illustrative example.

time of travel for particles not on the centerline would be greater than those located on the centerline. While the design presented was not prepared as definitive, it does show all of the features which could be expected for an actual configuration.

Fig. 9. Plume characteristics: illustrative design.

Table 4. Illustrative design physical characteristics

Length of zone of flow establishment, m	34.4
Transition from jet solution to plume solution, m	47.2
Virtual source for jet, m	-2.22
Virtual source for plume, m	-1.65
Time of travel — centerline values	
Zone of flow establishment, s	68.8
Jet, s	7.1
Plume ($Z_\infty = 900$ m), s	1194.9
Total time, min	(21.2)

4.3 Discussion and Conclusions

The design of buoyant jets can now be performed using a theoretical model to provide needed information on the physical behavior of the plume. Using the criteria of minimal dilution, design variables were identified that provide the desired results. The model used for the design, however, has several inherent assumptions which need to be reviewed in terms of their effect on the results.

The assumptions which are most likely to be violated in an ocean environment are the absence of stratification and the absence of any currents in the receiving waters. Any density stratification would increase the time of travel and potentially prevent the discharge plume from reaching the ocean floor. The plume width would expand as the stratification increases. Currents would act to increase dilution and to cause the discharge plume to bend. If the currents are sufficiently strong, the plume would be prevented from reaching the ocean floor. The effect of transverse currents on plume behavior is the subject of current research (Yih, 1979), and future developments can be expected to provide a more complete description of their effects.

The effect of the ocean floor has been neglected in the design. Although the ocean floor could be introduced into the theory by the use of the method of images, such added effort would only demonstrate the horizontal motion of the plume as it neared the ocean floor. This effect, if included, would have no influence on the selection of design variables.

The theory which has been developed provides valuable information useful for design and results which are validated by existing computer codes and experimental data. With an improved understanding of the interaction of adjacent buoyant jets from multiple port discharges, a similar theory could be developed for multiple port discharges. Multiple port discharges could then be developed to meet specified design objectives with a more complete understanding of their physical behavior.

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