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**SUPERDAN: Computer Programs for
Calculating the Dancoff Factor of Spheres,
Cylinders, and Slabs**

J. R. Knight

Prepared for the U.S. Nuclear Regulatory Commission
Office of Nuclear Regulatory Research
Under Interagency Agreement DOE 40-550-75

Contract No. W-7405 eng 26

COMPUTER SCIENCES DIVISION

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Date Published - March 1978

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UNION CARBIDE CORPORATION, NUCLEAR DIVISION
operating the
Oak Ridge Gaseous Diffusion Plant . Oak Ridge National Laboratory
Oak Ridge Y-12 Plant . Paducah Gaseous Diffusion Plant
for the
DEPARTMENT OF ENERGY

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ABSTRACT

Resonance integral calculations for a reactor lattice require the use of the Dancoff factor. In the past, tables and approximate expressions have been used to obtain this value. In order to reduce the uncertainty in the use of approximations and to avoid the inconvenience of tables, a series of computer programs has been written to calculate the Dancoff factor.

Three computer programs have been prepared, one each for infinitely long cylinders, spheres, and infinite slabs. The program for cylinders includes corrections for the effect of partial shadowing of a rod by adjacent rods and for the effect of clad surrounding the rods. The effect of clad is also considered in the slab program. Although these corrections can be applied to sphere arrays, they have not been included in the sphere program. It was felt that the infrequent use of spherical arrays did not justify the additional time and effort to include these corrections.

INTRODUCTION

In a lattice of rods, the presence of nearby units reduces the flux at a given resonance energy which is incident on a given rod. The Dancoff factor is used to determine this flux reduction in the resonance integral calculations. In the past, tables of the Dancoff factor published in ANL-5800² have been used extensively. In addition, several approximate expressions for the Dancoff factor, such as the one given by Sauer,³ have been used. In order to reduce the uncertainty in the use of approximations and to avoid the inconvenience of tables, a series of computer programs has been written to calculate the Dancoff factor.

Three computer programs have been prepared, one each for infinitely long cylinders, spheres, and infinite slabs. The program for cylinders includes corrections for the effect of partial shadowing of a rod by adjacent rods and for the effect of clad surrounding the rods. The effect of clad is also considered in the slab program. Although these corrections can be applied to sphere arrays, they have not been included in the sphere program. It was felt that the infrequent use of spherical arrays did not justify the additional time and effort to include these corrections.

I. CYLINDRICAL LATTICES

For the case of two cylindrical rods, Dancoff and Ginsburg¹ give the expression:

$$C = \frac{\rho}{\pi^2} \iint d\theta d\theta' \frac{\vec{x} \cdot \vec{x}'}{|\vec{\rho} - \vec{\rho}'|^3} \text{Ki}_3 \left(\frac{|\vec{\rho} - \vec{\rho}'|}{\lambda} \right), \quad (1)$$

where:

C = Dancoff factor,

ρ = rod radius,

θ = azimuth angle in cylinder 1,

θ' = azimuth angle in cylinder 2,

$x = d \cos \theta - \rho - \rho \cos (\theta + \theta')^*$,

$x' = d \cos \theta' - \rho - \rho \cos (\theta + \theta')$,

$|\vec{\rho} - \vec{\rho}'| = \sqrt{d^2 - 2d\rho (\cos \theta + \cos \theta') + 2\rho^2 [1 + \cos (\theta + \theta')]} ,$

d = pitch,

λ = moderator mean free path,

$\text{Ki}_3(x)$ = Bickley function⁴.

Figure 1 shows the cross section of the two cylinders projected on a plane which is perpendicular to the axis of the cylinders. The limits of integration cover all paths between the circles passing through moderator only. The limiting values of θ are from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The limiting values of

*This expression is erroneously given as $x = d \cos \theta' - \rho - \rho \cos (\theta + \theta')$ in Reference 1.

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Figure 1. Limits of integration.

θ' for a given θ which satisfy the stipulated conditions are from $-\theta_1'$ to θ_2' . Expressions for θ_1' and θ_2' are developed in Appendix A. Equation (1) then becomes:

$$C = \frac{2\rho}{\pi^2} \int_0^{\pi/2} d\theta \int_{\theta_1'}^{\theta_2'} d\theta' \frac{\mathbf{x}\mathbf{x}'}{|\rho-\rho'|^3} K_3\left(\frac{|\rho-\rho'|}{\lambda}\right). \quad (2)$$

Reference 2 gives values of C from Equation (2) in Table 4-25.

In problems of practical interest, we have an array of cylinders instead of the two considered so far. The total Dancoff correction to apply to a given cylinder is the sum of the values of C obtained for all other cylinders in the array. However, for the moderating materials usually encountered, the value of C decreases rapidly with distance. For practical purposes, only the nearest and next nearest cylinders give significant contributions to the total correction. For sufficiently close packed arrays, each of these cylinders can be partially obscured from a given unit by adjacent cylinders in the array. When this occurs, the

limits of integration for Equation (2) must be modified to exclude the obscured portions from the calculations. The top limit of the integral can be reduced from $\frac{\pi}{2}$ to some $\theta_{\text{Max}} < \frac{\pi}{2}$, and the values of θ_1 and θ_2 can also be modified. The conditions that require these changes and expressions for the corrected limits are given in Appendix B for square pitch arrays and in Appendix C for triangular pitch arrays.

When the cylindrical rods in an array are covered with a clad, the path between rods must pass through the clad as well as the moderator. Since the mean free path of the clad is generally not the same as that of the moderator, that portion of Equation (2) involving the mean free path must be modified. The effects of clad and gaps are discussed in Appendix D.

A computer program to calculate the Dancoff factor of cylindrical rods in arrays has been prepared. It uses FORTRAN IV, double precision, and is designed to operate on IBM-360 or IBM-370 series computers. Equation (2), with appropriate modifications discussed here, is calculated by a double numerical integration over the values of θ and θ' . A Gaussian quadrature scheme is used to perform the integrations. A higher order of quadrature and closer intervals are used for smaller values of θ , where the contributions to the value of C are larger. As θ increases, lower orders and somewhat wider interval spacings are used. This procedure allows the calculations to proceed more rapidly. The accuracy of the integrations was tested by varying the order of quadrature and interval spacing for typical problems and observing no significant change in the results. Values of the $Ki_3(x)$ function were obtained from the rational approximations of Gargantini and Pomentale.⁵ Typical running times on the IBM-360,

Model 91, are two seconds per array for bare rods and five seconds for rods with clad.

II. SPHERES

For two lumps of arbitrary geometry, Dancoff and Ginsburg¹ give the expression:

$$C = \frac{1}{\pi \Sigma} \iint d\Sigma d\Sigma' \cos \alpha \cos \alpha' \frac{e^{-\frac{|\vec{r}-\vec{r}'|}{\lambda}}}{|\vec{r}-\vec{r}'|^2}, \quad (3)$$

where:

C = Dancoff factor,

Σ = surface area of first lump,

Σ' = surface area of second lump,

α = angle between normal to lump 1 and the path between lumps,

α' = angle between normal to lump 2 and the path between lumps,

$|\vec{r}-\vec{r}'|$ = length of a given path between the lumps,

λ = mean free path of the moderator.

Figure 2 shows the cross section of two spheres projected onto a plane passing through the sphere centers. The limits of integration cover all paths between the spheres which pass through moderator only. The element of surface area on sphere 1, $d\Sigma$, is obtained by considering a circular strip passing through the point P. This strip has a radius of $\rho \sin \theta$, a width $\rho d\theta$, and a length $2\pi \rho \sin \theta$. Thus:

$$d\Sigma = 2\pi \rho^2 \sin \theta d\theta.$$

Similarly, the element of surface area on sphere 2 is a circular strip passing through the point P'. This strip has a radius of $\rho \sin (\theta' - \phi)$

Figure 2. Elements of surface area.

and a width of $\rho d\theta'$. For some cases, this strip will be interrupted by the requirement that the path between the spheres must pass through moderator only. If ϕ is the angle around the circular path starting from $\phi = 0$ at P' , an angle $\phi_M \leq \pi$ is the limiting angle reached. The circular strip length is

$$2 \int_0^{\phi_M} \rho \sin (\theta' - \epsilon) d\phi.$$

Thus:

$$d\Sigma' = 2 \int_0^{\phi_M} d\phi \rho^2 \sin (\theta' - \epsilon) d\theta'. \quad (4)$$

The expression for ϵ is given in Appendix A, Equation (A2), and the expression for ϕ_M is given in Appendix E, Equation (E8).

$$\cos \alpha = \frac{x}{|\rho - \rho'|},$$

where x is given in Appendix E, Equation (E7), and shown in Figure E1.

$$\cos \alpha' = \frac{x'}{|\vec{\rho} - \vec{\rho}'|},$$

$$x' = d \cos \theta' - \rho - \rho \cos (\theta + \theta'),$$

$$|\rho - \rho'| = |\mathbf{r} - \mathbf{r}'| = \sqrt{d^2 - 2d\rho (\cos \theta + \cos \theta') + 2\rho^2 [1 + \cos (\theta + \theta')] }.$$

The limits of the θ integration are from 0 to $\frac{\pi}{2}$, and the limits of the θ' integration are from ϵ to θ'_2 , where θ'_2 is the same as for cylinders, Equation (A4).

The integration over values of the angle ϕ may be carried out, as shown in Equation (E9). When all of the terms for the spherical geometry are substituted into Equation (3), we get:

$$C = \frac{\rho^2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \int_{\epsilon}^{\theta'_2} \frac{d\theta' \sin \theta \sin (\theta' - \epsilon)}{|\vec{\rho} - \vec{\rho}'|^4} \{ [d \cos \theta - \rho - \rho \cos (\theta + \epsilon) \cos (\theta' - \epsilon)] \phi_M + \rho \sin (\theta + \epsilon) \sin (\theta' - \epsilon) \sin \phi_M \} [d \cos \theta' - \rho - \rho \cos (\theta + \theta')] \exp \left(\frac{-|\vec{\rho} - \vec{\rho}'|}{\lambda} \right). \quad (5)$$

A computer program, which calculates the Dancoff factors of spheres from Equation (5) has been written. It uses FORTRAN IV, double precision, and is designed to run on IBM-360 or IBM-370 series computers. The double numerical integration over θ and θ' is done by a Gaussian quadrature scheme. A higher order of quadrature and closer intervals are used for smaller values of θ where the contributions to the total value of C are larger. As θ increases, lower orders and somewhat wider interval spacings are used. This procedure allows faster running time on the computer

without loss in accuracy. Typical running times on the IBM-360, Model 91, are one second per value of C .

No attempt has been made to account for the effects of shadowing of spheres in arrays, as was done for cylinders. In addition, the effects of clad and gaps have not been considered. Since spherical arrays are seldom encountered in practice, the extra time and effort did not appear to be justified. For cylindrical arrays, only the nearest and next nearest units were considered. For spherical arrays, the third nearest units are close enough so that they probably should also be included in the total value of C for an array.

III. SLAB ARRAYS

Section 4 of ANL-5800² gives the expression for the Dancoff factor of infinite slabs, as follows:

$$1 - C = 1 - 2E_3 (\Sigma_S d), \quad (6)$$

where:

C = Dancoff factor,

Σ_S = cross section in the moderator,

d = distance between slabs,

$E_3(x)$ = exponential integral.

This equation may be written in the form:

$$C = 2 E_3 \left(\frac{d}{\lambda} \right), \quad (7)$$

where λ = moderator mean free path. The method for computing the values of $E_3(x)$ is given in Appendix F.

If the slabs in a given array are covered with clad, Equation (7) becomes:

$$C = 2 E_3 \left(\frac{d_m}{\lambda} + \frac{d_c}{\lambda_c} \right), \quad (8)$$

where:

d_m = distance through moderator,

d_c = distance through clad,

λ_c = mean free path in clad.

A computer program to calculate the Dancoff factor of slabs from Equation (8) has been completed. It uses FORTRAN IV, double precision, and is designed to operate on IBM-360 or IBM-370 series computers. Typical running time on the IBM-360, Model 91, is a fraction of a second per value of C .

IV. RESULTS

The values of the Dancoff factors determined by the computer programs have been checked by two methods. The first method was a direct comparison between the computed quantities and existing tables, and the second was a comparison between these computed values and those obtained from Monte Carlo calculations. The values determined for two cylinders were compared with Table 4-25 of Reference 2 over a wide range of parameters, and in all cases the agreement was within the quoted accuracy of the table. The effect of shadowing for cylindrical arrays was checked by comparison with results from a modified version of the MORSE-SCG⁶ Monte Carlo computer program. Test problems in both square pitch and triangular pitch arrays gave satisfactory agreement. For two spheres, the agreement with Monte Carlo results was also satisfactory. The slab results were checked by comparison with values of the $E_3(x)$ function given in Table 5.4, Reference 7. The agreement here is also good.

Since no previous results for the Dancoff factors of two spheres have been found, a short table of values is given in Table 1.

TABLE 1. DANCOFF FACTORS FOR SPHERES

$\rho RHO / \lambda$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.00	.07559	.07296	.07047	.06813	.06592	.06383	.06185	.05997	.05820	.05651
2.10	.06650	.06338	.06044	.05769	.05511	.05268	.05039	.04824	.04621	.04429
2.20	.05532	.05186	.04863	.04564	.04284	.04024	.03781	.03555	.03343	.03144
2.30	.04340	.03969	.03628	.03314	.03024	.02757	.02509	.02280	.02068	.01872
2.40	.03840	.03453	.03100	.02779	.02485	.02216	.01970	.01745	.01539	.01349
2.50	.03412	.03014	.02655	.02331	.02038	.01772	.01532	.01314	.01116	.00936
2.60	.03042	.02637	.02275	.01952	.01662	.01403	.01170	.00962	.00774	.00606
2.70	.02719	.02310	.01948	.01624	.01345	.01103	.00887	.00696	.00529	.00384
2.80	.02435	.02024	.01665	.01340	.01064	.00831	.00621	.00434	.00270	.00127
2.90	.02183	.01772	.01417	.01098	.00821	.00587	.00386	.00210	.00057	.00000
3.00	.02959	.02550	.02159	.01798	.01469	.01177	.00922	.00701	.00514	.00361
3.10	.02759	.02352	.01970	.01624	.01305	.01021	.00772	.00551	.00364	.00211
3.20	.02578	.02175	.01806	.01469	.01158	.00881	.00636	.00414	.00227	.00074
3.30	.02416	.02016	.01658	.01331	.01030	.00761	.00521	.00300	.00113	.00000
3.40	.02268	.01874	.01524	.01207	.00915	.00654	.00419	.00200	.00013	.00000
3.50	.02134	.01744	.01402	.01095	.00812	.00559	.00334	.00113	.00000	.00000
3.60	.02012	.01627	.01291	.00994	.00720	.00475	.00250	.00013	.00000	.00000
3.70	.01900	.01521	.01194	.00907	.00642	.00405	.00170	.00000	.00000	.00000
3.80	.01797	.01424	.01105	.00818	.00562	.00325	.00088	.00000	.00000	.00000
3.90	.01703	.01335	.01024	.00737	.00490	.00253	.00016	.00000	.00000	.00000
4.00	.01616	.01254	.00954	.00667	.00430	.00193	.00000	.00000	.00000	.00000
4.10	.01535	.01179	.00884	.00597	.00360	.00123	.00000	.00000	.00000	.00000
4.20	.01461	.01110	.00824	.00537	.00300	.00063	.00000	.00000	.00000	.00000
4.30	.01391	.01047	.00770	.00483	.00246	.00009	.00000	.00000	.00000	.00000
4.40	.01327	.00988	.00717	.00430	.00193	.00000	.00000	.00000	.00000	.00000
4.50	.01267	.00934	.00663	.00376	.00139	.00000	.00000	.00000	.00000	.00000
4.60	.01211	.00884	.00613	.00326	.00089	.00000	.00000	.00000	.00000	.00000
4.70	.01159	.00837	.00566	.00279	.00042	.00000	.00000	.00000	.00000	.00000
4.80	.01110	.00793	.00522	.00235	.00000	.00000	.00000	.00000	.00000	.00000
4.90	.01064	.00753	.00482	.00195	.00000	.00000	.00000	.00000	.00000	.00000
5.00	.01021	.00715	.00444	.00157	.00000	.00000	.00000	.00000	.00000	.00000
5.10	.00981	.00680	.00409	.00120	.00000	.00000	.00000	.00000	.00000	.00000
5.20	.00942	.00647	.00376	.00083	.00000	.00000	.00000	.00000	.00000	.00000
5.30	.00907	.00616	.00345	.00046	.00000	.00000	.00000	.00000	.00000	.00000
5.40	.00873	.00587	.00316	.00009	.00000	.00000	.00000	.00000	.00000	.00000
5.50	.00841	.00560	.00293	.00000	.00000	.00000	.00000	.00000	.00000	.00000
5.60	.00810	.00534	.00267	.00000	.00000	.00000	.00000	.00000	.00000	.00000
5.70	.00782	.00510	.00241	.00000	.00000	.00000	.00000	.00000	.00000	.00000
5.80	.00755	.00487	.00215	.00000	.00000	.00000	.00000	.00000	.00000	.00000
5.90	.00729	.00466	.00198	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.00	.00704	.00446	.00182	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.10	.00681	.00427	.00167	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.20	.00659	.00409	.00152	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.30	.00638	.00392	.00137	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.40	.00618	.00375	.00122	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.50	.00599	.00360	.00107	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.60	.00581	.00346	.00092	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.70	.00563	.00332	.00077	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.80	.00547	.00319	.00062	.00000	.00000	.00000	.00000	.00000	.00000	.00000
6.90	.00531	.00307	.00047	.00000	.00000	.00000	.00000	.00000	.00000	.00000

D = Pitch.

Rho = Sphere radius.

Lambda = Moderator mean free path.

TABLE 1. (cont.)

RH ₀ /LAMBDA 0/RH ₀	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
7.00	.00516	.00295	.00169	.00097	.00055	.00032	.00018	.00010	.00006	.00003
7.10	.00501	.00284	.00161	.00091	.00052	.00029	.00017	.00010	.00005	.00003
7.20	.00487	.00273	.00153	.00086	.00048	.00027	.00015	.00009	.00005	.00003
7.30	.00474	.00263	.00146	.00081	.00045	.00025	.00014	.00008	.00004	.00002
7.40	.00461	.00253	.00139	.00077	.00042	.00023	.00013	.00007	.00004	.00002
7.50	.00448	.00244	.00133	.00072	.00039	.00021	.00012	.00006	.00004	.00002
7.60	.00437	.00235	.00127	.00068	.00037	.00020	.00011	.00006	.00003	.00002
7.70	.00425	.00227	.00121	.00064	.00034	.00018	.00010	.00005	.00003	.00002
7.80	.00414	.00219	.00115	.00061	.00032	.00017	.00009	.00005	.00003	.00001
7.90	.00404	.00211	.00110	.00058	.00030	.00016	.00008	.00004	.00002	.00001
8.00	.00394	.00204	.00105	.00055	.00028	.00015	.00008	.00004	.00002	.00001
8.10	.00384	.00196	.00101	.00052	.00026	.00014	.00007	.00004	.00002	.00001
8.20	.00375	.00190	.00096	.00049	.00025	.00013	.00006	.00003	.00002	.00001
8.30	.00366	.00183	.00092	.00046	.00023	.00012	.00006	.00003	.00002	.00001
8.40	.00357	.00177	.00088	.00044	.00022	.00011	.00005	.00003	.00001	.00001
8.50	.00348	.00171	.00084	.00041	.00020	.00010	.00005	.00002	.00001	.00001
8.60	.00340	.00166	.00081	.00039	.00019	.00009	.00005	.00002	.00001	.00001
8.70	.00333	.00160	.00077	.00037	.00018	.00009	.00004	.00002	.00001	.00000
8.80	.00325	.00155	.00074	.00035	.00017	.00008	.00004	.00002	.00001	.00000
8.90	.00318	.00150	.00071	.00034	.00016	.00008	.00004	.00002	.00001	.00000
9.00	.00311	.00145	.00068	.00032	.00015	.00007	.00003	.00002	.00001	.00000
9.10	.00304	.00141	.00065	.00030	.00014	.00007	.00003	.00001	.00001	.00000
9.20	.00297	.00136	.00062	.00029	.00013	.00006	.00003	.00001	.00001	.00000
9.30	.00291	.00132	.00060	.00027	.00012	.00006	.00003	.00001	.00001	.00000
9.40	.00285	.00128	.00057	.00026	.00012	.00005	.00002	.00001	.00001	.00000
9.50	.00279	.00124	.00055	.00025	.00011	.00005	.00002	.00001	.00000	.00000
9.60	.00273	.00120	.00053	.00023	.00010	.00005	.00002	.00001	.00000	.00000
9.70	.00267	.00116	.00051	.00022	.00010	.00004	.00002	.00001	.00000	.00000
9.80	.00262	.00113	.00049	.00021	.00009	.00004	.00002	.00001	.00000	.00000
9.90	.00256	.00109	.00047	.00020	.00009	.00004	.00002	.00001	.00000	.00000
10.00	.00251	.00106	.00045	.00019	.00008	.00003	.00001	.00001	.00000	.00000
10.10	.00246	.00103	.00043	.00018	.00008	.00003	.00001	.00001	.00000	.00000
10.20	.00241	.00100	.00041	.00017	.00007	.00003	.00001	.00001	.00000	.00000
10.30	.00237	.00097	.00040	.00016	.00007	.00003	.00001	.00000	.00000	.00000
10.40	.00232	.00094	.00038	.00016	.00006	.00003	.00001	.00000	.00000	.00000
10.50	.00228	.00092	.00037	.00015	.00006	.00002	.00001	.00000	.00000	.00000
10.60	.00224	.00089	.00035	.00014	.00006	.00002	.00001	.00000	.00000	.00000
10.70	.00219	.00086	.00034	.00013	.00005	.00002	.00001	.00000	.00000	.00000
10.80	.00215	.00084	.00033	.00013	.00005	.00002	.00001	.00000	.00000	.00000
10.90	.00211	.00082	.00032	.00012	.00005	.00002	.00001	.00000	.00000	.00000

TABLE 1. (cont.)

RM/LAMBDA	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
D/RHC										
2.00	.05451	.05338	.05193	.05055	.04923	.04798	.04678	.04563	.04453	.04348
2.10	.04248	.04077	.03915	.03762	.03616	.03478	.03347	.03223	.03105	.02992
2.20	.03358	.03184	.03021	.02868	.02724	.02588	.02461	.02341	.02228	.02121
2.30	.02650	.02521	.02364	.02218	.02082	.01956	.01838	.01728	.01626	.01530
2.40	.02175	.02016	.01869	.01734	.01609	.01494	.01388	.01291	.01201	.01117
2.50	.01773	.01625	.01489	.01366	.01254	.01152	.01058	.00973	.00895	.00824
2.60	.01454	.01318	.01195	.01084	.00984	.00894	.00813	.00739	.00672	.00612
2.70	.01200	.01075	.00965	.00866	.00777	.00699	.00628	.00565	.00508	.00458
2.80	.00954	.00842	.00742	.00655	.00577	.00509	.00448	.00394	.00347	.00304
2.90	.00828	.00726	.00637	.00560	.00492	.00433	.00381	.00335	.00295	.00260
3.00	.00691	.00600	.00521	.00453	.00394	.00343	.00299	.00260	.00227	.00198
3.10	.00580	.00498	.00428	.00368	.00317	.00273	.00235	.00203	.00175	.00151
3.20	.00487	.00414	.00352	.00300	.00255	.00218	.00185	.00158	.00135	.00115
3.30	.00411	.00346	.00291	.00245	.00206	.00174	.00147	.00124	.00105	.00089
3.40	.00348	.00289	.00241	.00201	.00167	.00140	.00117	.00097	.00081	.00068
3.50	.00295	.00243	.00200	.00165	.00136	.00112	.00093	.00077	.00064	.00053
3.60	.00250	.00204	.00166	.00136	.00111	.00091	.00074	.00061	.00050	.00041
3.70	.00213	.00172	.00135	.00112	.00091	.00073	.00059	.00049	.00039	.00032
3.80	.00182	.00145	.00116	.00093	.00074	.00059	.00048	.00038	.00031	.00025
3.90	.00155	.00123	.00097	.00077	.00061	.00048	.00038	.00030	.00024	.00019
4.00	.00133	.00104	.00081	.00064	.00050	.00039	.00031	.00024	.00019	.00015
4.10	.00114	.00088	.00068	.00053	.00041	.00032	.00025	.00019	.00015	.00012
4.20	.00098	.00075	.00057	.00044	.00034	.00026	.00020	.00015	.00012	.00009
4.30	.00084	.00064	.00048	.00037	.00028	.00021	.00016	.00012	.00009	.00007
4.40	.00072	.00054	.00041	.00031	.00023	.00017	.00013	.00010	.00007	.00006
4.50	.00062	.00046	.00034	.00026	.00019	.00014	.00011	.00008	.00006	.00004
4.60	.00054	.00040	.00029	.00021	.00016	.00012	.00009	.00006	.00005	.00003
4.70	.00046	.00034	.00025	.00018	.00013	.00010	.00007	.00005	.00004	.00003
4.80	.00040	.00029	.00021	.00015	.00011	.00008	.00006	.00004	.00003	.00002
4.90	.00035	.00025	.00018	.00013	.00009	.00006	.00005	.00003	.00002	.00002
5.00	.00030	.00021	.00015	.00011	.00008	.00005	.00004	.00003	.00002	.00001
5.10	.00026	.00018	.00013	.00009	.00006	.00004	.00003	.00002	.00002	.00001
5.20	.00023	.00016	.00011	.00008	.00005	.00004	.00003	.00002	.00001	.00001
5.30	.00020	.00013	.00009	.00006	.00004	.00003	.00002	.00001	.00001	.00001
5.40	.00017	.00012	.00008	.00005	.00004	.00002	.00002	.00001	.00001	.00001

TABLE 1. (cont.)

PM1/LAMBDA D/RHO	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
2.00	.04248	.C4152	.C4059	.C3971	.03886	.03804	.03725	.03650	.03577	.03507
2.10	.02885	.C2783	.02686	.C2593	.02504	.02420	.02339	.02262	.02188	.02117
2.20	.02021	.C1926	.C1936	.01752	.01672	.01596	.01524	.01456	.01391	.01330
2.30	.01441	.C1357	.C1279	.C1206	.01138	.01074	.01014	.00957	.00904	.00855
2.40	.01040	.C0969	.C0903	.C0842	.00785	.00733	.00684	.00639	.00597	.00558
2.50	.00758	.C0699	.C0644	.00594	.00548	.00505	.00467	.00431	.00398	.00368
2.60	.00557	.C0508	.00463	.00422	.00385	.00352	.00321	.00294	.00268	.00245
2.70	.00413	.C0372	.C0335	.C0303	.00273	.00247	.00223	.00201	.00182	.00165
2.80	.00307	.C0274	.C0244	.C0218	.00195	.00174	.00156	.00139	.00124	.00111
2.90	.00230	.C0203	.C0179	.C0158	.00140	.00123	.00109	.00097	.00086	.00076
3.00	.00173	.C0151	.C0132	.C0115	.00101	.00088	.00077	.00067	.00059	.00052
3.10	.00130	.C0112	.00097	.00084	.00073	.00063	.00055	.00047	.00041	.00036
3.20	.00095	.C0084	.C0072	.C0062	.00053	.00045	.00039	.00033	.00029	.00024
3.30	.00075	.C0063	.C0054	.C0045	.00038	.00033	.00028	.00023	.00020	.00017
3.40	.00057	.C0048	.C0040	.00034	.00028	.00024	.00020	.00017	.00014	.00012
3.50	.00044	.C0036	.C0030	.C0025	.00021	.00017	.00014	.00012	.00010	.00008
3.60	.00033	.C0027	.C0022	.00018	.00015	.00012	.00010	.00008	.00007	.00006
3.70	.00026	.C0021	.C0017	.C0014	.00011	.00009	.00007	.00006	.00005	.00004
3.80	.C0020	.C0016	.00013	.C0010	.00008	.00007	.00005	.00004	.00003	.00003
3.90	.C0015	.C0012	.C0010	.C0008	.00006	.00005	.00004	.00003	.00002	.00002
4.00	.C0012	.C0009	.00007	.C0006	.00005	.00004	.00003	.00002	.00002	.00001
4.10	.00009	.C0007	.00006	.C0004	.00003	.00003	.00002	.00002	.00001	.00001
4.20	.00007	.C0005	.C0004	.C0003	.00002	.00002	.00001	.00001	.00001	.00001
4.30	.00005	.C0004	.C0003	.C0002	.00002	.00001	.00001	.00001	.00001	.00000
4.40	.00004	.C0003	.C0002	.00002	.00001	.00001	.00001	.00001	.00000	.00000
4.50	.00003	.C0002	.00002	.C0001	.00001	.00001	.00001	.00001	.00000	.00000
4.60	.00003	.C0002	.00001	.C0001	.00001	.00001	.00001	.00001	.00000	.00000
4.70	.00002	.C0001	.00001	.C0001	.00001	.00001	.00001	.00001	.00000	.00000
4.80	.00002	.C0001	.00001	.C0001	.00001	.00001	.00001	.00001	.00000	.00000
4.90	.00001	.C0001	.C0001	.C0000	.00000	.00000	.00000	.00000	.00000	.00000

ACKNOWLEDGMENTS

The author wishes to acknowledge the generous assistance of L. M. Petrie in preparing and checking these computer programs. He made the necessary modifications to the MORSE-SGC⁶ Monte Carlo program and performed the calculations used to verify the correctness of the Dancoff factors generated. He also supplied the derivation of the expression for $\cos \alpha$ for spheres given in Appendix E. R. M. Westfall offered many helpful suggestions and supplied the routines used to calculate the K_{i3} function.⁵

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APPENDIXES

APPENDIX A
CYLINDER JANCOFF LIMITS OF INTEGRATION

Consider two cylinders of radius ρ and having a center-to-center distance (pitch) d . The angle between a given radius and the line connecting the centers is θ in cylinder 1 and θ' in cylinder 2. For a given angle θ in cylinder 1, the limits of integration in cylinder 2 are θ'_1 and θ'_2 . These limits are the largest angle below and above the line of centers for which the path of neutrons between the cylinder passes only through moderator. This condition is met when the neutron path is tangent to either cylinder 1 or cylinder 2. For small values of θ , this path is tangent to cylinder 2; and for large values of θ , it is tangent to cylinder 1. As a limiting value, θ_L , the path is tangent to both cylinders. The expression for θ_L may be obtained from Figure A1.

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Figure A1. Limiting angle θ_L .

Due to symmetry, the distance from the intersection of the neutron path with the line of centers to the center of either cylinder is $\frac{d}{2}$.

Thus:

$$\cos \theta_L = \frac{2\rho}{d} . \quad (A1)$$

The case for θ less than θ_L is considered in Figure A2.

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Figure A2. θ less than θ_L .

The distance h from a point at angle θ on cylinder 1 to the center of cylinder 2 is:

$$h = \sqrt{(d - \rho \cos \theta)^2 + (\rho \sin \theta)^2},$$

$$h = \sqrt{d^2 - 2 d \rho \cos \theta + \rho^2} . \quad (A2)$$

$$\sin \epsilon = \frac{\rho \sin \theta}{h} , \quad \cos \beta = \frac{\rho}{h} .$$

$$\theta'_1 = \beta - \epsilon , \quad \theta'_2 = \beta + \epsilon .$$

Thus:

$$\theta_1' = \cos^{-1}\left(\frac{\rho}{h}\right) - \sin^{-1}\left(\frac{\rho \sin \theta}{h}\right), \quad (\text{A3})$$

$$\theta_2' = \cos^{-1}\left(\frac{\rho}{h}\right) + \sin^{-1}\left(\frac{\rho \sin \theta}{h}\right). \quad (\text{A4})$$

Next, consider the case for θ greater than θ_L , shown in Figure A3.

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Figure A3. θ greater than θ_L .

$$\theta = \theta_1' + \delta.$$

$$c = (d - \rho \sec \theta) \cos \theta,$$

$$c = d \cos \theta - \rho.$$

$$\cos \delta = \frac{c}{\rho} = \frac{d \cos \theta}{\rho} - 1.$$

$$\theta_1' = \theta - \delta = \theta - \cos^{-1}\left(\frac{d \cos \theta}{\rho} - 1\right). \quad (\text{A5})$$

The expression for θ_2' is the same as for θ less than θ_L . If $\theta = \theta_L$, either expression for θ_1' is correct.

APPENDIX B

CHANGES IN THE LIMITS OF INTEGRATION FOR
SQUARE PITCH ARRAYS OF CYLINDERS DUE TO SHADOWING EFFECTS

Consider a square array of cylinders of radius ρ and having a pitch d , shown in Figure B1.

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Figure B1. Square array.

For purposes of computing the Dancoff factor, only the four nearest cylinders, designated R1, and the four next nearest cylinders, designated R2, are considered. The limits of integration, θ'_1 and θ'_2 , developed in Appendix A are correct for cylinders R1. However, for sufficiently close packed arrays, cylinders R2 will be partially obscured by cylinders R1, and the expressions for θ'_1 and θ'_2 must be modified. In addition, the top limit for the θ integration (above which none of cylinders R2 can be "seen" only through moderator) must be reduced from $\frac{\pi}{2}$ to some $\theta_{\text{Max}} < \frac{\pi}{2}$.

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The limiting spacing for shadowing of θ_2^1 is shown in Figure B2.

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Figure B2. Limiting spacing for shadowing of θ_2^1 .

From Figure B2, it can be seen that:

$$\sqrt{2} \, d = 4 \, \rho,$$

$$\frac{d}{\rho} = \frac{4}{\sqrt{2}}.$$

(B1)

For $\frac{d}{\rho} < \frac{4}{\sqrt{2}}$, and sufficiently large values of θ , θ_2^1 must be modified from the expression given in Appendix A.

Next, consider the limiting spacing for θ_i^1 shadowing, shown in Figure B3.

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Figure B3. Limiting spacing for θ_i^1 shadowing.

From Equation (A1), $\theta_L = \cos^{-1} \left(\frac{2\rho}{d} \right)$.

Thus:

$$\beta = \frac{\pi}{4} + \theta_L = \frac{\pi}{4} + \cos^{-1} \left(\frac{2\rho}{d} \right) \quad . \quad (B2)$$

The value of β can also be obtained from Equations (A2) and (A3) for an angle $\theta = 0$ and the distance of separation $= \sqrt{2} d$ ($\theta < \theta_L$):

$$h = \sqrt{2 d^2 - 2 \sqrt{2} d \rho + \rho^2} ,$$

$$h = \sqrt{2} d - \rho ,$$

$$\beta = \cos^{-1} \left(\frac{\rho}{\sqrt{2} d - \rho} \right) . \quad (B3)$$

For the limiting separations considered here, these two expressions for β must be equal.

$$\frac{\pi}{4} + \cos^{-1} \left(\frac{2\rho}{d} \right) = \cos^{-1} \left(\frac{\rho}{\sqrt{2} d - \rho} \right) .$$

Noting that:

$$\sin^{-1} \sqrt{1 - \frac{4\rho^2}{d^2}} = \cos^{-1} \left(\frac{2\rho}{d} \right) ,$$

and using the expression for the cosine of the sum of two angles leads to:

$$\frac{1}{\sqrt{2}} \left(\frac{2\rho}{d} \right) - \frac{1}{\sqrt{2}} \sqrt{1 - \frac{4\rho^2}{d^2}} = \frac{\rho}{\sqrt{2} d - \rho} ,$$

$$\frac{1}{\sqrt{2}} \left(\frac{2}{d/\rho} \right) - \frac{1}{\sqrt{2}} \sqrt{1 - \frac{4}{(d/\rho)^2}} = \frac{1}{\sqrt{2} d/\rho - 1} .$$

(B4)

This expression was solved for $\frac{d}{\rho}$ by an iterative procedure. From this:

$$\frac{d}{\rho} \approx 2.0558 .$$

For values of $\frac{d}{\rho}$ less than this, and for sufficiently small values of θ , θ_1^1 will be shadowed and must be modified.

The limiting values of θ for θ_2^1 shadowing $\left(\frac{d}{\rho} < \frac{4}{\sqrt{2}}\right)$ are shown in Figure B4.

Figure B4. Limiting values of θ for θ_2^1 shadowing.

From Equation (A1), $\theta_L = \cos^{-1} \left(\frac{2\rho}{d} \right)$.

$$\theta_{\text{Max}} = \frac{\pi}{4} + \theta_L = \frac{\pi}{4} + \cos^{-1} \left(\frac{2\rho}{d} \right). \quad (\text{B5})$$

For values of $\theta \geq \theta_{\text{Max}}$, cylinder R2 is totally obscured from C. Thus, the upper limit of the θ integration is reduced from $\frac{\pi}{2}$ to θ_{Max} . The expression

for θ_{Min} may be obtained from Equation (A5) for an angle θ_{Max} and a distance of $\sqrt{2} d$:

$$\theta_{\text{Min}} = - \left[\theta_{\text{Max}} - \cos^{-1} \left(\frac{\sqrt{2} d}{\rho} \cos \theta_{\text{Max}} - 1 \right) \right]. \quad (\text{B6})$$

For $\theta > \theta_{\text{Min}}$, θ_2' shadowing occurs, and the expression for θ_2' must be modified.

The limiting value of θ for θ_1' shadowing $\left(\frac{d}{\rho} < \sim 2.0558 \right)$ is shown in Figure B5.

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Figure B5. Limiting value of θ for θ_1' shadowing.

From Equation (A1), $\theta_L = \cos^{-1} \frac{2\rho}{d}$,

$$\theta = \frac{\pi}{4} + \theta_L = \frac{\pi}{4} + \cos^{-1} \left(\frac{2\rho}{d} \right).$$

Considering θ_{Lim} as a limiting angle as seen from R2 at an angle θ and at a distance $\sqrt{2} d$, Equation (A5) gives:

$$\theta_{Lim} = \frac{\pi}{4} + \cos^{-1} \left(\frac{2\rho}{d} \right) - \cos^{-1} \left\{ \frac{\sqrt{2} d}{\rho} \cos \left[\frac{\pi}{4} + \cos^{-1} \left(\frac{2\rho}{d} \right) \right] - 1 \right\}. \quad (B7)$$

for $\theta < \theta_{Lim}$, θ_1' shadowing occurs and the expression for it must be modified.

Now we need the correct expressions for θ_1' and θ_2' to satisfy the shadowing conditions outlined here. The case for θ_2' is shown in Figure B6.

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Figure B6. Limiting value, θ_2' .

The angle ψ_2 can be obtained from Equations (A2) and (A4), for an angle $\frac{\pi}{4} - \theta$, and:

$$h_2 = \sqrt{d^2 - 2 d \rho \cos\left(\frac{\pi}{4} - \theta\right) + \rho^2}.$$

Thus we have:

$$\psi_2 = \cos^{-1}\left(\frac{\rho}{h_2}\right) + \sin^{-1}\left[\frac{\rho \sin\left(\frac{\pi}{4} - \theta\right)}{h_2}\right]. \quad (B8)$$

Likewise, χ_2 can be obtained from Equation (A5) for an angle $\frac{\pi}{2} - \psi_2$.

$$\chi_2 = \frac{\pi}{2} - \psi_2 - \cos^{-1}\left[\frac{d}{\rho} \cos\left(\frac{\pi}{2} - \psi_2\right) - 1\right], \quad (B9)$$

and:

$$\theta'_2 = \frac{\pi}{4} + \chi_2. \quad (B10)$$

By using a similar path of reasoning for obtaining the expression for θ'_1 , we can say that for this case the angle ψ_1 is obtained from Equations (A2) and (A4), for an angle $\frac{\pi}{4} + \theta$, and:

$$h_1 = \sqrt{d^2 - 2 d \rho \cos\left(\frac{\pi}{4} + \theta\right) + \rho^2}$$

$$\psi_1 = \cos^{-1}\left(\frac{\rho}{h_1}\right) + \sin^{-1}\left[\frac{\rho \sin\left(\frac{\pi}{4} + \theta\right)}{h_1}\right]. \quad (B11)$$

$$\chi_1 = \frac{\pi}{2} - \psi_1 - \cos^{-1}\left[\frac{d}{\rho} \cos\left(\frac{\pi}{2} - \psi_1\right) - 1\right], \quad (B12)$$

$$\theta'_1 = \frac{\pi}{4} + \chi_1. \quad (B13)$$

To summarize the results, we have:

1. No R2 shadowing for $\frac{d}{\rho} \geq \frac{4}{\sqrt{2}}$.
2. For $\frac{d}{\rho} < \frac{4}{\sqrt{2}}$ and $\theta_{\text{Min}} < \theta < \theta_{\text{Max}}$, $\theta_2' = \frac{\pi}{4} + \chi_2$, and upper limit of θ is θ_{Max} .
3. No R2 shadowing of θ_1' for $\frac{d}{\rho} \geq \sim 2.0558$.
4. For $\frac{d}{\rho} < \sim 2.0558$ and $\theta < \theta_{\text{Lim}}$, $\theta_1' = \frac{\pi}{4} + \chi_1$.

APPENDIX C

CHANGES IN THE LIMITS OF INTEGRATION FOR
TRIANGULAR PITCH ARRAYS OF CYLINDERS DUE TO SHADOWING EFFECTS

Consider a triangular array of cylinders of radius ρ and having a pitch d , shown in Figure C1.

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Figure C1. Triangular array of cylinders.

For the purpose of computing Dancoff factors, only the six nearest cylinders, designated R1, and the six next nearest cylinders, designated R2, are considered. For sufficiently close packed arrays, each cylinder R1 will be partially obscured by the R1 cylinders adjacent to it. Likewise, for spacing below some other limit, cylinders R2 will be partially obscured

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by cylinders R1. (In the extreme case, where the cylinders touch each other, the cylinders R2 are completely obscured.) Under these conditions, the expressions for θ_1' and θ_2' must be modified. In addition, the top limit of the θ integration (above which the cylinder is totally obscured) must be reduced from $\frac{\pi}{2}$ to some $\theta_{\text{Max}} < \frac{\pi}{2}$. The limiting spacing for shadowing of θ_2' for R1 cylinders is considered in Figure C2. No θ_1' shadowing occurs for R1 cylinders.

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Figure C2. Limiting spacing for shadowing of θ_2' .

The altitude of the triangle shown is 2ρ . This altitude is also $\frac{\sqrt{3}}{2} d$, thus:

$$2\rho = \frac{\sqrt{3}}{2} d, \quad \frac{d}{\rho} = \frac{4}{\sqrt{3}}. \quad (C1)$$

For $\frac{d}{\rho} < \frac{4}{\sqrt{3}}$ and sufficiently large values of θ , the θ_2' expression must be modified.

The limiting values of θ for θ_2' shadowing $\left(\frac{d}{c} < \frac{4}{\sqrt{3}}\right)$ for R1 cylinders is shown in Figure C3.

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Figure C3. Limiting values of θ for θ_2' shadowing.

Equation (A1) gives $\theta_L = \cos^{-1}\left(\frac{2\rho}{d}\right)$.

$$\theta_{Max} = \frac{\pi}{3} + \theta_L = \frac{\pi}{3} + \cos^{-1}\left(\frac{2\rho}{d}\right). \quad (C2)$$

For $\theta \geq \theta_{Max}$, the cylinder R1 is totally obscured, and the upper limit of the θ integration must be reduced from $\frac{\pi}{2}$ to θ_{Max} .

If we consider θ_{Min} as a limiting angle as seen from R1 an angle θ_{Max} , Equation (A5) gives:

$$\theta_{\text{Min}} = - \left[\theta_{\text{Max}} - \cos^{-1} \left(\frac{d}{\rho} \cos \theta_{\text{Max}} - 1 \right) \right]. \quad (\text{C3})$$

For $\theta > \theta_{\text{Min}}$, θ'_2 shadowing occurs and the expression for θ'_2 must be modified. This expression may be obtained from Figure C4.

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Figure C4. Limiting value, θ'_2 .

The correct expression for θ'_2 for R1 which satisfies the condition may be obtained as follows. The angle ψ can be obtained from Equations (A2) and (A4), for an angle $\frac{\pi}{3} - \theta$:

$$h = \sqrt{d^2 - 2 d \rho \cos \left(\frac{\pi}{3} - \theta \right) + \rho^2},$$

$$\psi = \cos^{-1} \left(\frac{\rho}{h} \right) + \sin^{-1} \left[\frac{\rho \sin \left(\frac{\pi}{3} - \theta \right)}{h} \right]. \quad (\text{C4})$$

Likewise, χ can be obtained from Equation (A5) for an angle $\frac{\pi}{3} - \psi$:

$$\chi = \frac{\pi}{3} - \psi - \cos^{-1} \left[\frac{d}{\rho} \cos \left(\frac{\pi}{3} - \psi \right) - 1 \right], \quad (C5)$$

and

$$\theta_2' = \frac{\pi}{3} + \chi. \quad (C6)$$

For cylinders R2, the limiting spacing for θ_2' shadowing is shown in Figure C5.

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Figure C5. Limiting spacing for θ_2' shadowing.

From Figure C5, it can be seen that:

$$d = 4\rho,$$

(C7)

$$\frac{d}{\rho} = 4.$$

For $\frac{d}{\rho} < 4$ and sufficiently large values of θ , the θ_2' expression must be modified.

Next, consider the limiting spacing for θ_1' shadowing of R2 cylinders, shown in Figure C6.

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Figure C6. Limiting spacing for θ_1' shadowing.

From Equation (A1), $\theta_L = \cos^{-1}\left(\frac{2\rho}{d}\right)$.

Thus:

$$\beta = \frac{\pi}{6} + \theta_L = \frac{\pi}{6} + \cos^{-1} \left(\frac{2\rho}{d} \right). \quad (C8)$$

The value of β can also be obtained from Equations (A2) and (A3), using an angle $\theta = 0$ and a distance of $\sqrt{3} d$:

$$\begin{aligned} h &= \sqrt{3 d^2 - 2 \sqrt{3} d \rho + \rho^2}, \\ h &= \sqrt{3} d - \rho, \\ \beta &= \cos^{-1} \left(\frac{\rho}{\sqrt{3} d - \rho} \right). \end{aligned} \quad (C9)$$

For the limiting separation considered here, these two expressions for β must be equal.

$$\frac{\pi}{6} + \cos^{-1} \left(\frac{2\rho}{d} \right) = \cos^{-1} \left(\frac{\rho}{\sqrt{3} d - \rho} \right).$$

Noting that:

$$\sin^{-1} \sqrt{1 - \frac{4\rho^2}{d^2}} = \cos^{-1} \left(\frac{2\rho}{d} \right),$$

and using the expression for the cosine of the sum of two angles leads to:

$$\begin{aligned} \frac{\sqrt{3}}{2} \left(\frac{2\rho}{d} \right) - \frac{1}{2} \sqrt{1 - \frac{4\rho^2}{d^2}} &= \frac{\rho}{\sqrt{3} d - \rho}, \\ \frac{\sqrt{3}}{d/\rho} - \frac{1}{2} \sqrt{1 - \frac{4}{(d/\rho)^2}} &= \frac{1}{\sqrt{3} d/\rho - 1}. \end{aligned} \quad (C10)$$

This expression was solved for $\frac{d}{\rho}$ by an iterative procedure. From this:

$$\frac{d}{\rho} \approx 2.8388.$$

For values of $\frac{d}{\rho}$ less than this and for sufficiently small values of θ , the expression for θ_i must be modified.

The limiting values of θ for θ_2^i shadowing ($\frac{d}{\rho} < 4$) are shown in Figure C7.

Figure C7. Limiting values of θ for θ_2^i shadowing.

From Equation (A1), $\theta_L = \cos^{-1} \left(\frac{2\rho}{d} \right)$.

$$\theta_{Max} = \frac{\pi}{6} + \theta_L = \frac{\pi}{6} + \cos^{-1} \left(\frac{2\rho}{d} \right). \quad (C11)$$

For values of $\theta \geq \theta_{\text{Max}}$, cylinder R2 is totally obscured. Thus, the upper limit of the θ integration is reduced from $\frac{\pi}{2}$ to θ_{Max} .

Considering θ_{Min} as a limiting angle as seen from R2 at an angle θ_{Max} and a distance of $\sqrt{3} d$, Equation (A5) gives:

$$\theta_{\text{Min}} = - \left[\theta_{\text{Max}} - \cos^{-1} \left(\frac{\sqrt{3} d}{r} \cos \theta_{\text{Max}} - 1 \right) \right]. \quad (\text{C12})$$

For $\theta > \theta_{\text{Min}}$, θ_2' shadowing requires the expression for θ_2' to be modified.

The limiting value of θ for θ_1' shadowing of R2 cylinders $\left(\frac{d}{\rho} < \sim 2.8388 \right)$ is shown in Figure C8.

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Figure C8. Limiting value of θ for θ_1' shadowing.

From Equation (A1), $\theta_L = \cos^{-1} \left(\frac{2\rho}{d} \right)$,

$$\beta = \frac{\pi}{6} + \theta_L = \frac{\pi}{6} + \cos^{-1} \left(\frac{2\rho}{d} \right).$$

Considering θ_{Lim} as the limiting angle as seen from R2 at an angle ϱ and at a distance of $\sqrt{3} d$, Equation (A5) gives:

$$\theta_{Lim} = \frac{\pi}{6} + \cos^{-1} \left(\frac{2\rho}{d} \right) - \cos^{-1} \left\{ \frac{\sqrt{3} d}{\rho} \cos \left[\frac{\pi}{6} + \cos^{-1} \left(\frac{2\rho}{d} \right) \right] - 1 \right\}. \quad (C13)$$

For $\theta < \theta_{Lim}$, θ'_1 shadowing requires that the expression for θ'_1 be modified.

Now we need the corrected expressions for θ'_1 and θ'_2 to satisfy the shadowing conditions described here. The case for θ'_2 is shown in Figure C9.

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Figure C9. Limiting value, θ'_2 .

The angle ψ_2 can be obtained from Equations (A2) and (A4), for an angle $\frac{\pi}{6} - \theta$:

$$h_2 = \sqrt{d^2 - 2 d \rho \cos \left(\frac{\pi}{6} - \theta \right) + \rho^2}.$$

Thus we have:

$$\psi_2 = \cos^{-1} \left(\frac{\rho}{h_2} \right) + \sin^{-1} \left[\frac{\rho \sin \left(\frac{\pi}{6} - \theta \right)}{h_2} \right]. \quad (C14)$$

Similarly, χ_2 can be obtained from Equation (A5) for an angle $\frac{2\pi}{3} - \psi_2$:

$$\chi_2 = \frac{2\pi}{3} - \psi_2 - \cos^{-1} \left[\frac{d}{\rho} \cos \left(\frac{2\pi}{3} - \psi_2 \right) - 1 \right], \quad (C15)$$

and:

$$\theta_2' = \frac{\pi}{6} + \chi_2. \quad (C16)$$

By following a similar path of reasoning, we can find the expression for θ_1' . The angle ψ_1 is obtained from Equations (A2) and (A5) for an angle $\frac{\pi}{6} + \theta$:

$$h_1 = \sqrt{d^2 - 2 d \rho \cos \left(\frac{\pi}{6} + \theta \right) + \rho^2},$$

$$\psi_1 = \cos^{-1} \left(\frac{\rho}{h_1} \right) + \sin^{-1} \left[\frac{\rho \sin \left(\frac{\pi}{6} + \theta \right)}{h_1} \right], \quad (C17)$$

$$\chi_1 = \frac{2\pi}{3} - \psi_1 - \cos^{-1} \left[\frac{d}{\rho} \cos \left(\frac{2\pi}{3} - \psi_1 \right) - 1 \right], \quad (C18)$$

$$\theta_1' = \frac{\pi}{6} + \chi_1. \quad (C19)$$

To summarize the results, we have:

1. No R1 shadowing for $\frac{d}{\rho} \geq \frac{4}{\sqrt{3}}$.
2. For $\frac{d}{\rho} < \frac{4}{\sqrt{3}}$ and $\theta_{\text{Min}} < \theta < \theta_{\text{Max}} \text{ (R1)}$, $\theta_2' = \frac{\pi}{3} + \chi$, and the upper limit of θ is θ_{Max} .
3. No R2 shadowing for $\frac{d}{\rho} \geq 4$.
4. For $\frac{d}{\rho} < 4$ and $\theta_{\text{Min}} < \theta < \theta_{\text{Max}} \text{ (R2)}$, $\theta_2' = \frac{\pi}{6} + \chi_2$, and the upper limit of θ is θ_{Max} .
5. No R2 shadowing of θ_1' for $\frac{d}{\rho} \geq \sim 2.8388$.
6. For $\frac{d}{\rho} < \sim 2.8388$ and $\theta < \theta_{\text{Lim}} \text{ (R2)}$, $\theta_1' = \frac{\pi}{6} + \chi_1$.

APPENDIX D

THE EFFECT OF CLAD ON THE DANCOFF FACTOR
IN CYLINDRICAL GEOMETRY

The expression for the Dancoff factor contains a term which takes into account the effect of the moderating material separating the two cylinders. This is the Bickley function term, which is:

$$Ki_3 \left(\frac{|\vec{\rho} - \vec{\rho}'|}{\lambda} \right),$$

where $|\vec{\rho} - \vec{\rho}'|$ is the length of the path through the moderator and λ is the mean free path of the moderator. As long as no clad is present, this term takes into account the attenuation in neutron flux at resonance energy due to the moderator. When clad is present, this term must be modified.

With clad surrounding the rods, the neutrons pass through a total clad path, T_c , and through a total moderator thickness, $|\vec{\rho} - \vec{\rho}'| - T_c$. If the mean free path of the neutrons in the clad is λ_c , the Bickley function term may be written:

$$Ki_3 \left(\frac{|\vec{\rho} - \vec{\rho}'| - T_c}{\lambda} + \frac{T_c}{\lambda_c} \right). \quad (D1)$$

Thus, the total clad path must be determined for each neutron track considered.

The case for clad present is shown in Figure D1. The thickness of the clad has been exaggerated to make the figure easier to interpret.

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Figure D1. Cylinder with clad.

The quantities in Figure D1 are:

ρ = rod radius,

ρ_c = clad outer radius,

α = angle between the normal to the surface and the neutron path to the other cylinder.

T_1 = path length through clad of cylinder 1.

From the law of sines, we have:

$$\frac{\rho_c}{\sin(\pi - \alpha)} = \frac{\rho}{\sin \eta} = \frac{T_1}{\sin(\alpha - \eta)}.$$

Since $\sin(\pi - \alpha) = \sin \alpha$, this can be written:

$$\frac{\rho_c}{\sin \alpha} = \frac{\rho}{\sin \eta} = \frac{T_1}{\sin(\alpha - \eta)} \quad (D2)$$

Dancoff and Ginsburg's expression for α is:

$$\alpha = \cos^{-1} \left(\frac{d \cos \theta - \rho [1 + \cos(\theta + \theta')]}{|\rho - \rho'|} \right).$$

We also have:

$$\eta = \sin^{-1} \left(\frac{\rho}{\rho_c} \sin \alpha \right),$$

$$T_1 = \frac{\rho_c \sin(\alpha - \eta)}{\sin \alpha}. \quad (D3)$$

By a similar reasoning, the path length through the clad on cylinder 2 is:

$$T_2 = \frac{\rho_C \sin (\alpha' - \eta')}{\sin \alpha'}, \quad (D4)$$

where:

$$\alpha' = \cos^{-1} \left\{ \frac{d \cos \theta' - \rho [1 + \cos (\theta + \theta')]}{|\rho - \rho'|} \right\},$$

$$\text{and } \eta' = \sin^{-1} \left(\frac{\rho}{\rho_C} \sin \alpha' \right).$$

The total clad path is:

$$T_C = T_1 + T_2 = \rho_C \left[\frac{\sin (\alpha - \eta)}{\sin \alpha} + \frac{\sin (\alpha' - \eta')}{\sin \alpha'} \right]. \quad (D5)$$

When a gap is between the rod and the clad, the above expression must be modified. We have:

$$T_{G1} = \frac{\rho_G \sin (\alpha - \eta_G)}{\sin \alpha},$$

where ρ_G = gap outer radius,

T_{G1} = path length through the gap, and:

$$\eta_G = \sin^{-1} \left(\frac{\rho}{\rho_G} \sin \alpha \right).$$

Combining this with the expression for the gap in cylinder 2, T_{G2} , we obtain:

$$T_G = T_{G_1} + T_{G_2} = \rho_G \left[\frac{\sin(\alpha - \eta_G)}{\sin \alpha} + \frac{\sin(\alpha' - \eta_{G'})}{\sin \alpha'} \right]. \quad (D6)$$

The modified expression for the total clad path, with a gap present, is:

$$T_C = \rho_C \left[\frac{\sin(\alpha - \eta)}{\sin \alpha} + \frac{\sin(\alpha' - \eta')}{\sin \alpha'} \right] - T_G, \quad (D7)$$

and the corrected Bickley function term becomes

$$Ki_3 \left(\frac{|\rho - \rho'|}{\lambda} - T_C - T_G + \frac{T_C}{\lambda_C} \right). \quad (D8)$$

APPENDIX E

DERIVATION OF THE EXPRESSION FOR $\cos \alpha$
FOR THE SPHERICAL DANCOFF FACTOR

Consider a plane passing through the centers of two spheres. A path from a point on sphere 1 to a point P' on sphere 2 makes an angle α with respect to the normal on sphere 1. In Figure E1, the path and the point P are above the plane described. Dark lines in the figure indicate lines above this plane.

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Figure E1. $\cos \alpha$ for two spheres.

From Figure E1, we have:

$$a = \rho \sin \omega', \quad (E1)$$

$$\omega' = \theta' - \epsilon.$$

From Equation (A2), we have:

$$\epsilon = \sin^{-1} \left(\frac{\rho \sin \theta}{\sqrt{d^2 - 2d\rho \cos \theta + \rho^2}} \right).$$

$$b = a \cos \phi = \rho \sin \omega' \cos \phi. \quad (E2)$$

$$e = \rho \cos \omega'. \quad (E3)$$

Noting that:

$$\epsilon + \gamma + \psi = \frac{\pi}{2} - \theta,$$

we have:

$$\psi = \frac{\pi}{2} - \theta - \epsilon - \gamma. \quad (E4)$$

$$u = g \sin \psi = g \sin \left[\frac{\pi}{2} - \theta - \epsilon - \gamma \right].$$

Therefore:

$$u = g \cos [\theta + \epsilon + \gamma]. \quad (E5)$$

$$\gamma = \sin^{-1} \left(\frac{b}{g} \right) = \cos^{-1} \left(\frac{e}{g} \right).$$

If we take the expression for the cosine of the sum of two angles, where $\theta + \epsilon$ is taken as one angle and γ is taken as the other, we get:

$$u = g \cos (\theta + \epsilon) \cos \gamma - g \sin (\theta + \epsilon) \sin \gamma,$$

$$u = g \cos (\theta + \epsilon) \left(\frac{\rho \cos \omega'}{g} \right) - g \sin (\theta + \epsilon) \left(\frac{\rho \sin \omega' \cos \phi}{g} \right),$$

$$u = \rho \cos (\theta + \epsilon) \cos \omega' - \rho \sin (\theta + \epsilon) \sin \omega' \cos \phi. \quad (E6)$$

The expression for α is:

$$\cos \alpha = \frac{x}{|\rho - \rho'|},$$

$$x = d \cos \theta - \rho - u, \text{ and, since } \omega' = \theta' - \epsilon,$$

$$x = d \cos \theta - \rho - \rho \cos (\theta + \epsilon) \cos (\theta' - \epsilon)$$

$$+ \rho \sin (\theta + \epsilon) \sin (\theta' - \epsilon) \cos \phi. \quad (E7)$$

When we evaluate the expression used to obtain the Dancoff factor, it must be integrated over all values of the angle ϕ . For small values of θ and θ' , the integral covers the limits of θ from 0 to 2π . However, for some limiting larger values of θ or θ' , the lower part of the circular path cannot be "seen" through moderator only, and the ϕ integration must be of the form:

$$2 \int_0^{\phi_M} d\phi,$$

where ϕ_M is the limiting value of ϕ for the path passing through moderator only. This condition is met when:

$$\phi_M \leq \pi, \text{ and } \cos \alpha = 0.$$

This second condition implies that the path is tangent to sphere 1, and any further increase in ψ is not possible.

From Equation (E7), we have:

$$x = d \cos \theta - \rho - \rho \cos (\theta + \epsilon) \cos (\theta' - \epsilon) \\ + \rho \sin (\theta + \epsilon) \cos (\theta' - \epsilon) \cos \psi .$$

For $\cos \chi = 0$, $x = 0$, and $\psi = \psi_M$:

$$\psi_M = \cos^{-1} \left[\frac{\rho + \rho \cos (\theta + \epsilon) \cos (\theta' - \epsilon) - d \cos \theta}{\rho \sin (\theta + \epsilon) \sin (\theta' - \epsilon)} \right] . \quad (E8)$$

The expression for x [Equation (E7)] is the only term in the spherical Dancoff equation which depends on the value of ψ . Therefore, the ψ integration can be carried out as follows:

$$\int_0^{\psi_M} d\psi [d \cos \theta - \rho - \rho \cos (\theta + \epsilon) \cos (\theta' - \epsilon) \\ + \rho \sin (\theta + \epsilon) \sin (\theta' - \epsilon) \cos \psi] \\ = [d \cos \theta - \rho - \rho \cos (\theta + \epsilon) \cos (\theta' - \epsilon)] \psi_M \\ + \rho \sin (\theta + \epsilon) \sin (\theta' - \epsilon) \sin \psi_M . \quad (E9)$$

APPENDIX F

METHOD FOR CALCULATING VALUES OF THE $E_3(x)$ FUNCTION

The expression used to calculate the $E_3(x)$ function was obtained from AMS 55,⁷ Equation (5.1.12).

$$E_n(x) = \frac{(-x)^{n-1}}{(n-1)!} [-\ln x + \psi(n)] - \sum_{\substack{m=0 \\ m \neq n-1}}^{\infty} \frac{(-x)^m}{(m-n+1)m!},$$

$$\psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m}.$$

In the second expression, γ is Euler's constant. For the case of interest here, $n=3$, and:

$$E_3(x) = \frac{(-x)^2}{2!} [-\ln x + \psi(3)] - \sum_{\substack{m=0 \\ m \neq 2}}^{\infty} \frac{(-x)^m}{(m-2)m!}, \quad (F1)$$

$$\psi(3) = -\gamma + \sum_{m=1}^2 \frac{1}{m} = -\gamma + \frac{3}{2}. \quad (F2)$$

This infinite series converges rapidly over the range of values of x usually encountered in calculating the Dancoff factors of slabs. The value of γ is given to 24 significant figures in Table 1.1 of AMS 55. Equation (F1) will not yield the function value for $x=0$. However, from AMS 55, Equation (5.1.23),

$$E_n(0) = \frac{1}{n-1} \quad (n < 1),$$

$$E_3(0) = \frac{1}{2}. \quad (F3)$$