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GASEOUS DIFFUSION AT MODERATE FLOW RATES IN CIRCULAR CONDUITS

By
George Roley
R. W. Fahien

July 1960

Ames Laboratory
Iowa State University
Ames, Iowa

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GASEOUS DIFFUSION AT MODERATE FLOW RATES IN CIRCULAR CONDUITS*

by
George Roley and R.W. Fahien

ABSTRACT

Mass transfer studies were made for an air-carbon dioxide gas system flowing through a 4-inch cylindrical conduit at Reynolds numbers of 1,500, 5,000, and 10,000. It was the purpose of this work to investigate turbulent transport mechanisms in order that basic theory in this area be expanded. The experimental technique consisted of introducing a carbon dioxide tracer into the center of a moving air stream and determining concentration and velocity distributions at various downstream positions. Gas samples were removed by means of a pitot tube and analyzed by a thermal conductivity cell. A circular five-loop hot-wire anemometer was used to establish the velocity profiles.

A dual treatment of the data was provided by fitting the concentration and velocity distributions to two solutions of the basic diffusion equation,

$$\frac{\partial}{\partial r} \left(E_r r \frac{\partial C}{\partial r} \right) = r \frac{\partial}{\partial z} \left(C u - E_a \frac{\partial C}{\partial z} \right)$$

Certain discrepancies arise in both solutions because of different simplifying assumptions used. These factors are pointed out while considerable attention is devoted to theoretical explanations of features exhibited by the results.

Point and average values of total diffusivity, mass transfer Peclet number, and eddy viscosity were determined for the diffusion of carbon dioxide in air. Mass and momentum transfer analogies are illustrated by comparing the dimensionless groups of Peclet number to Schmidt number and eddy mass diffusivity to eddy momentum diffusivity. ✓

*This report is based on an M.S. thesis by George Roley submitted July, 1960, to Iowa State University, Ames, Iowa. This work was done under contract with the Atomic Energy Commission.

Equations were derived to enable the importance of axial diffusion to be estimated, a term that has been generally neglected in mass transfer studies. A characteristic diffusivity has been defined which permits a general correlation of total diffusivity with both position and flow rate.

INTRODUCTION

The mechanisms involved in heat, mass, and momentum exchange processes between fluids have been subjects of considerable scientific and technological interest. The chemical engineer is interested in these processes because of their application in many of the unit operations and unit processes where successful design requires a knowledge of the exact mechanisms. For example, it is important to know whether a fluid is in laminar or turbulent flow in an item of equipment since the transfer rates depend on this condition. Prediction of mass transfer is of particular interest in equipment design since it will necessarily influence the size of equipment required for a specified application. Designers in this area have been handicapped by lack of important theoretical concepts and, consequently, they have relied heavily upon empirical data, experience, and intuition to solve many of their problems. Mass transfer calculations based on overall considerations for many years have proven to be unsuccessful so that a better understanding of the internal effects has become apparent. In packed towers, for instance, too little is known regarding the flow through packing so that effects of mixing, uneven flow, and diffusion are absorbed in some empirical coefficient. In fact, engineering design could justly be called a "science of coefficients". This empirical method by which the problems were handled often resulted in conflicting conclusions with little or no correlation of results. Improved design practices can be gained through a better understanding of the transport processes involved. Furthermore, a study of one of the transport processes will furnish information regarding the other two since some basic similarities

exist. Although a great deal of work has been exerted in this field, little has been accomplished in the way of expanding the basic theory involved. One important development concerns the relationship between heat, mass, and momentum transfer. It now appears that these three processes may be treated by a single theory.

The mass transfer process has been suggested to obey the relationship

$$N_A = D \nabla C + E_T \nabla C$$

where D and E_T are the molecular and eddy diffusivities respectively.

Mass transfer by diffusion then results from a combination of molecular migrations and macroscopic motions of eddies. Further mixing may occur because of gross motion of the fluid but this is considered as a bulk mass transport process as opposed to diffusional mass transfer. While a great deal is known about molecular diffusion in view of aspects of the kinetic theory of gases, relatively little is known about turbulent diffusion. Molecular and turbulent diffusivities have been found to depend upon different physical variables so that failure of attempts to "lump" them together is not surprising. The importance of developing an adequate theory is evident.

Because of many theoretical gaps regarding the nature of the turbulent exchange process, prediction of its behavior cannot be accomplished without recourse to experiment. It was for this reason that the present study was undertaken. The purpose of this work was to determine the varying contributions afforded the mass transfer process by molecular and eddy mechanisms in gaseous systems. An additional objective included determining the variation of total diffusivity with radial position. The flow rates

investigated ranged from the low end of the turbulent region where eddy diffusion is the primary mode of transport through the intermediate and highly laminar range where molecular diffusion becomes important. A further point considered was evaluating the importance of axial diffusion in these flow ranges.

The experimental technique involved introducing a tracer gas coaxially into a moving air stream in an empty conduit. Concentration profiles at various downstream positions were established by removing samples by means of a pitot probe and analyzing them with a thermal conductivity cell. Velocity distributions were determined by a hot-wire anemometer. Although the important practical problems deal with transfer of material from one phase to another, a study of this type can provide valuable information regarding the basic mechanisms involved. These data were treated by two different solutions of the general diffusion equation to obtain values of diffusivity and mass transfer Peclet numbers.

LITERATURE SURVEY

General

The groundwork for present day studies of turbulence was formed from theoretical concepts initiated by Prandtl, von Karman, and Taylor. A comprehensive discussion of these basic findings is presented by Bakhmeteff (1), while a more recent version has been made available by Bird, et al. (4).

The application of molecular and turbulent diffusion in the field of chemical engineering has become increasingly important so that introductory treatments can be found in basic chemical engineering texts such as Sherwood and Pigford (42), Coulson and Richardson (5), and Treybal (54). Perry's Chemical Engineers' Handbook (34) also gives a comprehensive discussion.

Theories relating the molecular diffusivities of gases to molecular properties were first proposed by Maxwell (29) and Jeans (18) using principles based on the kinetic theory. Because of difficulty in eliminating convection currents and faith in the kinetic theory, molecular diffusivities are usually calculated rather than determined experimentally. No theoretical relationships comparable to those for gases are available for liquids. Additional complications of molecular attraction and effects of solute concentration complicates the analysis of liquid diffusion.

Concepts of the kinetic theory have also been helpful in developing a theory for turbulent motion. The eddies that cause mixing in turbulent fluids have been compared with the similar function of molecules in molecular diffusion. These analogies are useful in visualizing the eddy diffusion process but lead to false interpretations if carried too far.

Taylor (47) suggested a theory of turbulent diffusion by continuous movements. Unlike molecular motion where collisions between molecules result in changes in the magnitudes and directions of the velocity vectors, turbulent motion is not characterized by sharply definable masses of fluids so that "collisions between masses" result in a continuous movement of the fluid. This continuous eddying has been correlated with the speed of a particle before and after some time interval and the displacement occurring during this time interval. A definition of scale of turbulence was proposed by Taylor (48, p.426) and cited by Dryden (11) in terms of the correlation existing between fluctuations at neighboring points.

It is clear that whatever we may mean by the diameter of an eddy (scale of turbulence), a high degree of correlation must exist between the velocities at two points which are close together when compared to this diameter. On the other hand, the correlation is likely to be small between the velocity at two points situated many eddy diameters apart. If, therefore, we imagine that the correlation R_y between the values of the speed u at two points distant y apart in the direction of the y coordinate has been determined for various values of y , we may plot a curve of R_y against y , and this curve will represent, from the statistical point of view, the distribution of u along the y axis. If R_y falls to zero at, say, $y = Y$, then a length l_2 can be defined such that

$$l_2 = \int_0^{\infty} R_y dy = \int_0^Y R_y dy$$

This length may be taken as a possible definition of the "average size of eddies".

Turbulence denotes a condition of irregular motion of the instantaneous fluid velocity about some mean value. If the average velocities

of a fluid in the x, y, and z directions are given by U, V, and W and the components of deviating velocity are denoted by u, v, and w, then the intensity of turbulence is measured by the root-mean-square values $(\bar{u})^{1/2}$, $(\bar{v})^{1/2}$, and $(\bar{w})^{1/2}$ or more simply u' , v' , and w' . Virtually all theories of turbulence assume that the eddy diffusion coefficient, in terms of its statistical properties, is proportional to the product of "scale" and "intensity". The analogous relationships for molecular diffusion are the mean free path of the molecules and their root-mean-square velocity.

Dryden (11) gives an excellent summary of theories on turbulence while devoting considerable attention to its definition. Since the irregular random fluctuations associated with turbulence occur in laminar flow on a molecular scale, Dryden sets a lower limit as that at which the fluctuations cannot be detected by common measuring instruments. Practical means of promoting turbulence in continuous processing units are discussed.

Previous Work

Until the past few years, experimental studies of eddy diffusion have been limited due to the mild interest in this area and difficulties in the techniques involved. Recent concern, however, has prompted several laboratory investigations with the experimental work being focused on three main systems:

1. gas and liquid streams in empty ducts
2. wetted-wall columns and
3. packed beds for both liquid and gaseous systems

These primary systems are those of greatest industrial importance so data

representative of these methods account for a substantial part of the literature.

In an early work on eddy diffusion, Towle and Sherwood (52) made mass transfer measurements utilizing a tracer gas technique. Carbon dioxide and hydrogen were introduced at the axis of a horizontal duct confining an air stream in turbulent motion. Concentration traverses were made over a central third of the pipe diameter at several positions downstream from an egg shaped injector. Values of eddy diffusivity were calculated using the mass transfer counterpart of an equation originally derived by Wilson (55) for the diffusion of heat from a point source into a flowing stream of infinite extent. Values of eddy diffusivity thus obtained were found to be in the order of 100 times the molecular diffusivity. Eddy diffusivities were the same for both hydrogen and carbon dioxide leading to the conclusion that the molecular weight of the diffusing gas is not important. The increase in asymptotic values of eddy diffusivity with distance from the injector was approximately proportional to the increase in Reynolds number.

In a later study, Towle, et al. (53) employed a similar technique for determining the effect of a wire screen grid on the turbulence of an air stream. Eddy diffusivities were determined at various positions downstream from the wire screen grid. The eddy diffusivities in the first 15 diameters downstream were less than half the values obtained with no grid present, however, these values increased rapidly and approached an asymptote at about 45 diameters downstream. This asymptotic value corresponded to that obtained with no grid present at the same

Reynolds number. The decrease in turbulence observed with the wire screen grid was explained by visualizing the air stream as being "strained". This "straining" produced smaller eddies than exist in normal flow thereby reducing the rate of diffusion.

Sherwood and Woertz (43) investigated eddy diffusion between a liquid surface and a turbulent gas stream. Employing a falling film technique, water was transferred from one wall of a rectangular duct across a turbulent gas stream to a film of calcium chloride solution on the opposite wall. At high Reynolds numbers the eddy diffusivity was nearly constant over 80 to 90 per cent of the cross section. A plot of eddy diffusivity versus Reynolds number for three main gas stream materials (helium, air, and carbon dioxide) showed the eddy diffusivities were inversely proportional to the gas densities. The diffusivity of water vapor through air in a rectangular duct agreed well with the results of Towle for the diffusivity of carbon dioxide and hydrogen in a turbulent air stream in a circular duct. Contrary to the simple film concept, much of the resistance to diffusion was noted in the eddy zone or "core" of the turbulent stream. The resistance offered by the "core" varied from 28 to 57 per cent depending on the Reynolds number. The remaining resistance was divided about equally between the two narrow films at the duct walls.

Sherwood (41) presents an excellent summary of the relationships between the interphase transfer of heat, mass, and friction.

Dhanak (9) studied eddy diffusion of mass and momentum in a wetted-wall channel in which rippling of the liquid film was eliminated. A correlation of eddy diffusivity to Reynolds number confirmed the conclusion drawn by Towle and Sherwood that the eddy diffusivity shows an

approximately linear relationship with Reynolds number. Eddy diffusivities were found to remain fairly constant within the main portion of the turbulent core. Values of eddy viscosities were compared to those obtained by Woertz in similar fashion and were found to be 50 per cent lower. This was attributed to the less turbulent conditions occurring near the wall in the case where rippling of the liquid film was eliminated.

The first studies of turbulent diffusion generated in beds of packed solids were reported by Bernard and Wilhelm (3). A detailed discussion of turbulent motion for fluid flowing around a bluff object is presented. Liquid and gas phase experiments were performed using methylene blue solution and carbon dioxide tracers respectively. The differential equation for diffusion-convection was solved assuming the injector to be a point source and the u/E_t ratio to be constant. Modified Peclet numbers were correlated with modified Reynolds numbers for various packings by use of the relationship

$$Pe' = m(Re')^n$$

where n is a function of the D_p/D_t ratio and m is a function of the D_p/D_t ratio and a shape factor. Pressure-drop measurements were made for the various packing sizes to obtain friction factors as a function of Reynolds number. These relationships indicated that turbulence in packed beds is a function of the packing size and is therefore interstitial in nature.

Fahien and Smith (12) studied mass transfer in packed beds for a variety of pipe and packing sizes using a gas injector technique. The differential equation for mass transfer was solved allowing both diffusivity and velocity to vary with radial position. The solution, considering boundary conditions of the wall and a finite size injector tube,

was consummated by replacing the differential equation with a set of homogeneous linear difference equations containing eigen functions and eigen values. Peclet numbers were found to increase as the pipe wall was approached especially when the particle to tube diameter ratio was greater than .05. The particle size did not have any noticeable effect on the Peclet number at the center of the pipe but did affect the eddy diffusivity.

Dorweiler (10) illustrated the interaction between molecular and eddy diffusivities by extending the mass transfer data for packed beds to low flow rates including the "transition" and "laminar" regions. Molecular and eddy Peclet numbers were defined and correlated with Reynolds number. The calculations were performed in accordance with the techniques developed by Bernard (2) and Fahien in Fahien and Smith (12).

Results of mass transfer in gases flowing through empty tubes at low flow rates were reported by Frandolig (14). A large portion of turbulent exchange was observed in flow ranges where molecular diffusion was expected to be the controlling mechanism. Significant radial variation of diffusivity was observed. Near the region of the pipe wall, the total diffusion coefficient was found to approach a value corresponding to the molecular diffusivity.

Lynn, et al. (26) made experimental studies on the coaxial mixing of natural gas and air at Reynolds numbers of 44,000 and 79,000. Total diffusivities for the blending of gas in air were computed. Particular emphasis was given to establishing behavior of the centerline diffusivity. Concentrations and velocities were expressed as functions of $(r/r_o)^2$ rather than r/r_o . Use of the $(r/r_o)^2$ variable permitted the diffusivity

to be evaluated at the center, $(r/r_0 = 0)$, since derivatives of composition with respect to $(r/r_0)^2$ are not zero at this point.

Kalinske and Robertson (20) have reported findings on turbulence generated by water flowing in an open rectangular channel. Motion pictures were taken of the dispersion of a carbon tetrachloride and benzene mixture. These films enabled vertical and lateral displacements (intensity of turbulence) to be determined at various points downstream. Using Taylor's equation

$$\frac{d\bar{Y}^2}{dx} = \frac{2\bar{v}^2}{U^2} \int_0^x R dx$$

where R is defined as $\bar{v}_x^2/\bar{v}^2$ and the definition of diffusion coefficient as $\frac{U}{2} \left(\frac{d\bar{Y}^2}{dx} \right)$ for large values of x , values of diffusivity were obtained from the slope of a $\bar{Y}^2$ versus x plot. Lateral turbulence was significantly greater than vertical turbulence near the surface.

Kalinske and Pien (19) reported results of a similar study. Taylor's theory of eddy diffusion was confirmed by experiments on the diffusion of foreign matter in a turbulent water stream.

Measurements on mixing of binary gas mixtures flowing in beds of porous solids were reported by Deisler and Wilhelm (6). A frequency response technique was employed to measure interparticle and intraparticle diffusion effects simultaneously. This technique involves introducing a binary gas mixture into a packed bed at constant flow rate but varying periodically and sinusoidally in composition. A shift in phase and decrease in the composition amplitude results by the various diffusion mechanisms acting in the bed. The diffusion constant within the spherical packing (porous pellet) was only 3 per cent of the molecular diffusivity.

Of particular interest, axial diffusion was found to be significantly greater than molecular diffusion for Reynolds numbers between 4 and 50.

Similar studies on axial mixing in packed beds were made by McHenry and Wilhelm (30). The ratio of the inlet and outlet amplitudes of a sinusoidal concentration wave was used to compute axial Peclet numbers, $d_p U/E_z$, where d_p is the particle diameter and E_z is the axial diffusivity. A solution of the equation

$$E_z \frac{\partial^2 x}{\partial z^2} - \frac{U \partial x}{\partial z} = \frac{\partial x}{\partial t}$$

where x is the mole fraction of one gas in a binary mixture, z is the axial coordinate, and t is time, resulted in an approximate exponential solution. Working at Reynolds numbers between 100 and 400 with H_2-N_2 and $C_2H_4-N_2$ gas systems and using a weighted average of 21 determinations gave an axial Peclet number of 1.88 ± 0.15 . The contribution of mass transfer by axial diffusion was found to be approximately sixfold that of radial diffusion.

Application of the dynamic response method to gases flowing in empty tubes was made by Keyes (21) for flow rates corresponding to Reynolds numbers from 4,000 to 50,000. Mass transfer film coefficients were determined through evaluation of an "equivalent gas film thickness". The semi-theoretical equation of Martinelli

$$\frac{B_F}{d} = \frac{32.95 R_c \propto N_{SC} + \ln(1 + 50 N_{SC}) + (1/2) \ln 2.53 \cdot 10^{-3} N_{RE}^{0.9}}{\propto N_{SC} N_{RE}^{0.9}}$$

where

B_F = equivalent film thickness

d = inside tube diameter

R_c = ratio of mean to maximum concentration difference

α = ratio of eddy heat or mass diffusivity to eddy momentum diffusivity

were found to correlate the results reasonably well.

Measurements of gas absorption in falling films were made by Stirba and Hurt (45) for liquid-film Reynolds numbers under 1,800 to 2,000. Mass transfer rates were observed to be many times greater than could be theoretically predicted on the basis that molecular diffusion was the only transfer process. The presence of turbulence in the so-called laminar film was suggested when the large increase in transfer rate could not be explained by the rippling action of the liquid film alone. It was concluded that turbulence has a greater effect on mass transfer than either momentum or heat transfer.

Schwarz and Hoelscher (39) reported mass transfer measurements in a wetted-wall column for a single Reynolds number of 25,000. Water vapor concentration profiles were used to compute eddy diffusivities for mass and momentum transfer as a function of radial position. A maximum in eddy diffusivity for both mass and momentum was observed at the approximate mid-radius.

Hanratty (15) presents an interesting article on turbulent exchange of mass and momentum at a boundary surface. The experimental data of many investigators has led to doubt on the existence of the classical boundary layer. A new theory has been postulated whereby masses of fluid near the wall form a discontinuous film that is continually moving

to and from the wall. The exchange processes are then a function of the contact time which can be expressed mathematically.

Hanratty, et al. (16) studied turbulent diffusion by measuring the mixing of a tracer dye in glass-sphere beds fluidized in water. Behavior of eddy diffusivity is described as a function of time and a displacement factor, reaching a constant value for large diffusion times.

Klinkenberg, et al. (23) attacked the problem of eddy diffusion by an entirely mathematical approach. A point source of solute on the axis of a tube was used as a model to derive eddy diffusion constants. The relative importance of axial diffusion, radial diffusion, and diffusion in infinite space were shown to be determinable by use of graphs utilizing parameters of the experimental system.

The transfer of natural gas into a turbulent air stream flowing in a horizontal duct was investigated by Schlinger and Sage (37) for gas velocities of 25, 50, and 100 feet per second. Concentration profiles, determined at several positions downstream, were noticeably altered by gravitational forces acting on the gases. Calculations of total diffusivities were made assuming independence of radial position.

Data on the evaporation of liquids from surfaces of cylinders, spheres, disks, and flat plates were reported by Maisel and Sherwood (27). Heat, mass, and friction transfer were correlated with Reynolds number using the "j" factor concept. In a similar study the same authors (28) related turbulent intensity to the mass transfer coefficients.

Longwell and Weiss (25) studied the effects of turbulence on the distribution of fuel injected into high-velocity air streams. Methods presented for predicting these distributions, using principles of turbulent

diffusion, gave an average error of less than 7 per cent based on the experimentally observed distributions.

Michacek, et al. (51) present an analysis on axial mixing in straight pipes. The effects of axial mixing are shown to become increasingly important as the laminar flow range is approached. This is a result of the relative motion of the fluid elements for conditions where flat velocity profiles do not exist. Pipe roughness caused little effect on axial mixing.

Prausnitz and Wilhelm (35) investigated concentration fluctuations in a packed bed by measuring the spreading of an acid tracer solution. Concentration fluctuations were explained in terms of the turbulence parameters, scale and intensity.

EXPERIMENTAL

Description of Apparatus

Air-carbon dioxide flow system

The flow system employed in this investigation is shown schematically in Figure 1 and photographically in Figure 2. The primary features include:

1. air purification section
2. air metering section
3. test section
4. carbon dioxide metering section
5. gas analyzing section
6. velocity measuring section

The main air stream was delivered from a high pressure air source through a 3/4-inch air line. Small amounts of oil entrainment carried over from the compressing units were removed by passing the air through a double filter. The first, in the form of a one-foot section of 1 1/4-inch pipe packed with steel wool, was used as a coarse filter. The second, a Fisher Governor Type 361 porous stone air filter, acted as a final filter. The method of gas analysis used in this study requires that the carrier gas be quantitatively free of moisture and carbon dioxide. These contaminants were removed by passing the air through two 6-foot purification towers of 4-inch flanged pipe. The first contained 3- to 8-mesh granules of silica gel for removal of moisture while the second contained 8-mesh soda lime to remove carbon dioxide. Use of small particle sizes afforded a large contact area but presented the problem of containing

fluidized beds. Steel wool plugs covered with 100-mesh screen stationed at both ends of each column were found to contain the packing satisfactorily. Both towers were packed to a height of approximately three feet.

The air flow was metered by a series of three Brooks rotameters covering flow ranges from 4.62 SCFM to 27.2 SCFM at 14.7 psia and 70°F. Originally the system was designed so that only a single rotameter could be inserted at any one time. However, this was later arranged so that all three rotameters could be easily interchanged through a valve system as shown in the flow diagram. The rotameters were operated at a pressure of 18 psig using calibration curves supplied by the manufacturer and ideal gas law pressure corrections. Flow rates were adjusted by a 3/4-inch gate valve located downstream from the rotameters. Pressure reduction and regulation were controlled by a Reynolds spring-loaded diaphragm pressure regulator MR-8250-4. Bourdon-tube pressure gages were placed at points immediately preceding the rotameter tubes and purification beds.

The air was next discharged into the bottom of a variable-height diffusion column constructed of 4-inch nominal diameter galvanized steel pipe. A calming section having a length of 45 pipe diameters was allowed to precede the test section.

A carbon dioxide tracer was supplied from commercial fifty-pound cylinders of liquid CO₂ with vapor pressures of approximately 900 psig. The supply pressure was reduced and regulated by use of a standard reducing valve. Vaporization of the carbon dioxide resulted in considerable cooling of the diaphragm reducing valve; consequently, the reducing valve and first two feet of carbon dioxide line were heated with asbestos insulated

AIR-CARBON DIOXIDE FLOW SYSTEM

Figure 1. Air-carbon dioxide flow system.

Figure 2. Photographs of apparatus used in diffusion study
 (a) Over-all view of control panel and diffusion column
 (b) View of diffusion column with various-height column sections
 (c) Closeup of sampling section (d,e) Two views of hot-wire
 anemometer

nichrome wire heating tape. Ends of the wire were attached to the outlet taps of a Variac powerstat which was used to regulate the temperature. Since the carbon dioxide passed through 50 to 60 feet of 1/4-inch soft copper tubing before reaching the injection tube, a temperature equilibrium was assumed to exist. The heat transfer surface in this length of tubing was great enough to bring the gas to or near room temperature. The flow of carbon dioxide, regulated by a 3/8-inch needle valve, was metered by a pair of Brooks rotameter tubes covering ranges from 5,500 to 20,000 cubic centimeters per minute. In order to maintain a stable rotameter reading, a supply pressure of 18 psig was used. This pressure provided stable readings at low flow rates and did not require more than one or two adjustments during a run. At higher flow rates cooling of the carbon dioxide reduced the supply pressure to the extent that more frequent adjustments of the reducing valve were necessary. It was later noted that more stable flow rates were obtained when the nichrome heating tape was turned on approximately one-half hour before a run was started. The rotameters were calibrated against a standard model wet-test meter. Calibrations were performed while operating at the same pressure used in the experimental runs with temperature, pressure, and water vapor corrections applied as directed by the manufacturer.

Details of the test section are illustrated in Figure 3. Carbon dioxide was admitted to the column through a long, 13/32-inch i.d., hard copper injection tube extending the entire length of the calming section. The injection tube was supported at the base by a circular spoked support disk of 75 per cent free area. Four-armed spiders of fine wire supports

DETAILS OF TEST SECTION

Figure 3. Details of test section

were placed at 5-foot intervals to maintain the injection tube in a coaxial position. A spoked-wheel device that located the injection tube centrally was slipped over the tube end periodically as a check for proper alignment. Flow disturbances at the carbon dioxide-air boundary were minimized by tapering the end of the injection tube at an angle of 75 degrees to within one-half the wall thickness.

Gas samples were removed through a 5/32-inch i.d. pitot sampler housed in a 6-inch section of 4-inch pipe. The pitot probe was constructed of thin-walled copper tubing, its tip being machined to a fine knife edge. A graduated scale on the exterior section gave the exact radial position of the pitot probe while at the same time serving as a guide for maintaining proper angular alignment of the pitot. A 4-foot discharge section was placed over the pitot assembly to minimize discharge effects and prevent air currents from affecting the flow pattern. At low flow rates ($N_{RE} = 1,500$) it was necessary to partially close the gate valve on the discharge section. This provided a slight pressure buildup to force gas samples through the pitot tube. The sampler was connected with tygon tubing to 1/4-inch soft copper tubing which joined the metering tube. After flowing through the rotameter, the gas sample was passed through a thermal conductivity cell. The experimental technique includes analysis of the average effluent concentration of the mixed gases. This was accomplished by packing a portion of the discharge section with woven wire so as to create a large number of eddies to insure thorough mixing of the gases. The average concentration was sampled at the point indicated in Figure 3 after which the excess gas mixture was discharged to

the atmosphere. The height of the diffusion column could be varied by adding sections of various heights between the calming section and pitot assembly. Column sections were held in place by 3/8-inch threaded rods extending through lugs spaced at 90-degree intervals. The metal to metal joints were given an air tight seal by using Permatex number 2 gasket compound. A thin film of the Permatex compound was spread over the tongue and groove joints and allowed to air dry a few minutes until a "tacky" nature was observed. The column sections were then assembled and tightened. Air leaks were detected by spreading soap solution over the joints. The operating pressure within the column was reproduced from run to run by measuring the pressure with a manometer and adjusting accordingly with the gate valve on the discharge section. This procedure was necessary as it was discovered that calibration of the gas analyzer depends on the pressure within its chambers which in turn is a function of the column pressure. Very reproducible zero points were obtained in this manner.

Gas analysis

Carbon dioxide-air gas samples were analyzed by means of a Gow-Mac Model 30-S fast acting thermal conductivity cell. The cell is essentially a Wheatstone bridge containing two wire filaments in each of four arms of a bridge. The circuit diagram is shown in Figure 4. A set of four automotive batteries provided an operating current of 138.00 milliamperes, with the exact value of current determined by measuring the drop across a standard 10.00-ohm resistance. The voltage drop across the standard resistance was read with a Rubicon, high precision, type B potentiometer

CIRCUIT DIAGRAM FOR THERMAL CONDUCTIVITY CELL

Figure 4. Circuit diagram for thermal conductivity cell

with a portable lamp and scale galvanometer. Use of this instrument permitted adjustment of current to within ± 0.5 per cent which was necessary to obtain precise measurements. Current was adjusted by manipulation of two 10-turn heliopot of 200 and 25 ohms. The desired zero point, unbalanced e.m.f. resulting when reference air is passed through both cell passages, was adjusted by use of a variable 2-ohm resistance connected across two arms of the cell bridge. This feature was incorporated after abnormally high zero point readings resulted during a test run. At first it was thought that the cell filaments had become damaged; however, a second thermal conductivity cell offered the same problem resulting in the installation of the zero point adjustor. Gas samples were removed by the pitot sampler and passed through one arm of the thermal conductivity cell while reference air was passed through the other arm. A small plug of pyrex wool was inserted in the gas lines prior to the cell to prevent any chemical drier from entering the cell chambers. The resulting impulse data were measured and recorded by a Brown strip chart recorder, Model No. 153X12V-X-30, equipped with a dual range. Samples containing less than approximately 2.4 per cent carbon dioxide were measured on a 0-10 millivolt range while higher concentrations were measured by a 0-20 millivolt range.

The extreme sensitivity of the thermal conductivity cell required effective control procedures for optimum performance. Regulation of gas flow was of major importance in avoiding "flow effects", consequently, rigid monitoring of the gas rates was maintained throughout the operating procedure. Accuracy of the cell also depends on the maintenance of a constant temperature of the heat reservoir. Precautions suggested by

Gow-Mac were observed. A glass wool sheathing was placed inside the steel housing to provide an insulation for the heat reservoir. The entire T/C cell was then installed in an insulated box. Thermal stability of the heat reservoir was aided by operating the cell continuously.

Calibration of the cell was accomplished by measuring the e.m.f. resulting when a known concentration of carbon dioxide in air was passed through one side of the cell. The concentrations were adjusted by rates of flow of both gases and thoroughly mixed in the woven wire packed portion of the discharge section. Different concentrations were obtained by changing the carbon dioxide rate while the main air stream rate was held constant. The difference between the unbalanced e.m.f. and the zero point e.m.f. plotted against volume per cent carbon dioxide gave a good straight line on a log-log plot. The calibration curve is shown in Figure 5. On a relative basis, the relationship between concentration and e.m.f. is of the form

$$C/C_A = (E/E_A)^n$$

where n is the slope of a $\log C$ versus $\log E$ plot. Experimentally, E and E_A can be measured. The desired quantity, C/C_A , can be determined by evaluating the cell constant, n . Calibration of the cell and application of a least squares procedure resulted in values of n equal to .952, .952, and .961 respectively for the three flow rates studied.

Velocity measurement

Velocity profiles were established by use of hot-wire techniques. Hot-wire methods are based on the fact that the resistance of a wire is a

function of its temperature. Flow of a fluid past a heated wire will reduce its temperature thus changing its resistance, the magnitude of which depends on the speed of the ambient fluid. A measure of the change in resistance provides a measure of the speed. The laws governing the convective cooling of electrically heated wires by streams of fluids have been investigated most extensively by King (22). He showed both theoretically and experimentally that the heat loss per unit length H was closely approximated by the relation

$$H = i^2 R = BV^{1/2} + C$$

where B and C are constants depending upon the wire, temperature, and ambient fluid, and V is the velocity of the fluid. Wires may be maintained at constant resistance or at constant current. In the constant resistance method, B and C remain constant with changes in fluid velocity, and the anemometer may be calibrated by a current versus velocity plot. B and C are found to vary with the fluid velocity when the anemometer is run at constant current. In this method the voltage drop across the wire is a measure of the velocity. In the immediate region of the leads and potential terminals, heat is conducted away from the wire at such a rate that the wire temperature at these extremities may be considered very nearly at air temperature. These terminal errors are not important as long as the anemometer is directly calibrated under conditions similar to actual service. Platinum has been the most popular metal used for anemometer wires primarily because of its chemical inertness at elevated temperatures. The wire diameter used is of importance. Fine wires are more susceptible to aging but are more sensitive and require less current

Figure 5. Thermal conductivity cell calibration curve

than heavier wires. The final choice of diameter will depend upon the particular application.

A circular, five-loop, hot-wire anemometer of the constant current type was used in this investigation and is shown in the photographs of Figure 2 and schematically in Figure 6. The anemometer was modeled after those used by Swartz (38) in accordance with the procedures recommended by Ower (33) and King (22). Five circular heating elements were constructed from .006-inch platinum wire, the relatively heavier wire being used because of its ability to maintain a circular shape better than the thinner wires. Thin mica strips cemented to a Plexiglas mount supported the five concentric wire loops in the proper radial position. The platinum wire was "sandwiched" in grooves cut into heavy copper electrodes and then spot welded for permanence of resistance. A short vertical run of the copper leads was permitted before passing radially to the mount. The entire hot-wire assembly was housed in a section of 4-inch brass pipe. Anemometer loops were located at radii calculated from the equation

$$\frac{r}{r_o} = \left[\frac{2n-1}{10} \right]^{1/2}$$

where

n = anemometer no. (1, 2, 3, 4, or 5)

r = radius of anemometer

r_o = radius of pipe.

This arrangement permitted the anemometer loops to be placed at positions surrounded by equal areas. With this spacing of anemometers the total flow rate could be approximated simply by arithmetically averaging the

Figure 6. Schematic of hot-wire anemometer

flow rates at each anemometer and multiplying by the cross-sectional area of the pipe.

The power supply shown in the circuit diagram of Figure 7 consisted of two to three 6-volt automotive batteries connected in series. Voltage drops across the anemometer wires were measured and recorded by a Moseley X-Y Recorder with ranges from 5 millivolts to 150 volts. Voltage drops were measured by the Y terminal of the recorder. Motion of the carriage in the x-direction was provided by supplying the X terminal of the recorder with a constant varying input signal. This was accomplished in the following way: A synchronous motor was used to turn a 25-ohm, 10-turn, linearly calibrated helipot resulting in a constant change in voltage drop. This impulse was fed to the X terminal of the recorder.

Calibration of the anemometer is based on the principles of fluid flow through pipes. For the laminar range of Reynolds numbers below 2,100, the flow was assumed to be parabolic according to the relation

$$u = 2V[1 - (\frac{r}{r_0})^2]$$

where u is the velocity at radius r and V is the average velocity. The calibration was extended into the high flow ranges (average velocities greater than 2.5 feet per second) by introducing a flat velocity profile in the system using methods suggested by Stoker (46). Two 100-mesh screens were placed in the system, one an inch below the anemometer and one directly above the anemometer. Teago (50) found that the velocity at the four inner loops could be calculated by dividing the average velocity by .942 while the outer loop velocity was the average velocity multiplied by

CIRCUIT DIAGRAM FOR HOT-WIRE ANEMOMETER

Figure 7. Circuit diagram for hot-wire anemometer

.800. Prior to calibration the platinum anemometer loops were annealed at a dull red heat in still air for six hours. The anemometer was calibrated by placing it on the calming section of the diffusion column after the long carbon dioxide injection tube had been removed. The discharge section was then placed over the anemometer housing to minimize convection currents. Flow rates were adjusted to the desired levels and the anemometer brought into circuit, the power being supplied to the anemometers through a set of phone-jacks. Current was adjusted by measuring the drop across a standard resistance of approximately one-sixth of an ohm until the voltage drop was equal to .35500 volts as measured by a potentiometer. The value .35500 is the voltage drop which would cause the anemometers to be at a dull red heat when placed in still air. The voltage drop across the anemometer loop was then measured by the X-Y recorder. Each anemometer was run through the entire laminar range before the next anemometer was calibrated. After calibration in the laminar range had been completed, the screens were inserted and calibration in the higher flow ranges were performed in the same manner. The laminar range ends at an average velocity of approximately one foot per second while the screened data starts at an average velocity of 2.5 feet per second. The gap left in between these two procedures was filled by fitting the two sets of data into a smooth curve. Matching of the two curves was taken as a criterion of the agreement of both calibration procedures. The calibration curve for the hot-wire anemometer is shown in Figure 8.

Procedure

The experimental technique consisted of injecting the carbon dioxide

Figure 8. Hot-wire anemometer calibration curve

tracer into a moving air stream and establishing concentration and velocity profiles at various positions downstream. A knowledge of the concentration and velocity distributions provides sufficient information to calculate point values of diffusivities. The main air stream rate was set and allowed approximately one-half hour for equilibrium to establish. Once fixed, the main air stream flow remained quite constant and did not require further attention during the experimental run. Periodic checks, however, were made. The desired zero point of the thermal conductivity cell was set and recorded. Carbon dioxide was admitted to the column and allowed 15 to 20 minutes to attain a temperature equilibrium. To minimize artificial mixing of the two streams due to jetting effects, the carbon dioxide was admitted at a speed corresponding to the center-line air velocity under empty tube conditions. Linear gas velocities were used, calculated as the volumetric flow rates divided by the cross-sectional area. Equilibrium was established and the cell response for the average concentration was recorded. Point values of concentration were then determined by withdrawing samples through the pitot probe, taking care to keep the sampling rate below the rate of the main air stream velocity to prevent sucking around the neighborhood of the pitot probe. Seventeen points along a diameter were traversed and analyzed. At the end of each run the average concentration and zero point readings were redetermined. In order to insure that representative concentration profiles were obtained, three diameter traverses spaced at 120-degree intervals were made for each height. This procedure gave six values of concentration for each radial position. Figure 11 suggests the pattern followed. Diameter traverses

were made in regular order from wall to center to wall, sampling each radial position over an 8 to 16 minute time interval. The corresponding cell impulse was obtained by a graphical integration of the strip chart with a planimeter. After the position of the pitot probe had been changed, the flow rates and cell operating current were checked and adjusted accordingly.

Velocities at five radial positions were measured by replacing the pitot probe with the hot-wire anemometer. Duplicate determinations were made for each of the flow rates and heights investigated. Air was passed through the injection tube to simulate the flow effect of the carbon dioxide. Air rather than carbon dioxide was used because of the different cooling effects on the anemometers due to differences in thermal conductivity. Identical velocity ratios were used. The main air rate was set first to blow off any particles that may have deposited on the anemometer wires. The proper value of current was established and the voltage drop across each loop measured. Velocities corresponding to these voltage drops were obtained from an enlarged calibration curve.

The concentration and velocity data provided enough information to calculate values of diffusivities. These data were treated by the two mathematical procedures described in the next section.

Scope of Work

The range of experimental work for which concentration and velocity distributions were determined is shown in Table 1.

Table 1. Variables investigated

1. Tube size: 4 inches	
2. Injector size: .40625 inches i.d.	
3. Reynolds number	Height, z (inches)
1,500	10
1,500	15
1,500	20
5,000	50
5,000	70
5,000	90
10,000	50
10,000	70
10,000	90

Three diameter traverses were made for each height and flow rate studied when obtaining concentration data while velocity determinations were made in duplicate. All the experimental data in this work were taken at room temperature and pressure. The temperature remained constant at $23.4 \pm 1.2^{\circ}\text{C}$ throughout the investigation and was regarded only in the calculation of flow rates. Effects of temperature on rates of diffusion were not considered since the maximum temperature variation was only 2.4°C .

MATHEMATICAL TREATMENT

A nucleus from which studies of diffusion can be developed is provided by the partial differential equations used to describe laws governing material transfer by molecular and eddy mechanisms. Progress in this area has been limited due to difficulty in effecting satisfactory solutions to these equations.

Mass transfer in a moving fluid generally is composed of the simultaneous contributions of molecular and eddy processes and is usually expressed in terms of an effective diffusivity defined mathematically by the expression

$$N = (D_v + E_T) \nabla C = E \nabla C$$

where N = rate of mass transfer, moles/sq ft/sec

D_v = molecular diffusivity, sq ft/sec

E_T = eddy or turbulent diffusivity, sq ft/sec

E = total effective diffusivity, sq ft/sec

∇C = concentration gradient, moles/cu ft/ft

The differential mass transfer equation presupposes that the above relationship is true. In the following pages a summary of the mathematical methods used by previous investigators is presented. A later section of this paper presents some modifications and new derivations which have been developed during the course of this investigation.

Development of the Basic Diffusion Equation

Consider the diffusion of a tracer material from an injector tube source in a cylindrical system. Under steady state conditions, a mass

balance over the differential element of volume shown in Figure 9a is:

$$dN_1 + dN_3 = dN_2 + dN_4 \quad (1)$$

Diffusion is the only mechanism involved in radial mass transfer since net flow in this direction is not possible. The defining equation for E can be used to obtain the moles transferred radially, dN_1 , into the element. The moles leaving or dN_2 must equal dN_1 plus the change occurring in dN_1 over the distance dr . For a differential element

$$dN_1 = -E_r dA_r \frac{\partial C}{\partial r} \quad (2)$$

$$dN_2 = dN_1 + \frac{\partial}{\partial r} (-E_r dA_r \frac{\partial C}{\partial r}) dr \quad (3)$$

where E_r is the radial effective diffusivity. Transport of mass due to gross motion of the fluid must be considered in addition to diffusion for mixing in the axial direction. That is

$$dN_3 = Cu dA_z + (-E_a dA_z \frac{\partial C}{\partial z}) \quad (4)$$

$$dN_4 = dN_3 + \frac{\partial}{\partial z} (Cu dA_z - E_a dA_z \frac{\partial C}{\partial z}) dz \quad (5)$$

where E_a is the axial effective diffusivity. Substitution of Equation 2, 3, 4, and 5 into Equation 1 gives

$$\frac{\partial}{\partial r} (E_r dA_r \frac{\partial C}{\partial r}) = \frac{\partial}{\partial z} (Cu dA_z - E_a dA_z \frac{\partial C}{\partial z}) \quad (6)$$

The areas through which mass transfer takes place are given by

$$dA_r = 2\pi r dz \quad (7)$$

$$dA_z = 2\pi r dr \quad (8)$$

Substitution into Equation 6 and simplifying establishes the basic

Figure 9a. Cylindrical shell of infinitesimal volume

Figure 9b. Partial differential functions in a cylindrical system

diffusion equation.

$$\frac{\partial}{\partial r} (E_r r \frac{\partial C}{\partial r}) = r \frac{\partial}{\partial z} (C u - E_a \frac{\partial C}{\partial z}) \quad (9)$$

The assumptions involved in the development of Equation 9 thus far are:

1. angular symmetry exists
2. the system is at steady state
3. no chemical reaction is occurring, and
4. mass transfer can be described in terms of an effective diffusivity.

To date, a rigorous solution of Equation 9 has not been formulated. Instead, investigators have made various simplifying assumptions in order to reduce the complex relationships existing between E_r , E_a , r , and z . As the study of turbulent transfer continues, the mathematical treatment is becoming more refined to the extent that fewer assumptions are being made. However, the methods are necessarily more involved.

In addition to the various assumptions, certain limiting boundary conditions which describe the physical system are imposed to solve Equation 9.

I. At the plane of the injector tube, ($z = 0$), no radial diffusion has taken place.

$$C(r, 0) = C_f \quad (0 < r < t)$$

$$C(r, 0) = 0 \quad (t < r < r_o)$$

II. At infinite height the concentration at any radial position approaches the mean concentration.

$$C(r, \infty) = C_M$$

III. No diffusion can occur through the confining walls of the conduit.

$$\frac{\partial C}{\partial r} = 0 \text{ at } r = r_0$$

IV. Radial symmetry exists about the tube center.

$$\frac{\partial C}{\partial r} = 0 \text{ at } r = 0$$

Bessel Solution

Bernard (2) in developing an analytical solution to Equation 9 assumed the effective diffusivity and velocity to be constant throughout the system, i.e., $E_r = E_a \neq f(r, z)$ and $u \neq f(r, z)$. While these assumptions are not rigorous, the solution obtained is useful for obtaining average values of diffusivity and Peclet number. With these considerations Equation 9 reduces to

$$\frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} - \alpha \frac{\partial C}{\partial z} + \frac{\partial^2 C}{\partial z^2} = 0 \quad (10)$$

where $\alpha = u/E$. Observation of Equation 10 suggests that the concentration can be expressed as a function of radius and height. A solution of the form

$$C = R(r) Z(z) \quad (11)$$

is assumed where $R(r)$ is a function of r only and $Z(z)$ is a function of z only. Substitution of the above expression in Equation 10 permits the variables to be separated.

$$\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) = \frac{1}{Z} \left(\alpha \frac{dZ}{dz} - \frac{d^2 Z}{dz^2} \right) \quad (12)$$

R is independent of z and so as z varies the left hand side of the above equation remains constant. Similarly, the right hand side remains constant as r varies; therefore, this equation can be set equal to a constant, $-\beta^2$.

$$\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) = -\beta^2 = \frac{1}{Z} \left(\alpha \frac{dZ}{dz} - \frac{d^2 Z}{dz^2} \right) \quad (13)$$

The above expression is equivalent to two independent differential equations and the problem becomes one of solving two equations, one with radial terms and one with axial terms. The solution of the equality given by the left hand side of Equation 13 is

$$R = C_1 J_0(\beta r) + C_2 Y_0(\beta r) \quad (14)$$

where $J_0(\beta r)$ and $Y_0(\beta r)$ are Bessel functions of zero order of the first and second kind respectively. The concentration of a tracer material at any point where $r = 0$ must be finite (boundary condition II or IV), therefore, the coefficient C_2 must equal zero since $Y_0(\beta r)$ is infinite at this point. Equation 14 then reduces to

$$R = C_1 J_0(\beta r) \quad (15)$$

The solution of the equality given by the right hand side of Equation 13 is

$$Z = C_3 e^{[\alpha - (\alpha^2 + 4\beta^2)^{1/2}]z/2} + C_4 e^{[\alpha + (\alpha^2 + 4\beta^2)^{1/2}]z/2} \quad (16)$$

Boundary condition II requires that $C_4 = 0$ since the concentration must remain finite with increasing z. Equation 16 becomes

$$Z = C_3 e^{[\alpha - (\alpha^2 + 4\beta^2)^{1/2}]z/2} \quad (17)$$

and the particular solution of Equation 13 becomes

$$C = A e^{[\alpha - (\alpha^2 + 4\beta^2)^{1/2}]z/2} \cdot J_0(\beta r) \quad (18)$$

The constant A must be determined such that boundary condition I is satisfied. It is evident that no single value of A will satisfy this requirement for all values of r and the general solution must be taken as a combination of the particular solutions which results in the infinite series given by

$$C = \sum_{n=0}^{\infty} A_n e^{[\alpha - (\alpha^2 + 4\beta_n^2)^{1/2}]z/2} \cdot J_0(\beta_n r) \quad (19)$$

Imposing boundary condition III

$$\left(\frac{\partial C}{\partial r}\right)_{r=r_0} = 0 = \sum_{n=0}^{\infty} A_n e^{[\alpha - (\alpha^2 + 4\beta_n^2)^{1/2}]z/2} \cdot J_1(\beta_n r_0)(-\beta_n) \quad (20)$$

since

$$J'_0(\beta_n r) = -\beta_n J_1(\beta_n r) \quad (21)$$

where $J_1(\beta_n r)$ is a Bessel function of the first order. Equation 20 can be satisfied for all values of z only if $J_1(\beta_n r_0) = 0$; i.e., $\beta_0, \beta_1, \beta_2, \dots, \beta_n$ are the positive roots in ascending order of Equation 20. The final solution is obtained by using a Fourier-Bessel expansion to express $C(r, 0)$ as the infinite series indicated by Equation 19 and is given by

$$\frac{C}{C_A} = \frac{C_M}{C_A} \sum_{n=0}^{\infty} \frac{\alpha e^{[\alpha - (\alpha^2 + 4\beta_n^2)^{1/2}]z/2}}{J_0^2(\beta_n r_0) (\alpha^2 + 4\beta_n^2)^{1/2}} \cdot J_0(\beta_n r) \quad (22)^*$$

* C_M is the integral average concentration while C_A is the measured effluent concentration. These cannot be equal if C_M is calculated by using the assumption that velocity is independent of radial position. When experimental velocities are used, C_M and C_A will differ slightly due to analytical errors.

Equation 22 was derived on the basis that the injector tip could be considered a point source and does not involve an appreciable error if the ratio of t to r_o is less than 0.25. Also the mean integral concentration is assumed to be the same at any cross-section. When the injector tube cannot be considered a point source the solution is

$$C = 2C_f \frac{t}{r_o^2} \sum_{n=0}^{\infty} \frac{1}{\beta_n} \left(\frac{J_1(\beta_n t)}{J_0^2(\beta_n r_o)} \right) e^{[\alpha - (\alpha^2 + 4\beta_n^2)^{1/2}]z/2} \cdot J_0(\beta_n r) \quad (23)$$

A mass balance at $z = 0$ gives

$$C_f \pi t^2 = C_A \pi r_o^2 \quad (24)$$

Solving for C_f and substitution into Equation 23 results in

$$\frac{C}{C_A} = \frac{2}{t} \sum_{n=0}^{\infty} \frac{1}{\beta_n} \left(\frac{J_1(\beta_n t)}{J_0^2(\beta_n r_o)} \right) e^{[\alpha - (\alpha^2 + 4\beta_n^2)^{1/2}]z/2} \cdot J_0(\beta_n r) \quad (25)$$

Fortunately, the infinite expansion of Equation 22 and 25 converges rapidly when the ratio of $C_{\max}/C_{\min}$ is small. Terms beyond the third are unimportant when the ratio is 6:1 and only two terms are required when a 3:1 ratio is used. With this consideration Equation 22 reduces to

$$\frac{C}{C_A} = 1 + \frac{\alpha}{J_0^2(\beta_1 r_o) (\alpha^2 + 4\beta_1^2)^{1/2}} \cdot e^{[\alpha - (\alpha^2 + 4\beta_1^2)^{1/2}]z/2} \cdot J_0(\beta_1 r) \quad (26)$$

Bernard fitted his experimental data to Equation 26 by using a trial and error procedure to calculate the α term.

Kurihara (23) used a modified form of Equation 25 for the case where the finite size of the injector tube becomes important. The solution is similar to that of Bernard up to the point of Equation 19 written in the form

$$\frac{C}{C_A} = \sum_{n=0}^{\infty} A_n R_n Z_n = A_0 R_0 Z_0 + A_1 R_1 Z_1 + \dots \quad (27)$$

Multiplying both sides of Equation 27 by $R_0 r$ and making use of the orthogonal properties of the R functions

$$\int_0^{r_0} \frac{C}{C_A} R_0 r \, dr = \int_0^{r_0} A_0 R_0^2 Z_0 r \, dr + \int_0^{r_0} \sum_{n=1}^{\infty} A_n R_n Z_n R_0 r \, dr \quad (28)$$

The orthogonality principle requires that all summations in the second term on the right hand side of Equation 28 be zero since all contain a cross product of the R functions. Equation 28 then becomes

$$\int_0^{r_0} \frac{C}{C_A} R_0 r \, dr = \int_0^{r_0} A_0 R_0^2 Z_0 r \, dr \quad (29)$$

The first series constant, A_0 , can be determined from Equation 29 noting that the R and Z functions are both unity at $z = 0$.

$$A_0 = \frac{\int_0^{r_0} \frac{C}{C_A} r \, dr}{\int_0^{r_0} r \, dr} = \frac{2}{r_0^2} \int_0^{r_0} \frac{C}{C_A} r \, dr \quad (30)$$

The constant A_1 can be determined by a similar procedure. Multiplying Equation 27 by $R_1 r$ and imposing the orthogonality principle gives

$$\int_0^{r_0} \frac{C}{C_A} R_1 r \, dr = A_1 \int_0^{r_0} R_1^2 Z_1 r \, dr \quad (31)$$

At $z = 0$ Equation 31 becomes

$$\int_0^t \frac{C_f}{C_A} R_1 r \, dr = A_1 \int_0^{r_0} R_1^2 r \, dr \quad (32)$$

from which

$$A_1 = \frac{\int_0^t \frac{C_f}{C_A} J_0(\beta_1 r) r dr}{\int_0^{r_0} J_0^2(\beta_1 r) r dr} \quad (33)$$

The general equation for A_n can be written

$$A_n = \frac{\int_0^t \frac{C_f}{C_A} J_0(\beta_n r) r dr}{\int_0^{r_0} J_0^2(\beta_n r) r dr} \quad (34)$$

where β_n are the positive roots of $J_1(\beta_n r_0) = 0$. Substitution of the constants given by Equation 30 and 34 into Equation 27 yields

$$\frac{C}{C_A} = \frac{2}{r_0^2} \int_0^{r_0} \frac{C}{C_A} r dr + \sum_{n=1}^{\infty} \frac{\int_0^t \frac{C_f}{C_A} J_0(\beta_n r) r dr}{\int_0^{r_0} J_0^2(\beta_n r) r dr} R_n Z_n \quad (35)$$

A mean integral average concentration is defined by

$$\frac{C_M}{C_A} = \frac{\int_0^{r_0} \frac{C}{C_A} r dr}{\int_0^{r_0} r dr} = \frac{2}{r_0^2} \int_0^{r_0} \frac{C}{C_A} r dr \quad (36)$$

It is of interest to note that the mean integral concentration, Equation 36, is identical to the first series constant, A_0 , as given by Equation 30. Multiply Equation 35 by C_A/C_M

$$\frac{C}{C_M} = \frac{C_A}{C_M} \frac{2}{r_o^2} \int_0^{r_o} \frac{C}{C_A} r \, dr + \frac{C_A}{C_M} \frac{\infty}{\sum_{n=1}} \frac{\int_0^t \frac{C_f}{C_A} J_0(\beta_n r) r \, dr}{\int_0^{r_o} J_0^2(\beta_n r) r \, dr} R_n Z_n \quad (37)$$

Substitute Equation 36

$$\frac{C}{C_M} = 1 + \frac{C_A}{C_M} \frac{\infty}{\sum_{n=1}} \frac{\int_0^t \frac{C_f}{C_A} J_0(\beta_n r) r \, dr}{\int_0^{r_o} J_0^2(\beta_n r) r \, dr} R_n Z_n \quad (38)$$

A mass balance at $z = 0$

$$\frac{C_f}{C_A} \int_0^t r \, dr = \int_0^{r_o} \frac{C}{C_A} r \, dr = \frac{C_M}{C_A} \frac{r_o^2}{2} \quad (39)$$

$$\frac{C_f}{C_A} = \frac{C_M}{C_A} \frac{r_o^2}{t^2} \quad (40)$$

Substitution of Equation 40 in Equation 38 yields

$$\frac{C}{C_M} = 1 + \frac{r_o^2}{t^2} \frac{\infty}{\sum_{n=1}} \frac{\int_0^t J_0(\beta_n r) r \, dr}{\int_0^{r_o} J_0^2(\beta_n r) r \, dr} R_n Z_n \quad (41)$$

Using $R_n = J_0(\beta_n r)$ and the relations

$$\int_0^t J_0(\beta_n r) r dr = \frac{t}{\beta_n} J_1(\beta_n t) \quad (42)$$

and

$$\int_0^{r_0} J_0^2(\beta_n r) r dr = \frac{r_0^2}{2} J_0^2(\beta_n r_0) \quad (43)$$

Equation 41 becomes

$$\frac{C}{C_M} = 1 + \frac{2}{t} \sum_{n=1}^{\infty} \frac{J_1(\beta_n t) J_0(\beta_n r)}{\beta_n J_0^2(\beta_n r_0)} Z_n \quad (44)$$

The Z term has been evaluated as $Z = e^{[\alpha - (\alpha^2 + 4\beta_n^2)^{1/2}]z/2}$. A binomial expansion of the square root in the exponent takes the form

$$Z_n = e^{[\alpha - \alpha(1 + 1/2 \frac{4\beta_n^2}{\alpha^2} - 1/8(\frac{4\beta_n^2}{\alpha^2})^2 + \dots)]z/2} \quad (45)$$

$$= e^{[\alpha - \alpha - 1/2 \frac{4\beta_n^2}{\alpha} + \frac{2\beta_n^4}{\alpha^3} + \dots]z/2} \quad (46)$$

$$= e^{-\beta_n^2 z/\alpha} \quad (47)$$

This approximation is justified when β_n is much smaller than α . With this simplification Equation 44 becomes

$$\frac{C}{C_M} = 1 + \frac{2}{t} \sum_{n=1}^{\infty} \frac{J_1(\beta_n t) J_0(\beta_n r)}{\beta_n J_0^2(\beta_n r_0)} e^{-\beta_n^2 z/\alpha} \quad (48)$$

This series converges fast enough so that only one variable term is necessary. The final solution is

$$\frac{C}{C_M} = 1 + \frac{2}{t\beta_1} \frac{J_1(\beta_1 t)}{J_0^2(\beta_1 r_0)} J_0(\beta_1 r) e^{-\beta_1^2 z/\alpha} \quad (49)$$

The concentration data of this work were fitted to Equation 49 to calculate values of diffusivity and mass transfer Peclet numbers. Sample calculations are presented in the appendix.

Semi-numerical Solution

Fahien and Smith (11) presented a semi-numerical solution to the diffusion equation by allowing velocity and diffusivity to vary with radial position. This method involves resolving the equation into two parts; one of which can be solved analytically, the other requiring numerical methods for its solution. In addition to the assumptions involved in the derivation of Equation 9 the following are included:

1. velocity is not a function of height and
2. diffusion in the axial direction is negligible.

With these assumptions Equation 9 reduces to

$$\frac{\partial}{\partial r} \left(Er \frac{\partial C}{\partial r} \right) - ur \frac{\partial C}{\partial z} = 0 \quad (50)$$

It is convenient to express this equation in terms of a dimensionless quantity, θ , defined as $\theta = r/r_0$. With this substitution Equation 50 becomes

$$\frac{\partial}{\partial \theta} \left(E\theta \frac{\partial C}{\partial \theta} \right) - r_0^2 u \theta \frac{\partial C}{\partial z} = 0 \quad (51)$$

A solution of the form

$$C = R(r) Z(z) \quad (52)$$

is assumed and substituted into Equation 51 resulting in the following separation of variables.

$$\frac{1}{u\theta R} \frac{d}{d\theta} \left(E\theta \frac{dR}{d\theta} \right) = -\lambda = \frac{r_o^2}{Z} \frac{dZ}{dZ} \quad (53)$$

An analytical solution of the second equality is available and is given by

$$Z = e^{-\lambda z / r_o^2} \quad (54)$$

The constant coefficient is neglected here since Z is later expressed as a product containing another constant. A solution of the first equality is more difficult due to the complex relationships existing between u, E, and θ . This may be rewritten in the form

$$\frac{d}{d\theta} \left(E\theta \frac{dR}{d\theta} \right) + \lambda u\theta R = 0 \quad (55)$$

Numerical methods are applied to Equation 55 to evaluate the functions $\frac{dR}{d\theta}$ and $\frac{d}{d\theta} \left(E\theta \frac{dR}{d\theta} \right)$. These derivatives can be approximated by slopes of straight lines through points on the curve shown in Figure 9b where the variation of R and $E\theta \frac{dR}{d\theta}$ with θ is geometrically represented. Although only approximate due to the size of the finite interval chosen, the slope will be accurately represented if the interval is chosen sufficiently small and if the function is nearly straight over the interval. From Figure 9b, a central difference formula written about some general point, say (k,h), results in

$$\frac{d}{d\theta} (E\theta \frac{dR}{d\theta}) (k,h) = \frac{(E\theta \frac{dR}{d\theta}) (k+1/2)(h) - (E\theta \frac{dR}{d\theta}) (k-1/2)(h)}{h} \quad (56)$$

and at the neighboring 1/2 increments

$$(\frac{dR}{d\theta}) (k+1/2)(h) = \frac{R (k+1)(h) - R (k)(h)}{h} \quad (57)$$

$$(\frac{dR}{d\theta}) (k-1/2)(h) = \frac{R (k)(h) - R (k-1)(h)}{h} \quad (58)$$

Terms associated with the quantities (k,h) are evaluated at the point (k,h) and are not multiplied by this quantity. For example, $R(k-1)(h)$ is to be read as "the value of R at the point $(k-1)(h)$ ". Substitution of Equations 57 and 58 into Equation 56 results in

$$\begin{aligned} \frac{d}{d\theta} (E\theta \frac{dR}{d\theta}) (k,h) &= (E\theta) (k+1/2)(h) \frac{[R (k+1)(h) - R (k)(h)]}{h^2} \\ &\quad - (E\theta) (k-1/2)(h) \frac{[R (k)(h) - R (k-1)(h)]}{h^2} \end{aligned} \quad (59)$$

Solving Equation 55 for $\frac{d}{d\theta} (E\theta \frac{dR}{d\theta})$ after evaluating at the point (k,h) and equating to Equation 59 gives

$$\begin{aligned} -\lambda u(k,h) \theta(k,h) R(k,h) &= \frac{1}{h^2} (E\theta) (k+1/2)(h) [R (k+1)(h) - R (k)(h)] \\ &\quad - \frac{1}{h^2} (E\theta) (k-1/2)(h) [R (k)(h) - R (k-1)(h)] \end{aligned} \quad (60)$$

Factoring Equation 60 for corresponding values of the eigen vectors, R , results in

$$\begin{aligned}
& R(k-1)(h) E(k-1/2)(h) \theta(k-1/2)(h) + \\
& R(k,h) [\lambda h^2 u(k,h) \theta(k,h) - E(k+1/2)(h) \theta(k+1/2)(h) \\
& \quad - E(k-1/2)(h) \theta(k-1/2)(h)] + \\
& R(k+1)(h) E(k+1/2)(h) \theta(k+1/2)(h) = 0
\end{aligned} \tag{61}$$

The E terms are written at the mid-point of an interval while R terms are written at integral multiples of an interval. The θ terms are written at either point since they are associated with both E and R. It is convenient to introduce the weight functions, H and P, at this point defined as $H = E\theta$ and $P = u\theta$. With this notation Equation 61 can be written

$$\begin{aligned}
& R(k-1)(h) H(k-1/2)(h) + \\
& R(k,h) [\lambda h^2 P(k,h) - H(k+1/2)(h) - H(k-1/2)(h)] + \\
& R(k+1)(h) H(k+1/2)(h) = 0
\end{aligned} \tag{62}$$

Equation 62 is valid for any interior point except at the tube center ($\theta = 0$) and at the tube wall ($u = 0$). The equation for the tube center can be obtained by expanding Equation 55, observing the limit as θ approaches zero, and substituting the weight functions to arrive at

$$R(0) \left[\frac{\lambda h^2}{4} P\left(\frac{h}{2}\right) - H\left(\frac{h}{2}\right) \right] + R(h) H\left(\frac{h}{2}\right) = 0 \tag{63}$$

Similarly for the tube wall,

$$R(Nh) \left[\frac{\lambda h^2}{2} P(N-1/2)(h) - H(N-1/2)(h) \right] + R(N-1)(h) H(N-1/2)(h) = 0 \tag{64}$$

where N is the total number of intervals. The equations representing conditions at the tube center, tube interior, and tube wall can be used to write $(N + 1)$ equations about $(N + 1)$ points resulting in $(N + 1)$ unknown eigen vectors, R, and an unknown eigen value, λ . The resulting system of $(N + 1)$ linear equations containing $(N + 2)$ unknowns can be solved

simultaneously in terms of the eigen vector to get $(N + 1)$ solutions. The general solution is the sum of the individual solutions and is given by

$$C = \sum_{n=0}^{\infty} A_n R_n Z = A_0 R_0 Z_0 + A_1 R_1 Z_1 + \dots \quad (65)$$

where n designates the set of orthogonal functions involved.

Constants in the series of Equation 65 can be determined by application of boundary condition I and use of the orthogonal properties of the eigen vectors. Multiplying both sides of Equation 65 by the quantity $P_k R_{k,0}$ and summing over k gives

$$\sum_{k=0}^N C P_k R_{k,0} = \sum_{k=0}^N \sum_{n=0}^{\infty} A_n R_{k,n} Z P_k R_{k,0} \quad (66)$$

where $R_{k,n}$ refers to the value of R at increment k and n th eigen value.

Expanding on n

$$\sum_{k=0}^N C P_k R_{k,0} = \sum_{k=0}^N A_0 R_{k,0}^2 Z P_k + \sum_{k=0}^N \sum_{n=1}^{\infty} A_n R_{k,n} Z P_k R_{k,0} \quad (67)$$

All eigen functions in the second term of the right hand side of Equation 67 are now orthogonal (summation is zero since $n \neq 0$) and Equation 67 can be written in the form

$$\sum_{k=0}^N C P_k R_{k,0} = \sum_{k=0}^N A_0 R_{k,0}^2 Z P_k \quad (68)$$

The constant A_0 can be evaluated by imposing boundary condition I. Since $Z = 1.0$ at $z = 0$ and $R_{k,0} = 1.0$ for all k , Equation 68 becomes

$$\sum_{k=0}^t C P_k = A_0 \sum_{k=0}^N P_k \quad (69)$$

from which

$$A_0 = \frac{\sum_{k=0}^t C_k P_k}{\sum_{k=0}^t P_k} \quad (70)$$

A mass balance over the cylindrical system will show the relationship between A_0 and the mean concentration, C_M . By definition

$$C_M = \frac{\int_0^1 C u \theta \, d\theta}{\int_0^1 u \theta \, d\theta} = \frac{\sum_{k=0}^N C_k P_k}{\sum_{k=0}^N P_k} \quad (71)$$

Assume that C_M is constant for all z . Then at $z = 0$,

$$C_M = \frac{\sum_{k=0}^t C_k P_k}{\sum_{k=0}^t P_k} \quad (72)$$

The significance of the constant A_0 can be observed by comparing Equations 70 and 72 to get

$$A_0 = C_M \quad (73)$$

The general expression for any constant in the series solution can be obtained in a similar manner and is given by

$$A_n = C_M \frac{\sum_{k=0}^N P_k}{t} \cdot \frac{\sum_{k=0}^t P_k R_{k,n}}{\sum_{k=0}^N R_{k,n}^2 P_k} \quad (74)$$

Substitution of Equation 74 into Equation 65 will give the general solution for the concentration at any point, $C_{k,z}$.

$$C_{k,z} = C_M + \sum_{n=1}^{\infty} \left[\frac{\sum_{k=0}^N P_k \sum_{k=0}^t P_k R_{k,n}}{\sum_{k=0}^N P_k \sum_{k=0}^N R_{k,n}^2} R_{k,n} Z_n \right] \quad (75)$$

The negative exponential term contained in the Z function converges the infinite series of Equations 65 and 75 to two terms if z is chosen sufficiently great, i.e.,

$$C = A_0 + A_1 R_1 Z_1 = C_M + A_1 R_1 Z_1 \quad (76)$$

Multiplying Equation 76 by $R_{k,1} P_k$ and summing over k

$$\sum_{k=0}^N C R_{k,1} P_k = \sum_{k=0}^N A_0 R_{k,1} P_k + A_1 \sum_{n=0}^N (R_{k,1})^2 P_k Z_1 \quad (77)$$

The orthogonal properties of the eigen vectors require that the first term on the right hand side of Equation 77 be zero so the expression can be reduced to

$$\sum_{k=0}^N C R_{k,1} P_k = A_1 \sum_{n=0}^N (R_{k,1})^2 P_k Z_1 \quad (78)$$

Imposing boundary condition I

$$\sum_{k=0}^t C R_{k,1} P_k = A_1 \sum_{k=0}^N R_{k,1}^2 P_k \quad (79)$$

Solving Equation 76 for R_1 , substitution into Equation 79, and solving for Z_1 results in

$$Z_1 = \frac{\sum_{k=0}^N P_k (C_k - C_M)^2}{C_f \sum_{k=0}^N P_k (C_k - C_M)} \quad (80)$$

Solving Equation 72 for C_f and substitution of Equation 71 yields

$$C_F = C_M \frac{\sum_{k=0}^N P_k}{t} = \frac{\sum_{k=0}^N C_k P_k}{N} \frac{\sum_{k=0}^N P_k}{t} = \frac{\sum_{k=0}^N C_k P_k}{\sum_{k=0}^N P_k} \quad (81)$$

Substitution of Equation 81 into Equation 80 results in

$$Z_1 = \frac{\sum_{k=0}^N P_k (C_k - C_M)^2}{\sum_{k=0}^N P_k (C_k - C_M)} \cdot \frac{t}{N} \quad (82)$$

All the terms in Equation 82 are obtainable from experimental data on concentration and velocity. Referring to Equation 54, the eigen value can be evaluated from the expression

$$\lambda_1 = -\frac{r_o^2}{z} \ln Z_1 \quad (83)$$

The eigen values can be used to calculate point values of total diffusivity in terms of the weight function H. Multiplying Equation 76 by the proper $R_{k,n} P_k$ product and imposing orthogonality allows the following relations to be established

$$R_{0,1} = \frac{C_0 - C_M}{A_1 Z_1} ; R_{1,1} = \frac{C_1 - C_M}{A_1 Z_1} ; R_{2,1} = \frac{C_2 - C_M}{A_1 Z_1} \dots \quad (84)$$

where the first subscript designates the number of increment k and the second determines the eigen value. Substituting the $R_{0,1}$ and $R_{1,1}$ relations of Equation 84 into the equation for the tube center and solving for the H term

$$H\left(\frac{h}{2}\right) = \frac{\lambda_1 h^2}{4} P\left(\frac{h}{2}\right) \frac{C_0 - C_M}{C_0 - C_1} \quad (85)$$

Substitution of the three R relations of Equation 84 into Equation 62 written for the point $k = 1$ and solving for the H terms results in

$$H\left(\frac{3h}{2}\right) = \frac{H\left(\frac{h}{2}\right) [(C_1 - C_M) - (C_0 - C_M)] - (C_1 - C_M) \lambda h^2 P(h)}{(C_2 - C_M) - (C_1 - C_M)} \quad (86)$$

Substitution of Equation 85 into Equation 86 and simplifying yields

$$\begin{aligned} H\left(\frac{3h}{2}\right) &= \frac{\lambda_1 h^2 \left[\frac{P(h)}{4} (C_M - C_0) - P(h) (C_1 - C_M) \right]}{(C_2 - C_1)} \\ &= \frac{\lambda_1 h^2 [P(0) (C_0 - C_M) + P(h) (C_1 - C_M)]}{(C_1 - C_2)} \end{aligned} \quad (87)$$

The general expression for Equation 87 is given by

$$H(k+1/2)(h) = \frac{\lambda_1 h^2}{C_k - C_{k+1}} \sum_{k=0}^k P_k (C_k - C_M) \quad (88)$$

or, expressing in terms of E,

$$E(k+1/2)(h) = \frac{\lambda_1 h^2 \sum_{k=0}^k P_k (C_k - C_M)}{(C_k - C_{k+1}) \Theta(k+1/2)(h)} \quad (89)$$

An alternate boundary condition may be used to evaluate the Z_1 function and eigen value. The boundary condition used in the previous method was taken at $z = 0$ (plane of injector tip). Thus the diffusivity calculated by this method at various heights above the injector will represent an average diffusivity over this height interval and will include effects of mixing due to presence of the injector tube. However, if the calculations are performed using a boundary condition at some

height say $z = z_0$ instead of $z = 0$, this effect can be minimized. Writing Equation 78 at a height $z = z_0$ gives

$$\sum_{k=0}^N C_k(z_0) P_k R_{k,l} = A_1 \sum_{k=0}^N (R_{k,l})^2 P_k Z_1(z_0) \quad (90)$$

Substitution of the R function defined by Equation 84 at z_1 gives

$$\sum_{k=0}^N C_k(z_0) \left[\frac{C_k(z_1) - C_M}{A_1 Z_1(z_1)} \right] P_k = A_1 \sum_{k=0}^N \left[\frac{C_k(z_1) - C_M}{A_1 Z_1(z_1)} \right]^2 P_k Z_1(z_0) \quad (91)$$

Solving for $\frac{Z_1(z_1)}{Z_1(z_0)}$ results in

$$\frac{Z_1(z_1)}{Z_1(z_0)} = \frac{\sum_{k=0}^N [C_k(z_1) - C_M]^2 P_k}{\sum_{k=0}^N C_k(z_0) [C_k(z_1) - C_M] P_k} \quad (92)$$

Equation 53 can be written

$$\int_{Z_1(z_0)}^{Z_1(z_1)} \frac{dZ}{Z} = - \frac{\lambda}{r_0^2} \int_{z_0}^{z_1} dz \quad (93)$$

Integrating

$$\ln \frac{Z_1(z_1)}{Z_1(z_0)} = - \frac{\lambda}{r_0^2} (z_1 - z_0) \quad (94)$$

$Z_1(z_1)/Z_1(z_0)$ can be determined from experimental data according to the

relationship shown by Equation 92. The eigen value can be then obtained by solving Equation 94

$$\lambda_1 = - \frac{r_o^2}{z_1 - z_o} \ln \frac{z_1(z_1)}{z_1(z_o)} \quad (95)$$

Both boundary conditions were used to calculate values of diffusivity and Peclet number, the results of which are tabulated in Tables 5a and 5b. Of course, no results can be calculated for the first height using the alternate boundary condition since this solution depends on data at a previous height.

To apply the above solution, concentration and velocity distributions across the radius must be known. These were obtained from smoothed data (Tables 6 and 11) obtained from plots of the experimental data (Tables 8 and 10). In this work the radius was divided into eight equal increments. The working equations were modified slightly so that values of concentration and velocity could be expressed as C/C_A and u/V ratios. The $C_k - C_{k+1}$ data required in Equation 89 were smoothed by subtracting values of concentration at adjoining increments, plotting against position increment, and fitting a smooth curve through the result. This method, illustrated in the sample calculations, did not change the diffusivity and Peclet number distribution. A sample calculation of the semi-numerical method is illustrated in Tables 4a, 4b, 4c, and 4d of the appendix.

RESULTS

Experimental Considerations

A logical order in the examination of results begins initially with a review of the experimental work. This analysis provides a basis by which the reliability of other dependent results can be determined and shows how the experimental data fit the calculation techniques.

Height of test section

Heights above the injector tube for which experimental data were determined were obtained from a deliberation of two factors. First, the infinite series arising in the semi-numerical solution converges rapidly with increasing height so that high section heights were preferable. Second, the broader concentration profiles that result with increasing heights were used as a criterion for determining the maximum permissible section heights. At very high heights flat profiles resulted which were unsatisfactory as the semi-numerical solution involves a division by the concentration differences at adjoining radial increments. This method is subject to large errors if flat concentration profiles are used where the difference in concentration between neighboring increments is very small. Conversely, a ratio of $C_{\max}/C_{\min}$ of not more than 3:1 is required in order for the Bessel solution to be applicable. Thus it became necessary to compromise between these two effects. The final heights decided upon were determined experimentally. Preliminary test runs were made to determine the maximum heights usable while yet maintaining a roughly parabolically distributed concentration profile. It was found that greater heights could be used with higher flow rates (Table 1).

Concentration data

All concentrations were expressed as a ratio with respect to the average effluent concentration, C_A . Use of this concentration term as a reference value permitted more accurate data to be obtained as slight changes in daily temperature, variation of moisture content, and other uncontrollable factors could be minimized. Also, large relative errors that may occur at very low concentrations (Figure 5) were reduced. Use of the effluent concentration as a reference value merely acted as a multiplying factor and did not change the concentration distribution.

Concentration analyses as determined by the thermal conductivity cell were thought to be accurate within 0.01 per cent although the method was not checked by a chemical analysis of duplicate samples. The precision of the gas analyzer was very high as could be seen by comparing the thermal conductivity cell impulse data (Table 7) for the average effluent concentrations. As observed in Table 7, the e.m.f. corresponding to the average effluent concentration for each of three diameters at the start and end of a run agreed quite well. The slight discrepancies present were probably due to small variations in gas flow rates during the course of an experimental run.

Concentrations at any one point on the sampling traverse did not remain exactly constant but fluctuated in a random manner about a mean equilibrium value as small variations in the flow pattern occurred. Changes occurring in concentration could easily be followed by visual observation of the recorder strip chart data, portions of which are reproduced in Figure 10. The magnitude of fluctuation of e.m.f. and hence concentration about some mean value is exhibited in Figure 10a where the

Figure 10. Recorder traces of cell impulse

variation appears to follow a more or less sinusoidal relationship. This phenomenon was observed at all test conditions. Figure 10b represents the e.m.f. corresponding to the average concentration. The nearly horizontal line depicted is an indication of the thorough mixing occurring within the wire packed portion of the discharge section. The abrupt change occurring in concentration after the pitot sampler had been moved one radial increment, $(1/8 r_o)$, is shown in Figure 10c. After the sampling position had been changed, approximately 3 to 4 minutes were required for the break to occur. Flushing of the sample lines and thermal conductivity cell of the old sample was responsible for the lag in response. The recorder strip charts proved a valuable asset in judging when equilibrium conditions were established and aided in locating and subsequently neglecting some of the large non-representative fluctuations that occurred at infrequent intervals.

Angular variations of concentration at constant radial positions for different diameter traverses are shown in Figure 11. At positions of low concentration ($\phi = 0.938$) the maximum deviation based on a mean average is approximately 15 per cent. Because of this variation and since the calculation methods assume complete angular symmetry, it was decided to make three diameter traverses for all runs. The final values used were an average of the three traverses.

Figure 12 is another plot of the same data shown to illustrate reproducibility of the concentration profiles for three different diameters. The variation exhibited is typical for all the flow rates studied. Averaging the values of C/C_A for identical radial positions on the three diameters gave an equally weighted average of C/C_A based on six points.

Figure 11. Angular variation of concentration

Figure 12. Reproducibility of concentration profiles

Since each point was sampled over an 8 to 16 minute interval, the sampling time for each final C/C_A value was approximately one hour. Results of this technique are shown in Figure 13 where the averaged C/C_A values are shown as a function of θ .

Use of the function θ as a variable to express concentration and velocity distribution data does not permit direct evaluation of the diffusivity at the tube center. This is a consequence of the manner in which E is defined, that is,

$$E = \frac{N_A}{-\frac{1}{r_o} \frac{\partial C}{\partial \theta}}$$

since $\partial C / \partial \theta = 0$ at $\theta = 0$, E has no meaning at this point. Frandolig (14) suggested that the center-line diffusivity be defined by

$$E_{\theta=0} = \lim_{\theta \rightarrow 0} \frac{N_A}{-\frac{1}{r_o} \frac{\partial C}{\partial \theta}}$$

In this way E is an everywhere continuous function in θ . Lynn (26) resolved this problem by expressing concentrations and velocities as functions of θ^2 since derivatives with respect to this variable do not become zero. Use of θ^2 as the functional variable suggests that sampling intervals equally spaced in θ^2 rather than θ be used. Points on composition and velocity graphs would then be more equally spaced. However, this procedure would put more emphasis at points near the tube wall thus amplifying effects of the pipe wall and roughness on flow conditions. Also, fewer points are sampled near the region of the tube center thereby not allowing this part of the profile to be determined accurately. In

this work sampling positions equally spaced in θ were used although both procedures have been found to be satisfactory.

Velocity data

A similar procedure for minimizing errors was used for the velocity data where point values of velocity were expressed as u/V ratios where V is the average velocity. Velocity data were determined for each height and flow rate investigated substituting the carbon dioxide flow in the injector tube by air flow. The same flow rate ratios were used. In this way an evaluation of the relative cooling effect of each of the gases on the hot-wire anemometer was eliminated and the flow pattern was effectively simulated. Data taken with and without the injector tube present are given in the appendix. Observation of these results indicate that velocity does not depend on height, one of the assumptions involved in both calculation procedures used. This effect was expected at high heights where disrupted flow patterns caused by the injector tube had time to re-establish themselves to empty tube flow conditions. However, even at low heights used in the $N_{RE} = 1,500$ series no effect of height or the air injection system on the velocity profiles could be detected. This could possibly be due to the fact that the smallest platinum anemometer loop occurs at a value of $\theta = 0.316$ which is incapable of perceiving flow disturbances occurring at smaller radial positions (for injector tube, $\theta = 0.1009$). Because of these factors the velocity profiles for all heights were averaged for each Reynolds number. This involved averaging the measured voltage drops and reading the corresponding velocities from an enlarged calibration curve. Experimentally determined velocity

Figure 13. Averaged concentration profiles

Figure 14. Experimental velocity profiles

profiles are shown in Figure 14 for the three mass velocities investigated.

Random velocity fluctuations occurring in the flow pattern were a cause of mild concern. Since the voltage drop across the anemometers was high in comparison to the total voltage drop in the hot-wire circuit, a variation of current occurred due to the changing resistance of the wires with fluctuation in air velocity. This was observed on the potentiometer where the voltage drop across a standard resistance was measured. The change in potential across the standard resistance was small and generally did not reflect on the X-Y plotter where the potential across the hot wires was recorded. After some preliminary operating experience with the equipment, good average values of potential drop were easily obtainable. The delicate nature of the hot-wire anemometer did not permit this instrument to maintain its calibration for more than two weeks of rigorous use.

Accuracy of the velocity data was checked by an integration of the point velocities over the entire cross-sectional area. By definition

$$V\pi r_o^2 = \int_0^{r_o} u 2\pi r dr$$

or, in terms of θ quantities,

$$V = 2 \int_0^1 u \theta d\theta = \int_0^1 u d(\theta)^2$$

The five point velocities determined for each flow rate were plotted against θ^2 and the resulting curve was graphically integrated to determine the average velocity. Comparison of these averages with those given by

the rotameters resulted in deviations of 1.37, 3.27, and -5.31 per cent respectively for the three flow rates studied. The increased deviation occurring with increased flow rate could be expected by observation of the hot-wire anemometer calibration curve (Figure 8) where decreased sensitivity is shown at high velocities.

Gas stream velocity ratio

One important factor in the consideration of mixing in coaxial gas streams concerns the relative velocity of the two streams. It was desired to admit the carbon dioxide stream at such a rate to maintain an undisturbed velocity profile. Under empty tube conditions in the laminar flow range the velocity profile will be parabolic in nature according to the relation

$$u = 2V [1 - (r/r_o)^2]$$

The laminar velocity profile for flow in an annulus is given by

$$u = \frac{2V}{\left(\frac{1-K^4}{1-K^2}\right) - \left(\frac{1-K^2}{\ln 1/K}\right)} \left[1 - (r/r_o)^2 + \frac{1-K^2}{\ln 1/K} \ln r/r_o\right]$$

where K is the small tube to large tube radius ratio. At the end of the injector tip the flow pattern will be somewhere between that of an empty tube and that of flow in an annulus. Theoretical considerations would predict a carbon dioxide velocity which is twice the average air velocity for empty tube conditions in the laminar range and a velocity which is 1.56 times the average air velocity for flow in an annulus.

The effect of carbon dioxide to air velocity ratios on the center concentration was determined for heights of 4.75, 10, 20, and 30 inches

at a Reynolds number of 1500. The results are shown graphically in Figure 15. It is observed that at all heights the center concentration, C_O/C_A , reaches a minimum which occurs at a velocity ratio of approximately 1.9 for heights from 10 to 30 inches. This velocity ratio was used in the $N_{RE} = 1,500$ series where the smallest section height was 10 inches. At higher flow rates, $N_{RE} = 5,000$ and $N_{RE} = 10,000$, the heights above the injector tube were great enough to permit empty tube velocity distributions to re-establish; therefore, the carbon dioxide was released at a rate corresponding to the center-line velocity in empty tubes.

Theoretical Considerations

A theoretical analysis of the data is considered as a second point in the discussion of results. The opportunity is taken to compare theoretical predictions with observed results.

Concentration data

Two pertinent features can be noticed in the concentration profiles shown in Figure 13. It will be observed the concentration distributions for the three heights at each flow rate cross at a value of approximately 1.0 on the ordinate and a value of .64 on the abscissa. These phenomena can be explained by resorting to the mathematical techniques of the Bessel solution. Equation 49 can be written in the form

$$\frac{C}{C_M} = 1 + A_1 J_0(\beta_1 r) Z_1$$

The concentration distribution across the duct radius will vary from a

Figure 15. Variation of center concentration with carbon dioxide to air velocity ratio

value of C/C_M greater than one to a value of C/C_M less than one. However,

$$\frac{C}{C_M} = 1$$

only when $A_1 J_0(\beta_1 r) Z_1$ becomes zero. The first root for which $J_0(\beta_1 r)$ is zero occurs at a value of

$$\beta_1 r = 2.4048$$

from which

$$r = \frac{2.4048}{\beta_1} = \frac{2.4048}{1.904} = 1.263 \text{ inches}$$

and

$$\theta = \frac{1.263}{2.013} = .627$$

which compares reasonably well with the experimental results. The slight departure between the predicted and the experimental values can be attributed to the assumptions of constant velocity and diffusivity which are not strictly true.

For the semi-numerical method, $C/C_M = 1 + A_1 R_1(\theta) Z_1$ for which $R_1(\theta) = 0$ at the cross-over point. A θ value even more agreeable with the experimental value would necessarily be obtained by this method since it involves fewer assumptions than the Bessel technique.

Bessel plots

The data of this work were fitted to Equation 49 using a least squares method to establish the best linear relationship existing between C/C_A and $J_0(\beta_1 r)$. Values of diffusivity and Peclet number calculated in this manner were then limited to the accuracy of this procedure. A

graphical representation of Equation 49 is shown in Figure 16 where C/C_A is plotted against $J_0(\beta_1 r)$. It is evident that such plots do not form straight lines but exhibit definite degrees of curvature. Again these phenomena are results of the various assumptions involved, namely, constant diffusivity and plug-like velocity flow. The curves are shown to become more linear with increasing Reynolds number indicating applicability of this calculation method to conditions of turbulent flow. In fact, under highly turbulent isotropic conditions, the Bessel equation would be a rigorous description of the diffusion process. Unfortunately, practical application of this equation is limited.

As a point of interest Figure 17 has been drawn to illustrate the variation of the $J_0(\beta_n r)$ function with radial position. The β values were calculated from the roots of Equation 20, i.e., $J_1(\beta_n r_0) = 0$ where r_0 is the radius of the diffusion column. For $\beta = 0$, $J_0(\beta_0 r) = 1$ for all values of r and for this reason is not shown. Each β function originates at $r = 0$, $J_0(\beta r) = 1$ and oscillates about the value $J_0(\beta r) = 0$ with steadily decreasing amplitude approaching zero as a final limit. The amplitudes of the plots (for corresponding cycles) are identical and vary in period only. Ascending in order of β simply decreases the period in an inverse ratio to the β values.

Variation of diffusivity with radial position

The variation of total diffusivity with radial position as calculated by the semi-numerical method is shown graphically in Figure 18. In addition to significant radial variation, the diffusivity is seen to vary somewhat with height and decreases in magnitude as the wall is approached,

Figure 16. Variation of concentration with zero-order Bessel function of the first kind

Figure 17. Variation of $J_0(\beta r)$ with column radius

Figure 18. Variation of eddy mass diffusivity with radial position

finally reaching the value of the molecular diffusivity as a limit.

Shapes of the E versus θ curves of Figure 18 can be explained by theories postulated by von Karman, Prandtl, and Deissler which will be briefly reviewed at this point. One should bear in mind that these theories are designed to predict eddy momentum transfer and not eddy mass transfer. Application of these theories to mass diffusion is permissible only if mass and momentum are transferred by identical mechanisms. Some analogies do exist between these processes although not to the degree once believed.

When momentum is transferred in a fluid flowing through a pipe, two types of shear stress can occur. The shear stress produced by the molecular viscosity of the fluid is given by $\tau_v g_c = \mu du/dy$ and the turbulent shear stress is given by $\tau_t g_c = \rho \ell du/dy$ so that the expression for total shear can be written as

$$\tau g_c = \mu du/dy + \rho \ell du/dy \quad (96)$$

where ℓ is the eddy momentum diffusivity. At a distance y from a wall von Karman has postulated that the shear stress produced by a flowing fluid can be expressed by

$$\tau g_c = \rho K^2 \frac{\left(\frac{du}{dy}\right)^4}{\left(\frac{d^2u}{dy^2}\right)^2} \quad (97)$$

where K is a constant of proportionality. Neglecting the viscous shear stress, a combination of Equations 96 and 97 produces

$$\ell = \frac{K^2 \left(\frac{du}{dy}\right)^3}{\left(\frac{d^2u}{dy^2}\right)^2} \quad (98)$$

This equation is valid and can be used to describe transport phenomena except at regions near the wall where viscosity effects become important.

According to the Prandtl mixing length theory, eddy diffusivity is proportional to a deviating velocity, v' , and a mixture length, l . Now if it can be assumed that these two quantities are in turn proportional to u and y respectively, then

$$\mathcal{E} = n^2 u y \quad (99)$$

where n^2 is a proportionality constant. Experimental measurements have shown that this equation does not give an accurate prediction of eddy diffusivity for distances very near the wall. Near the wall, the observed decrease is greater than that predicted by Equation 99. A consideration of this feature has been made in a recent analysis of transport phenomena by Deissler (8).

Deissler (8) in his analysis has assumed that the factors upon which the eddy momentum diffusivity depends can be expressed by

$$\mathcal{E} = f(u, y, \mu/\rho, du/dy, d^2u/dy^2 \dots) \quad (100)$$

Equation 98 could also be obtained from a dimensional analysis of the functional relation shown by Equation 100 if effects of u , y , and μ/ρ are assumed negligible. However, in regions of low turbulence (near the wall), the effects of kinematic viscosity cannot be neglected since viscous effects become appreciable in relation to the inertia effects, and the functional relationship assumed is

$$\mathcal{E} = f(u, y, \mu/\rho) \quad (101)$$

The derivatives are here neglected since in nearly laminar flow du/dy approaches u/y which is already included in the relation, and d^2u/dy^2 approaches zero. From a dimensional analysis of Equation 101 and other considerations, Deissler has presented the following form for the eddy momentum diffusivity:

$$\epsilon = n^2 u y (1 - e^{-n^2 u y \rho / \mu}) \quad (102)$$

The term, $1 - e^{-n^2 u y \rho / \mu}$, is a correction factor used to account for the exponential decrease in eddy diffusivity near the wall. For conditions of highly turbulent flow, Equation 98 holds while Equation 102 is applicable to low levels of turbulence.

These equations can be used to qualitatively predict the variation of E with radial position if momentum and mass transfer processes are assumed to occur by similar mechanisms. Application of Equations 98 and 102 to Figure 18 can now be illustrated. Consider first the curves drawn for Reynolds numbers of 1,500 and 5,000 where relatively low levels of turbulence are present and Equation 102 can be used to describe the process. At the tube center, the diffusivity will be a maximum since u and y are large and the negative exponential term is unimportant at this point. With further increase in radial position (decreasing in y) the exponential term increases (approaching one as a limit) and the eddy diffusivity drops toward zero. In Figure 18 where total diffusivities are used, the limiting value at the wall is that given by the molecular contribution, D_v .

If mass and momentum diffusivities are for the moment assumed equal, Equation 98 can be used to account for the variation of E shown

in the $N_{RE} = 10,000$ curves. In the area about the tube center $(du/dy)^3$ increases faster than $(d^2u/dy^2)^2$ for increasing values of θ (decreasing y) so that E increases steadily. As the tube wall is approached, the flow becomes less turbulent and so, the mass transfer process becomes influenced by the relationship shown in Equation 102. A point is finally reached at the approximate mid-radius when the effects of Equations 98 and 102 counterbalance each other and a peak in the radial distribution of diffusivity occurs. Beyond the mid-radius the phenomena become controlled by Equation 102 where an exponential decrease is observed with further increase in radial position.

Peclet number curves

A dimensionless quantity known as the Peclet number frequently arises in mass transfer studies. This group gives a measure of the ratio of eddy momentum diffusivity to eddy mass diffusivity and is defined mathematically by

$$N_{PE} = D_t u / E$$

where D_t is the tube diameter. According to Prandtl's mixing length theory, diffusivity is proportional to the product of scale and intensity, i.e.,

$$E \propto l v'$$

where l is the mixture length (scale of turbulence) and v' is the deviating velocity (intensity of turbulence). If l and v' are in turn proportional to D_t and u respectively, E will also be proportional to these quantities and

$$N_{PE} = \frac{D_t u}{K D_t u} = \text{constant} \quad (103)$$

where K is a constant of proportionality. This statement says in effect that if D_t and u correctly measure the scale and intensity of turbulence the ratio of eddy momentum to eddy mass diffusivity will be constant. This assumption has a rough theoretical basis and can be tested by comparing the average Peclet number with Reynolds number. Experimentally, however, the assumption is found to be valid under certain turbulent conditions only as will be shown in a later discussion.

Figure 19 displays the variation of point values of mass transfer Peclet number with radial position. The variation shown can be explained by a simple analysis of the velocity distribution also keeping in mind arguments given in the discussion of the diffusivity variation. The increase in Peclet number with radial position is due to the fact that the turbulent diffusivity is decreasing at a faster rate than the velocity until some peak value is obtained where these two effects are equal. With a further increase in radial position the velocity rapidly falls off toward zero while the total diffusivity approaches the molecular diffusivity. The maximum Peclet points resulting from this phenomenon could not be detected experimentally as they occurred very near the tube wall. For this reason, the maximums are represented by dotted lines on the plots.

A striking comparison of Peclet numbers between gaseous and liquid systems is illustrated in Figure 20 as a function of flow rate. Data for the liquid system were obtained from a work by Seagrave (40). Calculations in this work and that of Seagrave's were made using the mathematical

Figure 19. Variation of mass transfer Peclet number with radial position

Figure 20. Variation of Peclet number with Reynolds number for gas and liquid systems

technique developed by Fahien and Smith (12) so that Peclet numbers shown in Figure 20 then were calculated in an identical manner using average values of velocity and the arithmetic average of the point values of diffusivity. A constant and equal value of Peclet number is approached in both systems as the Reynolds number increases into the turbulent range in accordance with the predictions of the mixing length theory, Equation 103. The near equality of Peclet numbers at this flow rate is not accidental but has theoretical basis. At high flow rates the shear stress in a moving fluid is due primarily to the eddy momentum transfer process and can be expressed as

$$\tau_o g_c = \epsilon \rho \frac{\int_0^{u_o} du}{\int_0^{D_t/2} dy} = \epsilon \rho \frac{V}{y} = \frac{2 \epsilon \rho V}{D_t}$$

Here τ_o and $\epsilon \rho$ are assumed constant over the tube cross section for illustrative purposes. Also from empirical measurements,

$$\tau_o = \frac{f V^2 \rho}{2 g_c}$$

Combining

$$\frac{f V}{2} = \frac{2 \epsilon}{D_t}$$

Rearranging

$$\frac{D_t V}{\epsilon} = \frac{4}{f} = N_{PE} \text{ (momentum)}$$

If as the mixing length theory postulates, momentum transfer occurs by the same mechanism as mass transfer, then the mass transfer Peclet number should be the same as the momentum transfer Peclet number. Furthermore, the Peclet number will vary as the friction factor varies according to the relation above. At high values of Reynolds numbers ($N_{RE} = 10,000$), the friction factor versus Reynolds number curve becomes flat so the asymptotic limit approached by the Peclet numbers in Figure 20 is answered satisfactorily. This reasoning explains why the Peclet numbers for the two systems reach a constant value and also why they are nearly equal. This is due to the fact that under highly turbulent conditions, the eddy transport properties are not functions of physical properties of the fluid as viscosity and density as is evident from friction factor plots which correlate many types of fluids to a single curve. The experimental values of N_{PE} at $N_{RE} = 10,000$ are very nearly equal with the discrepancy probably due to experimental error. Seagrave in his work used a smooth brass pipe (4-inch diameter) while galvanized iron was used in this work so that differences in pipe roughness would then account for some of the variation. Based on a mean average, the deviation between the Peclet number values at $N_{RE} = 10,000$ for the two systems is 11 per cent which is about the same degree of certainty available on friction factor curves. As the Reynolds number decreases, the Peclet numbers for the liquid system rise while those for the gas system drop. This is because molecular effects are becoming important and physical properties of the fluid are beginning to have their influence. Also, molecular diffusivities for gases are much higher than those for liquids so that at very low rates

of flow Peclet numbers in gas systems will always be lower than those in liquid systems.

An average of the calculated E values at $N_{RE} = 1,500$ was used to determine the Peclet number at this flow rate. It will be shown in the section "Special Considerations" that these values are somewhat high; however, they were used since they do give a qualitative illustration of the N_{PE} variation for gaseous systems. Seagrave also experienced abnormally high E 's at $N_{RE} = 1,500$ which resulted in a correspondingly low value of N_{PE} . Since this value of N_{PE} was not consistent with his other measurements and is not representative of the N_{PE} variation in liquid systems, it has been excluded from Figure 20.

Position Peclet number curves

Point values of momentum diffusivity and mass diffusivity can be evaluated by defining a modified position Peclet number as

$$N_{PE}' = \frac{yu}{E}$$

where y is the distance from the wall and is taken as a measure of the scale of turbulence. In the previous definition of Peclet number, the tube diameter was used as a measure of the size of eddies (scale of turbulence). This approximation is reasonably justified at distances removed from the wall; however, at distances very near the wall the scale would be expected to depend not so much on the tube size but more on the actual distance from the wall boundary. As before, the intensity of turbulence is assumed proportional to the velocity. Since $y = r_0 - r$, the relationship between the position Peclet number and the conventional Peclet number can be written

Figure 21. Variation of position Peclet number with radial position

$$N_{PE}' = 1/2 N_{PE} (1 - \theta)$$

The variation of N_{PE}' with θ is illustrated in Figure 21. If N_{PE} is a constant with respect to θ , then N_{PE}' should decrease linearly with θ . Such a linear relationship is observed on curves drawn for $N_{RE} = 5,000$ and $N_{RE} = 10,000$ at high values of θ . A similar but somewhat less noticeable effect is present in the $N_{RE} = 1,500$ series. This phenomenon is not to be taken as a validation of the initial assumptions since it has been shown that N_{PE} is not a constant with θ ; therefore, the linear decrease observed is not a consequence of this premise.

If in the defining equation for N_{PE}' , y and u correctly measure the scale and intensity of turbulence, N_{PE}' would be expected to be constant except at regions removed from the wall where the size of eddies are not controlled by the distance from the wall. As evident from Figure 21, no such relationship exists suggesting either that y , u , or both y and u are not correct measures of the scale and intensity or perhaps the omission of important terms. A consideration of the velocity gradients, du/dy and d^2u/dy^2 , as proposed by von Karman, is seen to provide a more realistic description of the turbulent mechanisms for the central portion of a tube. At regions very near the wall, the molecular viscosity and density become important as shown by Deissler. This accounts for the fact that N_{PE}' is not constant in this region either.

A modified position Reynolds number can be defined in a manner similar to the modified position Peclet number. This number gives a measure of point flow conditions (i.e. the ratio of point eddy momentum diffusivity to molecular diffusivity) and is defined by

$$N_{RE}' = \frac{y u \rho}{\mu} = 1/2 N_{RE} (1 - \theta)$$

The variation of N_{PE}' with N_{RE}' is shown in Figure 22. In order to explain the curves drawn, it will be necessary to consider Deissler's equation for eddy momentum diffusivity at regions near the wall, i.e.,

$$\epsilon = n^2 u y (1 - e^{-n^2 u y \rho / \mu})$$

A series expansion of the exponential terms takes the form

$$e^{-n^2 u y \rho / \mu} = 1 + (-n^2 u y \rho / \mu) + \frac{(-n^2 u y \rho / \mu)^2}{2} + \dots$$

For very small values of u and y ,

$$e^{-n^2 u y \rho / \mu} \approx 1 - n^2 u y \rho / \mu$$

so that

$$\epsilon = n^4 u^2 y^2 \rho / \mu \quad (104)$$

Rearranging leads to

$$\frac{u y}{\epsilon} = \frac{\mu}{n^4 u y \rho}$$

or

$$(N_{PE}')_{\text{momentum}} = \frac{1}{n^4 N_{RE}'}$$

This equation predicts that N_{PE}' will vary inversely with N_{RE}' at regions near the tube wall or for conditions of low flow rate where molecular effects are important. Such a region corresponds to low values of N_{RE}' .

Figure 22. Variation of position Peclet number with position Reynolds number

in Figure 22 where it is noticed that no inverse relationship is observable. This is probably because eddy effects are still significant even in this so-called laminar region and this phenomenon occurs so near the tube wall that it evades experimental detection.

At larger distances from the wall, the exponential term in Deissler's equation becomes unimportant and so

$$N_{PE'} = \frac{1}{n^2} = \text{constant}$$

Such a region of constant position Peclet number is shown in the $N_{RE} = 5,000$ curve to a moderate degree while a more sharply definable region is evident in the $N_{RE} = 10,000$ plot.

At still larger distances, turbulence depends upon the velocity gradients as given by Equation 98 and $N_{PE'}$ will behave according to

$$N_{PE'} = \frac{yu \left(\frac{d^2u}{dy^2} \right)^2}{K^2 \left(\frac{du}{dy} \right)^3}$$

For $N_{RE} = 1,500$, d^2u/dy^2 is a constant while y and du/dy are linear functions of θ as can be observed from the parabolic flow equation so that $N_{PE'}$ will vary as $yu/(du/dy)^3$. It will be noticed that y and u increase with N_{RE} while $(du/dy)^3$ decreases with N_{RE} . The resulting increase of $N_{PE'}$ with N_{RE} is evident. At Reynolds numbers of 5,000 and 10,000, $(du/dy)^3$ increases faster than $(d^2u/dy^2)^2$ for increasing values of θ (decreasing values of N_{RE}) so $N_{PE'}$ decreases exponentially with N_{RE} .

It has been mentioned that the increase in eddy diffusivity as prophesied by Deissler's equation (Equation 102) for increasing y (decreasing θ) was detected in this work, to some extent, for all flow conditions studied. The von Karman expression (Equation 98) for distances away from the wall, qualitatively predicts the observed increase in eddy diffusivity with θ in the $N_{RE} = 10,000$ runs. However, this equation also states that $\epsilon = 0$ at the tube center where $du/dy = 0$ and $d^2u/dy^2 \neq 0$. This is not possible since

$$\lim_{\theta \rightarrow 0} (\mu + \rho \epsilon) = \lim_{\theta \rightarrow 0} \frac{\tau_o \theta g_c}{\frac{du}{dy}} = \lim_{\theta \rightarrow 0} \frac{-\tau_o g_c}{\frac{d^2u}{dy^2} r_o} \neq 0$$

If applied at the other boundary, the tube wall, the von Karman equation also fails since it predicts that $\epsilon \neq 0$ at this boundary. The inconsistencies just pointed out can be reconciled by a combination of the von Karman and Deissler equations written as

$$\epsilon = \left[k^2 \frac{(du/dy)^3}{(d^2u/dy^2)^2} + n^2 u y \right] [1 - e^{-n^2 u y \rho / \mu}] \quad (105)$$

The first term containing the velocity derivatives may be regarded as a "gradient generated turbulence" while the second term may be thought of as a "tube generated turbulence". The exponential term becomes important at distances very close to the wall and drops out as the tube center is approached. This equation gives the proper form for diffusivity at all points and permits N_{PE} , (momentum) to vary with θ and N_{RE} , in a manner consistent with the data of this work (Figures 21 and 22).

Eddy viscosity

Since great mention has been made of the fact that the eddy momentum diffusivity (also eddy viscosity) behaves in a manner similar to eddy mass diffusivity, it will be worth while to investigate this point further.

For turbulent flow

$$\tau g_c = (\mu + \rho \epsilon) \frac{du}{dy} = \tau_o g_c \frac{r}{r_o} \quad (106)$$

and the empirical correlation for shear stress in round pipes has the form

$$\tau_o = \frac{f v^2 \rho}{2 g_c} \quad (107)$$

where f is a dimensionless friction factor. An expression for eddy momentum diffusivity can be obtained by solving Equation 106 for ϵ ,

$$\epsilon = \frac{\tau_o g_c \frac{r}{r_o} - \mu \frac{du}{dy}}{\rho \frac{du}{dy}} \quad (108)$$

or in terms of $du/d\theta$

$$\epsilon = \frac{\tau_o g_c \frac{r}{r_o} + \mu \frac{du}{d\theta}}{-\rho \frac{du}{d\theta}} \quad (109)$$

Equation 109 was used to calculate values of eddy viscosity, $\rho \epsilon$, which are shown as a function of radial position in Figure 23. Eddy viscosity is shown to be nearly constant over the central portion of the tube and decreases toward zero as the wall is approached. This distribution can be predicted by observation of Equation 109. The term $\mu du/d\theta$ in the

Figure 23. Variation of eddy viscosity with radial position

numerator of Equation 109 is very small and need not be considered for small values of θ . Since $du/d\theta$ is proportional to r at distances near the tube center, the eddy viscosity will be constant in this region. At higher values of θ (for $N_{RE} = 5,000$), however, $\mu du/d\theta$ becomes significant since $du/d\theta$ increases rapidly in a negative direction so that a decrease in eddy viscosity is noted with further increase in θ . A similar effect is present in the $N_{RE} = 10,000$ curve, the difference being that the $\mu du/d\theta$ term is negligible through most of the tube radius. The eddy viscosity profiles are then very similar to the reciprocal of $du/d\theta$.

An analogy between the transfer of momentum and the transfer of mass can be made by a comparison of Figures 23 and 18. Curves for the two sets of data bear a reasonable qualitative relationship so that on this basis, momentum transfer can be described as being analogous to mass transfer. Actually this relationship would probably be better defined if the eddy viscosity could be determined more accurately. This point will be made clear by considering the method used for calculating eddy viscosities. The shear stress at the tube wall, τ_o , was evaluated from Equation 107 using friction factor correlations given by Perry (34 p.383). Since data points for the velocity distribution at points very near the tube wall were not available, a severe extrapolation of the velocity derivatives was necessary at this point. To aid in this extrapolation and to maintain consistent boundary limits, use of Equations 106 and 107 was made. The shear stress at the wall was evaluated from Equation 107 and substituted into Equation 106 to calculate du/dy at $\theta = 1$ also noting that $\varepsilon = 0$ at this point. Values of velocity derivatives at

other radial positions were determined from experimental data. These derivatives were calculated graphically from an enlarged duplicate of Figure 14, plotted against θ , and fitted with a smooth curve. This procedure was used since the calculation given by Equation 109 is very sensitive to $du/d\theta$.

Schmidt number curves

A measure of the ratio of the eddy momentum transfer rate to the eddy mass transfer rate can be illustrated by a dimensionless group known as the turbulent Schmidt number. Mathematically it is defined by

$$N_{SC} = \frac{\epsilon}{E_t}$$

This ratio is quite important in developing analogies between momentum and mass transport processes. Deissler in his analysis of transport relationships assumed the Schmidt group was unity due to lack of experimental data. Schmidt numbers for $N_{RE} = 5,000$ and $N_{RE} = 10,000$ are shown in Figure 24 as a function of θ . Total effective diffusivities were used in calculating values of the Schmidt group. This procedure was thought to be reasonable since the molecular contribution was very small. The single curve drawn for each flow rate was obtained by using an arithmetic average of E for corresponding radial positions at each of the three heights. The departure from unity is evident in the $N_{RE} = 10,000$ curve while a remarkable relationship exists in the $N_{RE} = 5,000$ curve. It is possible that the Schmidt number curves (Figure 24) and the Peclet number curves (Figure 19) may be somewhat related since both groups may be a measure of the same ratio. This comparison provides a

Figure 24. Variation of turbulent Schmidt number with radial position

measure of the validity of the scale and intensity parameters, D and u , of the Peclet number. The $N_{RE} = 5,000$ curves for the two dimensionless groups agree qualitatively in that both groups are nearly constant at low values of θ and exhibit a maximum at higher radial positions. A similarity exists for the $N_{RE} = 10,000$ curves at points up to the mid-radius where a decreasing function is occurring. Beyond this point the analogy fails. However, if Equation 99 is valid under assumed conditions of turbulent flow, it can be combined with the defining equation for the turbulent Schmidt group to produce

$$\frac{uy}{E} = \frac{N_{SC}}{n} = N_{PE}'$$

Thus, N_{SC} would be expected to vary with θ in the same way that N_{PE}' varies. The N_{SC} versus θ curve at $N_{RE} = 10,000$ bears a strong resemblance to the N_{PE}' versus θ curve as both are decreasing functions with increasing θ . This indicates that a reasonable measure of the scale and intensity is provided by the y and u parameters of the position Peclet number at this flow rate.

The dimensionless groups of N_{PE}' and N_{SC} show absolutely no similarity for the $N_{RE} = 5,000$ curve. However, such a similarity cannot be expected since the assumptions involved do not apply. If the exponential term in Deissler's equation is important, then Equation 99 cannot be used to describe the process. Instead, the defining equation for N_{SC} must be combined with Equation 104 to produce

$$N_{PE}' = \frac{N_{SC}}{n^4 N_{RE}'}$$

The Schmidt group would be expected to be proportional to the product of N_{PE} and N_{RE} . A study of Figures 21 and 22 contradicts this reasoning which leads to the conclusion that at $N_{RE} = 5,000$, the product yu does not give an accurate measure of the turbulence generated. Since mass and momentum transport processes are a result of combined effects, no single group of variables can be expected to give a complete correlation of the data.

Special Considerations

A final point considered in this discussion is a presentation of the various features displayed by the results due to the assumptions used. The modification and development of equations for the purpose of estimating longitudinal diffusion and relating diffusivity to position and flow rate are presented.

Variation of diffusivity with longitudinal position

The variation E exhibits with height above the injector tube can be studied by referring to the calculation techniques used. In calculating E by the semi-numerical method the constant multiplying factor, λ , was evaluated from Equation 83. Values of E calculated by this method then represent an average E over the distance $z = 0$ to $z = z$. For example, an E value calculated at $N_{RE} = 5,000$ at $z = 50$ inches represents an average E between $z = 0$ and $z = 50$ inches. Similarly at 70 inches, an average between $z = 0$ and $z = 70$ inches is calculated. Now, if diffusivity actually varies with height, then, of course, different E values will be calculated in each case. Use of the alternate boundary condition described in the section "MATHEMATICAL TREATMENT" would serve to minimize this

effect. The same arguments can be used to explain the variation of E with z as calculated by the Bessel solution.

Another factor that may be responsible for the variation of E with z may be due to effects of the injector tube which might cause a variation of u with z . This possibility is probably unimportant since the velocity data did not show any abnormal features due to the presence of the injector tube, and the assumption that velocity is independent of height appears to be a reasonable one.

Finally, the effect of time on diffusion and the effects of longitudinal diffusion to be discussed shortly may account for some of the variation of E with z .

A plot of $\ln Z$ versus z provides a method by which the variation of E with z may be studied. If any of the preceding arguments are valid, a linear relationship will not exist between these quantities. Referring to Figure 25, it is obvious that some variation does exist. The deviation from linearity is extremely small at $N_{RE} = 10,000$ since a near perfect straight line can be fitted through all three data points. A somewhat noticeable though negligible deviation is shown by the $N_{RE} = 5,000$ data points while a marked departure from linearity is illustrated by the $N_{RE} = 1,500$ values.

Longitudinal diffusion

Diffusion in the direction of flow has been generally neglected in radial diffusion studies because of the intricacy it introduces in the mathematical treatment. Two factors seem to justify this assumption. First, axial diffusion is thought to be small due to lack of steep axial

gradients and second, mixing due to the bulk movement of gas in the direction of flow usually masks effects of axial diffusion. This is especially true at high flow rates, but not necessarily so at low rates of flow. A consideration of the mathematics involved will serve to illustrate this point.

A careful scrutiny of the eigen value as defined by Equation 53 is in order. The constancy of this value will be subject to justification of the various assumptions involved in the derivation of Equation 51; namely that axial diffusion is negligible. Rewriting the basic diffusion equation (Equation 9) in terms of θ quantities and rearranging

$$\frac{\partial}{\partial \theta} (E_r \theta \frac{\partial C}{\partial \theta}) = r_o^2 \theta \frac{\partial}{\partial z} (C_u - E_a \frac{\partial C}{\partial z}) \quad (110)$$

This equation makes no assumption regarding the absence of axial diffusion; hence, it may be used to define what may be called a "true" eigen value, λ_T . Again assume a solution of the form $C = R(r) Z(z)$, substitute into Equation 110, and separate the variables to get

$$\frac{d}{u \theta R d\theta} (E_r \theta \frac{dR}{d\theta}) = - \lambda_T = \frac{r_o^2}{Z} \frac{d}{dz} (Z - \frac{E_a}{u} \frac{dZ}{dz}) \quad (111)$$

Now if E_r is not a function of z and $E_a Z''/u$ can be assumed negligible, the eigen value as defined by Equation 111 will be identical to that as given by Equation 53. Furthermore, if the assumptions involved are valid then the eigen value will be a true constant, i.e., λ will be independent of height. In this work values of λ were calculated by assuming $E_a Z''/u$ is negligible and are tabulated in Table 5e of the appendix. Rearrangement of Equation 54 gives

$$\ln Z = - \frac{\lambda}{r_o^2} z \quad (112)$$

A curve of slope = $-\lambda/r_0^2$ results when $\ln Z$ is plotted versus z as shown in Figure 25 where it will be noted that the curve must pass through the point (0,0). Points at Reynolds numbers of 5,000 and 10,000 in addition to being equal, form a single curve of constant slope. Since the eigen values are both equal and constant, the neglect of longitudinal diffusion at these flow rates appears justified. Inspection of the curve drawn for a Reynolds number of 1,500 reveals a curve of constantly changing slope, suggesting that axial diffusion becomes important at this flow rate and should not be neglected (or possibly that the time allowed for diffusion is insufficient). These points will be discussed again where further evidence of this phenomenon is presented. It is possible to estimate the relative importance of longitudinal diffusion by a method to be described shortly.

One obvious feature regarding the calculated values of diffusivity concerns the relative magnitude of E for the different flow rates used. Much higher values of diffusivity were calculated for the low flow rate ($N_{RE} = 1,500$) which were entirely unexpected in view of present theories on turbulence. One would expect greater turbulence at higher rates of flow and therefore a correspondingly higher value of E . Arithmetic averages of diffusivities calculated by both solutions to the diffusion equation are shown as a function of Reynolds number in Figure 26. The expected linear relationship of E with V breaks off at a Reynolds number of approximately 5,000 and experiences a sharp increase. The apparent failure of this phenomenon to hold can be a result of two factors. First, it will be remembered that axial or longitudinal diffusion was neglected in the development of the semi-numerical solution. However, recent work by

Figure 25. Variation of $\ln Z$ with height

Tichacek, et al. (51) and McHenry (30) has indicated that this assumption is not valid; in fact, the axial contribution to diffusion may be many times that of radial diffusion at low rates of flow. A second factor considers the effect of time on rates of diffusion. Hanratty, et al. (16) have postulated that diffusion also depends on time, reaching a constant value for large diffusion times. Thus the diffusion times allowed in the $N_{RE} = 1,500$ series where low section heights were required may not have been sufficient to establish a steady-state condition. This same type of discrepancy was reported in a similar work by Frandolig (14).

The data of this study appear to be reasonably well correlated by both solutions to the general diffusion equation except for the lowest flow rate ($N_{RE} = 1,500$) where the assumptions involved are not legitimate. Apparent failure of the mathematical techniques in the laminar range is probably due to the omission of the axial term. It will be useful to estimate the relative importance of longitudinal diffusion at this flow rate since flow in this region is of great concern in many types of industrial equipment. As mentioned previously, the consideration of an axial diffusion term (Equation 9) greatly complicates the mathematical analysis but as the data of this study indicate, this problem cannot always be neglected.

The scope of this work was such that the critical range of Reynolds numbers at the highly laminar to the low end of the turbulent range could be studied. This range makes it possible to determine the importance of axial diffusion at low rates and justify its neglect at high rates of flow. Axial contributions to the diffusion process can be estimated by solving Equation 111 for the case where longitudinal diffusion is not assumed equal to radial diffusion but where E_r and E_a are independent of r and z , and

Figure 26. Variation of average diffusivity with Reynolds number

$u = V$, as in the Bernard solution. A comparison of this solution to that of the semi-numerical method will give an insight to the importance of the axial term. Using the assumptions stated, a comparison of the right hand members of Equations 53 and 111 will show the following relationships exist.

$$\lambda_T = \frac{r_o^2}{Z} \frac{d}{dz} \left(Z - \frac{E_a}{u} \frac{dZ}{dz} \right) \quad (113)$$

$$\lambda = \frac{r_o^2}{Z} \frac{dZ}{dz} \quad (114)$$

Now if axial diffusion is negligible, $E_a Z'/u = 0$ and Equation 113 becomes equivalent to Equation 114. Equation 113 can be used to estimate the axial diffusivity once the Z functions are evaluated. It will be convenient at this point to write Equation 113 as a logarithmic function of Z . Since

$$\frac{dZ}{dz} = Z \frac{d \ln Z}{dz}$$

it follows that

$$\frac{d^2 Z}{dz^2} = Z \frac{d^2 \ln Z}{dz^2} + Z \left(\frac{d \ln Z}{dz} \right)^2$$

With these substitutions Equation 113 becomes

$$\lambda_T = r_o^2 \left[\frac{d \ln Z}{dz} - \frac{E_a}{u} \left(\frac{d^2 \ln Z}{dz^2} + \left[\frac{d \ln Z}{dz} \right]^2 \right) \right] \quad (115)$$

Use of the $\ln Z$ term is more desirable than the Z function since $\ln Z$ versus z is more nearly linear thus enabling the derivatives to be determined more accurately. Solving Equation 115 for E_a results in

the form

$$E_a = \frac{\left[\frac{\lambda_T}{r_o^2} + \frac{d \ln Z}{dz} \right] (u)}{\frac{d^2 \ln Z}{dz^2} + \left(\frac{d \ln Z}{dz} \right)^2} \quad (116)$$

For Reynolds numbers of 5,000 and 10,000, $d \ln Z/dz$ is constant, therefore, $d^2 \ln Z/dz^2 = 0$ and $\lambda = \lambda_T$. Equation 116 will then reduce to

$$E_a = \frac{\left[\frac{\lambda}{r_o^2} + \frac{d \ln Z}{dz} \right] (u)}{\left(\frac{d \ln Z}{dz} \right)^2} \quad (117)$$

Solving Equation 112 for λ and substituting into Equation 117 gives

$$E_a = \frac{\left[\frac{d \ln Z}{dz} - \frac{d \ln Z}{dz} \right] (u)}{\left(\frac{d \ln Z}{dz} \right)^2} = 0$$

This states that omission of the axial term at Reynolds numbers of 5,000 and 10,000 is permissible since longitudinal diffusion is not important here. The constancy of the eigen value (Table 5e) is further prove that Equation 114 is valid at these flow rates.

At a Reynolds number of 1,500, however, $d \ln Z/dz$ is not a constant as evident from Figure 25. For purposes of estimation suppose that the $\ln Z$ curve for $N_{RE} = 1,500$ is straight in the interval from $z = 10$ inches to $z = 20$ inches. This assumption is reasonable in the small interval chosen. The slope of the curve in this range, as determined

from experimental data, is given by $d \ln Z/dz = -1.63 \text{ ft}^{-1}$. Also, $d^2 \ln Z/dz^2$ will be zero in view of the assumption used. Using an average eigen value of 0.0637 (Table 5e) and substituting values of the functional variables into Equation 116 result in

$$E_a = \frac{\left[\frac{.0637}{.1678^2} - 1.63 \right] (.7342)}{0 + (1.63)^2} = .173 \text{ ft}^2/\text{sec}$$

Comparing this value to the average value of E as calculated by the semi-numerical method results in the expression

$$\frac{E_a}{E_r} = \frac{0.173}{0.00421} = 41$$

The radial diffusivity used above actually contains effects of longitudinal diffusion; therefore, this ratio should be regarded as a minimum. If E_r at $N_{RE} = 1,500$ obeys the relationship

$$E_r = KV + D_v$$

as given by Figure 26, then the ratio of axial to radial diffusivity becomes

$$\frac{E_a}{E_r} = \frac{0.173}{0.00062} = 279$$

The purpose of the calculations presented here is directed toward establishing the importance of the axial coefficient. McHenry and Wilhelm (30) reported in their studies of mass transfer in packed beds that the axial diffusion contribution was approximately sixfold that of

radial diffusion for Reynolds numbers between 100 and 400. The results obtained in this work appear to be consistent with their values inasmuch as entirely different experimental and calculation techniques were employed.

Although the mathematical techniques applied to turbulent transfer processes have improved in recent years, a great deal of work is still required in this area. Great promise is afforded through modern day developments in the use of high-speed computers. Use of these machines is seen to provide a more basic technique for the rigorous treatment of heat, mass and momentum transfer data.

Folium curves

An interesting picture on how diffusivity varies with radial position can be obtained from Equation 51 where longitudinal diffusion has been neglected. Equation 51 can be written

$$\partial(E\theta \frac{\partial C}{\partial \theta}) = (r_o^2 u\theta \frac{\partial C}{\partial z}) \partial\theta \quad (118)$$

Integrating and solving for E gives

$$E = \frac{\int_0^\theta (r_o^2 u\theta \frac{\partial C}{\partial z}) \partial\theta}{\theta \frac{\partial C}{\partial \theta}} \quad (119)$$

It will now be shown that the above expression is equivalent to Equation 89 and can be used to define an average or characteristic value of E.

Differentiation of $C = C_M + A_1 R_1 Z_1$ with respect to z results in

$$\frac{\partial C}{\partial z} = A_1 R_1 \frac{\partial Z}{\partial z} \quad (120)$$

Since $Z = e^{-\lambda z/r_0^2}$ the preceding expression becomes

$$\frac{\partial C}{\partial z} = A_1 R_1 \frac{\partial(e^{-\lambda z/r_0^2})}{\partial z} = A_1 R_1 Z_1 \left(-\frac{\lambda}{r_0^2}\right) \quad (121)$$

Substitution of Equation 121 into Equation 119 gives

$$E = -\lambda A_1 Z_1 \frac{\int_0^\theta u \theta R_1 \partial \theta}{\theta \frac{\partial C}{\partial \theta}} \quad (122)$$

At sufficiently short distances

$$R_1 = \frac{C - C_M}{A_1 Z_1} \quad (123)$$

also

$$\int_0^\theta u \theta (C - C_M) \partial \theta = \Delta \theta \sum_0^k P_k (C_k - C_M) = h \sum_0^k P_k (C_k - C_M) \quad (124)$$

and

$$\theta \frac{\partial C}{\partial \theta} = \theta_k \left(\frac{C_{k+1} - C_k}{\Delta \theta} \right) = \theta_k \left(\frac{C_{k+1} - C_k}{h} \right) \quad (125)$$

Substitution of Equations 123, 124, and 125 into 122 results in

$$E = \frac{\lambda h^2 \sum_0^k P_k (C_k - C_M)}{\theta_k (C_k - C_{k+1})} \quad (126)$$

Thus, Equations 120 and 119 can be written

$$\frac{E}{V} = \frac{\frac{1}{V} \int_0^\theta (r_o^2 u \theta \frac{\partial C}{\partial z}) d\theta}{\theta \frac{\partial C}{\partial \theta}} = \frac{\lambda h^2 \sum_0^k \frac{P_k}{V} (\frac{C_k - C_M}{C_A})}{\theta_k (\frac{C_k - C_{k+1}}{C_A})} = \lambda h^2 \frac{y}{x} \quad (127)$$

where

$$y = \sum_0^k \frac{P_k}{V} (\frac{C_k - C_M}{C_A})$$

and

$$x = \theta_k (\frac{C_k - C_{k+1}}{C_A})$$

A plot of the summation term in the numerator against the denominator will produce a curve such as the one shown in Figure 27 where the experimental points are seen to produce geometrically similar loops. The slope of a line drawn from the origin to any point on a loop will obey the relationship

$$\text{slope} = \frac{y}{x} = \frac{E}{\lambda h^2 V} \quad (128)$$

Values of increasing θ in Figure 27 follow the loops in a clockwise fashion, originating and finally ending at the origin.

Equation 127 can also be written in terms of Bessel functions. Remembering that $P_k = u\theta$ and $C_k - C_M = A_1 J_0(\beta_1 r) Z_1$, the following alterations can be made.

Figure 27. Foliums of diffusivity

$$E = \frac{\lambda h^2 v \sum_0^k \frac{P_k}{V} \left(\frac{C_k - C_M}{C_A} \right)}{\theta_k \left(\frac{C_k - C_{k+1}}{C_A} \right)} = \frac{\lambda h v \int_0^{\theta} \frac{u \theta}{V} A_1 J_0(\beta_1 r) Z_1 d\theta}{-\theta \left(\frac{\partial C}{\partial \theta} \right) h} \quad (129)$$

The Bessel solution assumed the velocity to be everywhere constant so that $u/v = 1$. Use of this equality together with the relations

$$\int_0^{\theta} \theta J_0(\beta_1 r) d\theta = \frac{\theta J_1(\beta_1 r)}{\beta_1 r_0}$$

and

$$\frac{\partial C}{\partial \theta} = 0 = r_0 \frac{\partial}{\partial r} [C_M + A_1 J_0(\beta_1 r) Z_1] = -r_0 \beta_1 J_1(\beta_1 r) A_1 Z_1$$

reduces Equation 129 to

$$E_B = \frac{\lambda v}{\beta_1^2 r_0^2} \quad (130)$$

where E_B is the diffusivity as calculated by the β constant of the Bessel method. Solving Equations 128 and 130 for λ produces

$$\frac{E}{h^2 v \left(\frac{Y}{X} \right)} = \lambda = \frac{E_B \beta_1^2 r_0^2}{v} \quad (131)$$

The diffusivity given by the left-hand equality can be any value of E , that is, E may or may not be equal to E_B or E_C , a quantity that will be defined next.

A characteristic diffusivity, E_C , can be defined as that value of E

determined by the slope of a line drawn from the origin through the maximum y point of the loops. This maximum is determined by the same straight line on the curves drawn for Reynolds numbers of 1,500 and 5,000. Furthermore, the slope of this line is dictated by the slope a line would have using the Bessel analogy. That is, the diffusivity defined by Equation 130 is the same as the characteristic E for Reynolds numbers of 1,500 and 5,000. This fact becomes obvious by solving Equation 131 for (y/x)

$$\frac{y}{x} = \frac{E}{E_B^2 \beta_1^2 r_o^2 h^2}$$

In this work $h = 0.125$ and $\beta_1 r_o = 3.83$ so that if $E = E_C = E_B$ as assumed,

$$\frac{y}{x} = \frac{1}{(3.83)^2 (0.125)^2} = 4.36$$

A line of slope equal to 4.36 is shown on the curves of Figure 27. This line nearly passes through the maximum ordinate of the foliums for Reynolds numbers of 1,500 and 5,000 so that E_B not only equals E in general but more specifically $E_B = E_C$. Also, from Equation 128

$$\left(\frac{E_C}{\lambda V}\right) N_{RE} = 1,500 = \left(\frac{E_C}{\lambda V}\right) N_{RE} = 5,000$$

since $(y/x)_{\theta_C}$ and h^2 for $N_{RE} = 1,500$ and $N_{RE} = 5,000$ are identical. The subscript θ_C denotes the radial position at which the line of said slope passes through the maximum point on the folium to define E_C . This relationship between the $N_{RE} = 1,500$ and $N_{RE} = 5,000$ rates is astonishing when it will be remembered that a parallelism also exists between the

$N_{RE} = 5,000$ and $N_{RE} = 10,000$ eigen values as shown by the $\ln Z$ versus z curves (Figure 25). Since these eigen values are nearly the same, the following expression must be true.

$$\left(\frac{E}{V \frac{y}{x}}\right) N_{RE} = 5,000 = \left(\frac{E}{V \frac{y}{x}}\right) N_{RE} = 10,000$$

The characteristic diffusivity has been defined by

$$\frac{y_{\max}}{x} = \frac{E_c}{\lambda h^2 V}$$

where it has been shown that $y_{\max}/x$ is the same for all z at constant V . Hence λ can be modified by defining a new eigen value

$$\delta = \frac{\lambda V}{E_c}$$

Substitution of λ as given by this relationship in the diffusion equation after the variables are separated permits a dimensionless diffusivity to be written as

$$E' = \frac{E}{E_c}$$

Use of this parameter in future work will allow the mass transfer diffusion coefficients to be normalized in much the same way as are the u/V , r/r_0 , and C/C_A quantities.

The value of θ at which the maximum in the folium curves occurs can be obtained by equating the derivative of Equation 127 to zero, i.e.,

$$\frac{(r_0^2 u \theta \frac{\partial C}{\partial z}) \frac{\partial \theta}{\partial \theta}}{\theta \frac{\partial^2 C}{\partial \theta^2} + \frac{\partial C}{\partial \theta}} = 0$$

For non-zero values of θ , this expression can be zero only when $\partial C / \partial z = 0$. Again the significance of the concentration curves (Figure 13) becomes apparent because the value of radial position at which $\partial C / \partial z = 0$ must be that θ value at which the concentration profiles for different heights cross.

CONCLUSIONS

The major conclusions of this investigation are tabulated as follows:

1. Accurate means of studying turbulence, from an experimental point of view, must consider time-weighted or statistical measurements of the required variables. The randomness associated with turbulent diffusion makes such a procedure necessary.
2. To date, a completely rigorous theoretical and mathematical approach to the study of turbulence has not been conceived. This is due primarily to uncertainties in the nature of the turbulent exchange process and difficulties in solving the rigorous differential equation. Use of high-speed computers in this area will enable a more rigorous treatment to be applied.
3. Eddy diffusion is an important mechanism to the mass transfer process even in the pseudo-laminar region where the molecular contribution would be expected to provide the primary means of mixing.
4. The application of both mathematical techniques used in this investigation was found to give satisfactory explanations of many features exhibited by the data and the results. However, both methods proved inapplicable when applied to the laminar flow range because of failure to consider axial diffusion.
5. Total diffusivities vary in magnitude with both radial and longitudinal position.
6. Average Peclet numbers for gas and liquid systems increase and decrease respectively, with increasing Reynolds numbers and approach the same limiting value at high flow rates.

7. Mass and momentum transfer, in this study, have been shown to be analogous in nature from a qualitative viewpoint by the similarity of the eddy diffusivity and eddy viscosity curves.

8. The turbulent Schmidt and position Peclet groups for $N_{RE} = 10,000$ bear a strong resemblance in their variation with θ indicating that the y and u parameters of N_{PE} provide a reasonable measure of the scale and intensity parameters. At lower flow rates ($N_{RE} = 5,000$) the scale and intensity are better measured by D and u since the Peclet number distribution and Schmidt group show better agreement here.

9. Average values of total effective diffusivities in gas streams increase linearly with flow rate at Reynolds numbers greater than 5,000. Below this rate, the omission of the axial diffusion term causes an unexpected increase in diffusivity.

10. The contribution that axial diffusion affords the mass transfer process greatly exceeds that of radial diffusion for the laminar range of flow. The ratio of axial to radial diffusivity of greater than 40 was obtained in this work at a Reynolds number of 1,500.

11. Use of the characteristic diffusivity defined in this paper is proposed as a method of normalizing mass transfer diffusivities.

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NOMENCLATURE

A	Constant in series solution; area, ft^2
C	Concentration, volume per cent carbon dioxide in air
C_A	Measured average effluent concentration
C_f	Concentration of pure carbon dioxide in injector tube
C_k	Concentration at point k
C_M	Mean or integral average concentration
C_o	Concentration at tube center
d	Total differential operator
D_v	Molecular diffusivity, sq ft/sec
D_t	Tube diameter, in or ft
E	Total effective diffusivity, $E = (D_v + E_T)$, sq ft/sec ; thermal conductivity cell e.m.f., mv
E_a	Axial eddy diffusivity, sq ft/sec
E_r	Radial eddy diffusivity, sq ft/sec
E_T	Turbulent or eddy diffusivity, sq ft/sec
f	Friction factor, dimensionless
g_c	Conversion factor, $32.17 \text{ lb}_m \text{ ft/lb}_f \text{ sec}^2$
G	Mass velocity, lbs/hr/sq ft
h	Interval size in semi-numerical solution
H	Weight function in semi-numerical solution ($H = E\theta$), sq ft/sec
HWA-IR	Voltage drop across hot-wire anemometer, volts
J_o	Zero-order Bessel function of the first kind
J_1	First-order Bessel function of the first kind
k	Number of radial increment in semi-numerical solution

n	Index of summation; slope of log C versus log E plot for thermal conductivity cell calibration curve
N	Total number of intervals in semi-numerical solution, reciprocal of h
N_A	Rate of diffusion of component A, lbs/sec/sq ft
N_{PE}	Peclet number, $(D_t u/E)$, dimensionless
N_{PE}'	Modified position Peclet number, $1/2 N_{PE} (1 - \theta)$, dimensionless
N_{RE}	Reynolds number, $D_t V \rho / \mu$, dimensionless
N_{RE}'	Modified position Reynolds number, $1/2 N_{RE} (1 - \theta)$, dimensionless
N_{SC}	Turbulent Schmidt number, $(\mathcal{E}/E)$, dimensionless
P	Weight function in numerical solution, $(P = u\theta)$, ft/sec
r	Distance from center of pipe, in or ft
r_0	Radius of pipe, in or ft
R	Eigen function representing radial variation of concentration
t	Radius of injector tube in Bessel solution, in; ratio of injector tube radius to pipe radius in semi-numerical solution, dimensionless; time, sec
u	Point velocity in axial direction, ft/sec
V	Average velocity in axial direction, ft/sec
x	Rectangular coordinate
y	Rectangular coordinate; distance from tube wall, $(r_0 - r)$, in or ft
Y_0	Zero-order Bessel function of the second kind
z	Axial distance from tip of injector tube, in or ft

Z	Eigen function representing axial variation of concentration, dimensionless
α	Ratio of velocity to diffusivity, V/E , ft^{-1}
β	Constant coefficient of Bessel function, in^{-1}
Δ	Delta quantity
ε	Coefficient of eddy diffusivity for momentum, sq ft/sec
ε_0	Eddy viscosity, lb/ft-sec
λ	Eigen constant in semi-numerical solution, ft^{-1}
∂	Partial differential operator
∇	Gradient operator
θ	Radial position, r/r_0 , dimensionless
μ	Absolute viscosity, lbs/ft/sec
τ	Shear stress, $\text{lb}_f/\text{sq ft}$
τ_0	Shear stress at tube wall, $\text{lb}_f/\text{sq ft}$
ρ	Density, lbs/cu ft
Σ	Summation operator

APPENDIX A

Sample Calculation for Bessel Solution

Least squares application

Total average diffusivities and Peclet numbers were determined from Equation 49 written in the form:

$$C/C_M = 1 + \frac{2J_1(\beta_1 t)}{t\beta_1 J_0^2(\beta_1 r_0)} \cdot J_0(\beta_1 r) e^{-\beta_1^2 z D_t / N_{PE}} \quad (A-1)$$

Multiplying by C_M/C_A yields

$$C/C_A = C_M/C_A + C_M/C_A \frac{2J_1(\beta_1 t)}{t\beta_1 J_0^2(\beta_1 r_0)} \cdot J_0(\beta_1 r) e^{-\beta_1^2 z D_t / N_{PE}} \quad (A-2)$$

A linear relationship between C/C_A and $J_0(\beta_1 r)$ is suggested. Applying a least squares procedure to a plot of C/C_A versus $J_0(\beta_1 r)$ to fit the best straight line will give C_M/C_A as an intercept and a slope of $[C_M/C_A] \times$

$\frac{2J_1(\beta_1 t)}{t\beta_1 J_0^2(\beta_1 r_0)} e^{-\beta_1^2 z D_t / N_{PE}}$. Equation A-2 can be written in the form

$$y = ax + b$$

where a and b are the slope and intercept respectively. The least squares method for evaluating the slope, a , and the intercept, b or C_M/C_A , result in the following forms:

$$a = \frac{\sum C/C_A \sum J_0(\beta_1 r) - n \sum C/C_A J_0(\beta_1 r)}{[\sum J_0(\beta_1 r)]^2 - n \sum J_0^2(\beta_1 r)}$$

$$b = \frac{\sum J_0(\beta_1 r) \sum C/C_A J_0(\beta_1 r) - \sum C/C_A \sum J_0^2(\beta_1 r)}{[\sum J_0(\beta_1 r)]^2 - n \sum J_0^2(\beta_1 r)}$$

Evaluation of the constant, β

All the terms required in the least squares procedure can be evaluated using experimental concentration data once the constant, β_1 , is known. From boundary condition III, $\frac{\partial C}{\partial r} = 0$ at $r = r_0$. Therefore,

$$J_0'(\beta_1 r_0) = -\beta_1 J_1(\beta_1 r_0) = 0$$

where the primed quantity indicates a derivative. The first root for which $J_1(\beta_1 r_0) = 0$ is 3.8317 (17 p.157). Hence, $\beta_1 r_0 = 3.8317$ and $\beta_1 = 3.8317/r_0 = 1.904$ or $\beta_1 r = 3.8317 r/r_0$.

A sample calculation of a least squares application is presented in Table 2 for the following experimental conditions:

$$N_{RE} = 5,000$$

$$V = 2.4496 \text{ ft/sec}$$

$$z = 50 \text{ inches}$$

$$t = 0.20313 \text{ inches radius}$$

The slope and intercept are calculated as:

$$a = \frac{(10.8247)(3.2747) - (9)(5.8323)}{(3.2747)^2 - (9)(3.6269)} = .7775$$

$$b = \frac{(3.2747)(5.8323) - (10.8247)(3.6269)}{(3.2747)^2 - (9)(3.6269)} = .9199$$

The equation of the best straight line will then be

$$C/C_A = .7775 J_0(\beta_1 r) + .9199$$

Determination of C_0/C_M

Equation A-1 can be used to calculate the Peclet number and total eddy diffusivity once the ratio C_0/C_M is evaluated. At the tube center where $r = 0$ and $J_0(\beta_1 r) = J_0(0) = 1$, the above equation in terms of the slope and

Table 2. Application of least squares to experimental concentration data

θ	C/C_A	$\beta_1 r$	$J_0(\beta_1 r)$	$J_0^2(\beta_1 r)$	$C/C_A J_0(\beta_1 r)$
0.000	1.6407	0.0000	1.0000	1.0000	1.6407
0.062	1.6284	0.2414	0.9855	0.9712	1.6048
0.187	1.5887	0.7204	0.8744	0.7646	1.3892
0.312	1.4830	1.1993	0.6714	0.4508	0.9957
0.437	1.3229	1.6783	0.4105	0.1685	0.5431
0.562	1.1183	2.1572	0.1343	0.0180	0.1502
0.687	0.8824	2.6362	-0.1137	0.0129	-0.1003
0.812	0.6646	3.1152	-0.2965	0.0879	-0.1971
0.938	0.4959	3.5941	-0.3912	0.1530	-0.1940
$\Sigma = 10.8249$			$\Sigma = 3.2747$	$\Sigma = 3.6269$	$\Sigma = 5.8323$

intercept becomes

$$C_o/C_A = a + b = .7775 + .9199$$

The value of C_o/C_M can now be calculated as

$$C_o/C_M = \frac{C_o/C_A}{C_M/C_A} = \frac{a + b}{b} = \frac{.7775 + .9199}{.9199} = 1.8452$$

At the tube center Equation A-1 takes the form

$$C_o/C_M = 1 + \frac{2J_1(\beta_1 t)}{t\beta_1 J_0^2(\beta_1 r_o)} e^{-\beta_1^2 z D_t / N_{PE}}$$

Evaluation of the other functional constants and substitution into the above expression yields

$$1.8452 = 1 + 6.0488 e^{-729.372/N_{PE}}$$

from which $N_{PE} = 371$.

$$\text{Also, } E = \frac{D_t V}{N_{PE}} = \frac{(.3355)(2.4496)}{(371)} = 2.22 \times 10^{-3} \text{ ft}^2/\text{sec}$$

Results of Bessel Calculations

Results of the analytical solution for other rates investigated in this study are tabulated in Table 3.

Table 3. Results of Bessel solution

N_{RE}	z	C_M/C_A	C_O/C_A^a	N_{PE}	$E \times 10^3$
1,500	10	1.012	1.436	55	4.506
1,500	15	1.014	1.241	66	3.712
1,500	20	1.006	1.107	71	3.458
5,000	50	0.920	1.697	371	2.220
5,000	70	0.924	1.375	406	2.025
5,000	90	0.949	1.235	430	1.878
10,000	50	0.864	2.323	571	2.876
10,000	70	0.918	1.632	498	3.299
10,000	90	0.921	1.281	479	3.430

^aCenter concentration as determined by equation of best straight line

APPENDIX B

Sample Calculation for Semi-numerical Solution - Boundary Condition I

Table 4 illustrates the details involved in the semi-numerical method used to calculate point values of eddy diffusivities and Peclet numbers. A sample calculation is presented for the following experimental conditions:

$$N_{RE} = 5,000$$

$$V = 2.4496 \text{ ft/sec}$$

$$z = 50 \text{ inches}$$

$$t = 0.20313 \text{ inches}$$

Calculation of the mean concentration, C_M/C_A (Table 4a)

The mean concentration is given by

$$C_M/C_A = \frac{\sum_{k=0}^N C_k/C_A \cdot P_k/V}{\sum_{k=0}^N P_k/V} = \frac{3.8695}{3.8737} = 0.9989$$

Calculation of the eigen value, λ (Table 4b)

The eigen value can be calculated after the Z function is evaluated.

$$\sum_{k=0}^t P_k/V = 0.01967 + (0.1564) \frac{0.1009}{0.1250} = 0.14591$$

$$\sum_{k=0}^t P_k/V \left(\frac{C_k - C_M}{C_A} \right) = 0.01263 + (0.09649) \frac{0.1009}{0.1250} = 0.09051$$

Since t is a non-integral value of k , the summations performed above were carried to the integral value just above t and the result linearly

proportioned. Z is then given by

$$1/Z_1 = \frac{\sum_{k=0}^N C_k/C_A \cdot P_k/V}{\sum_{k=0}^N P_k/V} \times \frac{\sum_{k=0}^t P_k/V \left(\frac{C_k - C_M}{C_A} \right)}{\sum_{k=0}^N P_k/V \left(\frac{C_k - C_M}{C_A} \right)^2}$$

$$1/Z_1 = \frac{(3.8695)(0.09051)}{(0.1459)(0.49082)} = 4.89038$$

$$\lambda_1 = r_o^2/z \ln(1/Z_1) = \frac{(0.1678)^2}{(4.1667)} \ln(4.89038) = 0.01072 \text{ ft}$$

$$\lambda_1 h^2 v = (0.01072)(0.125)^2 (2.4496) = 0.00041031 \text{ ft}^2/\text{sec}$$

Calculation of point diffusivity and Peclet number (Table 4c)

Once the constant multiplying coefficient of $\lambda_1 h^2 v$ is evaluated, the H function, point values of diffusivity, and point values of Peclet number can be calculated using the following relationships.

$$H(k+1/2)h = \lambda_1 h^2 v \frac{\sum_{k=0}^k P_k/V \left(\frac{C_k - C_M}{C_A} \right)}{\frac{C_k - C_{k+1}}{C_A}}$$

$$E(k+1/2)h = \frac{H(k+1/2)h}{\theta(k+1/2)h}$$

$$N_{PE}(k+1/2)h = D_t u/E$$

Smoothed values of C/C_A and $(C_k - C_{k+1})/C_A$ used in the semi-numerical solution are presented in Tables 6a and 6b.

Table 4a. Calculation of mean concentration, C_M/C_A

k	θ	C_K/C_A	u_K/V	P_K/V	$C_K/C_A \cdot P_K/V$
0	0.000	1.641	1.261	0.0197 ^a	0.0323
	0.062	1.631	1.257		
1	0.125	1.616	1.251	0.1564	0.2527
	0.187	1.589	1.239		
2	0.250	1.545	1.225	0.3063	0.4732
	0.312	1.483	1.208		
3	0.375	1.408	1.188	0.4455	0.6273
	0.437	1.323	1.168		
4	0.500	1.227	1.143	0.5715	0.7012
	0.562	1.118	1.119		
5	0.625	0.999	1.094	0.6838	0.6831
	0.687	0.882	1.072		
6	0.750	0.768	1.037	0.7778	0.5970
	0.812	0.665	0.970		
7	0.875	0.575	0.867	0.7586	0.4362
	0.938	0.496	0.637		
8	1.000	0.431	0.000	0.1543 ^b	0.0665
				$\sum_{k=0}^N = 3.8737$	$\sum_{k=0}^N = 3.8695$

^aThe P function at $\theta = 0$ must represent a weighted average over the first one-half increment so P_0/V does not follow the normal description of velocity times radial position. By graphical definition

$$P_0/V = 1/2 P(h/4) = 1/2 u/V(h/4)\theta(h/4) \\ = (0.5) (1.259) (0.03125) = 0.01967$$

^bA similar analysis will show that

$$P_N/V = P_8/V = 1/2 P(N-1/4)h = 1/2 P(7 \ 3/4)h \\ = 1/2 u/V (7 \ 3/4 h)\theta(7 \ 3/4 h) \\ = (0.5) (0.3185) (0.96875) = 0.15427$$

Table 4b. Calculation of eigen value, λ_1

k	θ	$\frac{C_k - C_M}{C_A}$	$P_k/V \left(\frac{C_k - C_M}{C_A} \right)$	$P_k/V \left(\frac{C_k - C_M}{C_A} \right)^2$
0	0.000 0.062	0.642	0.0126	0.0081
1	0.125 0.187	0.617	0.0965	0.0595
2	0.250 0.312	0.546	0.1672	0.0913
3	0.375 0.437	0.409	0.1822	0.0745
4	0.500 0.562	0.228	0.1303	0.0297
5	0.625 0.687	0.000	0.0000	0.0000
6	0.750 0.812	-0.231	-0.1797	0.0415
7	0.875 0.938	-0.424	-0.3217	0.1364
8	1.000	-0.568	-0.0876	0.0498
				N $\Sigma = 0.4908$ k=0

Table 4c. Calculation of point diffusivity and Peclet number

k	θ	$\sum_{k=0}^k P_k/V$	$(\frac{C_k - C_M}{C_A})$	$\frac{C_k - C_{k+1}}{C_A}$ ^a	$H(k+1/2)h \cdot 10^3$	$E(k+1/2)h \cdot 10^3$	N_{PE}
0	0.000	0.0126		0.028			
	0.062				0.1851	2.961	349
1	0.125	0.1091		0.083			
	0.187				0.5394	2.877	354
2	0.250	0.2763		0.135			
	0.312				0.8399	2.688	369
3	0.375	0.4585		0.182			
	0.437				1.0338	2.363	406
4	0.500	0.5888		0.221			
	0.562				1.0932	1.944	473
5	0.625	0.5888		0.229			
	0.687				1.0551	1.535	574
6	0.750	0.4092		0.200			
	0.812				0.8395	1.033	771
7	0.875	0.0875		0.150			
	0.938				0.2394	0.255	2052
8	1.000	-0.0001					

^aThe $(C_k - C_{k+1})/C_A$ data used in the calculations were smoothed by plotting a bar graph of C/C_A against k . Values of $(C_k - C_{k+1})/C_A$ at the one-half intervals of k were obtained from the smooth curve drawn. This procedure, illustrated in Figure 28, did not change the resulting distributions for diffusivity and Peclet number.

Sample Calculation for Semi-numerical Solution - Alternate Boundary Condition

Use of the alternate boundary condition in the semi-numerical method does not change the calculation procedure except for evaluation of the eigen value. Calculation of the mean concentration, point diffusivities, and point Peclet numbers are the same as before and will not be considered

here. Table 4d illustrates the method for calculating the eigen value for the following experimental conditions:

$$N_{RE} = 5,000$$

$$V = 2.4496 \text{ ft/sec}$$

$$z = 70 \text{ inches}$$

$$t = 0.20313 \text{ inches}$$

z_0 refers to data for the 50-inch height while z_1 refers to data at the 70-inch height, both for Reynolds number of 5,000. Values of point diffusivity and Peclet number calculated by both boundary conditions are tabulated in Tables 5a and 5b.

Calculation of the eigen value, λ

The Z function is given by

$$\frac{Z_1(z_0)}{Z_1(z_1)} = \frac{\sum_{k=0}^N \frac{C_k}{C_A}(z_0) \cdot P_k / V \left(\frac{C_k - C_M}{C_A} \right) (z_1)}{\sum_{k=0}^N \frac{C_k}{C_A}(z_1) \cdot P_k / V \left(\frac{C_k - C_M}{C_A} \right) (z_1)}$$

$$= \frac{0.27945}{0.15863} = 1.76164$$

The expression for calculating the eigen value is

$$\lambda_1 = \frac{r_o^2 \ln \frac{Z_1(z_0)}{Z_1(z_1)}}{(z_1 - z_0)} = \frac{(0.1678)^2 (0.5662)}{(70 - 50)/12} = 0.009566 \text{ ft}$$

Table 4d. Calculation of the eigen value, λ_1 (alternate boundary condition)

k	$C_k/C_A(z_0)$	$P_k/V(\frac{C_k-C_M}{C_A})(z_1)$	$(C_k/C_A(z_0)) \cdot P_k/V(\frac{C_k-C_M}{C_A})(z_1)$
0	1.641	0.00740	0.01214
1	1.616	0.05677	0.09174
2	1.545	0.09679	0.14954
3	1.408	0.10247	0.14428
4	1.227	0.06915	0.08485
5	0.999	-0.00068	-0.00068
6	0.768	-0.09878	-0.07581
7	0.575	-0.18282	-0.10512
8	0.431	-0.04983	-0.02148
			N
			$\Sigma = 0.27945$
			0

Figure 28. Method for smoothing $(C_k - C_{k+1})/C_A$ data

Table 5a. Point values of diffusivity, (ft²/sec), calculated by semi-numerical solution

$N_{RE} = 1,500$						
θ	$z = 10$ inches		$z = 15$ inches		$z = 20$ inches	
	$E \times 10^3$ ^a	$\bar{E} \times 10^3$ ^b	$E \times 10^3$	$\bar{E} \times 10^3$	$E \times 10^3$	$\bar{E} \times 10^3$
0.0625	8.565	-	7.882	5.590	7.225	5.618
0.1875	8.403	-	7.731	5.483	7.066	5.494
0.3125	6.760	-	5.752	4.079	5.549	4.315
0.4375	5.310	-	4.472	3.171	4.275	3.324
0.5625	3.872	-	3.256	2.309	3.136	2.439
0.6875	2.372	-	2.048	1.453	2.024	1.574
0.8125	1.629	-	1.247	0.885	1.101	0.856
0.9375	0.559	-	0.422	0.300	0.311	0.242

$N_{RE} = 5,000$						
θ	$z = 50$ inches		$z = 70$ inches		$z = 90$ inches	
	$E \times 10^3$	$\bar{E} \times 10^3$	$E \times 10^3$	$\bar{E} \times 10^3$	$E \times 10^3$	$\bar{E} \times 10^3$
0.0625	2.961	-	2.618	2.407	2.116	1.668
0.1875	2.877	-	2.571	2.364	2.238	1.764
0.3125	2.688	-	2.486	2.286	2.192	1.728
0.4375	2.363	-	2.241	2.060	2.065	1.628
0.5625	1.944	-	1.884	1.732	1.836	1.447
0.6875	1.535	-	1.514	1.392	1.506	1.187
0.8125	1.033	-	0.977	0.898	0.977	0.770
0.9375	0.255	-	0.267	0.246	0.248	0.195

$N_{RE} = 10,000$						
θ	$z = 50$ inches		$z = 70$ inches		$z = 90$ inches	
	$E \times 10^3$	$\bar{E} \times 10^3$	$E \times 10^3$	$\bar{E} \times 10^3$	$E \times 10^3$	$\bar{E} \times 10^3$
0.0625	1.791	-	2.268	2.592	2.702	3.057
0.1875	2.762	-	3.190	3.647	3.377	3.821
0.3125	3.190	-	3.693	4.222	3.760	4.255
0.4375	3.594	-	3.749	4.286	3.869	4.378
0.5625	3.745	-	3.730	4.264	3.884	4.394
0.6875	3.444	-	3.570	4.081	3.735	4.226
0.8125	2.826	-	2.850	3.258	3.116	3.526
0.9375	1.400	-	1.090	1.246	1.403	1.587

^a E - diffusivity as calculated using boundary condition I

^b $\bar{E}$ - diffusivity as calculated using alternate boundary condition

Table 5b. Point values of Peclet number calculated by semi-numerical solution

$N_{RE} = 1,500$						
θ	$z = 10$ inches		$z = 15$ inches		$z = 20$ inches	
	N_{PE}^a	$\bar{N}_{PE}^b$	N_{PE}	$\bar{N}_{PE}$	N_{PE}	$\bar{N}_{PE}$
0.0625	57	-	62	87	67	86
0.1875	56	-	61	85	66	85
0.3125	65	-	76	108	79	102
0.4375	74	-	88	124	92	118
0.5625	85	-	101	143	105	135
0.6875	108	-	125	177	127	163
0.8125	105	-	157	193	155	200
0.9375	138	-	183	257	248	319
$N_{RE} = 5,000$						
θ	$z = 50$ inches		$z = 70$ inches		$z = 90$ inches	
	N_{PE}	$\bar{N}_{PE}$	N_{PE}	$\bar{N}_{PE}$	N_{PE}	$\bar{N}_{PE}$
0.0625	349	-	395	429	488	620
0.1875	354	-	396	431	455	577
0.3125	369	-	399	434	453	575
0.4375	406	-	428	466	465	589
0.5625	473	-	488	531	501	635
0.6875	574	-	582	633	585	742
0.8125	771	-	816	887	816	1035
0.9375	2052	-	1960	2128	2110	2684
$N_{RE} = 10,000$						
θ	$z = 50$ inches		$z = 70$ inches		$z = 90$ inches	
	N_{PE}	$\bar{N}_{PE}$	N_{PE}	$\bar{N}_{PE}$	N_{PE}	$\bar{N}_{PE}$
0.0625	1160	-	916	801	769	679
0.1875	740	-	647	566	611	540
0.3125	641	-	553	484	543	480
0.4375	559	-	536	468	519	459
0.5625	523	-	525	459	504	446
0.6875	547	-	528	462	504	446
0.8125	626	-	621	543	568	502
0.9375	935	-	1200	1050	933	824

N_{PE}^a - Peclet number as calculated using boundary condition I

$\bar{N}_{PE}^b$ - Peclet number as calculated using alternate boundary condition

Table 5c. Point values of position Peclet numbers and position Reynolds numbers calculated by semi-numerical solution

$N_{RE} = 1,500$						
θ	<u>$z = 10$ inches</u>		<u>$z = 15$ inches</u>		<u>$z = 20$ inches</u>	
	N_{PE}	N_{RE}	N_{PE}	N_{RE}	N_{PE}	N_{RE}
0.0625	26.5	1382	28.8	1382	31.5	1382
0.1875	22.6	1156	24.6	1156	26.9	1156
0.3125	22.4	919	26.3	919	27.2	919
0.4375	20.8	671	24.7	671	25.8	671
0.5625	18.7	440	22.2	440	23.1	440
0.6875	16.9	244	19.6	244	19.8	244
0.8125	9.8	97	12.8	97	14.5	97
0.9375	4.3	15	5.7	15	7.8	15
$N_{RE} = 5,000$						
θ	<u>$z = 50$ inches</u>		<u>$z = 70$ inches</u>		<u>$z = 90$ inches</u>	
	N_{PE}	N_{RE}	N_{PE}	N_{RE}	N_{PE}	N_{RE}
0.0625	164	2947	185	2947	229	2947
0.1875	144	2517	161	2517	185	2517
0.3125	127	2077	137	2077	156	2077
0.4375	114	1642	120	1642	130	1642
0.5625	104	1223	107	1223	109	1223
0.6875	139	837	91	837	91	837
0.8125	73	455	76	455	76	455
0.9375	64	100	61	100	66	100
$N_{RE} = 10,000$						
θ	<u>$z = 50$ inches</u>		<u>$z = 70$ inches</u>		<u>$z = 90$ inches</u>	
	N_{PE}	N_{RE}	N_{PE}	N_{RE}	N_{PE}	N_{RE}
0.0625	544	5923	429	5923	360	5923
0.1875	304	5103	263	5103	248	5103
0.3125	220	4273	190	4273	187	4273
0.4375	157	3435	151	3435	146	3435
0.5625	114	2605	115	2605	110	2605
0.6875	85	1791	82	1791	79	1791
0.8125	59	1009	58	1009	53	1009
0.9375	29	249	38	249	29	249

Table 5d. Point values of eddy viscosity, eddy momentum diffusivity, and turbulent Schmidt numbers calculated by semi-numerical solution

$N_{RE} = 5,000$						
θ	$\varepsilon_p \times 10^4$	$\varepsilon \times 10^3$	N_{SC} z=50"	N_{SC} z=70"	N_{SC} z=90"	N_{SC} avg
0.100	1.649	2.228	0.758	0.849	1.027	0.878
0.200	1.648	2.227	0.779	0.857	1.012	0.882
0.300	1.648	2.227	0.825	0.895	1.026	0.915
0.400	1.648	2.227	0.909	0.964	1.073	0.982
0.500	1.580	2.135	0.984	1.029	1.095	1.036
0.600	1.511	2.042	1.104	1.141	1.167	1.137
0.700	1.295	1.750	1.186	1.186	1.186	1.186
0.800	0.890	1.202	1.145	1.145	1.145	1.145
0.900	0.334	0.451	0.949	0.929	0.929	0.929
1.000	0.000	0.000	-	-	-	-

$N_{RE} = 10,000$						
θ	$\varepsilon_p \times 10^4$	$\varepsilon \times 10^3$	N_{SC} z=50"	N_{SC} z=70"	N_{SC} z=90"	N_{SC} avg
0.100	5.084	7.090	3.717	3.009	2.660	3.131
0.200	5.843	7.895	2.770	2.448	2.305	2.508
0.300	5.843	7.896	2.430	2.209	2.123	2.254
0.400	5.043	7.096	2.249	2.120	2.046	2.133
0.500	5.560	7.514	2.059	2.004	1.927	1.996
0.600	4.849	6.553	1.815	1.771	1.698	1.761
0.700	5.765	5.087	1.496	1.443	1.379	1.439
0.800	2.390	3.229	1.098	1.059	1.009	1.055
0.900	0.633	0.830	0.429	0.429	0.429	0.429
1.000	0.000	0.000	-	-	-	-

Table 5e. Other results of semi-numerical solution

N_{RE}	$z(in)$	C_M/C_A	$\lambda_1(rt)^a$	$\bar{\lambda}_1(rt)^b$
1,500	10	1.1216	0.0703	-
1,500	15	1.0699	0.0626	0.0444
1,500	20	1.0315	0.0582	0.0452
5,000	50	0.9989	0.0107	-
5,000	70	0.9692	0.0104	0.0096
5,000	90	0.9767	0.0100	0.0079
10,000	50	0.9411	0.0096	-
10,000	70	0.9598	0.0100	0.0114
10,000	90	0.9412	0.0104	0.0117

^a
 λ_1 - eigen value as calculated using boundary condition I

^b
 $\bar{\lambda}_1$ - eigen value as calculated using alternate boundary condition

Table 6a. Smoothed values of C/C_A used in semi-numerical solution calculations

	$N_{RE} = 1,500$			$N_{RE} = 5,000$			$N_{RE} = 10,000$		
0	z=10"	z=15"	z=20"	z=50"	z=70"	z=90"	z=50"	z=70"	z=90"
0.000	1.395	1.230	1.100	1.641	1.345	1.222	2.437	1.660	1.296
0.062	1.391	1.223	1.098	1.631	1.345	1.216	2.360	1.637	1.282
0.125	1.385	1.216	1.097	1.616	1.332	1.208	2.258	1.594	1.269
0.187	1.376	1.207	1.096	1.589	1.310	1.195	2.122	1.537	1.243
0.250	1.358	1.194	1.089	1.545	1.285	1.175	1.955	1.466	1.195
0.312	1.329	1.178	1.075	1.483	1.248	1.151	1.746	1.375	1.150
0.375	1.291	1.155	1.072	1.408	1.199	1.121	1.540	1.280	1.101
0.437	1.244	1.128	1.064	1.323	1.147	1.085	1.366	1.184	1.051
0.500	1.193	1.104	1.050	1.227	1.090	1.049	1.172	1.088	1.005
0.562	1.136	1.072	1.036	1.118	1.029	1.009	0.980	0.989	0.956
0.625	1.067	1.035	1.015	0.999	0.968	0.970	0.816	0.907	0.910
0.687	0.989	0.996	0.998	0.882	0.904	0.936	0.685	0.839	0.875
0.750	0.913	0.965	0.982	0.768	0.842	0.897	0.582	0.777	0.848
0.812	0.837	0.938	0.970	0.665	0.781	0.861	0.498	0.727	0.827
0.875	0.813	0.918	0.960	0.575	0.728	0.831	0.424	0.686	0.807
0.938	0.797	0.901	0.951	0.496	0.683	0.805	0.356	0.651	0.792
1.000	0.783	0.885	0.944	0.431	0.646	0.782	0.297	0.621	0.779

Table 6b. Smoothed values of $(C_k - C_{k+1})/C_A$ used in semi-numerical solution calculations

	$N_{RE} = 1,500$			$N_{RE} = 5,000$			$N_{RE} = 10,000$		
k	z=10"	z=15"	z=20"	z=50"	z=70"	z=90"	z=50"	z=70"	z=90"
0	0.013	0.007	0.003	0.028	0.018	0.014	0.195	0.075	0.033
1	0.037	0.020	0.009	0.083	0.053	0.038	0.338	0.145	0.074
2	0.069	0.039	0.017	0.135	0.083	0.057	0.410	0.180	0.093
3	0.100	0.056	0.025	0.182	0.107	0.071	0.390	0.195	0.100
4	0.126	0.069	0.031	0.221	0.125	0.077	0.340	0.101	0.091
5	0.148	0.077	0.034	0.229	0.127	0.075	0.274	0.143	0.070
6	0.099	0.058	0.028	0.200	0.117	0.068	0.186	0.101	0.047
7	0.037	0.025	0.012	0.150	0.080	0.048	0.070	0.048	0.019
8									

APPENDIX C

Thermal Conductivity Cell Impulse Data

The experimental values (uncorrected) of cell impulse as obtained by graphical integration of the recorder strip charts are listed in Table 7. The C/C_A values were obtained by averaging the zero point at the start and end of a run and subtracting this value from each of the e.m.f.'s listed to get E . Similarly, the zero point correction was subtracted from the average of E_A (start) and E_A (end) to get E_A . The following relationship was used to establish the C/C_A ratios.

$$C/C_A = (E/E_A)^n$$

Table 7. Thermal conductivity cell impulse data (millivolts)

$N_{RE} = 1,500$									
θ	<u>$z = 10$ inches</u>			<u>$z = 15$ inches</u>			<u>$z = 20$ inches</u>		
	AA'	BB'	CC'	AA'	BB'	CC'	AA'	BB'	CC'
0.938	6.71	6.82	7.30	7.63	7.60	7.78	8.58	8.04	8.26
0.812	6.88	7.40	7.54	7.89	8.24	8.14	8.77	8.39	8.39
0.687	8.31	8.83	8.53	8.27	8.55	8.73	9.01	8.59	8.70
0.562	9.86	10.06	9.50	8.98	9.33	9.32	9.32	8.98	9.07
0.437	11.04	11.02	10.26	9.57	9.66	9.68	9.51	9.25	9.32
0.312	11.82	11.72	10.92	9.82	10.18	10.08	9.64	9.42	9.34
0.187	12.34	11.90	11.36	10.12	10.42	10.30	9.75	9.54	9.56
0.062	12.52	12.12	11.50	10.32	10.42	10.40	9.72	9.60	9.64
0.000	12.42	12.16	11.58	10.40	10.56	10.48	9.78	9.54	9.62
0.062	12.54	12.02	11.42	10.36	10.56	10.44	9.78	9.56	9.60
0.187	12.32	11.78	11.62	10.18	10.36	10.32	9.70	9.56	9.48
0.312	11.92	11.40	11.12	10.06	10.08	10.00	9.56	9.40	9.36
0.437	10.98	10.66	10.58	9.68	9.70	9.44	9.51	9.30	9.28
0.562	10.04	9.74	9.80	9.15	9.15	8.99	9.27	9.07	9.02
0.687	8.69	8.50	8.65	8.73	8.61	8.25	8.91	8.78	8.80
0.812	7.30	7.22	7.43	8.22	8.07	7.67	8.74	8.50	8.58
0.938	7.08	7.11	6.74	7.97	7.81	7.59	8.57	8.43	8.53
ZP (start)	0.60	0.61	0.59	0.60	0.65	0.60	0.65	0.62	0.64
ZP (end)	0.68	0.70	0.37	-	0.90	0.43	0.95	0.75	0.82
E _A (start)	8.71	8.65	8.74	8.60	8.68	8.59	8.69	8.64	8.72
E _A (end)	8.85	8.76	8.33	8.19	8.87	8.40	9.00	8.88	8.98

Cell constant, $n = 0.952$

Table 7. (Continued)

$N_{RE} = 5,000$									
θ	<u>$z = 50$ inches</u>			<u>$z = 70$ inches</u>			<u>$z = 90$ inches</u>		
	AA ^a	BB ^a	CC ^a	AA ^a	BB ^a	CC ^a	A ^a A ^a	BB ^a	CC ^a
0.938	3.12	3.22	3.58	4.01	4.15	4.48	5.07	4.83	5.13
0.812	4.00	4.25	4.50	4.54	4.75	5.00	5.43	5.16	5.38
0.687	5.20	5.48	5.69	5.22	5.48	5.75	5.76	5.53	5.80
0.562	6.54	6.80	7.20	5.95	6.25	6.42	6.30	5.96	6.21
0.437	7.71	8.03	8.26	6.67	6.84	7.09	6.66	6.42	6.60
0.312	8.77	8.90	9.21	7.20	7.45	7.76	6.98	6.78	6.98
0.187	9.30	9.58	9.72	7.67	7.52	8.11	7.19	7.07	7.17
0.062	9.64	9.78	9.84	8.01	7.75	8.28	7.22	7.24	7.33
0.000	9.76	9.76	9.88	7.98	7.71	8.27	7.26	7.27	7.37
0.062	9.72	9.72	9.66	7.99	7.68	8.19	7.27	7.25	7.28
0.187	9.49	9.50	9.37	7.91	7.48	8.01	7.07	7.18	7.17
0.312	8.99	8.76	8.66	7.53	7.12	7.49	6.80	6.95	6.86
0.437	8.17	7.82	7.73	7.05	6.52	6.91	6.37	6.57	6.47
0.562	7.01	6.67	6.45	6.30	5.89	6.21	5.86	6.09	6.06
0.687	5.81	5.29	5.15	5.69	5.22	5.41	5.49	5.76	5.67
0.812	4.52	4.07	3.94	4.97	4.57	4.74	4.98	5.38	5.12
0.938	3.60	3.23	2.93	4.50	4.04	4.09	4.58	5.06	4.88
ZP (start)	0.64	0.67	0.65	0.67	0.67	0.65	0.61	0.61	0.65
ZP (end)	0.70	0.73	0.73	0.42	0.41	0.66	0.64	0.58	0.64
E _A (start)	6.13	6.03	6.09	6.08	6.18	6.04	5.95	6.03	6.05
E _A (end)	6.13	6.12	6.12	5.87	5.83	6.03	6.05	6.03	6.07

Cell constant, $n = 0.952$

^aDue to a physical obstruction this diameter traverse was made in the reverse order, i.e., from far wall to center to near wall.

Table 7. (Continued)

$N_{RE} = 10,000$									
θ	$z = 50$ inches			$z = 70$ inches			$z = 90$ inches		
	AA'	BB'	CC'	AA'	BB'	CC'	A'A ^a	BB'	CC'
0.938	2.27	2.21	2.35	3.99	4.26	4.22	5.06	4.88	5.20
0.812	2.99	3.09	3.13	4.47	4.67	4.71	5.44	5.00	5.49
0.687	4.11	4.08	4.24	5.00	5.29	5.39	5.82	5.33	5.71
0.562	5.88	5.97	5.94	6.03	6.13	6.47	6.34	5.84	6.21
0.437	8.24	8.29	8.08	7.00	7.28	7.64	6.94	6.37	6.90
0.312	10.72	10.66	10.46	8.42	8.65	8.87	7.43	7.11	7.66
0.187	13.22	13.34	12.68	9.38	9.74	10.08	8.03	7.69	8.20
0.062	15.12	14.90	14.02	10.22	10.48	10.56	8.17	8.05	8.35
0.000	15.58	15.64	15.40	10.60	10.68	10.60	8.14	8.23	8.54
0.062	15.60	15.40	15.26	10.52	10.58	10.54	8.07	8.18	8.47
0.187	14.60	14.44	13.00	9.84	10.18	9.92	7.70	8.05	8.19
0.312	12.64	11.84	10.72	9.31	9.36	8.96	7.08	7.66	7.52
0.437	9.98	9.55	8.68	7.81	8.06	7.67	6.64	7.08	6.94
0.562	7.46	7.05	6.21	6.67	6.89	6.57	6.12	6.51	6.43
0.687	5.44	5.27	4.58	5.95	6.02	5.62	5.65	6.01	5.99
0.812	4.23	4.05	3.48	5.14	5.22	4.97	5.34	5.72	5.81
0.938	3.29	3.15	2.65	4.62	4.83	4.49	5.15	5.58	5.66
ZP (start)	0.66	0.60	0.60	0.60	0.68	0.60	0.67	0.64	0.68
ZP (end)	0.73	0.61	0.63	0.63	0.61	0.65	0.75	0.68	0.79
E _A (start)	6.56	6.56	6.54	6.47	6.51	6.45	6.38	6.38	6.56
E _A (end)	6.73	6.56	6.26	6.59	6.55	6.60	6.56	6.47	6.68
Cell constant, $n = 0.961$									

^aDue to a physical obstruction this diameter traverse was made in the reverse order, i.e., from far wall to center to near wall.

Table 8. Experimental concentration data, C/C_A^a

$N_{RE} = 1,500$			
θ	$z = 10$ inches	$z = 15$ inches	$z = 20$ inches
0.000	1.395	1.230	1.097
0.062	1.391	1.223	1.098
0.187	1.376	1.207	1.096
0.312	1.329	1.178	1.075
0.437	1.244	1.128	1.064
0.562	1.136	1.072	1.036
0.687	0.989	0.996	0.998
0.812	0.837	0.938	0.970
0.938	0.797	0.901	0.951
$N_{RE} = 5,000$			
θ	$z = 50$ inches	$z = 70$ inches	$z = 90$ inches
0.000	1.641	1.345	1.222
0.062	1.628	1.345	1.216
0.187	1.589	1.310	1.195
0.312	1.483	1.248	1.151
0.437	1.323	1.147	1.085
0.562	1.118	1.029	1.009
0.687	0.882	0.904	0.936
0.812	0.665	0.781	0.861
0.938	0.496	0.683	0.805
$N_{RE} = 10,000$			
θ	$z = 50$ inches	$z = 70$ inches	$z = 90$ inches
0.000	2.437	1.660	1.296
0.062	2.360	1.637	1.282
0.187	2.122	1.537	1.243
0.312	1.746	1.388	1.150
0.437	1.366	1.170	1.051
0.562	0.980	0.989	0.956
0.687	0.685	0.839	0.875
0.812	0.498	0.727	0.827
0.938	0.356	0.651	0.792

^a Averaged for six angular positions

Hot-wire Anemometer Data

The experimental values of velocities as determined by the hot-wire technique are listed in Table 9. Duplicate determinations were made for nearly all flow rates and heights investigated. Difficulty was experienced in maintaining the calibration for anemometer number one. Consequently, the data for this anemometer are lacking in some cases. Since the anemometer data did not indicate any variation of velocity with height, the results at all heights were averaged for each Reynolds number. This involved averaging the HWA-IR drops and acquiring the corresponding velocities from the calibration curve. Results of this procedure are tabulated in Table 10. The best curve, Figure 14, was drawn through the points given in Table 10. Point values of velocity for the various values of θ used in the semi-numerical solution were obtained from this plot and divided by the average velocity, V . The u/V ratios thus obtained and used in the semi-numerical calculations are given in Table 11.

Table 9. Experimental velocity data

Empty Tube Data				
$N_{RE} = 1500$			$V = .734 \text{ ft/sec}$	
θ	HWA-IR	u	HWA-IR	u
.316	2.79	1.27	2.78	1.28
.548	5.10	1.01	5.10	1.01
.707	6.81	0.675	6.83	0.66
.837	8.45	0.44	8.45	0.44
.949	10.60	0.185	10.60	0.185
$N_{RE} = 5,000$			$V = 2.45 \text{ ft/sec}$	
θ	HWA-IR	u	HWA-IR	u
.316	2.24	2.84	2.25	2.80
.548	3.93	2.64	3.90	2.73
.707	5.05	2.43	5.05	2.43
.837	6.22	2.25	6.20	2.275
.949	7.84	1.44	7.88	1.41
$N_{RE} = 10,000$			$V = 4.90 \text{ ft/sec}$	
θ	HWA-IR	u	HWA-IR	u
.316	2.00	6.02	1.91	-
.548	3.34	5.42	3.31	5.68
.707	4.26	5.31	4.22	5.65
.837	5.25	4.80	5.25	4.80
.949	6.30	3.40	6.26	3.50

Table 9. (Continued)

Velocity Data with Injector Tube

 $N_{RE} = 1500$ $V = .734 \text{ ft/sec}$ $z = 10 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	2.79	1.27	2.80	1.26
.548	5.11	1.01	5.10	1.01
.707	6.71	0.72	6.70	0.725
.837	8.35	0.465	8.34	0.47
.949	10.50	0.20	10.50	0.20

 $z = 15 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	2.79	1.27	2.78	1.29
.548	5.12	1.00	5.08	1.03
.707	6.69	0.73	6.69	0.73
.837	8.33	0.47	8.29	0.48
.949	10.40	0.22	10.40	0.22

 $z = 20 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	2.75	1.36	2.72	1.42
.548	5.12	1.00	5.14	0.98
.707	6.70	0.725	6.71	0.72
.837	8.35	0.465	8.35	0.465
.949	10.50	0.20	10.50	0.20

Table 9. (Continued)

Velocity Data with Injector Tube

 $N_{RE} = 5,000$ $V = 2.45 \text{ ft/sec}$ $z = 50 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	2.12	-	2.12	-
.548	3.95	2.59	3.93	2.64
.707	5.02	2.49	4.96	2.62
.837	6.20	2.275	6.10	2.42
.949	7.86	1.43	7.83	1.45

 $z = 70 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	2.20	3.07	2.13	-
.548	3.88	2.79	3.83	2.96
.707	4.97	2.59	4.95	2.63
.837	6.07	2.46	6.05	2.49
.949	7.71	1.55	7.80	1.475

 $z = 90 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	2.14	-	2.16	-
.548	3.86	2.86	3.85	2.89
.707	5.01	2.51	4.97	2.59
.837	6.11	2.40	6.09	2.43
.949	7.86	1.43	7.88	1.41

Table 9. (Continued)

Velocity Data with Injection Tube

 $N_{RE} = 10,000$ $V = 4.90 \text{ ft/sec}$ $z = 50 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	1.82	-	1.81	-
.548	3.28	5.91	3.25	6.11
.707	4.24	5.46	4.21	5.72
.837	5.15	5.30	5.15	5.30
.949	6.25	3.53	6.27	3.47

 $z = 70 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	1.86	-	1.84	-
.548	3.30	5.75	3.28	5.91
.707	4.23	5.54	4.21	5.72
.837	5.16	5.25	5.12	5.46
.949	6.22	3.60	6.23	3.58

 $z = 90 \text{ inches}$

θ	HWA-IR	u	HWA-IR	u
.316	1.86	-	1.83	-
.548	3.28	5.91	3.25	6.11
.707	4.25	5.37	4.18	5.88
.838	5.16	5.25	5.11	5.52
.949	6.21	3.63	6.21	3.63

Table 10. Averaged velocity data^a

θ	$N_{RE} = 1500$	$N_{RE} = 5,000$	$N_{RE} = 10,000$
.316	1.30	2.96	6.20
.548	1.01	2.76	5.86
.707	0.72	2.53	5.58
.837	0.46	2.37	5.19
.947	0.20	1.45	3.54

^a Averaged for all heights

Table 11. Smoothed values of u/v used in semi-numerical solution calculations

θ	$N_{RE} = 1500$		$N_{RE} = 5,000$		$N_{RE} = 10,000$	
	u	u/v	u	u/v	u	u/v
0.000	1.455	1.982	3.09	1.261	6.20	1.266
0.062	1.445	1.968	3.08	1.257	6.19	1.264
0.125	1.425	1.941	3.065	1.251	6.175	1.260
0.187	1.395	1.900	3.035	1.239	6.155	1.256
0.250	1.355	1.846	3.000	1.225	6.125	1.250
0.312	1.310	1.784	2.960	1.208	6.09	1.243
0.375	1.240	1.689	2.910	1.188	6.045	1.234
0.437	1.170	1.594	2.860	1.168	5.985	1.222
0.500	1.080	1.471	2.800	1.143	5.920	1.208
0.562	0.985	1.342	2.740	1.119	5.835	1.191
0.625	0.8755	1.192	2.680	1.094	5.740	1.172
0.687	0.765	1.042	2.625	1.072	5.615	1.146
0.750	0.640	0.872	2.540	1.037	5.460	1.115
0.812	0.510	0.695	2.375	0.970	5.275	1.077
0.875	0.375	0.511	2.125	0.867	4.92	1.004
0.938	0.230	0.313	1.560	0.637	3.90	0.796
1.000	0.000	0.000	0.000	0.000	0.000	0.000