NON-LINEAR NUMERICAL STUDIES OF THE TEARING MODE

Dalton D. Schnack, Jr.
(Ph.D. Thesis)

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NON-LINEAR NUMERICAL STUDIES OF THE TEARING MODE*

by

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ABSTRACT

A non-linear, time dependent, hydromagnetic model is developed and applied to the tearing mode, one of a class of instabilities which can occur in a magnetically confined plasma when the constraint of infinite conductivity is relaxed. The model is based on the eight partial differential equations of resistive magnetohydrodynamics (MHD). The equations are expressed as a set of conservation laws which conserves magnetic flux, momentum, mass, and total energy. These equations are then written in general, orthogonal, curvilinear coordinates in two space dimensions, so that the model can readily be applied to a variety of geometries. No assumption about the ordering of terms is made. The resulting equations are then solved by the method of finite differences on an Eulerian mesh.

We develop spatial finite difference techniques which guarantee the simultaneous conservation of the desired physical quantities throughout the course of the calculation. Conservative boundary conditions on thermodynamic quantities at a conducting boundary are derived, and special algorithms

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are developed for advancing the solution at a singular boundary. For the temporal differencing, we use the Alternating Direction Implicit (ADI) method, which avoids the computational difficulties inherent in explicit techniques.

We apply our model to several geometries. For the case of the tearing mode in a sheet pinch in Cartesian coordinates, we find our model to give good agreement with previous analytic and numerical results for the linear regime. We also study the non-linear saturation of this mode. We find that the tearing mode will saturate when the magnetic island width becomes comparable with the width of the singular layer, in agreement with theoretical predictions. In addition, we study the linear and non-linear properties of the double tearing mode, which occurs when two singular surfaces are in close proximity. We find that the interaction of the singular surfaces can lead to linear growth rates which are greatly enhanced over that of the single tearing mode, while the non-linear evolution can result in magnetic field configurations which are consistent with enhanced particle and thermal transport in a direction perpendicular to the zero order magnetic field. For this case, approximately ten percent of the magnetic energy released goes into kinetic energy, and ninety percent goes into thermal energy.

We demonstrate the versatility of our model by studying the tearing mode in cylindrical geometry. In particular, we study the $m = 0, 1$, and 2 modes in a Reversed Field Pinch.

The equilibrium is taken to be the force free Bessel function model. For the axisymmetric case of $m = 0$, the independent variables are (r, z) . For $m > 0$, we employ a transformation to helical coordinates which allows us to solve the full three dimensional equations under the assumption that helical symmetry is preserved. For these cases we follow the $m = 1$ and $m = 2$ modes into the non-linear regime. We find that the $m = 2$ mode saturates quickly, while the $m = 1$ mode is more dangerous in that it can continue to grow to large amplitude.

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CHAPTER I - INTRODUCTION

1.1 Motivation

Among the greatest technological challenges facing man in the latter part of the twentieth century is the construction of a nuclear fusion reactor capable of generating more power than is required for its operation. To reach this goal it is necessary to bring together the nuclei of light elements (isotopes of hydrogen, i.e. deuterium and tritium) with sufficient relative energy to overcome the electrostatic repulsive force between them and allow the resulting energy producing fusion reactions to occur. If such relative energies can be achieved, then a criterion (the Lawson criterion) for the attainment of useable fusion power is that the product of the number density of nuclei, n , and the time interval over which they are allowed to react, τ , be greater than a prescribed value, which is found to be [1]

$$n\tau > 10^{14} \text{ sec-cm}^{-3}. \quad (1.1.1)$$

Since losses such as radiation from impurities, charge exchange, etc. are not included in the derivation of this expression, a significant margin above the Lawson value of $n\tau$ may be required in actual practice.

There are two obvious routes to surpassing the Lawson condition: a) make n large and keep the material together for a relatively short time; and b) contain a tenuous fuel for a relatively long time. Both of these must be carried

out at sufficiently high temperatures (~ 10 kev, or $\sim 10^8$ °K) for a significant number of nuclei to interact.

In one approach, the surface of a high density pellet of fuel (deuterium, say) is irradiated with energy from a laser, an ion beam, or other source. The resulting implosion causes spherical shock waves to propagate inward toward the center of the fuel pellet, producing a region of high density and temperature where fusion reactions may take place. Typical densities are on the order of 10^{25} cm⁻³, thus requiring energy pulses in the picosecond range. The development of high power energy sources is thus intimately tied to the achievement of fusion energy from such inertial confinement methods.

In the other approach, a low density ($n \sim 10^{15}$ cm⁻³) gas of fuel (deuterium, tritium, etc.) is enclosed in a container for a time on the order of a few tenths of a second. To achieve fusion reactions, however, the gas must be heated to a kinetic temperature on the order of that found in the sun. At these temperatures practically all of the atoms become ionized, and the gas becomes electrically conducting. This ionized gas (or plasma) must be kept from contacting the walls of the chamber. This is not so much to protect the walls themselves, which have a large heat capacity relative to the low density plasma, but to keep impurities (i.e., heavy, or high Z, atoms) from being introduced into the plasma due to local rapid heating (sputtering) of the metal walls. These impurities, which become quickly ionized, lead to significant energy losses from the fuel plasma due to

bremsstrahlung, with rapid cooling and loss of plasma as the end result. The containment can be accomplished by means of a magnetic field, which can exert a force on the moving charged particles, and acts much like a low density fluid with pressure proportional to the square of the magnetic field strength. Designs based on this scenario are called magnetic confinement schemes, and it is to such devices that this work is addressed.

A large group of possible designs for a magnetic fusion reactor have been based on the fact that a current flowing in an ionized gas, or plasma, will interact with its own magnetic field in such a way as to cause the plasma to contract in the dimension transverse to the direction of the current [2]. Thus a cylindrical plasma with current flowing in the axial direction will tend to diminish in radius. This pinch effect [3] would cause a plasma column to be pulled away from an enclosing container in just the desired fashion. Unfortunately, such Z-pinch configurations have been found to be unstable to small perturbations [4] so that the plasma breaks up before the Lawson criterion is exceeded. Much effort has gone into studying means of stabilizing a pinched plasma.

The instabilities which plague the current carrying plasma column are of three basic types: a) the sausage mode, in which the plasma column breaks up into "links" (like a string of sausages) thus stopping the current and destroying the confinement; b) the kink mode, in which the

column wriggles, or kinks, until it strikes the wall; and c) flute modes, in which the surface of the plasma becomes distorted like the flutes on a Roman column which grow in time, eventually destroying the plasma. These basic instabilities are sketched in Figure (1.1a-c). It has been found that the sausage and kink modes can be stabilized by the application of a strong magnetic field in the axial, or current carrying, direction [5]. This field acts in much the same way that a set of rubber bands under tension might act: it resists lateral (kink) and compressional (sausage) displacements. In addition, the kink mode can be stabilized by the presence of an electrically conducting enclosure in which image currents are spontaneously generated in such a way as to retard the wriggling motion.

In many ways the confinement of a plasma by a magnetic field is analogous to supporting a dense fluid by a less dense one, a situation which is known to be unstable (the Rayleigh-Taylor instability): the system can lower its potential energy by interchanging the two fluids. The flute modes are of this type, i.e., the positions of the plasma and the magnetic field are interchanged [4,6]. These modes can be stabilized by magnetic shear, but their growth rate (a measure of the time for a significant distortion of the plasma to occur) may be so small, or they may saturate (stop growing) at such a small amplitude, that they do not affect the containment of the plasma as dictated by the Lawson criterion.

A magnetic field is said to be sheared if its orientation in a given plane changes in a direction perpendicular to that plane. For example, the configuration $B_x = \sin y$, $B_z = \cos y$ is sheared in the y -direction, since its orientation $\theta = \tan^{-1}(B_x/B_z)$ in the (X,Z) plane is a function of y . That magnetic shear can stabilize interchange modes is seen by noting that an interchange of two field lines in the direction of shear will cause the field lines to bend, thus increasing the magnetic energy $B^2/8\pi$. This is achieved automatically in the cylindrical pinch where B_z (the applied axial field) and B_θ (the azimuthal field due to plasma currents) are in general functions of the radial coordinate r , so that the field lines tend to wrap themselves helically around cylindrical surfaces (called flux surfaces), the pitch of the helix varying with radius (see Figure 1.2). It is still possible for an interchange mode to develop if the flutes of the perturbation also wrap themselves about the plasma with the same pitch as the field lines. Such modes would be localized in radius, and for a perfectly conducting plasma can be stabilized by line-tying [7], i.e., holding the ends of the field lines fixed.

In most currently popular pinch experiments, the plasma is in the shape of a torus, or donut, as opposed to being cylindrical. The basic situation is the same as described above, but now there are some important restrictions placed on certain parameters.

In the infinitely long cylindrical pinch, the helical

perturbations of the column could have arbitrary pitch. This is not so in the toroidal case, where only an integral number of wavelengths are allowed to fit around the torus, i.e., the perturbation must be a single-valued function of the toroidal angle. This is equivalent to applying periodic boundary conditions at the end of a cylinder of length $2\pi R$, where R is the major radius of the torus. Let the wave vector of the perturbation be

$$\underline{k} = k_{\theta} \hat{\theta} - k_z \hat{z} \quad (1.1.2)$$

and let a displacement on a surface of constant r be

$$\underline{s} = (r\theta) \hat{\theta} + z \hat{z} \quad (1.1.3)$$

where $\hat{\theta}$ and $\hat{z}$ are unit vectors in the θ and z directions. Then the phase of the perturbation is

$$\phi = \underline{k} \cdot \underline{s} = k_{\theta} r \theta - k_z z \quad (1.1.4)$$

Periodicity in θ requires $k_{\theta} r = m$, while periodicity in z requires $k_z = 2\pi/\lambda = n/R$, where m and n are integers. We thus have

$$\frac{rk_{\theta}}{Rk_z} = - \frac{m}{n} \quad (1.1.5)$$

as a relation to be satisfied by the components of the wave vector.

The magnetic field lines need not close upon themselves after a finite number of turns around the torus, and in general will fill a given toroidal, or cylindrical, flux surface ergodically. The pitch of the helix formed by the

field lines is related to the ratio of field components B_0/B_z , and is in general a continuous function of r . In light of previous discussion, we expect interchange modes to occur when the field lines are parallel to the wavefronts of the perturbation, which are in turn perpendicular to the wave vector. Then, from (1.1.5), wherever the relation

$$\frac{rB_z}{RB_0} = \frac{m}{n} \quad (1.1.6)$$

is satisfied we have possible instability. The function $q(r) = rB_z/RB_0$ is called the safety factor, and surfaces where equation (1.1.6) is satisfied are called mode-rational, or singular surfaces. It is clear that it is on such surfaces that the field lines close upon themselves.

The most dangerous mode to plasma confinement is the kink mode. It is characterized by a poloidal mode number $m = 1$ and is the only mode that involves a gross displacement of the plasma across the minor axis ($r = 0$ in a cylinder). To stabilize against this mode we are thus required to maintain $q > 1$ everywhere in the plasma, which restricts the fields to be such that

$$\frac{B_0}{B_z} < \frac{a}{R} \quad (1.1.7)$$

where "a" is the minor radius of the torus. The quantity $A = R/a$ is the aspect ratio and typically takes on values between 3 and 10, so that the externally applied axial field must be significantly greater than the self-field. This in turn places a restriction on the amount of current that can

be maintained in the discharge.

In the pinch effect, the azimuthal field B_θ is responsible for plasma confinement. The axial field B_z is externally imposed for stability purposes. A useful parameter for comparing controlled fusion devices is the ratio of the plasma pressure to the energy in the confining magnetic field. This parameter

$$\beta = \frac{8\pi p}{B^2} \quad (1.1.8)$$

is a measure of how much energy must be expended (e.g., in creating the magnetic field) to confine a given amount of plasma. Obviously, large values of β are desirable. For the case discussed above, this ratio for the confining field is

$$\beta_\theta = \frac{8\pi p}{B_\theta^2} \sim 1 \quad (1.1.9)$$

since p and $B_\theta^2/8\pi$ must be comparable magnitude for equilibrium. However, when the stabilizing field is taken into account, we find the overall ratio to be

$$\beta = \frac{8\pi p}{B^2} \sim \left(\frac{a}{R}\right)^2 \quad (1.1.10)$$

where we have used equation (1.1.7). Thus operation of these reactors, called Tokamaks, is expected to be limited to β values of a few percent, so that we must expend considerable energy to confine a given amount of plasma. (It should be mentioned that another operating regime, the Reversed Field Pinch, exists for toroidal Z-pinch devices, which permits

significantly higher values of β . This will be discussed in more detail in Chapter V.)

The modes discussed above exist in a plasma of infinite conductivity and, in spite of the operating limitations, can be effectively stabilized. Thus the goal of long confinement times would seem to be close at hand. However, the introduction of even the smallest amount of plasma resistivity introduces modes which have no counterpart in ideal (infinitely conducting) plasma theory, and which can destroy the confinement of the "stabilized" plasma discussed above. It is the behavior of one of these resistive instabilities, and in particular its non-linear properties, that is the subject of this thesis.

The resistivity of a plasma is due to the retardation of the motion of electrons in an electric field by collisions. By equating the change in momentum of such an electron between collisions to the impulse of the electric force, we arrive at

$$\eta = \frac{mv}{ne^2} \quad (1.1.11)$$

as an approximate expression for resistivity. Here, m , e , and n are the electron mass, charge, and number density, and v is the frequency of collisions. These collisions are generally thought of as occurring between particles. Hence, for a collisionless plasma (where the mean free path is significantly greater than the size of the container) we might expect η to approach zero. However, it is possible

for wave-particle interactions to produce effective collision frequencies (even in the case of a collisional plasma) which can exceed the classical values. These anomalous resistivity mechanisms have been discussed in detail by Davidson [8]. We thus expect all plasmas to have some finite value of electrical conductivity.

That the introduction of resistivity can change plasma behavior so drastically as to destabilize previously stable configurations may be surprising. To understand this behavior, we consider an appropriate combination of the Maxwell equations and Ohm's law

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B}) + \frac{c^2 \eta}{4\pi} \nabla^2 \underline{B} \quad (1.1.12)$$

in Gaussian units. Here, $\underline{B}$ is the magnetic field, $\underline{V}$ is the fluid velocity, and η is the resistivity. This equation will be derived in Chapter II. For now, it suffices to say that there is an equation of identical form in hydrodynamics [9], namely

$$\frac{\partial \underline{\zeta}}{\partial t} = \nabla \times (\underline{V} \times \underline{\zeta}) + \nu \nabla^2 \underline{\zeta} \quad (1.1.13)$$

where $\underline{\zeta} = \nabla \times \underline{V}$ is the fluid vorticity, and ν is the viscosity. It is well known that in inviscid flow ($\nu = 0$), this equation implies that the vorticity is co-moving with the fluid. Thus, vortex lines (lines everywhere tangent to the vorticity vector) can be labelled by the material fluid elements which initially reside on them, as the two must move about in space together. Put another way, fluid elements initially on a

given vortex line remain on that vortex line forever. The effect of finite viscosity is to destroy this coupling and allow the vortex lines to slip across the fluid. Then in an analogous manner, we expect that in a perfectly conducting ($\eta = 0$) plasma, the magnetic field lines will be "frozen in" the fluid, and must remain so for all time. The introduction of even the smallest amount of resistivity relaxes this constraint and allows relative motion between the fluid and the field. In particular, field lines need not retain their integrity, but may now break and reconnect, changing the topology of the configuration. It is through this change in topology, namely the reconnection of magnetic field lines which originally were confined to concentric flux surfaces of different radii, that resistive instabilities manifest themselves.

That these instabilities are a serious problem to plasma containment can be seen when we recall that particles can move much more easily along field lines than across them. Thus in our initial configuration particles tended to remain on nested, concentric surfaces. The change in topology now allows particles to stream from one surface to another, degrading the confinement. In addition, the evolution of these modes may cause the safety factor to drop below unity, triggering the onset of the dangerous kink instability. An understanding of the linear and non-linear properties of these instabilities is thus of considerable importance to the development of a controlled fusion reactor, and it is

to the furtherance of this understanding that this work addresses itself.

1.2 The Tearing Mode

The tearing mode is one of a class of finite resistivity instabilities first discussed in detail by Furth, Killeen, and Rosenbluth [10], and is characterized by the formation of magnetic islands as a result of the field line reconnection discussed above. In addition to this mode, whose source of free energy is the gross configuration of the magnetic field, this class includes the rippling mode, which is driven by resistivity gradients, and the gravitational mode, which is driven by an external "gravitational" force. We shall be concerned only with the tearing instability. In this section, we present a heuristic discussion of the physics of the tearing mode [10]. A more detailed discussion of the analytic theory of this mode as given by Furth, Killeen, and Rosenbluth (FKR) can be found in Appendix A.

In its most basic sense, the tearing mode is simply a manifestation of the fact that parallel current carrying wires are attracted to each other [11]. In Figure (1.3a) we show an equally spaced array of parallel wires carrying equal current in the same direction, along with the resulting magnetic field. Each wire is attracted by its neighbor, and an equilibrium results when all these forces are balanced. For analogy with an infinitely conducting plasma, we can think of the wires as being held rigidly in place. The

effect of resistivity is to remove this constraint, allowing the wires to move. Thus a small perturbation in the position of one of the wires will cause an imbalance in the forces, and the wires will tend to bunch together. The final configuration is shown in Figure (1.3b), where the magnetic islands are evident. The formation of such an island is characterized by the generation of a component of the magnetic field perpendicular to the original magnetic field, in, and symmetric about, the plane of the wires.

To see that this instability has no counterpart in an ideal ($\eta \rightarrow 0$) plasma [11], we introduce

$$\underline{B} = \underline{B}_0(y) + \underline{B}_1(x, y, z, t) \quad (1.2.1a)$$

$$\underline{V} = \underline{V}_1(x, y, z, t) \quad (1.2.1b)$$

$$\underline{B}_0(y) = B_{x0}(y)\hat{i} + B_{z0}(y)\hat{k} \quad (1.2.1c)$$

into equation (1.1.12) and expand the vector operation. Here $\underline{B}_0$ is a zero-order equilibrium field, compared to which $\underline{B}_1$ and $\underline{V}_1$ are considered to be small. Then ignoring terms which are second order in small quantities, and assuming

$\nabla \cdot \underline{B}_1 = \nabla \cdot \underline{V}_1 = 0$, we have

$$\frac{\partial \underline{B}_1}{\partial t} = \underline{B}_0 \cdot \nabla \underline{V}_1 - \underline{V}_1 \cdot \nabla \underline{B}_0 + \frac{c^2 \eta}{4\pi} \nabla^2 \underline{B}_1. \quad (1.2.2)$$

We now assume the first order quantities can be written in the form

$$f_1(x, y, z, t) = f_1(y) e^{i(\omega t + k_x x + k_z z)}. \quad (1.2.3)$$

When we take the y-component of the resulting equation, we get

$$\omega B_{y1} = i(\underline{k} \cdot \underline{B}_0) V_{y1} + \frac{c^2 \eta}{4\pi} \left(\frac{d^2 B_{y1}}{dy^2} - k^2 B_{y1} \right) \quad (1.2.4)$$

where $\underline{k} = k_x \hat{i} + k_z \hat{k}$ is the wave vector of the perturbation.

Now let $\underline{k}$ be such that the equation

$$F(y) = \underline{k} \cdot \underline{B}_0 = k_x B_{x0} + k_z B_{z0} = 0 \quad (1.2.5)$$

has solutions y_s which lie within the plasma. Clearly, on the surfaces $y = y_s$, the field lines of the equilibrium field are parallel to the wavefronts of the perturbation, so that these are the singular surfaces discussed in section 1.1.

A magnetic island will be characterized by values of B_{y1} which are non-zero on these surfaces (where $\underline{k} \cdot \underline{B} = 0$), and symmetric across these surfaces, as described above, and from equation (1.2.4) we see that, for $\eta = 0$, this requires an infinite value for V_{y1} on the singular surface. Such modes are thus forbidden in a perfectly conducting fluid. The inclusion of even the smallest amount of resistivity relaxes this constraint and allows the tearing mode to develop.

We now re-orient our coordinate system so that the x-axis is parallel to $\underline{k}$ and $y = 0$ defines the singular surface. Then the component of $\underline{B}_0$ perpendicular to $\underline{k}$ does not enter equation (1.2.4) and therefore plays no role in the instability. It simply behaves like an independent, unidirectional field. The process is therefore two-dimensional, and the analysis

can be carried out in the x-y plane (x being the direction of the wave vector), where it is the vanishing of the x-component of the zero order field which is significant. This situation is depicted in Figure (1.4).

Now look at an element of fluid lying in the region $y > 0$ where $B_{x0} > 0$, and let us imagine that we can move this fluid element at velocity $\underline{V}_1$ without perturbing the magnetic field, i.e., for the motion we are considering, $\partial \underline{B} / \partial t = 0$. Then Faraday's law states that the electric field $\underline{E} = 0$. From Ohm's law there will be a current density

$$\underline{J}_1 = \frac{\underline{V}_1 \times \underline{B}_0}{\eta c} \quad (1.2.6)$$

associated with this motion, with a resulting body force on the fluid element given by

$$\underline{F}_1 = \frac{1}{c} \underline{J}_1 \times \underline{B}_0 = \frac{(\underline{V}_1 \cdot \underline{B}_0) \underline{B}_0 - B_0^2 \underline{V}_1}{\eta c^2}$$

or

$$\begin{aligned} \underline{F}_1 &= \frac{V_{1\parallel} B_0^2 \hat{i} - B_0^2 (V_{1\parallel} \hat{i} + V_{1\perp} \hat{j})}{\eta c^2} \\ &= - \frac{V_{1\perp} B_0^2 \hat{j}}{\eta c^2} \end{aligned} \quad (1.2.7)$$

which opposes motion across the field. We see that for $\eta \rightarrow 0$ (i.e., for an ideal plasma), this force can become arbitrarily large, and the plasma cannot flow separately from the field. This is just the usual statement that in an infinitely conducting plasma the field is "frozen-in"

the fluid. Now, since the component of $\underline{B}_0$ perpendicular to $\underline{k}$ is ignorable (as discussed above), we can take B_0 to be B_{x0} . Then allowing for finite resistivity, we find that the restoring force is still large throughout most of the plasma, but can become weak near a singular surface (where $B_0 = 0$). If there is a driving force in the opposite direction of the restoring force, it can become dominant in the region near the singular surface and an instability can develop.

Let "a" be a scale length for changes in the zero order magnetic field, i.e., the effective half-width of the current sheet. In light of the above discussion we expect there to be detached flow in some region of half-width ϵa , where $\epsilon \ll 1$, centered about the singular surface. If $\underline{F}_d$ is the (as yet unspecified) driving force for the instability, then inside this region it is approximately equal in magnitude and opposite in direction to the restoring force. The power generated by this force is

$$P = \underline{F}_d \cdot \underline{V}_1 \approx - \underline{F}_1 \cdot \underline{V}_1$$

From (1.2.7)

$$F_1 = - \frac{V_1 B_0^2}{\pi c^2} \approx - \frac{V_1 (B_0')^2 (\epsilon a)^2}{\pi c^2}$$

where we have written $B_0 = B_0'(\epsilon a)$ inside the detached region, and the notation $()'$ signifies differentiation with respect

to y . Thus

$$P \sim \frac{v_1^2 (B_0')^2 (\epsilon a)^2}{nc^2} \quad (1.2.8)$$

Now, since the fluid is incompressible in our model,

$$\frac{\partial v_{x1}}{\partial x} + \frac{\partial v_{y1}}{\partial y} = 0.$$

Assuming that the derivatives of velocity scale like

$\partial/\partial x \sim k$, $\partial/\partial y \sim (\epsilon a)^{-1}$, we find

$$\left| \frac{v_{x1}}{v_{y1}} \right| \sim \frac{1}{\epsilon a k} \gg 1$$

so that the kinetic energy of the fluid in the neighborhood of the singular layer is

$$E_K \sim \rho v_{x1}^2 \sim \frac{\rho v_{y1}^2}{k^2 (\epsilon a)^2}.$$

When we equate the rate of change of this quantity to the power developed by the driving force, we can solve for the thickness of the region of detached flow. The result is

$$\epsilon a = \left[\frac{\omega \rho n c^2}{k^2 (B_0')^2} \right]^{\frac{1}{2}} \quad (1.2.9)$$

To find an appropriate driving force we note that there are really two contributions to the current in Ohm's law:

a) a part due to the relative motion between the fluid and the field, which we have seen to be responsible for the restoring force, and b) a part due to the electric field induced by the perturbed magnetic field, B_{y1} . By Lenz's law, this electric field will be such as to oppose the change

in flux linking the two sides of the singular surface. We expect that regions which have fluid moving toward the singular surface will have an increase in this flux because the field lines are effectively frozen in the fluid and the flux is thus convected inward, while those regions where the fluid is moving away from the singular surface will show a decrease. It is easy to see that in regions of increasing (decreasing) flux, an electric field must be induced in the negative (positive) z-direction. The resulting Lorentz force due to this induced electric field will oppose the restoring force, and may dominate near the singular surface.

More specifically, the z-component of the perturbed Ohm's law is

$$\eta_0 J_{z1} = E_{z1} - \frac{1}{c} v_{y1} B_{x0} . \quad (1.2.10)$$

The perturbed electric field, as obtained from Faraday's law, is

$$E_{z1} \approx \frac{\omega B_{y1}}{ck} . \quad (1.2.11)$$

while the perturbed current is obtained from Ampere's law and $\nabla \cdot \underline{B} = 0$:

$$J_{z1} = \frac{ic}{4\pi k} (k^2 B_{y1} - B_{y1}'') .$$

When we note that the scale length in the y-direction is much less than that in the x-direction, we can write

$$J_{z1} \approx \frac{c}{4\pi k} B_{y1}'' . \quad (1.2.12)$$

Using equations (A5), (A12), and (A15) (See Appendix A), we express B''_{y1} , as

$$B''_{y1} \sim \frac{B'_{y1}}{\epsilon a} \sim \frac{B_{y1}}{\epsilon k a^3} \quad (1.2.13)$$

so that

$$J_{z1} = \frac{c B_{y1}}{4 \pi \epsilon k a^3} \quad (1.2.14)$$

Within the region of detached flow near the singular surface we want the current to be determined primarily by the induced electric field, so that

$$\eta_0 J_{z1} \sim E_{z1}$$

This allows for the determination of ϵa . We find

$$\epsilon a = \frac{\eta c^2}{4 \pi k \omega a^2} \quad (1.2.15)$$

Then equating (1.2.9) and (1.2.15) we can obtain the growth rate as

$$\omega^5 = \frac{B_0^2 \eta^3 c^6}{(4 \pi)^4 \rho k^2 a^{10}} \quad (1.2.16)$$

where we have written $B'_0 \sim B_0/a$. In terms of the Alfvén transit time

$$\tau_H = \frac{V_A}{a} = \frac{B_0}{a \sqrt{4 \pi \rho}} \quad (1.2.17)$$

and the resistive diffusion time

$$\tau_R = \frac{4 \pi a}{c^2 \eta} \quad (1.2.18)$$

we have

$$\omega = \alpha^{-2/5} \tau_H^{-2/5} \tau_R^{-3/5} \quad (1.2.19)$$

where $\alpha = ka$ is a non-dimensional wave number. We thus see that the tearing mode grows at a rate part-way between the two relevant time scales in the problem, containing $2/5$ of the Alfvén time and $3/5$ of the resistive diffusion time.

When the growth rate is measured in units of the resistive diffusion time, we arrive at the well known formula

$$p = \alpha^{-2/5} S^{2/5} \quad (1.2.20)$$

which was obtained by Furth, Killeen, and Rosenbluth in 1963.

Here $p = \omega \tau_R$, and $S = \tau_R / \tau_H$ is the magnetic Reynolds number.

Finally, we can solve for the width of the singular layer ϵ . Measuring lengths in units of "a", we find

$$\epsilon \sim \alpha^{-3/5} S^{-2/5} \quad (1.2.21)$$

We see that ϵ can decrease dramatically for high S . Typical values of S are $10^2 \sim 10^5$ in laboratory plasmas to $10^9 \sim 10^{12}$ in space plasmas. Thus for practical problems ϵ is quite small and presents considerable computational difficulty.

In the above discussion the plasma has been treated as a conducting fluid, as it will be throughout this work, and certain objections may be raised to this. First of all, this approach implies that all relevant scale lengths in the problem are macroscopic, i.e., so large that individual particle behavior can be treated in an averaged way. In

particular, it implies that the particle gyroradius is smaller than all relevant scale lengths. However, in the case of a reversed (as opposed to sheared) field, which is of interest in space physics, the gyroradius becomes large near the singular surface. Secondly, our treatment has been for a collisional plasma, and there are many cases of practical interest (e.g., solar flares, the geomagnetic tail, certain fusion experiments, etc.) where the plasma is either collisionless or nearly so. These questions are addressed in Appendix B, where we review the kinetic theory of the tearing instability [12], [13] and find that a kinetic treatment yields results which are consistent with the predictions of fluid theory.

1.3 Previous Work

In this section, we briefly review previous analytical, experimental, and computational studies of the tearing mode.

1.3.1 Analytic

That magnetic field reconnection could take place at a neutral line separating two regions of oppositely directed magnetic field was first proposed by Dungey [14] in connection with current discharges in solar flares. Instead of an instability, he envisioned a steady state process whereby the dissipated magnetic field was responsible for the acceleration of particles to high energy.

The first systematic analysis of an instability of this type was done by Furth, Killeen, and Rosenbluth [10], who derived a dispersion relation for the case of a sheet pinch

by matching solutions to the infinite conductivity equations, as applied outside the singular layer, to approximate solutions to the resistive equations obtained within the layer. The resulting growth rate (see Appendix A) is given by equation (1.2.20). In addition, the tearing mode was found to be unstable only when

$$\Delta' > 0, \quad \alpha < 1 \quad (1.3.1a,b)$$

where Δ' is the discontinuity in the logarithmic derivative of the perturbed magnetic field across the singular surface, i.e.,

$$\Delta' = \frac{1}{B_{y1}} \left. \frac{\partial B_{y1}}{\partial y} \right|_{y_s^+}^{y_s^-}, \quad (1.3.2)$$

and $\alpha = ka$ is the non-dimensional wave number.

Johnson, Greene, and Coppi [15,16] extended this work to cylindrical geometries and obtained a tearing instability with the same growth rate and stability criterion as found in FKR. In addition, they obtained overstable resistive interchange modes with growth rates proportional to $\eta^{1/3}$. Interchange modes which are not localized have been found by Roberts and Taylor [17], and Dagazian [18]. These modes resemble those found in Bénard's problem [19], with the additional complexity that the characteristic convective roll must now be "twisted" due to magnetic shear.

Rutherford and Furth [20] considered the situation in which the plasma became so hot due to Ohmic heating that the growth rate of the tearing mode is comparable to the diamagnetic drift frequency. For this "drift-tearing" mode,

magnetic viscosity in the ion stress tensor and the Hall term and electron pressure gradient in Ohm's law should be included.

The behavior of the tearing mode in a cylindrical Tokamak has been investigated by Furth, Rutherford, and Selberg [21], who constructed stability diagrams by numerically computing Δ' (equation (1.3.2)) for various current density profiles. Also, by considering Ohm's law in the form

$$E_{\parallel} = \eta J_{\parallel} + \frac{m_e}{ne^2} \left(\frac{\partial J_{\parallel}}{\partial t} - \mu \nabla_{\perp}^2 J_{\parallel} \right) \quad (1.3.3)$$

where μ is the electron viscosity, and $(\)_{\parallel}$ and $(\)_{\perp}$ refer to directions parallel and perpendicular to the magnetic field, they found new "tearing" modes with growth rates varying like $\mu^{1/3}$ independent of resistivity.

Toroidal effects in Tokamak magnetic fields have been studied by Glasser, Greene, and Johnson [22], who found them stabilizing. In place of equation (1.3.1a), the new instability criterion is

$$\Delta' > \Delta_c > 0 \quad (1.3.4)$$

where Δ_c is a constant depending on the toroidal curvature.

The effect of a zero order flow velocity on the linear growth of the tearing mode has been investigated by Furzer, et al [24] for a cylindrical Reversed Field Pinch, and by Dobrott, et al [25] for a sheet pinch. They find the effect to be stabilizing. For the latter case, Killeen and Shestakov [26] have determined numerically that, for velocities

on the order of those characterized by resistive diffusion, this effect is small.

A consistent picture of the connection between the resistive and ideal internal kink modes in Tokamaks has recently been given by Coppi, et al [23]. They obtain an expression for the growth rate which reduces to the ideal [33] and tearing mode values in the proper limits.

There have been several attempts to understand the tearing mode from the framework of kinetic theory. Furth [12] solved the Vlasov equation for a region where the magnetic field suffered a reversal and determined that kinetic theory predicted the same behavior as fluid theory for this mode. More recently, two studies have dealt with understanding this mode across the entire range of collisionality. Hazeltine, Dobrott, and Wang [48] used a variational solution of the guiding center kinetic equation with Fokker-Planck collision term to obtain a form of Ohm's law which is uniformly valid for any ratio of wave frequency to collision frequency. The theory requires that the particle gyroradius be negligible compared to macroscopic scale lengths, and this may be violated in the tearing mode. Drake and Lee [13] dispensed with the guiding center approximation and obtained growth rates which are valid in a sheared magnetic field for all degrees of collisionality regardless of the size of the gyroradius. A more complete discussion of this work is found in Appendix B.

Historically, steady state magnetic reconnection [14,36-41]

has been advanced as a mechanism for energy release in solar flares [14,37,38,42,44,87] and for the occurrence of geomagnetic sub-storms [42,43,45]. Cross and Van Hoven [46] suggested that the single tearing mode could be important in solar flares. By Fourier analyzing in the y-direction (transverse to the singular surface), they found growth rates $\sim S^{.57}$, much higher than those predicted by FKR. However, it seems likely that, due to the periodicity in the y-direction, they were observing the double tearing mode [35], in which growth rates of this order and faster can occur (see Chapter IV of this thesis). More recently, a solar flare model has been presented [47] in which resistive instabilities play an important role: the occurrence of double (or multiple) tearing modes, coupled with the generation of an anomalously high resistivity, leading to rapid releases of energy in amounts consistent with observation.

The non-linear behavior of the tearing mode has been investigated by Rutherford [27], who determined that the mode would saturate when the width of the magnetic island becomes comparable to the width of the singular layer (e.g., equation (1.2.9)). White, et al [28], have used a reduced set of quasi-linear equations [29] to study the growth and saturation of tearing modes with poloidal mode numbers $m > 2$ in Tokamaks. They find the island width w to evolve according to the equation

$$\frac{dw}{dt} = 1.66n[\Delta_1'(w) - \alpha w] \quad (1.3.5)$$

where α is a constant, η is the resistivity, and $\Delta_1'(w)$ is the perturbation in Δ' (equation (1.3.2)) due to the growth of the island.

1.3.2 Experimental

Early experimental evidence of the tearing mode was obtained in configurations where the magnetic field suffered a reversal (i.e., the magnetic field vanished at the singular surface), such as a reversed field theta-pinch [93,94]. Perhaps the most striking direct evidence of the occurrence of the tearing mode has been found by Anderson and Kunkel [30], who observed the breakup of the current layer in a hard core Z-pinch into parallel filaments.

More recently, the occurrence of the "minor disruption" in Tokamaks [31] has been explained in terms of tearing modes [32]. The disturbance is characterized by sawtooth oscillations in the soft x-ray emission from the plasma interior. The infinite conductivity kink mode for this plasma has too small a growth rate [31] and saturates at too low a level [33] to account for the ramp of the sawtooth. It has, however, been determined [23,34] that the $m = 1$ tearing mode (i.e., the resistive internal kink mode) can grow to a large enough amplitude fast enough to explain the rise. In this model, the temperature of the plasma core rises due to Ohmic heating. The resulting decrease in resistivity causes an increase in current and a corresponding decrease in the safety factor. When q falls below unity, an $m = 1$ magnetic island forms and grows. The structure of the island is such as to flatten the temperature

and current profiles, and when it fills the entire core of the plasma, will result in a cooler plasma, a reduced current, and a safety factor greater than unity. This in turn causes the instability to abruptly cease. Ohmic heating then starts the cycle over again.

Other characteristics of Tokamak experiments are the observation that the current penetration during the initial phases of the pinch proceeds faster than expected, and the occurrence of the so-called "major disruption", or disruptive instability which is characterized by negative spikes on the loop-voltage signal followed by violent plasma motion and destruction of the discharge. It has been recently proposed [35] that both of these events can be explained by the double-tearing mode, in which two or more singular surfaces (probably with poloidal mode number $m > 1$) lie in close proximity. The resulting interaction could lead to enhanced growth rates, which could cause a sudden redistribution of plasma current, accounting for the observed rapid current penetration. If this occurred later in the discharge the resulting rapid change in magnetic flux could produce voltage spikes and enhanced radial transport, leading to the disruption of the plasma.

1.3.3 Computational

With the advances in speed and memory size of modern computers, it has become feasible to determine the linear growth rates of plasma instabilities by numerical, rather than analytical, techniques. For the case of ideal MHD,

such computations have followed two paths: calculations based on the energy principle [49,50], which have yielded significant information about the spectrum of ideal MHD activity in plasmas [51-53]; and calculations based on the solution of an initial value problem in which the linearized fluid equations are advanced in time until an exponentially growing solution appears. Codes of this type have recently appeared in two [54] and three [55] space dimensions, and have been successful in determining the stability of various plasma configurations against infinite conductivity modes [56].

Killeen [57] was the first to apply numerical methods to the study of the linear properties of the tearing mode. He solved the linearized equations as derived in FKR (see Appendix A) as an initial value problem and obtained growth rates as a function of magnetic Reynolds number in good agreement with FKR. Wesson [58] used a shooting method to solve the associated eigenvalue problem and obtained similar results. Dibiase [59] extended Killeen's work and conducted extensive parametric studies of the tearing mode in various magnetic field configurations [60]. Much of this work included the effects of viscosity and thermal conductivity. This code has also been used in connection with the tearing mode in Reversed Field Pinch experiments [61-63].

The codes that have just been described yield information about the linear behavior of unstable modes. In order to study the long-time, large amplitude behavior of plasma instabilities, and to simulate experimental devices, the non-linear fluid

equations must be solved. In general, such a calculation requires the simultaneous advancement in time of eight non-linear partial differential equations (the magnetohydrodynamic, or MHD, equations) in several spatial dimensions. Early codes of this type are reviewed by Roberts and Potter [64].

Codes which model the behavior of an infinitely conducting plasma in three spatial dimensions have recently appeared. Brackbill [65] has used a moving computational grid along with implicit methods [67] in extensive simulations of the Scyllac experiment. Bateman, et al [66] have used explicit methods on an Eulerian mesh to study fixed boundary MHD instabilities.

More realistic modelling of plasma behavior requires that transport coefficients, such as resistivity, be included. Explicit codes in two [95] and three [68,96] spatial dimensions are now in existence. However, the inclusion of resistivity changes the equations from hyperbolic to parabolic, with the result that explicit methods may not be adequate for reasons of economics (computer time) and numerical stability. Lindemuth and Killeen [69] were the first to use implicit finite difference methods for non-linear multidimensional MHD problems. Later versions of this code have successfully simulated Z-pinch implosions [70] by excluding two components of the magnetic field and one component of the velocity from the model. Lui and Chu [71,72] have used implicit methods to simulate the implosion phase of cylindrical and belt pinch experiments. Finan [73] is using implicit methods to solve the full set of resistive MHD equations in three space dimensions. In all of

these codes, an Eulerian mesh is employed.

There have been few numerical studies of the non-linear evolution of the tearing mode, and none with great generality. Van Hoven and Cross [74] observed saturation by considering the non-linear generation of neighboring harmonics. More recently, two dimensional calculations have been carried out in connection with the Tokamak minor disruption [32,34]. In this model, the plasma parameters are assumed to possess Tokamak ordering (high aspect ratio, $B_z \gg B_\theta$, $\beta \ll 1$), as discussed in section 1.1. The MHD equations are expanded in powers of the inverse aspect ratio $\epsilon = a/R \ll 1$, retaining only terms of lowest order in ϵ , and the fluid is assumed to be incompressible in the poloidal plane. This results in the reduced set of equations [29]

$$\frac{d\psi}{dt} = \eta(\nabla^2\psi + \frac{2kB_z}{m}) \quad (1.3.6a)$$

and

$$\rho \frac{d}{dt} \nabla^2 A = - \hat{z} \cdot (\nabla\psi \times \nabla \nabla^2\psi) \quad (1.3.6b)$$

where ψ is the helical flux function, A is the velocity stream function, and $\hat{z}$ is a unit vector in the axial direction. The radial and azimuthal components of the magnetic field are then given by

$$B_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad B_\theta = - \frac{\partial \psi}{\partial r} - \frac{krB_z}{m} \quad (1.3.7a,b)$$

while the velocity is determined from

$$\underline{v} = \nabla A \times \hat{z} \quad (1.3.8)$$

This model allows for rapid computation, but the results are limited to the specific case of Tokamak ordering. This work is currently being extended to three space dimensions [75,97,98].

1.4 Scope of the Present Work

As discussed in the section 1.3.3, previous computational analysis of the non-linear properties of the tearing mode has applied only to magnetoplasma systems possessing Tokamak ordering. In the present work we apply a more basic model: we numerically solve all eight non-linear resistive MHD equations in two dimensions as an initial value problem. No assumption about the ordering of various terms is made, rendering our results more general than those previously obtained.

We develop a model in which the relevant equations are expressed as a set of conservation laws, the conserved quantities being magnetic flux through any coordinate plane, momentum, mass, and total energy. These equations are then written in general orthogonal curvilinear coordinates, so that the model can be readily applied to a variety of geometries. The resulting equations are then solved by the method of finite differences on an Eulerian mesh.

The spatial discretization of the differential equations is performed in such a way as to guarantee the simultaneous conservation of the desired physical quantities throughout the course of the calculation. To this end, special boundary conditions on thermodynamic properties are derived which assure total conservation. The temporal discretization is implicit,

making use of the Alternating Direction Implicit (ADI) method [81,82].

Not the least among the problems of computational magnetohydrodynamics is the treatment of singular boundaries (e.g., $r = 0$ in cylindrical or spherical coordinates). For cases in which there is no gross motion of fluid across the origin, a modification of the conservative boundary conditions is applied. For the more difficult case characterized by flow across the origin (e.g., the $m = 1$ mode in a cylinder), we derive and apply special boundary conditions based on certain planar symmetries in the problem.

We apply our model to several geometries. For comparison with analytic results, we first study the tearing mode in a sheet pinch, for which Cartesian coordinates are used. We use eigenfunctions obtained from a code which solves the linearized MHD equations [57] as initial conditions for our two dimensional, non-linear code. When the initial amplitude of the mode is small enough, we find that the mode grows exponentially at a rate in good agreement with previous analytic [10] and numerical [57] results. When the mode is allowed to evolve to large amplitude, non-linear saturation is observed. A limited parametric study of the amplitude at saturation as a function of magnetic Reynolds number shows good agreement with analytic predictions [27].

We have also studied the linear and non-linear behavior of the double-tearing mode [35] in Cartesian geometry. This mode occurs when two singular surfaces lie in close proximity.

We find that the interaction between the singular surfaces can lead to linear growth rates which are greatly enhanced over those of the single tearing mode. We also find that the non-linear evolution of this mode can lead to magnetic field configurations which may result in enhanced cross-field particle and thermal transport. For this case, we find that approximately ten percent of the dissipated magnetic energy goes into kinetic energy, and ninety percent into thermal energy.

To demonstrate the versatility of our general model, we apply our code to cylindrical geometry, where we study the $m = 0, 1$, and 2 modes in a Reversed Field Pinch. The equilibrium is taken to be the force free Bessel function model, and, in a manner similar to that employed for the sheet pinch calculations, we use a linear code [60] to obtain tearing mode eigenfunctions, which we use as input. For the axisymmetric case of $m = 0$, the independent variables are taken to be (r, z) . For $m > 0$, we employ a transformation to helical coordinates which allows us to solve the three dimensional equations under the assumption that helical symmetry is preserved. For these cases we follow the $m = 1$ and $m = 2$ modes into the non-linear regime. We find that the $m = 2$ mode saturates quickly, while the $m = 1$ mode is more dangerous in that it can continue to grow to large amplitude.

Fig. 1.1a The sausage mode ($m = 0$)

Fig. 1.1b The kink mode ($m = 1$)

Fig. 1.1c The flute mode ($m > 1$. Shown here with $m = 4$)

Fig. 1.2 A magnetic field line wound helically on a cylindrical flux surface

Fig. 1.3a Equally spaced array of parallel wires carrying equal amounts of current, along with the resulting magnetic field

Fig. 1.3b The result of a perturbation of the equilibrium shown in Fig. 1.3a.
Note the magnetic islands

Fig. 1.4 Equilibrium magnetic field for the tearing mode

CHAPTER II - MATHEMATICAL MODEL

In Chapter I and Appendices A and B, we discuss the relevant physics of the tearing mode from both the kinetic and fluid points of view. We have found that these two approaches to the problem yield similar results even under circumstances in which the use of fluid equations may not be completely justified. In view of this, and because of its relative simplicity, we adopt the continuum approach.

In this chapter we will describe in some detail the mathematical model employed in this work. We will discuss the relevant equations and the assumptions implicit in their use, and will derive the form of these equations which is particularly well suited to the problem at hand. The method of computational solution, and the appropriate boundary conditions to be imposed, will be discussed in Chapter III.

2.1 The Resistive MHD Equations

We will consider our plasma to be modelled as a continuous conducting fluid moving in a magnetic field. In addition, we assume that the electric charge density vanishes at every point in the fluid. This implies that the scale lengths of interest in the problem are much larger than the Debye length, the length over which non-uniformities in charge density are expected to be noticeable. If there are no sources or sinks of matter, then mass will be conserved and we have the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0 \quad (2.1.1)$$

where ρ is the mass density of the plasma and $\underline{v}$ is the fluid velocity at a point in space.

The equation of motion for a infinitesimal element of fluid is simply

$$\rho \frac{d\underline{v}}{dt} = \underline{F} \quad (2.1.2)$$

where $\underline{F}$ is the force per unit volume acting on the fluid element. For the case of a conducting fluid, and neglecting gravity, this force consists of two parts: a) a force due to gradients in hydrodynamic pressure, and b) the Lorentz force arising from the interaction of the magnetic field with electric currents that may be flowing in the plasma. The Lagrangian equation of motion for a fluid element is then

$$\rho \frac{d\underline{v}}{dt} = - \nabla p + \frac{1}{c} \underline{J} \times \underline{B} \quad , \quad (2.1.3)$$

which in Eulerian form (i.e., referenced to a fixed reference frame) becomes [76]

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = - \nabla p + \frac{1}{c} \underline{J} \times \underline{B}. \quad (2.1.4)$$

Note that the electric force $\sim q\underline{E}$ is absent, not because the electric field vanishes, but because each fluid element possesses zero net charge.

We write the heat balance for the fluid element as it moves in space as

$$\rho T \frac{ds}{dt} = \eta J^2 \quad (2.1.5)$$

where T is the temperature, s is the entropy per unit mass, η is the resistivity of the fluid, and J is the magnitude of the current density. The heat increase per unit volume is thus due only to Joule heating, the viscous dissipation and thermal conduction being ignored. When we use the thermodynamic relation

$$Tds = p d\left(\frac{1}{\rho}\right) + de \quad (2.1.6)$$

where e is the internal energy per unit mass and $\tau = 1/\rho$ is the specific volume, along with the continuity equation (2.1.1), equation (2.1.5) becomes

$$\rho \frac{de}{dt} = - p \nabla \cdot \underline{v} + \eta J^2 \quad (2.1.7)$$

The first term on the right hand side is the change in internal energy density due to compression or dilation of the fluid element (PV work), while the second term is the rate of Joule heating ($I^2 R$). In Eulerian form, this energy equation is

$$\frac{\partial (\rho e)}{\partial t} = - p \nabla \cdot \underline{v} - \nabla \cdot (\rho e \underline{v}) + \eta J^2 \quad (2.1.8)$$

We have so far specified how the conducting fluid moves in the presence of an electromagnetic field. To obtain a self-consistent description we must also describe how the motion of the fluid affects the fields themselves. This is done through the Maxwell equations

$$\nabla \times \underline{E} = - \frac{1}{c} \frac{\partial \underline{B}}{\partial t} \quad (2.1.9a)$$

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} \quad (2.1.9b)$$

$$\nabla \cdot \underline{B} = 0 \quad (2.1.9c)$$

$$\nabla \cdot \underline{E} = 0 \quad (2.1.9d)$$

and Ohm's law

$$\eta \underline{J} = \underline{E} + \frac{1}{c} \underline{V} \times \underline{B} \quad (2.1.10)$$

In writing these equations we have assumed that any electric fields are of the order of induced effects and that characteristic velocities are not relativistic, so that we can neglect the displacement current in Ampere's law, equation (2.1.9b).

Equations (2.1.1), (2.1.4), (2.1.8), (2.1.9a), (2.1.9b), and (2.1.10), along with an equation of state $p = p(\rho, e)$, constitute a set of 15 equations in the 15 unknowns $\underline{V}$, $\underline{B}$, $\underline{E}$, $\underline{J}$, ρ , e , and p , and represent the basic physical model for our problem. This set can be greatly reduced by combining (2.1.9a), (2.1.9b), and (2.1.10) to eliminate the six unknowns $\underline{J}$ and $\underline{E}$. The resulting set of equations is

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B} - \frac{c^2 \eta}{4\pi} \nabla \times \underline{B}) \quad (2.1.11a)$$

$$\rho \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right) = - \nabla p + \frac{1}{4\pi} (\nabla \times \underline{B}) \times \underline{B} \quad (2.1.11b)$$

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \underline{V}) \quad (2.1.11c)$$

$$\frac{\partial (\rho e)}{\partial t} = - p \nabla \cdot \underline{V} - \nabla \cdot (\rho e \underline{V}) + \eta \left(\frac{c}{4\pi} \nabla \times \underline{B} \right)^2 \quad (2.1.11d)$$

which consists of eight equations in eight unknowns: three

components of the magnetic flux density $\underline{B}$, three components of the velocity $\underline{V}$, the density ρ , and the internal energy per unit mass e . The third thermodynamic variable, the pressure p , is eliminated by employing an equation of state. For our model we assume that the perfect gas law is valid, so that

$$e = \frac{p/\rho}{\gamma-1} \quad (2.1.12)$$

where $\gamma = C_p/C_v$ is the ratio of specific heats of the plasma. Equations (2.1.11) constitute the basic equations of resistive magnetohydrodynamics (MHD).

Equation (2.1.11a) is basically Ampere's law and as such assures the conservation of magnetic flux, and equation (2.1.11c) is a statement that mass is conserved. If equation (2.1.11b) is combined with the continuity equation, and the vector identities

$$(\nabla \times \underline{B}) \times \underline{B} = \underline{B} \cdot \nabla \underline{B} - \nabla (\frac{1}{2} B^2)$$

$$\nabla \cdot (\underline{B}\underline{B}) = \underline{B} \cdot \nabla \underline{B} + \underline{B} \nabla \cdot \underline{B}$$

are used along with $\nabla \cdot \underline{B} = 0$, we obtain

$$\frac{\partial (\rho \underline{V})}{\partial t} = - \nabla \cdot [\rho \underline{V}\underline{V} - \frac{\underline{B}\underline{B}}{4\pi} + (p + \frac{B^2}{8\pi}) \underline{I}] \quad (2.1.13)$$

which expresses the conservation of momentum. Here $\underline{I}$ denotes the unit dyad. In this equation, $\rho \underline{V}\underline{V}$ is the flux of fluid momentum convected into a volume element by fluid motion

(the Reynolds stress),

$$\frac{1}{4\pi}(\underline{B}^2 \underline{I} - \underline{B}\underline{B})$$

is the flux of momentum in the electromagnetic field convected into the volume elements by the Poynting vector, and ∇p is the net fluid pressure force acting on the faces of the volume element.

An equation expressing the conservation of total energy can be found [77,78] by combining equations (2.1.11a), (2.1.11c), (2.1.11d), and (2.1.13) in the form

$$\frac{\underline{B}}{4\pi} \cdot \frac{\partial \underline{B}}{\partial t} - \frac{1}{2} |\underline{V}|^2 \frac{\partial \rho}{\partial t} + \frac{\partial (\rho e)}{\partial t} + \underline{V} \cdot \frac{\partial (\rho \underline{V})}{\partial t}$$

to obtain the conservation law

$$\begin{aligned} \frac{\partial u}{\partial t} = & - \nabla \cdot [(u+p)\underline{V}] + \frac{1}{8\pi} (\underline{B}^2 \underline{I} - 2\underline{B}\underline{B}) \cdot \underline{V} \\ & + \left(\frac{c}{4\pi}\right)^2 \eta (\underline{B} \cdot \nabla \underline{B} - \nabla \underline{B} \cdot \underline{B}) \end{aligned} \quad (2.1.14)$$

where $u = \frac{1}{2} \rho V^2 + B^2/8\pi + \rho e$ is the total energy density of the fluid. The term $u\underline{V}$ represents the flux of energy density into or out of the volume element, the term $p\underline{V}$ represents the rate at which the fluid pressure forces do work at the boundary of the volume element, and the term

$$\frac{1}{8\pi} (\underline{B}^2 \underline{I} - 2\underline{B}\underline{B}) \cdot \underline{V} + \left(\frac{c}{4\pi}\right)^2 \eta (\underline{B} \cdot \nabla \underline{B} - \nabla \underline{B} \cdot \underline{B})$$

is the flux of electromagnetic energy flowing into the volume element by means of the Poynting vector

$$\underline{S} = \frac{c}{4\pi} (\underline{E} \times \underline{B}) .$$

From equation (2.1.11a) we see that there are two characteristic time scales in the problem: a diffusion time scale arising from the term

$$\frac{c^2}{4\pi} \nabla \times (\eta \nabla \times \underline{B}) ;$$

and a convective, or wave-like, time scale, whose origin is the term

$$\nabla \times (\underline{V} \times \underline{B}) .$$

Depending on the relative magnitude of various parameters, these time scales may be widely separated. In such cases (e.g., viscous hydrodynamics), it proves convenient to introduce a set of dimensionless variables in such a way as to isolate the effect of the separation of the time scales. To this end we non-dimensionalize our set of equations by means of the transformation

$$\underline{x}/a \rightarrow \underline{x}, \quad t/t_H \rightarrow t, \quad \underline{B}/B_0 \rightarrow \underline{B},$$

$$\underline{V}/V_A \rightarrow \underline{V}, \quad \rho/\rho_0 \rightarrow \rho, \quad p/p_0 \rightarrow p,$$

$$T/T_0 \rightarrow T, \quad u/u_0 \rightarrow u, \quad \eta/\eta_0 \rightarrow \eta.$$

Here, subscripts ()₀ refer to characteristic values of various quantities, "a" is a characteristic length for changes in the magnetic field, $V_A = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén velocity, and $t_H = a/V_A$ is the hydromagnetic (or Alfvén) transit time, i.e., the time it would take an Alfvén wave

to propagate a distance "a". The normalization of the thermodynamic quantities is chosen such that

$$u_0 = p_0 = B_0^2/8\pi = \rho v_A^2/2.$$

Then in terms of these variables our basic set of equations is

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B} - \frac{\eta}{S} \nabla \times \underline{B}) \quad (2.1.15a)$$

$$\frac{\partial (\rho \underline{v})}{\partial t} = -\nabla \cdot [\rho \underline{v} \underline{v} - \underline{B} \underline{B} + \frac{1}{2} (p + B^2) \underline{I}] \quad (2.1.15b)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v}) \quad (2.1.15c)$$

$$\begin{aligned} \frac{\partial u}{\partial t} = & -\nabla \cdot [(u+p) \underline{v} + (B^2 \underline{I} - 2 \underline{B} \underline{B}) \cdot \underline{v} \\ & + \frac{2\eta}{S} (\underline{B} \cdot \nabla \underline{B} - \nabla \underline{B} \cdot \underline{B})] \end{aligned} \quad (2.1.15d)$$

In these equations the non-dimensional quantity S , defined as

$$S \equiv t_R/t_H = 4\pi a v_A/c^2 \eta_0, \quad (2.1.16)$$

is the magnetic Reynolds number, and is the ratio of the resistive diffusion time scale t_R to the hydromagnetic (or convective) time scale t_H . This is the same quantity that appeared in the heuristic discussion of the tearing mode, as described in Chapter I, and in the FKR analysis (see Appendix A). At first glance it would appear that by making S large enough we can make the effects of resistivity so small that they can be ignored. However, as was seen in Chapter I and Appendix A, even though S is large and the

diffusive terms may be small over most of the plasma, there may be localized regions (e.g., a singular surface) where they become dominant.

2.2 A Convenient Model

Equations (2.1.15), along with appropriate boundary and initial conditions, define an initial value problem which can be used to follow the time history of a given plasma perturbation. The general form of the boundary conditions will be discussed in Chapter III, while a discussion of the generation of initial conditions and specialized boundary conditions will be found in later chapters. Here we will describe the specific form of the equations to be solved. Spatial and temporal finite difference techniques as applied to these equations will be discussed in Chapter III.

2.2.1 Conservation Form

Equations (2.1.15) form a set of conservation laws. Equations (2.1.15c) and (2.1.15d) are of the form

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot \underline{F} \quad (2.2.1)$$

where ρ is a scalar density and $\underline{F}$, the flux of ρ , is a vector. Equation (2.1.15b) is of the form

$$\frac{\partial \underline{V}}{\partial t} = - \nabla \cdot \underline{T} \quad (2.2.2)$$

where $\underline{V}$ is a vector density and $\underline{T}$, the flux of $\underline{V}$, is a

symmetric second rank tensor, or dyad. An equation of the form (2.2.1) is called a scalar conservation law, while an equation of the form (2.2.2) is called a vector conservation law. Equation (2.1.15a) appears in the form

$$\frac{\partial \underline{w}}{\partial t} = \nabla \times \underline{G} \quad (2.2.3)$$

where $\underline{w}$ is a pseudovector and $\underline{G}$ is a vector. To complete the formal transformation to flux-divergence form, we rewrite this equation as

$$\frac{\partial \underline{w}}{\partial t} = \nabla \cdot \underline{A} \quad (2.2.4)$$

where $\underline{A}$ is an antisymmetric second rank tensor ($A_{ij} = -A_{ji}$) whose components are related to those of the vector $\underline{G}$ by

$$G_1 = A_{32}, \quad G_2 = A_{13}, \quad G_3 = A_{21} \quad (2.2.5)$$

as may be verified by direct calculation. Equations of the form (2.2.3) or (2.2.4) will be called pseudovector conservation laws.

The meaning of scalar and vector conservation laws is found by integrating over a volume element and applying the divergence theorem to the right hand side of (2.2.1) and (2.2.2). In terms of Cartesian components we have, for the scalar law (2.2.1),

$$\frac{\partial}{\partial t} \int_V \rho d^3 \underline{r} = - \oint_S \underline{F} \cdot \hat{n} da = - \sum_i \int_{S_i} F_i da_i \quad (2.2.6)$$

where F_i is the component of the flux normal to S_i , the i^{th}

face of the cubical volume element, da_i is the differential area element of the i^{th} face, and the sum extends over all faces. We thus see that the rate of change of the quantity

$$M = \int_V \rho d^3 \underline{r}$$

inside the volume is given by the net flux of M (measured in units of M per unit time) entering (or leaving) through the surface bounding the volume. Similarly, for the vector conservation law, we find

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial t} \int_V v_i d^3 \underline{r} = - \sum_j \int_{S_j} T_{ij} da_j \quad (2.2.7)$$

where T_{ij} now represents the flux of the i^{th} component of the vector

$$\underline{v} = \int_V \underline{v} d^3 \underline{r}$$

through the j^{th} face of the cube.

However, for the pseudovector conservation law the correct physical interpretation is found by applying Stokes' theorem to equation (2.2.3). The result is

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial t} \int_S \underline{w} \cdot d\underline{a} = \oint_C \underline{G} \cdot d\underline{l} \quad (2.2.8)$$

where the closed curve C bounds the open surface S . The scalar $\phi = \int \underline{w} \cdot d\underline{a}$ is called a flux, and the vector $\underline{w}$ is termed a flux density. We see that changes in ϕ are driven by the circulation of the vector $\underline{G}$.

2.2.2 Transformation to Orthogonal Curvilinear Coordinates

The set of equations (2.1.15), as written, is valid in

any coordinate system. Computationally, we must specify a coordinate system and then expand the indicated vector operations to obtain a set of differential equations to which finite difference techniques may be applied. In this work we choose to keep these equations as general as possible by specifying the metric as

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2 \quad (2.2.9)$$

where x_1 , x_2 , and x_3 are orthogonal coordinates and the scale factors $h_i = h_i(x_1, x_2, x_3)$ are defined such that $h_i dx_i$ represents a differential arc length along the i^{th} coordinate. In this coordinate system the volume element is given by

$$d\tau = h_1 h_2 h_3 dx_1 dx_2 dx_3 .$$

Many coordinate systems of practical interest are of the form (2.2.9). For example we have:

a) Cartesian coordinates:

$$\begin{aligned} x_1 &= x , \quad x_2 = y , \quad x_3 = z \\ h_1 &= h_2 = h_3 = 1 , \\ d\tau &= dx dy dz \end{aligned} \quad (2.2.10a)$$

b) Cylindrical coordinates:

$$\begin{aligned} x_1 &= r , \quad x_2 = \theta , \quad x_3 = z \\ h_1 &= 1 , \quad h_2 = r , \quad h_3 = 1 \\ d\tau &= r dr d\theta dz \end{aligned} \quad (2.2.10b)$$

c) Spherical coordinates

$$x_1 = r, \quad x_2 = \theta, \quad x_3 = \phi$$

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin \theta$$

$$d\tau = r^2 \sin \theta dr d\theta d\phi \quad (2.2.10c)$$

d) Polar toroidal coordinates

$$x_1 = r, \quad x_2 = \theta, \quad x_3 = \phi$$

$$h_1 = 1, \quad h_2 = r, \quad h_3 = R + r \cos \theta$$

$$d\tau = r(R + r \cos \theta) dr d\theta d\phi \quad (2.2.10d)$$

where r is a "minor radius" coordinate, θ is the poloidal angle, ϕ is the toroidal angle, and R is a constant measuring the major radius of the torus. In this work application will be made only to Cartesian and cylindrical systems.

In terms of the general metric (2.2.9), the gradient operator becomes

$$\nabla = \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial x_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial x_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial x_3} \quad (2.2.11)$$

where $\hat{e}_i$ is a unit vector in the direction of the i^{th} coordinate. In expanding the indicated vector operations in dyadic notation, i.e.,

$$\underline{\underline{S}} = \sum_{i=1}^3 \sum_{j=1}^3 S_{ij} \hat{e}_i \hat{e}_j \quad (2.2.12)$$

we use the formulas [79]

$$\frac{\partial \hat{e}_1}{\partial x_1} = - \frac{\hat{e}_2}{h_2} \frac{\partial h_1}{\partial x_2} \quad (2.2.13a)$$

$$\frac{\partial \hat{e}_1}{\partial x_2} = \frac{\hat{e}_2}{h_1} \frac{\partial h_2}{\partial x_1} \quad (2.2.13b)$$

$$\frac{\partial \hat{e}_1}{\partial x_3} = \frac{\hat{e}_3}{h_1} \frac{\partial h_3}{\partial x_1} \quad (2.2.13c)$$

$$\frac{\partial \hat{e}_2}{\partial x_1} = \frac{\hat{e}_1}{h_2} \frac{\partial h_1}{\partial x_2} \quad (2.2.13d)$$

$$\frac{\partial \hat{e}_2}{\partial x_2} = - \frac{\hat{e}_1}{h_1} \frac{\partial h_2}{\partial x_1} \quad (2.2.13e)$$

$$\frac{\partial \hat{e}_2}{\partial x_3} = \frac{\hat{e}_3}{h_2} \frac{\partial h_3}{\partial x_2} \quad (2.2.13f)$$

$$\frac{\partial \hat{e}_3}{\partial x_1} = 0 \quad (2.2.13g)$$

$$\frac{\partial \hat{e}_3}{\partial x_2} = 0 \quad (2.2.13h)$$

$$\frac{\partial \hat{e}_3}{\partial x_3} = - \frac{\hat{e}_1}{h_1} \frac{\partial h_3}{\partial x_1} - \frac{\hat{e}_2}{h_2} \frac{\partial h_3}{\partial x_2} \quad (2.2.13i)$$

Implicit in this and what follows is the assumption that all physical quantities and scale factors are functions of x_1 and x_2 only (although we have allowed the unit vectors to depend on all three coordinates). For tearing mode calculations this assumption is expected to be good since, as was seen in Chapter I, this mode is essentially two dimensional. Then the expression for the divergence of a vector becomes

$$\nabla \cdot \underline{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 F_1) + \frac{\partial}{\partial x_2} (h_1 h_3 F_2) \right], \quad (2.2.14)$$

the divergence of an antisymmetric tensor is

$$\begin{aligned} \nabla \cdot \underline{A} = & - \frac{\hat{e}_1}{h_2 h_3} \frac{\partial}{\partial x_2} (h_3 A_{12}) + \frac{\hat{e}_2}{h_1 h_3} \frac{\partial}{\partial x_1} (h_3 A_{12}) \\ & + \frac{\hat{e}_3}{h_1 h_2} \left[\frac{\partial}{\partial x_1} (h_2 A_{13}) + \frac{\partial}{\partial x_2} (h_1 A_{23}) \right], \quad (2.2.15) \end{aligned}$$

while the divergence of a symmetric tensor can be written as

$$\begin{aligned} \nabla \cdot \underline{T} = & \frac{\hat{e}_1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 T_{11}) + \frac{\partial}{\partial x_2} (h_1 h_3 T_{12}) \right. \\ & \left. + h_3 \frac{\partial h_1}{\partial x_2} T_{12} - h_3 \frac{\partial h_2}{\partial x_1} T_{22} - h_2 \frac{\partial h_3}{\partial x_1} T_{33} \right] \\ & + \frac{\hat{e}_2}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 T_{12}) + \frac{\partial}{\partial x_2} (h_1 h_3 T_{22}) \right. \\ & \left. + h_3 \frac{\partial h_2}{\partial x_1} T_{12} - h_3 \frac{\partial h_1}{\partial x_2} T_{11} - h_1 \frac{\partial h_3}{\partial x_2} T_{33} \right] \\ & + \frac{\hat{e}_3}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 T_{13}) + \frac{\partial}{\partial x_2} (h_1 h_3 T_{23}) \right. \\ & \left. + h_2 \frac{\partial h_3}{\partial x_1} T_{13} + h_1 \frac{\partial h_3}{\partial x_2} T_{23} \right]. \quad (2.2.16) \end{aligned}$$

The terms in this expression which do not appear as perfect derivatives arise when the coordinates are not straight lines (i.e., the unit vectors depend on position, via equations (2.2.13)), and are known as Coriolis terms.

In terms of an antisymmetric tensor, the combination of Maxwell's equations and Ohm's law, equations (2.1.15a), is (see Appendix C)

$$\frac{\partial \underline{B}}{\partial t} = \nabla \cdot [\underline{B}\underline{V} - \underline{V}\underline{B} + \frac{\eta}{S} (\underline{V}\underline{B} - \underline{V}\underline{B}^{\dagger})] \quad (2.2.17)$$

where $\underline{V}\underline{B}^{\dagger}$ denotes the transpose of the tensor $\underline{V}\underline{B}$. Then making use of the expressions (2.2.14), (2.2.15), and (2.2.16), the fundamental set of equations (2.1.15) becomes,

in component form

$$\begin{aligned} \frac{\partial B_1}{\partial t} = & \frac{1}{h_2 h_3} \frac{\partial}{\partial x_2} \left\{ h_3 (v_1 B_2 - v_2 B_1) \right. \\ & \left. + \frac{1}{S} \frac{h_3 \eta}{h_1 h_2} \left[\frac{\partial}{\partial x_2} (h_1 B_1) - \frac{\partial}{\partial x_1} (h_2 B_2) \right] \right\} \quad (2.2.18a) \end{aligned}$$

$$\begin{aligned} \frac{\partial B_2}{\partial t} = & \frac{1}{h_1 h_3} \frac{\partial}{\partial x_1} \left\{ h_3 (v_2 B_1 - v_1 B_2) \right. \\ & \left. + \frac{1}{S} \frac{h_3 \eta}{h_1 h_2} \left[\frac{\partial}{\partial x_1} (h_2 B_2) - \frac{\partial}{\partial x_2} (h_1 B_1) \right] \right\} \quad (2.2.18b) \end{aligned}$$

$$\begin{aligned} \frac{\partial B_3}{\partial t} = & \frac{1}{h_1 h_2} \frac{\partial}{\partial x_1} \left\{ h_2 (v_3 B_1 - v_1 B_3) + \frac{1}{S} \frac{h_2 \eta}{h_1 h_3} \frac{\partial}{\partial x_1} (h_3 B_3) \right\} \\ & + \frac{1}{h_1 h_2} \frac{\partial}{\partial x_2} \left\{ h_1 (v_3 B_2 - v_2 B_3) \right. \\ & \left. + \frac{1}{S} \frac{h_1 \eta}{h_2 h_3} \frac{\partial}{\partial x_2} (h_3 B_3) \right\} \quad (2.2.18c) \end{aligned}$$

$$\begin{aligned} \frac{\partial (\rho v_1)}{\partial t} = & - \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x_1} \left[h_2 h_3 (\rho v_1^2 - B_1^2) \right] \right. \\ & + \frac{\partial}{\partial x_2} \left[h_1 h_3 (\rho v_1 v_2 - B_1 B_2) \right] \\ & + h_3 \frac{\partial h_1}{\partial x_2} (\rho v_1 v_2 - B_1 B_2) - h_3 \frac{\partial h_2}{\partial x_1} (\rho v_2^2 - B_2^2) \\ & \left. - h_2 \frac{\partial h_3}{\partial x_1} (\rho v_3^2 - B_3^2) \right\} - \frac{1}{2} \frac{1}{h_1} \frac{\partial}{\partial x_1} (p + B^2) \quad (2.2.18d) \end{aligned}$$

$$\begin{aligned}
 \frac{\partial(\rho V_2)}{\partial t} = & - \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x_1} [h_2 h_3 (\rho V_1 V_2 - B_1 B_2)] \right. \\
 & + \frac{\partial}{\partial x_2} [h_1 h_3 (\rho V_2^2 - B_2^2)] \\
 & + h_3 \frac{\partial h_2}{\partial x_1} (\rho V_1 V_2 - B_1 B_2) - h_3 \frac{\partial h_1}{\partial x_2} (\rho V_1^2 - B_1^2) \\
 & \left. - h_1 \frac{\partial h_3}{\partial x_2} (\rho V_3^2 - B_3^2) \right\} - \frac{1}{2} \frac{1}{h_2} \frac{\partial}{\partial x_2} (p + B^2)
 \end{aligned}
 \tag{2.2.18e}$$

$$\begin{aligned}
 \frac{\partial(\rho V_3)}{\partial t} = & - \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x_1} [h_2 h_3 (\rho V_1 V_3 - B_1 B_3)] \right. \\
 & + \frac{\partial}{\partial x_2} [h_1 h_3 (\rho V_2 V_3 - B_2 B_3)] \\
 & + h_2 \frac{\partial h_3}{\partial x_1} (\rho V_1 V_3 - B_1 B_3) + h_1 \frac{\partial h_3}{\partial x_2} (\rho V_2 V_3 - B_2 B_3) \left. \right\}
 \end{aligned}
 \tag{2.2.18f}$$

$$\frac{\partial \rho}{\partial t} = - \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x_1} (h_2 h_3 \rho V_1) + \frac{\partial}{\partial x_2} (h_1 h_3 \rho V_2) \right\}
 \tag{2.2.18g}$$

and

$$\begin{aligned}
 \frac{\partial u}{\partial t} = & - \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_1} \left\{ h_2 h_3 \left[(u+p) V_1 \right. \right. \\
 & - (B_1^2 - B_2^2 - B_3^2) V_1 - 2B_1 (B_2 V_2 + B_3 V_3) \\
 & - \frac{2\eta}{S} \left\{ \frac{B_3}{h_1 h_3} \frac{\partial}{\partial x_1} (h_3 B_3) + \frac{B_2}{h_1 h_2} \left[\frac{\partial}{\partial x_1} (h_2 B_2) \right. \right. \\
 & \left. \left. \left. - \frac{\partial}{\partial x_2} (h_1 B_1) \right] \right\} \right] \left. \right\}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_2} \left\{ h_1 h_3 \left[(u+p) v_2 \right. \right. \\
 & \quad \left. \left. + (B_1^2 - B_2^2 + B_3^2) v_2 - 2B_2 (B_1 v_1 + B_3 v_3) \right. \right. \\
 & \quad \left. \left. + \frac{2n}{s} \left\{ \frac{B_1}{h_1 h_2} \left[\frac{\partial}{\partial x_1} (h_2 B_2) - \frac{\partial}{\partial x_2} (h_1 B_1) \right] \right. \right. \right. \\
 & \quad \left. \left. \left. - \frac{B_3}{h_2 h_3} \frac{\partial}{\partial x_2} (h_3 B_3) \right] \right\} \right\} . \quad (2.2.18h)
 \end{aligned}$$

These are the equations which we solve numerically. The scale factors h_i and their derivatives are specified by function subroutines making any change of coordinate systems a relatively simple procedure. In some work we have used an extended set of equations which allow us to solve the full three-dimensional resistive MHD equations under the assumption that certain symmetries are preserved. This will be described in Chapter V.

CHAPTER III - COMPUTATIONAL TECHNIQUES

In Chapter II we described the basic physical and mathematical model employed in this work, and arrived at the set of equations (2.2.18) which we must solve numerically. In this chapter we describe the computational methods used to solve them.

3.1 Conservative Finite Difference Methods

Our basic equations form a set of Eulerian conservation laws in the quantities B_1 , B_2 , B_3 , ρV_1 , ρV_2 , ρV_3 , ρ , and u . The scalar conservation laws (2.2.18g,h) conserve total mass and total energy, and the pseudovector conservation laws (2.2.18a,b,c) conserve magnetic flux. The vector conservation laws (2.2.18d,e,f) conserve the components of momentum only in coordinate systems in which the scale factors are constants, i.e., Cartesian coordinates. This is because an inertial force (e.g., centrifugal force) is felt by a fluid element which is constrained to move along a path defined by a coordinate which is not a straight line, and we cannot expect to conserve all components of momentum (as referred to the curved coordinate system) in the presence of such a force. (The Cartesian components, however, are always conserved.)

When finite difference approximations are made to the differential operators appearing in our equations, it proves convenient to introduce them in such a way that the resulting difference equations have the same conservation properties

as the original differential equations. Consider a prototype scalar conservation law

$$\frac{\partial \rho}{\partial t} = - \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 F) + \frac{\partial}{\partial x_2} (h_1 h_3 G) \right] \quad (3.1.1)$$

where ρ is the scalar density and F and G are the x_1 and x_2 fluxes of ρ , respectively. Let this equation hold over a domain $x_{1\min} \leq x_1 \leq x_{1\max}$, $x_{2\min} \leq x_2 \leq x_{2\max}$, $0 \leq x_3 \leq 1$. Then multiplying both sides by the volume element $d\tau = h_1 h_2 h_3 dx_1 dx_2 dx_3$ and integrating over the entire domain yields

$$\begin{aligned} \frac{\partial}{\partial t} \int_{x_{1\min}}^{x_{1\max}} dx_1 \int_{x_{2\min}}^{x_{2\max}} dx_2 \rho h_1 h_2 h_3 = \\ - \int_{x_{2\min}}^{x_{2\max}} dx_2 \left[(h_2 h_3 F)_{x_{1\max}} - (h_2 h_3 F)_{x_{1\min}} \right] \\ - \int_{x_{1\min}}^{x_{1\max}} dx_1 \left[(h_1 h_3 G)_{x_{2\max}} - (h_1 h_3 G)_{x_{2\min}} \right] \end{aligned} \quad (3.1.2)$$

so that the rate of change of the quantity

$$M = \int \rho d\tau \quad (3.1.3)$$

integrated over the entire domain depends only on the values of the fluxes on the boundaries of the domain. If the boundary conditions are such that $h_2 h_3 F$ and $h_1 h_3 G$ vanish on the boundaries, or are periodic (so that

$(h_2 h_3 F)_{x_{1\max}} = (h_2 h_3 F)_{x_{1\min}}$, etc.), then M is conserved, i.e., is independent of time. We would like our finite difference approximation to have this same property.

To this end, consider the finite difference mesh as shown in Figure (3.1) with I points in the x_1 direction and J points in the x_2 direction. The coordinates of a given mesh point (i, j) are then (x_{1i}, x_{2j}) . We will consider each point to lie at the center of a cell with vertices at $(x_{1i-\frac{1}{2}}, x_{2j-\frac{1}{2}})$, $(x_{1i+\frac{1}{2}}, x_{2j-\frac{1}{2}})$, $(x_{1i+\frac{1}{2}}, x_{2j+\frac{1}{2}})$, and $(x_{1i-\frac{1}{2}}, x_{2j+\frac{1}{2}})$, where

$$f_{i+\frac{1}{2}} \equiv \frac{1}{2}(f_i + f_{i+1}) \quad (3.1.4)$$

We then integrate (3.1.1) over the volume of the cell centered at the mesh point (i, j) . We have

$$\begin{aligned} \int_{x_{1i-\frac{1}{2}}}^{x_{1i+\frac{1}{2}}} dx_1 \int_{x_{2j-\frac{1}{2}}}^{x_{2j+\frac{1}{2}}} dx_2 \frac{\partial \rho}{\partial t} h_1 h_2 h_3 = \\ - \int_{x_{2j-\frac{1}{2}}}^{x_{2j+\frac{1}{2}}} dx_2 \left[(h_2 h_3 F)_{i+\frac{1}{2}, j} - (h_2 h_3 F)_{i-\frac{1}{2}, j} \right] \\ - \int_{x_{1i-\frac{1}{2}}}^{x_{1i+\frac{1}{2}}} dx_1 \left[(h_1 h_3 G)_{i, j+\frac{1}{2}} - (h_1 h_3 G)_{i, j-\frac{1}{2}} \right]. \end{aligned} \quad (3.1.5)$$

We obtain a difference equation by appropriately approximating the integrals in this equation. We use the formulas

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f dx = \frac{1}{2} f_i (\Delta_+ x + \Delta_- x) + e_{T1} \quad (3.1.6a)$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} dx \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} dy f(x,y) = \frac{1}{4} f_{ij} (\Delta_+ x + \Delta_- x) (\Delta_+ y + \Delta_- y) + e_{T2} \quad (3.1.6b)$$

where Δ_+ and Δ_- are the forward and backward difference operators at point i , i.e.

$$\Delta_+ x = x_{i+1} - x_i$$

$$\Delta_- x = x_i - x_{i-1}$$

The truncation error in these equations is

$$e_{T1} = \frac{1}{48} f''_{ij} [(\Delta_+ x)^3 + (\Delta_- x)^3]$$

$$e_{T2} = \frac{1}{48} \{ f_{xxij} [(\Delta_+ x)^3 + (\Delta_- x)^3] + f_{yyij} [(\Delta_+ y)^3 + (\Delta_- y)^3] \}$$

The volume of the cell is found (by taking $f = h_1 h_2 h_3$) to be

$$\Delta \tau_{ij} = \frac{1}{4} (h_1 h_2 h_3)_{ij} (\Delta_+ x_1 + \Delta_- x_1) (\Delta_+ x_2 + \Delta_- x_2)$$

Then using (3.1.6) in (3.1.5), we obtain the spatially discretized equation

$$\frac{\partial \rho_{ij}}{\partial t} = - \frac{2}{(h_1 h_2 h_3)_{ij}} \left[\frac{(h_2 h_3 F)_{i+\frac{1}{2},j} - (h_2 h_3 F)_{i-\frac{1}{2},j}}{\Delta_+ x_1 + \Delta_- x_1} + \frac{(h_1 h_3 G)_{i,j+\frac{1}{2}} - (h_1 h_3 G)_{i,j-\frac{1}{2}}}{\Delta_+ x_2 + \Delta_- x_2} \right] \quad (3.1.7)$$

The temporal discretization will be discussed later in this chapter.

We now define the finite difference analogue of equation (3.1.3) as

$$\begin{aligned}
 M &= \sum_{i=2}^{I-1} \sum_{j=2}^{J-1} \rho_{ij} \Delta \tau_{ij} \\
 &= \frac{1}{4} \sum_{i=2}^{I-1} \sum_{j=2}^{J-1} \rho_{ij} (h_1 h_2 h_3)_{ij} (\Delta_+ x_1 + \Delta_- x_1) (\Delta_+ x_2 + \Delta_- x_2) .
 \end{aligned}$$

Then using (3.1.7), we find

$$\begin{aligned}
 \frac{\partial M}{\partial t} &= - \sum_{j=2}^{J-1} \frac{(\Delta_+ x_2 + \Delta_- x_2)}{2} [(h_2 h_3^F)_{I-\frac{1}{2},j} - (h_2 h_3^F)_{3/2,j}] \\
 &\quad - \sum_{i=2}^{I-1} \frac{(\Delta_+ x_1 + \Delta_- x_1)}{2} [(h_1 h_3^G)_{i,J-\frac{1}{2}} - (h_1 h_3^G)_{i,3/2}]
 \end{aligned}
 \tag{3.1.8}$$

where all the terms in the sums appearing at interior cell boundaries have cancelled exactly. Thus, if we choose the points $i = 3/2$, $i = I-\frac{1}{2}$, $j = 3/2$, $j = J-\frac{1}{2}$ to correspond to solid or periodic boundaries (at $x_1 = x_{1\max}$ and $x_{1\min}$, $x_2 = x_{2\max}$ and $x_{2\min}$), then the quantity M is identically conserved on the difference mesh. For the case of non-periodic boundaries, this would correspond to placing the wall midway between the two end mesh points. There are circumstances where this is neither possible nor desirable, in which case separate difference methods must be used to advance the boundary points. Such methods will be discussed later in this chapter. We note that the two sums appearing on the right hand side of (3.1.8) are simply the trapezoidal rule expressions for the integrals appearing on the right hand side of (3.1.2).

The differential equation for the l^{th} component of a

vector conservation law appears as

$$\frac{\partial v^\ell}{\partial t} = - \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 T^{1,\ell}) + \frac{\partial}{\partial x_2} (h_1 h_3 T^{2,\ell}) \right] + S^\ell \quad (3.1.9)$$

where S^ℓ represents the effective source of v^ℓ due to the Coriolis terms. In a manner similar to that just discussed, we find the conservation integral to be

$$\begin{aligned} \frac{\partial}{\partial t} \int_{x_{1\min}}^{x_{1\max}} dx_1 \int_{x_{2\min}}^{x_{2\max}} dx_2 v^\ell h_1 h_2 h_3 = & \\ & - \int_{x_{2\min}}^{x_{2\max}} dx_2 \left[(h_2 h_3 T^{1,\ell})_{x_{1\max}} - (h_2 h_3 T^{1,\ell})_{x_{2\min}} \right] \\ & - \int_{x_{1\min}}^{x_{1\max}} dx_1 \left[(h_1 h_3 T^{2,\ell})_{x_{2\max}} - (h_1 h_3 T^{2,\ell})_{x_{2\min}} \right] \\ & + \int_{x_{1\min}}^{x_{1\max}} dx_1 \int_{x_{2\min}}^{x_{2\max}} dx_2 h_1 h_2 h_3 S^\ell \end{aligned} \quad (3.1.10)$$

so that the quantity

$$V^\ell = \int_{x_{1\min}}^{x_{1\max}} dx_1 \int_{x_{2\min}}^{x_{2\max}} dx_2 v^\ell h_1 h_2 h_3$$

will not be conserved unless S^ℓ (or more precisely, the integral of S^ℓ) vanishes, as discussed previously. Using (3.1.6a,b) we find the finite difference approximation of (3.1.9) to be

$$\begin{aligned} \frac{\partial v_{ij}^{\ell}}{\partial t} = & - \frac{2}{(h_1 h_2 h_3)_{ij}} \left[\frac{(h_2 h_3 T^{1,\ell})_{i+\frac{1}{2},j} - (h_2 h_3 T^{1,\ell})_{i-\frac{1}{2},j}}{\Delta_+ x_1 + \Delta_- x_1} \right. \\ & \left. + \frac{(h_1 h_3 T^{2,\ell})_{i,j+\frac{1}{2}} - (h_1 h_3 T^{2,\ell})_{i,j-\frac{1}{2}}}{\Delta_+ x_2 + \Delta_- x_2} \right] \\ & + s_{i,j}^{\ell} \end{aligned} \quad (3.1.11)$$

with a corresponding conservation sum

$$\begin{aligned} \frac{\partial v^{\ell}}{\partial t} = & - \sum_{j=2}^{J-1} \frac{(\Delta_+ x_2 + \Delta_- x_2)}{2} \left[(h_2 h_3 T^{1,\ell})_{i-\frac{1}{2},j} \right. \\ & \left. - (h_2 h_3 T^{1,\ell})_{3/2,j} \right] \\ & - \sum_{i=2}^{I-1} \frac{(\Delta_+ x_1 + \Delta_- x_1)}{2} \left[(h_1 h_3 T^{2,\ell})_{i,J-\frac{1}{2}} \right. \\ & \left. - (h_1 h_3 T^{2,\ell})_{i,3/2} \right] \\ & + \frac{1}{4} \sum_{i=2}^{I-1} \sum_{j=2}^{J-1} s_{i,j}^{\ell} (h_1 h_2 h_3)_{i,j} (\Delta_+ x_1 + \Delta_- x_1) \\ & (\Delta_+ x_2 + \Delta_- x_2) : \end{aligned} \quad (3.1.12)$$

To obtain a conservative difference approximation to a pseudovector conservation law we must write out the components of equation (2.2.8) which is the result of the application of Stokes' theorem. Consider, for example, a small surface element lying in the $x_1 = \text{constant}$ plane. (See Figure (3.2)). The center of this element has coordinates (x_1, x_2, x_3) , and it has sides of length Δx_2 and Δx_3 . The curve C forms the boundary of the element. Then, for example, the x_1 component

of (2.2.8) is

$$\int_{x_2 - \frac{\Delta x_2}{2}}^{x_2 + \frac{\Delta x_2}{2}} dx_2 \int_{x_3 - \frac{\Delta x_3}{2}}^{x_3 + \frac{\Delta x_3}{2}} dx_3 h_2 h_3 \frac{\partial w_1}{\partial t} =$$

$$\int_{x_3 - \frac{\Delta x_3}{2}}^{x_3 + \frac{\Delta x_3}{2}} \left[(h_3 G_3)_{x_2 + \frac{\Delta x_2}{2}} - (h_3 G_3)_{x_2 - \frac{\Delta x_2}{2}} \right] dx_3$$

$$- \int_{x_2 - \frac{\Delta x_2}{2}}^{x_2 + \frac{\Delta x_2}{2}} \left[(h_2 G_2)_{x_3 + \frac{\Delta x_3}{2}} - (h_2 G_2)_{x_3 - \frac{\Delta x_3}{2}} \right] dx_2$$

(3.1.13)

where the integration has been taken over the surface element. Then using the formulas (3.1.6a,b) to approximate the integrals, we arrive at the difference formula

$$\frac{\partial w_1}{\partial t} = \frac{1}{h_2 h_3 \Delta x_2} \left[(h_3 G_3)_{j+\frac{1}{2}} - (h_3 G_3)_{j-\frac{1}{2}} \right]$$

$$- \frac{1}{h_2 h_3 \Delta x_3} \left[(h_2 G_2)_{k+\frac{1}{2}} - (h_2 G_2)_{k-\frac{1}{2}} \right] \quad (3.1.14)$$

where the center of the surface element has indices (i, j, k) and we have suppressed non-varying indices. The quantity to be conserved is now the flux through a surface of constant x_1 ($x_{2\min} \leq x_2 \leq x_{2\max}$, $x_{3\min} \leq x_3 \leq x_{3\max}$) consisting of many small elements such as the one just considered. We write this flux as

$$\phi = \sum_{j,k} \phi_{i,j,k} \quad (3.1.15)$$

The time derivative of this quantity is

$$\frac{\partial \phi}{\partial t} = \sum_{j,k} \frac{\partial \phi}{\partial t} = \sum_{j,k} h_2 h_3 \Delta x_2 \Delta x_3 \frac{\partial w_1}{\partial t} \quad (3.1.16)$$

Then using (3.1.14), we have

$$\begin{aligned} \frac{\partial \phi}{\partial t} = \Delta x_3 \sum_i [(h_3 G_3)_{J-\frac{1}{2}} - (h_3 G_3)_{3/2}] \\ - \Delta x_2 \sum_j [(h_2 G_2)_{K-\frac{1}{2}} - (h_2 G_2)_{3/2}] \end{aligned} \quad (3.1.17)$$

so that the rate of change of the total flux ϕ is determined by the circulation of $\underline{G}$ about the boundaries of the domain.

Note that if we have a system that is periodic in x_2 so that $(h_3 G_3)_{J-\frac{1}{2}} = (h_3 G_3)_{3/2}$, and is independent of x_3 , then ϕ is conserved identically on the difference mesh.

Such is the case, for example, in cylindrical coordinates ($x_1 = r$, $x_2 = \theta$, $x_3 = z$) when there is no axial dependence. Then radial flux is conserved.

3.2 Spatial Differencing

The system of conservation laws (2.2.18) can be formally written as

$$\frac{\partial \underline{U}}{\partial t} = - \nabla \cdot \underline{F} \quad (3.2.1)$$

where $\underline{U}$ is the eight component vector

$$\underline{U} = (B_1, B_2, B_3, \rho V_1, \rho V_2, \rho V_3, \rho, u) \quad (3.2.2)$$

and $\nabla \cdot \underline{F}$ represents the right hand side of (2.2.18). In

orthogonal curvilinear coordinates, and ignoring Coriolis terms, (3.2.1) becomes

$$\frac{\partial \underline{U}}{\partial t} = - \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_1 h_3 \underline{F}_1) + \frac{\partial}{\partial x_2} (h_1 h_3 \underline{F}_2) \right] \quad (3.2.3)$$

where the $\underline{F}_i$ are eight component vectors representing the fluxes of the components of $\underline{U}$. These have the general form

$$\underline{F}_i = \underline{A}_i \cdot \frac{\partial \underline{U}}{\partial x_1} + \underline{B}_i \cdot \frac{\partial \underline{U}}{\partial x_2} + \underline{C}_i \cdot \underline{U} \quad (3.2.4)$$

where $\underline{A}_i$, $\underline{B}_i$, and $\underline{C}_i$ are 8×8 matrices which serve to couple the various components of (3.2.3). We see that there are first derivatives, second derivatives, and mixed derivatives appearing in these equations. We shall now describe the treatment of first and second derivatives. The treatment of mixed derivatives is more closely tied to the temporal differencing and will be discussed later in this chapter.

One result of the treatment of the conservation laws as given in the previous section was to arrive at formulas for the spatial differencing of first derivatives of the fluxes. For example, we have for a non-uniform mesh

$$\frac{\partial F}{\partial x} = 2 \frac{F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}}{\Delta_+ x + \Delta_- x} \quad (3.2.5)$$

where we have used the identity

$$\frac{1}{2}(\Delta_+ x + \Delta_- x) = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}.$$

The truncation error is

$$e_T = - \frac{1}{24} F''' \frac{\Delta_+ x^2 - \Delta_- x^2}{\Delta_+ x + \Delta_- x} - \frac{1}{24} F'''' \frac{\Delta_+ x^3 + \Delta_- x^3}{\Delta_+ x + \Delta_- x} \quad (3.2.6)$$

so that (3.2.5) is formally first order accurate, becoming second order only on a uniform mesh. However, if the mesh spacing does not vary greatly from one mesh point to another, then the first term in (3.2.6) can be small, and effective second order accuracy can be maintained. For example, in this work the mesh spacing is varied such that

$$\frac{\Delta_+ x}{\Delta_- x} = r$$

where r is a constant slightly greater than unity. Then (3.2.6) becomes

$$e_T = -\frac{1}{24} F'' \frac{r^2 - 1}{r + 1} \Delta_- x - \frac{1}{24} F''' \frac{r^3 + 1}{r + 1} \Delta_- x^2.$$

If $r = 1 + \epsilon$, $\epsilon \ll 1$, the coefficient of the first order error term behaves like

$$\frac{2\epsilon + \epsilon^2}{2 + \epsilon} = \epsilon \quad (\epsilon \ll 1)$$

thus quantifying the statement that effective second order accuracy is achieved.

First order terms contained in F are differenced in a straightforward manner. We write

$$\begin{aligned} \frac{\partial}{\partial x}(af) &= 2 \frac{(af)_{j+\frac{1}{2}} - (af)_{j-\frac{1}{2}}}{\Delta_+ x + \Delta_- x} \\ &= \frac{(af)_{j+1} - (af)_{j-1}}{\Delta_+ x + \Delta_- x} \end{aligned} \quad (3.2.7)$$

where we have made use of (3.1.4).

Second derivatives are just as straightforward but slightly more complicated. We have

$$\begin{aligned}\frac{\partial}{\partial x}(a \frac{\partial f}{\partial x}) &= \frac{2}{\Delta_+ x + \Delta_- x} \left[(a \frac{\partial f}{\partial x})_{j+\frac{1}{2}} - (a \frac{\partial f}{\partial x})_{j-\frac{1}{2}} \right] \\ &= \frac{2}{\Delta_+ x + \Delta_- x} \left[a_{j+\frac{1}{2}} \frac{f_{j+1} - f_j}{\Delta_+ x} - a_{j-\frac{1}{2}} \frac{f_j - f_{j-1}}{\Delta_- x} \right]\end{aligned}\quad (3.2.8)$$

which is again consistent with the basic formula (3.2.5).

3.3 Temporal Differencing

So far we have only dealt with spatial discretization, i.e., the replacement of spatial differential operators with spatial finite difference operations. The equations we have derived have been of the form

$$\frac{\partial \underline{U}}{\partial t} = (D_1 + D_2 + D_{12}) \underline{U} \quad (3.3.1)$$

where D_1 represents first and second finite difference operations in the x_1 direction, D_2 represents first and second finite difference operations in the x_2 direction, and D_{12} represents the (as yet unspecified) finite difference approximation to the mixed derivatives. The replacement of the left hand side of this equation with a finite difference in time presents no difficulty. The problems arise in determining at which time level to place the spatial differences on the right hand side.

For the moment let us ignore the mixed derivatives. Then a general form for the temporal differencing of (3.3.1) is [67]

$$\frac{U^{n+1} - U^n}{\Delta t} = (\theta_1 D_1 + \theta_2 D_2) U^{n+1} + [(1 - \theta_1) D_1 + (1 - \theta_2) D_2] U^n \quad (3.3.2)$$

where θ_1 and θ_2 are constants in the range $0 \leq \theta_i \leq 1$, and the superscript n refers to values at the time

$$t^n = t^0 + n\Delta t.$$

Equation (3.3.2) thus defines a set of simultaneous algebraic equations for the unknown vector $\underline{u}^{n+1}$ in terms of the known vector $\underline{u}^n$. These equations are in general non-linear, since the D_i are non-linear operators.

Choosing $\theta_1 = \theta_2 = 0$ allows for $\underline{u}^{n+1}$ to be determined immediately. Techniques based on this choice are called explicit methods and, while they allow easy solution of the difference equations, are known to have upper limits on the size of the time increment Δt that can be used in order to maintain numerical stability [67]. The choice $\theta_1 = \theta_2 = 1$ requires the simultaneous solution of a set of non-linear equations. Methods which require such solution are known as implicit methods and, while they require more time to solve the equations at a given time level, they are not bound by any time step restriction. (This is rigorously true only for the case when the D_i are linear operators. While non-linear effects may introduce some time step limitation, it is generally less severe than that imposed by the use of explicit methods.) Thus effort lost in solving the equations at each time step can be more than made up by requiring fewer time steps to advance the solution for a given length of time. Unfortunately, this choice ($\theta_1 = \theta_2 = 1$) is only first order accurate in Δt [67]. A

choice that is both implicit (and linearly unconditionally stable) and second order accurate in Δt in the Crank-Nicholson weighting $\theta_1 = \theta_2 = \frac{1}{2}$ [80]. This results in a set of simultaneous non-linear algebraic equations of the form

$$[1 - \frac{\Delta t}{2}(D_1 + D_2)]\underline{U}^{n+1} = [1 + \frac{\Delta t}{2}(D_1 + D_2)]\underline{U}^n \quad (3.3.3)$$

There is one of these equations for each point on the mesh, and they are all coupled, since the operator $D_1 + D_2$ connects the point (i,j) with its four nearest neighbors $(i+1,j)$, $(i-1,j)$, $(i,j+1)$, and $(i,j-1)$. Equations (3.3.3) thus represents a set of $N = I \times J \times M$ simultaneous equations (where I is the number of mesh points in the x_1 direction, J is the number of mesh points in the x_2 direction, and M is the length of the vector $\underline{U}$). For a typical case, $I = J = 30$, $M = 8$, so that we may be required to invert a 7200×7200 matrix at each time step. Since the number of operations required to effect the inversion of an $N \times N$ matrix is of the order N^3 (e.g., Gaussian elimination), the practicality of (3.3.3) is highly questionable.

The size of the problem can be greatly reduced, and the desirable stability and accuracy properties maintained, by applying multi-step methods. In such methods the advancement of the solution from time t^n to time t^{n+1} is obtained via an intermediate timestep $t^{n+\frac{1}{2}}$. For example, if we advance equation (3.3.2) from t^n to $t^{n+\frac{1}{2}} = t^n + \Delta t/2$ with $\theta_1 = 1$,

$\theta_2 = 0$, and then advance from $t^{n+\frac{1}{2}}$ to t^{n+1} with $\theta_1 = 0$, $\theta_2 = 1$, we arrive at the scheme

$$(1 - \frac{\Delta t}{2} D_1) \underline{u}^{n+\frac{1}{2}} = (1 + \frac{\Delta t}{2} D_2) \underline{u}^n \quad (3.3.4a)$$

$$(1 - \frac{\Delta t}{2} D_2) \underline{u}^{n+1} = (1 + \frac{\Delta t}{2} D_1) \underline{u}^{n+\frac{1}{2}} \quad (3.3.4b)$$

This procedure is aptly called the Alternating Direction Implicit (ADI) method [81,82] since the different coordinate directions are treated implicitly on alternating steps. The set of equations (3.3.4a,b) is identical to the set (3.3.3) through terms of order Δt , and differs only slightly in terms of order Δt^2 , which is the order of accuracy of the overall temporal differencing. The advantage of using the split set of equations is seen by noting that on each partial step the unknown quantity is only coupled at three neighboring points. Thus on the first half step, equation (3.3.4) only couples unknowns at $(i+1,j)$, (i,j) , and $(i-1,j)$, while on the second half step the points $(i,j+1)$, (i,j) , and $(i,j-1)$ are coupled. On the first step we thus have I matrices of size $J \times M$ to invert, and on the second step we have J matrices of size $I \times M$. Thus, in our example of $I = J = 30$, $M = 8$, we would require about 8×10^8 operations, as opposed to the 4×10^{11} operations needed for the fully implicit method (3.3.3). Present high speed computers can easily attain on the order of 3×10^6 operations/sec. Thus the ADI scheme reduces the time required for matrix inversion from about 30 hours to several minutes. This time can be

even more dramatically reduced by taking advantage of the special structure of the matrices, as will be described later.

The inclusion of mixed derivative terms in the equations presents special problems to which no satisfactory solution has been found. The difficulty lies in the fact that the operator D_{12} couples mesh points in both coordinate directions so that it does not fit easily into the ADI scheme. This coupling is seen by a straightforward extension of (3.2.5). We have, for example,

$$\frac{\partial}{\partial x_2} \left(a \frac{\partial f}{\partial x_1} \right) = \frac{2}{\Delta_+ x_2 + \Delta_- x_2} \left[\left(a \frac{\partial f}{\partial x_1} \right)_{i,j+\frac{1}{2}} - \left(a \frac{\partial f}{\partial x_1} \right)_{i,j-\frac{1}{2}} \right]. \quad (3.3.5)$$

The terms $(a \partial f / \partial x_1)_{j \pm \frac{1}{2}}$ are the fluxes at the top and bottom of the cell. It is clear that when the derivatives on the right hand side are replaced by finite differences, neighboring points in both coordinate directions will be coupled. Marx [83] has used a scheme which allows splitting of this operator and still retains second order accuracy. He writes (e.g., for the first half time step)

$$\frac{\partial}{\partial x_2} \left(a \frac{\partial f}{\partial x_1} \right) = \frac{2}{\Delta_+ x_2 + \Delta_- x_2} \left[\left(a \frac{\partial f}{\partial x_1} \right)_{i,j+\frac{1}{2}}^n - \left(a \frac{\partial f}{\partial x_1} \right)_{i,j-\frac{1}{2}}^{n+\frac{1}{2}} \right] \quad (3.3.6)$$

which results in the finite difference approximation

$$\begin{aligned} \frac{\partial}{\partial x_2} \left(a \frac{\partial f}{\partial x_1} \right) = C & \left[a_{i,j+\frac{1}{2}}^n (f_{i+1,j+1}^n + f_{i+1,j}^n \right. \\ & - f_{i-1,j+1}^n - f_{i-1,j}^n) \\ & - a_{i,j-\frac{1}{2}}^{n+\frac{1}{2}} (f_{i+1,j}^{n+\frac{1}{2}} + f_{i+1,j-1}^{n+\frac{1}{2}} \\ & \left. - f_{i-1,j}^{n+\frac{1}{2}} - f_{i-1,j-1}^{n+\frac{1}{2}}) \right] \quad (3.3.7) \end{aligned}$$

where $C = (\Delta_+ x_1 + \Delta_- x_1)^{-1} (\Delta_+ x_2 + \Delta_- x_2)^{-1}$. At first glance, rows j and $j-1$ still appear to be coupled. However, if the algorithm proceeds row by row in increasing order of j , then the row $j-1$ at the advanced time level can be treated as known quantities, and only the $f_j^{n+\frac{1}{2}}$ are left as unknown. It is clear from (3.3.6) that this method is not conservative, since now the fluxes at the top and bottom of the cell are at different time levels, and we cannot achieve the exact cell to cell cancellation of fluxes required for conservation.

Thus in choosing a scheme for the mixed derivative we must strike a compromise between second order time accuracy and conservation (which is in itself a measure of accuracy). In this work we choose to retain conservation on the difference mesh by always treating the mixed derivatives explicitly (i.e., at the "old" time level). The difference approximations to the two types of mixed derivatives can be found from equation (3.3.5). We have

$$\begin{aligned} \frac{\partial}{\partial x_1} \left(a \frac{\partial f}{\partial x_2} \right) = C & \left[a_{i+\frac{1}{2},j} (f_{i+1,j+1} + f_{i,j+1} - f_{i+1,j-1} - f_{i,j-1}) \right. \\ & \left. - a_{i-\frac{1}{2},j} (f_{i,j+1} + f_{i-1,j+1} - f_{i,j-1} - f_{i-1,j-1}) \right] \quad (3.3.8a) \end{aligned}$$

and

$$\frac{\partial}{\partial x_2} \left(a \frac{\partial f}{\partial x_1} \right) = C \left[a_{i,j+\frac{1}{2}} (f_{i+1,j+1} + f_{i+1,j} - f_{i-1,j+1} - f_{i-1,j}) \right. \\ \left. - a_{i,j-\frac{1}{2}} (f_{i+1,j} + f_{i+1,j-1} - f_{i-1,j} - f_{i-1,j-1}) \right] \quad (3.3.8b)$$

where C has been defined after equation (3.3.7).

The Coriolis terms are time centered on each half time step, i.e.,

$$S_{i,j}^{(1)} = \frac{1}{2} (S_{i,j}^{n+\frac{1}{2}} + S_{i,j}^n) \\ S_{i,j}^{(2)} = \frac{1}{2} (S_{i,j}^{n+1} + S_{i,j}^{n+\frac{1}{2}})$$

so that the overall time centering of these terms is achieved.

The final set of difference equations can be written symbolically as

$$\left[1 - \frac{\Delta t}{2} (D_1 + S) \right] \underline{U}^{n+\frac{1}{2}} = \left[1 + \frac{\Delta t}{2} (D_2 + D_{12} + S) \right] \underline{U}^n \quad (3.3.9a)$$

$$\left[1 - \frac{\Delta t}{2} (D_2 + S) \right] \underline{U}^{n+1} = \left[1 + \frac{\Delta t}{2} (D_1 + D_{12} + S) \right] \underline{U}^{n+\frac{1}{2}} \quad (3.3.9b)$$

where the operator S represents the effect of the Coriolis terms, as described above.

3.4 Solution of the Difference Equations

Since the operators D_1 , D_2 , D_{12} , and S are in general non-linear, each step of the ADI algorithm, equations (3.3.9a,b),

represents a set of $M \times J$ non-linear algebraic equations to be solved on each row of the mesh. These equations must be linearized and solved iteratively. Lindemuth [70], and Finan [84], have used a multidimensional Newton-Raphson procedure to accomplish both linearization and solution. Here we use a more direct approach.

As described in section 3.2, we consider the momenta, not the velocities, to be the dependent variables. The velocities are considered to be auxiliary variables and are calculated as momentum divided by density. They are always treated as coefficients. For example, in the equations for the time rate of change of the magnetic field, there appear terms $V_i B_j$. These terms are always written as

$$(V_i B_j)^{(l)} = V_i^{(l-1)} B_j^{(l)}$$

where the superscript (l) refers to the l^{th} iteration of the solution of the non-linear equations. The pressure is also considered an auxiliary variable, and is calculated at each iteration as

$$p^{(l)} = (\gamma-1) \left[u^{(l)} - \frac{(\rho V^{(l)})^2}{\rho^{(l)}} - B^{(l)2} \right] .$$

The pressure gradient terms are thus treated "explicitly" (i.e., considered as known quantities at the last iteration of the new time step). Terms which appear as products of unknowns are linearized as

$$(fg)^{(l)} = \frac{1}{2} (f^{(l-1)} g^{(l)} + g^{(l-1)} f^{(l)})$$

where $f^{(l-1)}$ and $g^{(l-1)}$ are now considered as coefficients in the first and second terms, respectively. There are also terms which contain both products of unknowns and auxilliary variables. For example, one component of the Reynolds stress $\rho v v$ is written as

$$\rho v_1 v_2 = \frac{1}{3} \left[v_2^{(l-1)} (\rho v_1)^{(l)} + v_1^{(l-1)} (\rho v_2)^{(l)} + v_1^{(l-1)} v_2^{(l-1)} \rho^{(l)} \right]$$

Here, $v_2^{(l-1)}$, $v_1^{(l-1)}$, and $v_1^{(l-1)} v_2^{(l-1)}$ are treated as coefficients of the unknowns $(\rho v_1)^{(l)}$, $(\rho v_2)^{(l)}$, and $\rho^{(l)}$.

When equations (3.3.9) are linearized in this manner, there results a set of simultaneous linear algebraic equations to be solved on each row of the mesh. Since the operators on the left hand side of these equations couple at most three nearest neighbor points, the resulting system is block tridiagonal, and may be written in the form

$$-A_{i,j}^{(l-1)} \cdot U_{i+1,j}^{(l)} + B_{i,j}^{(l-1)} \cdot U_{i,j}^{(l)} - C_{i,j}^{(l-1)} \cdot U_{i-1,j}^{(l)} = D_{i,j}^{(l)}$$

$$i = 2, 3, \dots, I-1 \quad (3.4.1)$$

subject to the boundary condition [85]

$$G_1 \cdot U_{1,j}^{(l)} = H_1 \cdot U_{2,j}^{(l)} + J_1 \quad (3.4.2a)$$

at the left hand boundary, and

$$G_I \cdot U_{I,j}^{(l)} = H_I \cdot U_{I-1,j}^{(l)} + J_I \quad (3.4.2b)$$

at the right hand boundary. Here A , B , C , G , and H are

$M \times M$ matrices, $\underline{U}_{i,j}$ is the vector of length M of unknown quantities at the mesh point (i,j) , and $\underline{D}$ and $\underline{J}$ are vectors of length M . The notation $()^{(k)}$ indicates quantities at the k^{th} iteration. There is one equation for each interior mesh point on row j .

The solution to (3.4.1) is obtained by taking advantage of the block tridiagonal nature of the system. Since the boundary conditions give a linear relationship between two points, we inductively define the solution vector $\underline{U}_{i,j}$ to be given in terms of $\underline{U}_{i+1,j}$ by

$$\underline{U}_{i,j} = \underline{E}_i \cdot \underline{U}_{i+1,j} + \underline{F}_i. \quad (3.4.3)$$

We use this to eliminate $\underline{U}_{i-1,j}$ from (3.4.1). The result is a relationship of the same form as (3.4.3) from which we can identify

$$\underline{E}_i = (\underline{B}_{i,j} - \underline{C}_{i,j} \cdot \underline{E}_{i-1})^{-1} \cdot \underline{A}_{i,j} \quad (3.4.4a)$$

$$\underline{F}_i = (\underline{B}_{i,j} - \underline{C}_{i,j} \cdot \underline{E}_{i-1})^{-1} \cdot (\underline{D}_{i,j} - \underline{C}_{i,j} \underline{F}_{i-1}). \quad (3.4.4b)$$

These serve as recursion relations for the matrix $\underline{E}_i$ and the vector $\underline{F}_i$. The procedure is to use (3.4.4) to generate $\underline{E}$ and $\underline{F}$ for $i = 2, 3, \dots, I-1$, and then use (3.4.3) to find $\underline{U}_{i,j}$ for $i = I-1, I-2, \dots, 3, 2, 1$. In order to begin either sweep, boundary conditions must be used. A comparison of (3.4.2a) and (3.4.3) at $i = 1$ shows that

$$\underline{E}_1 = \underline{G}_1^{-1} \cdot \underline{H}_1 \quad (3.4.5a)$$

$$\underline{F}_1 = \underline{G}_1^{-1} \cdot \underline{J}_1, \quad (3.4.5b)$$

and the simultaneous solution of (3.4.3) at $i = I-1$ and (3.4.2b) yields

$$\underline{U}_{I,j} = (\underline{G}_I - \underline{H}_I \cdot \underline{E}_{I-1})^{-1} \cdot (\underline{H}_I \cdot \underline{F}_{I-1} + \underline{J}_I). \quad (3.4.6)$$

Then starting from (3.4.5a,b), equations (3.4.4) can be used to define $\underline{E}$ and $\underline{F}$ for $i = 2, 3, \dots, I-1$. Knowing these, and using (3.4.6), equation (3.4.3) then yields $\underline{U}_{i,j}$ for $i = I-1, I-2, \dots, 2, 1$.

For the case of periodic boundaries the solution proceeds in a somewhat different manner [86]. Now the boundary condition is that

$$\underline{U}_1 = \underline{U}_{I-1}, \quad \underline{U}_2 = \underline{U}_I \quad (3.4.7a,b)$$

so that $\underline{U}$ and its first derivative have the same value at $i = 3/2$ and $i = I-1/2$. For simplicity, we have dropped the index j .

To proceed, we write the system (3.4.1) on the interior of the mesh as

$$\begin{aligned} -\underline{C}_2 \cdot \underline{U}_1 + \underline{B}_2 \cdot \underline{U}_2 & - \underline{A}_2 \cdot \underline{U}_3 & = \underline{D}_2 \\ -\underline{C}_3 \cdot \underline{U}_2 & + \underline{B}_3 \cdot \underline{U}_3 - \underline{A}_3 \cdot \underline{U}_4 & = \underline{D}_3 \\ & \vdots & \\ -\underline{C}_{I-2} \cdot \underline{U}_{I-3} + \underline{B}_{I-2} \cdot \underline{U}_{I-2} - \underline{A}_{I-2} \cdot \underline{U}_{I-1} & = \underline{D}_{I-2} \\ & - \underline{C}_{I-1} \cdot \underline{U}_{I-2} + \underline{B}_{I-1} \cdot \underline{U}_{I-1} - \underline{A}_{I-1} \cdot \underline{U}_I = \underline{D}_{I-1} \end{aligned} \quad (3.4.8)$$

so that we have $I-2$ equations in the I unknowns $\underline{U}_1, \underline{U}_2, \dots, \underline{U}_I$.

We now make use of the boundary conditions (3.4.7) to rewrite this system as

$$\begin{aligned}
 \underline{B}_2 \cdot \underline{U}_2 - \underline{A}_2 \cdot \underline{U}_3 & & - \underline{C}_2 \cdot \underline{U}_{I-1} & = \underline{D}_2 \\
 - \underline{C}_3 \cdot \underline{U}_2 + \underline{B}_3 \cdot \underline{U}_3 - \underline{A}_3 \cdot \underline{U}_4 & & & = \underline{D}_3 \\
 & & - \underline{C}_4 \cdot \underline{U}_3 + \underline{B}_4 \cdot \underline{U}_4 - \underline{A}_4 \cdot \underline{U}_5 & = \underline{D}_4 \\
 & & \vdots & \vdots \\
 & & - \underline{C}_{I-2} \cdot \underline{U}_{I-3} + \underline{B}_{I-2} \cdot \underline{U}_{I-2} - \underline{A}_{I-2} \cdot \underline{U}_{I-1} & = \underline{D}_{I-2} \\
 - \underline{A}_{I-1} \cdot \underline{U}_2 & & - \underline{C}_{I-1} \cdot \underline{U}_{I-2} + \underline{B}_{I-1} \cdot \underline{U}_{I-1} & = \underline{D}_{I-1}
 \end{aligned}
 \tag{3.4.9}$$

which involves the $I-2$ unknowns $\underline{U}_2, \underline{U}_3, \dots, \underline{U}_{I-1}$. Except for the terms $-\underline{A}_{I-1} \cdot \underline{U}_2$ and $-\underline{C}_2 \cdot \underline{U}_{I-1}$, the system is block tri-diagonal.

We now note that we could solve the first equation for $\underline{U}_2$ in terms of $\underline{U}_3$ and $\underline{U}_{I-1}$. We could then use this value of $\underline{U}_2$ to solve the second equation for $\underline{U}_3$ in terms of $\underline{U}_4$ and $\underline{U}_{I-1}$, and could continue on through the system, solving the $\underline{U}_i$ in terms of $\underline{U}_{i+1}$ and $\underline{U}_{I-1}$. We will thus seek a solution of the form

$$\underline{U}_i = \underline{E}_i \cdot \underline{U}_{i+1} + \underline{S}_i \cdot \underline{U}_{I-1} + \underline{F}_i \quad . \quad (3.4.10)$$

When this is used to eliminate $\underline{U}_{i-1}$ from (3.4.1), we arrive at the recursion relations for $\underline{E}$, $\underline{S}$, and $\underline{F}$ analogous to equations (3.4.4a,b). They are

$$\underline{E}_i = (\underline{B}_i - \underline{C}_i \cdot \underline{E}_{i-1})^{-1} \cdot \underline{A}_i \quad (3.4.11a)$$

$$\underline{S}_i = (\underline{B}_i - \underline{C}_i \cdot \underline{E}_{i-1})^{-1} \cdot \underline{C}_i \cdot \underline{S}_{i-1} \quad (3.4.11b)$$

$$\underline{F}_i = (\underline{B}_i - \underline{C}_i \cdot \underline{E}_{i-1})^{-1} \cdot (\underline{D}_i + \underline{C}_i \cdot \underline{F}_{i-1}). \quad (3.4.11c)$$

From its construction, equation (3.4.10) is equivalent to the i^{th} equation of the set (3.4.9). In particular, when applied to the $(I-2)^{\text{nd}}$ equation of the set, we arrive at an expression for $\underline{U}_{I-2}$ in terms of $\underline{U}_{I-1}$ only. This value could be used to eliminate $\underline{U}_{I-2}$ from the $(I-3)^{\text{rd}}$ equation, giving a value for $\underline{U}_{I-3}$ in terms of $\underline{U}_{I-1}$ only. This procedure could be continued through the rest of the equations. In general, for the i^{th} equation, we would obtain relationship of the form

$$\underline{U}_i = \underline{T}_i \cdot \underline{U}_{I-1} + \underline{V}_i. \quad (3.4.12)$$

Using this to eliminate $\underline{U}_{i+1}$ from (3.4.10), we arrive at the recursion relations

$$\underline{T}_i = \underline{S}_i + \underline{E}_i \cdot \underline{T}_{i+1} \quad (3.4.13a)$$

$$\underline{V}_i = \underline{F}_i + \underline{E}_i \cdot \underline{V}_{i+1}. \quad (3.4.13b)$$

We have now expressed all our unknowns in terms of $\underline{U}_{I-1}$. To complete the solution, we apply (3.4.1) at $i = I-1$, and use (3.4.7b) and (3.4.12). The result is

$$\underline{U}_{I-1} = (\underline{B}_{I-1} - \underline{A}_{I-1} \cdot \underline{T}_2 - \underline{C}_{I-1} \cdot \underline{T}_{I-2})^{-1} \cdot (\underline{D}_{I-1} + \underline{A}_{I-1} \cdot \underline{V}_2 + \underline{C}_{I-1} \cdot \underline{V}_{I-2}) \quad (3.4.14)$$

We now need starting conditions for the recursion relations (3.4.11) and (3.4.13). At $i = 1$ we have, from (3.4.10)

$$\underline{U}_{1,j} = \underline{E}_1 \cdot \underline{U}_{2,j} + \underline{S}_1 \cdot \underline{U}_{I-1,j} + \underline{F}_1$$

This can be consistent with the boundary condition (3.4.7a) only if

$$\underline{E}_1 = \underline{\phi} \quad , \quad \underline{F}_1 = \underline{\phi} \quad , \quad \underline{S}_1 = \underline{I} \quad (3.4.15)$$

where $\underline{\phi}$ and $\underline{\phi}$ are the zero matrix and vector, and $\underline{I}$ is the unit matrix. Using these, equations (3.4.11) define $\underline{E}$, $\underline{S}$, and $\underline{F}$ for $i = 2, 3, \dots, I-2$. Evaluating (3.4.12) at $i = I-1$, we conclude

$$\underline{T}_{I-1} = \underline{I} \quad , \quad \underline{V}_{I-1} = \underline{\phi} \quad (3.4.16)$$

These starting conditions, along with the $\underline{E}_i$ and $\underline{F}_i$, can now be used in equations (3.4.13) to define $\underline{T}$ and $\underline{V}$, for $i = I-2, I-3, \dots, 3, 2$. These values of $\underline{T}$ and $\underline{V}$, along with (3.4.14), can then be used in equations (3.4.12) to solve for the $\underline{U}_{i,j}$ for $i = 2, 3, \dots, I-1$. The boundary conditions

(3.4.7) then define the remaining unknowns $\underline{U}_{1,j}$ and $\underline{U}_{I,j}$. Periodic boundary conditions thus require three sweeps of the mesh, whereas the regular boundary conditions (3.4.5) require only two.

Once the solution has been advanced to the new time level by these algorithms, the coefficients are updated and the procedure is repeated. This could be done until successive estimates for $\underline{u}^{n+1}$ converge to within a desired tolerance, but we have found that one update strikes a good compromise between accuracy and speed. All the results described in subsequent chapters were obtained with one iteration of the solution.

In examining the algorithms described above, we note that almost the only matrix inversion that need be done is that of $\underline{E} - \underline{C} \cdot \underline{E}_{j-1}$ at each point on the mesh. Thus by taking advantage of the tridiagonal nature of the difference equations we have reduced the problem from the inversion of one $(J \times M) \times (J \times M)$ matrix on each row to that of inverting $J(M \times M)$ matrices on each row. To advance a full time step now requires approximately $2J \times M^3$ operations. For an example of $J = 30$, $M = 8$, this is approximately 3.1×10^4 operations, as compared to 8×10^8 operations for direct solution of the split system, and 4×10^{11} for direct inversion of the original system. The required computer time is now on the order of seconds.

3.5 Boundary Conditions

In this section we will be concerned with boundary condi-

tions to be imposed on the solution of (2.1.15) at the outer boundary of the plasma, which we assume to be a perfectly conducting wall. The treatment of singular boundaries (e.g., the $r = 0$ point in cylindrical coordinates) will be left to Chapter V where cylindrical applications are discussed.

For simplicity we will consider the wall to be locally flat and construct a local Cartesian coordinate system with the y -axis perpendicular and the x and z axis parallel to the wall. We take the wall to be located at $y = y_w$. In inviscid flow the primary boundary condition at a non-porous wall is

$$\hat{n} \cdot \underline{v} = 0 \quad (3.5.1)$$

where $\hat{n}$ is a unit vector normal to the wall and $\underline{v}$ is the velocity vector. In our coordinate system this reduces to

$$v_y = 0 \quad \text{at} \quad y = y_w \quad (3.5.2)$$

The boundary conditions on the electromagnetic field at a perfectly conducting wall are

$$\hat{n} \cdot \underline{B} = 0 \quad (3.5.3)$$

$$\hat{n} \times \underline{E} = 0 \quad (3.5.4)$$

i.e., the normal component of the magnetic field and the tangential component of the electric field must vanish. From the first of these conditions we have

$$B_y = 0 \quad \text{at} \quad y = y_w \quad (3.5.5)$$

The boundary condition on the electric field can be used to derive conditions on the parallel components of the magnetic field. From Ohm's law and Ampere's law we have

$$\frac{\partial}{\partial t} \nabla \times \underline{B} = \underline{E} + \underline{v} \times \underline{B} . \quad (3.5.6)$$

Taking $\hat{n} \times$ this equation and using (3.5.4) we have

$$\frac{\partial}{\partial t} \hat{n} \times (\nabla \times \underline{B}) = \hat{n} \times (\underline{v} \times \underline{B}) . \quad (3.5.7)$$

Evaluating this at the wall, and recalling that, from (3.5.2),

$\partial v_y / \partial x = 0$ there, we arrive at

$$\frac{\partial B_x}{\partial y} = \frac{\partial B_z}{\partial y} = 0 \quad \text{at} \quad y = y_w . \quad (3.5.8a,b)$$

A boundary condition on pressure is obtained by considering the y-component of the momentum equation

$$\begin{aligned} \frac{\partial (\rho v_y)}{\partial t} = & - \frac{\partial}{\partial x} (\rho v_x v_y - B_x B_y) \\ & - \frac{\partial}{\partial y} [\rho v_y^2 + \frac{1}{2} (p + B_x^2 - B_y^2 + B_z^2)] . \end{aligned} \quad (3.5.9)$$

Evaluating at $y = y_w$, and using (3.5.2), (3.5.5), and (3.5.8),

we find

$$\frac{\partial p}{\partial y} = 0 \quad \text{at} \quad y = y_w . \quad (3.5.10)$$

Now consider the Lagrangian form of the equation of motion

$$\rho \frac{d\underline{v}}{dt} = - \nabla p + \underline{j} \times \underline{B} . \quad (3.5.15)$$

Taking the curl, and defining the vorticity as

$$\underline{\zeta} = \nabla \times \underline{v} ,$$

we have

$$\nabla \rho \times \dot{\underline{v}} + \rho \dot{\underline{\zeta}} = \nabla \times (\underline{J} \times \underline{B}) \quad (3.5.12)$$

where $(\dot{})$ signifies the time derivative. The x and z (i.e., parallel) components of this equation are

$$\begin{aligned} \dot{v}_y \frac{\partial \rho}{\partial x} - \dot{v}_x \frac{\partial \rho}{\partial y} + \rho \dot{\zeta}_z &= \frac{\partial}{\partial x} (J_z B_x - J_x B_z) \\ &- \frac{\partial}{\partial y} (J_y B_z - J_z B_y) \end{aligned} \quad (3.5.13a)$$

$$-\dot{v}_z \frac{\partial \rho}{\partial y} - \rho \dot{\zeta}_x = \frac{\partial}{\partial y} (J_x B_y - J_y B_x) . \quad (3.5.13b)$$

We now evaluate these expressions at $y = y_w$. We use (3.5.2), (3.5.5), (3.5.8), $J_x = J_z = 0$, and recall that $\nabla \cdot \underline{J} = 0$ implies that $\partial J_y / \partial y = 0$. Then, at $y = y_w$

$$-\dot{v}_x \frac{\partial \rho}{\partial y} + \rho \dot{\zeta}_z = 0 \quad (3.5.14a)$$

$$\dot{v}_z \frac{\partial \rho}{\partial y} + \rho \dot{\zeta}_x = 0 . \quad (3.5.14b)$$

If we assume that the fluid has an equation of state of the form

$$p \rho^n = \text{constant} \quad (3.5.15)$$

then

$$n p \frac{\partial \rho}{\partial y} + \rho^n \frac{\partial p}{\partial y} = 0 . \quad (3.5.16)$$

Evaluating at the wall, and using (3.5.10) we have

$$\frac{\partial \rho}{\partial y} = 0 \quad \text{at } y = y_w . \quad (3.5.17)$$

Using this in (3.5.14) we find that the parallel components of vorticity must vanish at the wall, or

$$\frac{\partial v_x}{\partial y} = 0 , \quad \frac{\partial v_z}{\partial y} = 0 \quad \text{at } y = y_w . \quad (3.5.18a,b)$$

When coupled with (3.5.17), we arrive at boundary conditions on the parallel components of the momentum. They are

$$\frac{\partial (\rho v_x)}{\partial y} = 0 , \quad \frac{\partial (\rho v_z)}{\partial y} = 0 \quad \text{at } y = y_w . \quad (3.5.19a,b)$$

The boundary condition on total energy density is obtained immediately. We have

$$u = \rho v^2 + \rho e + B^2 \quad (3.5.20)$$

where $e = e(p, \rho)$. Then evaluating the normal derivative of this expression at $y = y_w$, we find

$$\frac{\partial u}{\partial y} = 0 \quad \text{at } y = y_w . \quad (3.5.21)$$

These conditions can be expressed in our system of orthogonal curvilinear coordinates. We summarize them as

$$\begin{aligned} \frac{\partial}{\partial x_2} (h_1 B_1) &= 0 , \quad B_2 = 0 , \quad \frac{\partial}{\partial x_2} (h_3 B_3) = 0 , \\ \frac{\partial}{\partial x_2} (h_1 \rho v_1) &= 0 , \quad \rho v_2 = 0 , \quad \frac{\partial}{\partial x_2} (h_3 v_3) = 0 , \\ \frac{\partial \rho}{\partial x_2} &= 0 , \quad \frac{\partial u}{\partial x_2} = 0 \end{aligned} \quad (3.5.22)$$

where we have chosen x_2 as the coordinate to be locally

$$\begin{array}{c}
 - 90 - \\
 \underline{H}_J = \begin{pmatrix}
 h_{1J-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & h_{3J-1} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & h_{1J-1} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & h_{3J-1} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
 \end{pmatrix}
 \end{array}
 \quad (3.5.24b)$$

$$\underline{J}_J = \underline{\phi} \quad . \quad (3.5.24c)$$

An alternative boundary condition on the thermodynamic variables ρ and u may be found by insisting that mass and energy be conserved. In this case we place the wall at $j = J$ and consider a half-sized cell extending from $j = J-\frac{1}{2}$ to $j = J$ (see Figure (3.3)), and we consider only the flux in the x_2 direction entering the cell. If we define the "mass" in the interior cells (in one dimension) to be

$$M_{INT} = \sum_{j=1}^{J-1} \rho_j \Delta \tau_j = \frac{1}{2} \sum_j \rho_j h_1 h_2 h_3 (\Delta_+ x_2 + \Delta_- x_2) \quad (3.5.25,$$

where the sum extends over all cells up to $j = J-1$, then the rate of change of this quantity is

$$\frac{\partial M_{INT}}{\partial t} = - (h_1 h_3 F_2)_{J-\frac{1}{2}}^{n+1} \quad (3.5.26)$$

where we have used the results of section 3.1. We now advance the boundary cell in such a way as to exactly cancel

this change in mass on the interior of the mesh. We thus require

$$\frac{\partial \rho_J}{\partial t} \Delta \tau_J = (h_1 h_3 F_2)_{J-\frac{1}{2}}^{n+1} \quad (3.5.27)$$

With

$$\Delta \tau_J = \frac{1}{2} h_1 h_2 h_3 \Delta x_{2J} \quad (3.5.28)$$

as the volume of the boundary cell, we have

$$\frac{\partial \rho_J}{\partial t} = \frac{2}{(h_1 h_2 h_3)_J \Delta x_{2J}} (h_1 h_3 F_2)_{J-\frac{1}{2}}^{n+1} \quad (3.5.29)$$

as the equation of the evolution of the density at the wall.

This formulation again fits into our general boundary condition formulation. For example, the finite difference approximation to the continuity equation (where $F_2 = \rho v_2$) is

$$- C \cdot (\rho v_2)_J^{n+1} + \rho_J^{n+1} = C \cdot (\rho v_2)_{J-1}^{n+1} + \rho_J^n \quad (3.5.30)$$

with

$$C = \frac{\Delta t (h_1 h_3)_{J-\frac{1}{2}}}{(h_1 h_2 h_3)_J (\Delta x_2)_J} \quad .$$

The $\underline{G}$, $\underline{H}$ and $\underline{J}$ matrices in equations (3.5.24) must be modified to read

$$G_{7,5} = -C \quad , \quad G_{7,7} = 1 \quad (3.5.31a,b)$$

$$H_{7,5} = +C \quad (3.5.32)$$

$$J_7 = \rho_J^n \quad (3.5.33)$$

with all other elements remaining the same.

Similar expressions can be derived for the energy equation, but they are considerably more complicated and will not be given here.

Fig. 3.1 Finite difference mesh showing an interior cell with fluxes at each face.

Fig. 3.2 Closed curve in the (x_2, x_3) plane surrounding a mesh point. The application of Stokes' theorem to this configuration yields the proper finite difference equation.

Fig. 3.3 Conservative differencing at a solid boundary

CHAPTER IV - TEARING MODE IN A SHEET PINCH

4.1 Introduction

In this chapter we consider applications to Cartesian geometry: the linear and non-linear behavior of the tearing mode in a sheet pinch. This configuration will serve as a benchmark for our computational work, as it has been extensively studied both analytically [10] and numerically [26,57,58]. In addition, the non-linear behavior will be investigated. In section 4.2 we describe a linear computer code which we use to generate initial conditions for the non-linear code described in Chapter III. In section 4.3 we compare the results of the linear code with those obtained by running the non-linear code in the linear regime. We also describe the non-linear behavior of the mode. In section 4.4 we discuss the double tearing mode, which occurs when two singular surfaces lie in close proximity. We study the linear behavior with the linear code described above, and use our non-linear code to study the long time evolution of this mode.

4.2 Linear Calculations

In Appendix A we discuss the derivation of the basic set of linearized resistive MHD equations as presented in FKR. In that analysis, Fourier decomposition is done in two space dimensions (x and z) and in time. If the Fourier decomposition in time is not applied, then these equations can be written

$$\frac{\partial \psi}{\partial t} = -FW + \eta \left(\frac{\partial^2 \psi}{\partial y^2} - \alpha^2 \psi \right) \quad (4.2.1a)$$

$$\frac{\rho}{\alpha^2 S^2} \frac{\partial}{\partial t} \left(\frac{\partial^2 W}{\partial y^2} - \alpha^2 W \right) = F \left(\frac{\partial^2 \psi}{\partial y^2} - \alpha^2 \psi \right) - F'' \psi \quad (4.2.1b)$$

where the normalization is the same as that given in equations (A5), and we have written y in place of the variable μ .

Again, ψ is the first order y -component of magnetic field, W is the first order y -component of velocity, and F is the component of the zero order magnetic field in the (k, y) plane. The parameter $\alpha = ka$ is a non-dimensional wave number and S is the magnetic Reynolds number. These equations, together with the appropriate boundary and initial conditions, define an initial value problem in the unknowns ψ and W . One specifies these perturbation quantities at $t = 0$, and then advances (4.2.1) forward in time until an exponentially growing mode appears. The choice of the function F determines the background configuration from which unstable modes may arise.

This problem was first solved numerically by Killeen [57], who used implicit finite difference techniques and an equilibrium appropriate to the single tearing mode. For the present work, we choose the background field to be given by

$$F \equiv B_{x0} = \tanh \frac{\pi}{2} y \quad (4.2.2)$$

which results in a current sheet

$$J_z = -\frac{\pi}{2} \operatorname{sech}^2 \frac{\pi}{2} y \quad (4.2.3)$$

so that $y = 0$ defines the singular surface (i.e., the root

of $F = 0$). Conducting walls are placed at $y = \pm y_w$, and the appropriate boundary conditions are that both ψ and $\nabla\psi$ vanish there (see section 3.3).

The appearance of an eigenmode of equations (4.2.1) is determined by computing the growth rate

$$\omega_{T_R} \equiv p = \frac{1}{\psi} \frac{\partial \psi}{\partial t} \quad (4.2.4)$$

at each point in y on the finite difference mesh. When this quantity becomes constant to four significant figures in both space and time, an exponentially growing mode is said to have developed, and the growth rate is taken to be p . Since equations (4.2.1) are linear, and implicit finite difference techniques are used, numerical stability places no restriction on the time step Δt , and the temptation is to use a large time step to save computer time. One must be careful, however, that the time step does not exceed the e-folding time of the instability. If it does, then the growth rates obtained from (4.2.4) may be meaningless. A good rule of thumb is to require that the inequality

$$p\Delta t < 1 \quad (4.2.5)$$

hold. This must be verified after the growth rate is determined. In practice we have obtained a growth rate with a given time step, rerun the case with the time step cut in half, and compared the results. If they agree, then the growth rate obtained from (4.2.4) is deemed correct. If not, the time step is halved again until agreement is reached.

The eigenfunctions obtained from the linear code, RIPPLE3, along with the equilibrium conditions, are then used as the initial conditions for the non-linear code, MHDG. In generating these conditions we must convert from the FKR normalization (time measured in units of the resistive diffusion time, velocities measured in units of the diffusion velocity) to the normalization described in Chapter II (time measured in units of the Alfvén transit time, velocities measured in units of the Alfvén velocity). The proper conversion formulas for the Fourier amplitudes are

$$\tilde{B}_{y1} = \psi(y) \quad (4.2.6a)$$

$$\tilde{v}_{y1} = \frac{1}{\alpha S} W(y) \quad (4.2.6b)$$

Considered as functions of x and y , we have

$$B_{y1}(x, y) = \psi(y) \cos 2\pi x$$

$$v_{y1}(x, y) = - \frac{W(y)}{\alpha S} \sin 2\pi x$$

where we have measured x in units of the perturbation wavelength. The remaining components of $\underline{B}_1$ and $\underline{v}_1$ are found from the condition $\nabla \cdot \underline{B}_1 = \nabla \cdot \underline{v}_1 = 0$. They are

$$B_{x1}(x, y) = - \frac{1}{\alpha} \frac{\partial \tilde{B}_{y1}}{\partial y} \sin 2\pi x \quad (4.2.8a)$$

$$v_{x1}(x, y) = \frac{1}{\alpha} \frac{\partial \tilde{v}_{y1}}{\partial y} \cos 2\pi x \quad (4.2.8b)$$

4.3 Single Tearing Mode

We now consider in detail the background field given by (4.2.2). This configuration is called the single tearing mode because there is only one root of the equation $F = 0$ (i.e., there is only one singular surface within the computation domain). To complete the equilibrium we require the pressure, which is obtained from the zero order force balance condition. In our normalized units, this condition is

$$p + B_{x0}^2 = \beta + 1 \quad (4.3.1)$$

where

$$\beta = \frac{8\pi p_\infty}{B_{x_\infty}^2} \quad (4.3.2)$$

is the usual plasma beta (the ratio of the internal thermodynamic energy to magnetic field energy), with p_∞ and B_{x_∞} referring to values far from the current sheet. We see that pressure is peaked at the singular surface, and since we also assume that the plasma is initially of uniform density, there is a corresponding peak in temperature.

The linear code, RIPPLE3, was run with conducting walls at ± 5 and a uniform spatial mesh of 501 points giving a mesh spacing of $\Delta y = .02$. The initial conditions were

$$\psi_0 = 0$$

$$w_0 = y e^{-\alpha|y|}$$

from which exponentially growing eigenfunctions were obtained for a range of the magnetic Reynolds number $10^2 \leq S \leq 10^5$. All cases were run with wave number $\alpha = .5$

and $\beta = 1$.

Two different models for resistivity were used: 1) an analytic model, $\eta = \cosh^2 \frac{2\pi}{\Lambda} y$; and 2) Spitzer resistivity. The latter formula, in unnormalized units, is [88]

$$\eta = 1.435 \times 10^{-8} \frac{\ln \Lambda}{T^{3/2}} \text{ sec.} \quad (4.3.3)$$

where the non-dimensional parameter Λ is

$$\Lambda = 1.239 \times 10^4 \frac{T^{3/2}}{n^{1/2}}. \quad (4.3.4)$$

The normalized resistivity is then

$$\frac{\eta}{\eta_0} = \frac{\ln \Lambda(T)}{\ln \Lambda(T_0)} \left(\frac{T}{T_0} \right)^{-3/2}.$$

In this work we ignore the weak temperature dependence in $\ln \Lambda$, so that the non-dimensional Spitzer resistivity model is simply

$$\eta = T^{-3/2}. \quad (4.3.5)$$

Recalling that the temperature is peaked at $y = 0$, we see that for both models the resistivity is a minimum at the singular surface.

As described in section 4.2, the eigenfunctions obtained from the linear code were used as initial conditions for the non-linear code. As a check, several cases were run on the non-linear code with initial perturbation and time step small enough to assure that the mode was in the linear regime. To determine the growth rate in the non-linear code, the formula

$$p = \frac{1}{B_y} \frac{\partial B_y}{\partial t}$$

(i.e., the analogue of (4.2.4)) was found to be unsatisfactory because of oscillations due to Alfvén and magnetosonic waves set up by the initial perturbation. To isolate the resistive tearing mode from these ideal MHD effects we recall that B_y can be non-zero at the singular surface only in the presence of resistivity. We therefore consider the function

$$\phi_s(x) = \int_{-y_w}^0 B_x(0, y') dy' - \int_0^x B_y(x', 0) dx' \quad (4.3.6)$$

which is the magnetic flux at the singular surface as a function of x . In the absence of resistivity, this function must be a constant, and it is only dissipative effects (i.e., the tearing mode) which can cause it to have an x -variation. We therefore take

$$\Delta\phi_s = \phi_{s\max} - \phi_{s\min} \quad (4.3.7)$$

as a measure of the tearing mode, and compute the growth rate from

$$p = \frac{1}{\Delta\phi_s} \frac{\partial \Delta\phi_s}{\partial t} = \frac{\Delta\phi_s^{n+1} - \Delta\phi_s^{n-1}}{2\Delta\phi_s^n \Delta t} \quad (4.3.8)$$

Mesh points were placed on the walls, and conservative boundary conditions for density and energy were used, as described in section 3.3.

When eigenfunctions of the linearized equations were introduced into the non-linear code, the growth rate (4.3.8) was found to be constant to 4 significant figures for 50

time steps. This value, as a function of S , the magnetic Reynolds number, was then compared with results obtained from the linear code and those predicted by the linear analysis of FKR, and are presented in tables 4.1 and 4.2 and plotted in Figure (4.1). In studying these results it should be recalled that, in the present work, time is measured in units of the Alfvén transit time, while in FKR time is measured in units of the resistive diffusion time. Thus the FKR formula $p \sim S^{2/5}$ (equation (1.2.20)) must be converted to $p \sim S^{-3/5}$. This normalization more clearly displays the decrease in growth rate with decreasing resistivity (increasing S) as indicated by equation (1.2.16).

We find that the non-linear code and the linear code agree quite well for low S ($< 10^4$) and less well for higher S , for both resistivity models. For analytic resistivity, we find a least squares fit of the data over the range $10^2 \leq S \leq 10^5$ yields $p = .844 S^{-.494}$ for the linear code, and $p = 1.28 S^{-.559}$ for the non-linear code. For a lower range of S , $10^2 \leq S \leq 10^4$, we find that RIPPLE3 yields $p = .792 S^{-.463}$ while MHDG gives $p = .735 S^{-.473}$, demonstrating the close agreement between the two codes. These are to be compared with $p = 1.0 S^{-.6}$, the FKR result for this value of α . For Spitzer resistivity, a full range fit of the RIPPLE3 results has $p = .75 S^{-.542}$, while the low S fit is $p = .595 S^{-.504}$. The corresponding results from the non-linear code are $p = 1.23 S^{-.632}$ for full the range of S , and $p = .734 S^{-.548}$ for low S . Thus, while the linear code,

the non-linear code, and FKR agree quite well for low S , the present results actually agree better with FKR over a larger range of magnetic Reynolds number.

As has just been stated, the three sets of results are in better agreement for low S than for high S . It is felt that this is due to computational difficulties encountered as the width of the singular layer shrinks with increasing S , as discussed in Chapter I (see equation (1.2.21)). Typical initial profiles (i.e., the linear eigenfunctions produced by the linear code) are shown in Figures (4.2) and (4.3) for $S = 10^2$ and $S = 10^5$. We see that the momentum eigenfunctions ($\rho v_1 \equiv \rho v_x, \rho v_2 \equiv \rho v_y$) become quite rapidly varying near $y = 0$ for large S . As mentioned previously, the linear results were obtained on a uniform mesh with $\Delta y = .02$. For these cases the non-linear code was run with 32 points in x , and 101 points in y . The x -mesh was uniform, and the y -mesh was variable, with $\Delta y = .02$ near the singular surface and increasing away from it. While this resolution is suitable for lower values of S , it may be too coarse to accurately resolve the singular region for high S .

Next the size of the perturbation and time step were increased and the mode was allowed to run into the non-linear regime. The reconnected flux, $\Delta \phi_s$, as a function of time, for the case $S = 10^2$, is shown in Figure (4.4). This mode was found to grow exponentially for some time and then begin to saturate. The growth rate as a function of time is shown in Figure (4.5). Figure (4.6) shows the magnetic flux sur-

faces for the $S = 10^2$ case after 10 Alfvén transit times. This places it well into the non-linear regime. The magnetic island is evident. In Figure (4.7) we show the corresponding velocity field. Note the tight vortex motion which has been set up in the vicinity of the singular surface. The fluid convects magnetic field lines into the reconnection region at the x-type neutral point. There, field energy is converted into kinetic and thermal energy, and the fluid flows outward along the singular surface toward the center of the magnetic island where it meets oppositely directed flow from a neighboring neutral point. The fluid then is ejected out the sides of the island and is recirculated along field lines into the reconnection region. It is seen that the non-linear behavior of this mode does not cause drastic changes to the magnetic field structure away from the singular surface, i.e., the mode remains localized. However, since particles can easily stream along field lines, the change in topology has created a "short circuit" for particles to move from one side of the singular surface to the other, allowing for the possibility of an enhanced diffusion rate. This effect is more pronounced in the double (or multiple) tearing mode, and will be described later in this chapter.

In addition to the linear growth rate, an important parameter in studying instabilities is the level at which the mode saturates. A fast growing mode which saturates at a low amplitude may be less dangerous than a slower growing mode which exponentiates for a longer time.

Rutherford [27] has predicted that the tearing mode will saturate when the width of the magnetic island becomes comparable with the size of the singular layer. From FKR, this is (see equation (1.2.21))

$$\delta_s = a^{-3/5} s^{-2/5} \quad (4.3.9)$$

The width of the magnetic island can be determined by imposing instantaneous flux conservation within the island. The geometry is shown in Figure (4.8), where we present curves of constant flux ϕ in the x, y plane. Here

$$\phi(x, y) = \int_0^y B_x(0, y') dy' - \int_0^x B_y(x', y) dx' \quad , \quad (4.3.10)$$

and we have arbitrarily placed the value $\phi = 0$ at the center of the island. The value ϕ_s is the flux contour which determines the boundary of the island and is called the separatrix. It is an extremum of ϕ (with $\phi = 0$ being the other extremum), and is determined by

$$\phi_s = - \int_0^{x_s} B_y(x', 0) dx' \quad (4.3.11)$$

where x_s is the location of the x -type neutral point. This is equivalent to the amplitude of the reconnected flux $\Delta\phi_s$ (equations (4.3.6) and (4.3.7)) which is used to compute the growth rate. Then requiring flux conservation, we have

$$\Delta\phi_s = \int_0^{y_s} B_x(0, y') dy' \quad (4.3.12)$$

where y_s is the point where the separatrix intercepts the y -axis. Near the singular surface we write

$$B_x = B'_x y \quad (4.3.13)$$

where $()'$ denotes differentiation with respect to y . Then (4.3.12) yields

$$\Delta\phi_s = \frac{1}{2} B'_x y_s^2, \quad (4.3.14)$$

so that the half-width of the magnetic island is

$$y_s = \left(\frac{2\Delta\phi_s}{B'_x} \right)^{\frac{1}{2}}. \quad (4.3.15)$$

The Rutherford theory says that saturation will occur when $y_s \approx \delta_s$, as given in equation (4.3.9). We then arrive at a predicted saturation amplitude of

$$\Delta\phi_s = \frac{1}{2} B'_x \alpha^{-6/5} S^{-4/5}. \quad (4.3.16)$$

A series of runs were made with the non-linear code to numerically determine the saturation amplitude of the tearing mode. For these runs, in addition to the equilibrium field (4.2.2), we have imposed a constant z -component of the magnetic field of amplitude $B_z = 4$. This is similar to the fields found in a Tokamak of aspect ratio $A = 4$ (see Chapter I). The plasma beta defined in equation (4.3.2) is then analogous to the poloidal beta, while the total beta becomes $\beta = .0625$. All runs were made with $\alpha = .5$, $y_w = 5$, and conservative boundary conditions. The range of magnetic Reynolds numbers was $10^2 \leq S \leq 7 \times 10^2$, the upper limit being

due to the slow growth rate and the difficulty in resolving the singular layer.

Because of the relatively slow growth rate, the time step was chosen to be larger than that used in the linear check-out runs, and exact exponential growth was not obtained. Instead, the growth rate was found to rise slowly and then level off at a value somewhat larger than its initial value. The percentage rise ranged from less than a percent for $S = 10^2$ (5.27×10^{-2} to 5.31×10^{-2}), to approximately double for $S = 7 \times 10^2$ (1.05×10^{-2} to 2.17×10^{-2}). In all cases the growth rate then decreased as non-linear effects began to dominate. The determination of the onset of saturation is thus somewhat subjective, and we have arbitrarily taken the point where the growth rate began to decrease as the saturation amplitude. In Figures (4.9) and (4.10) we show the reconnected flux and growth rate as functions of time for these cases. From equation (4.3.16), with $\alpha = \frac{1}{2}$ and $B'_x = \pi/2$, we have the predicted behavior $\Delta\phi_S = 1.8 S^{-.8}$. A least squares fit of our results yields $\Delta\phi_S = .86 S^{-.69}$. These results are summarized in Table 4.3 and plotted in Figure (4.11). We conclude that the tearing mode saturates when the width of the magnetic island becomes of the order of the width of the singular layer, in agreement with the theory of Rutherford.

4.4 Double Tearing Mode

So far we have considered configurations in which there

exists only one root of the equation $\underline{k} \cdot \underline{B} = 0$. This need not be the case. For example, consider the situation in cylindrical geometry [21] where the magnetic fields are

$$B_z = B_{z0} = \text{constant} \quad (4.4.1a)$$

$$B_\theta = \frac{r^3}{(1+r^2)^2} \quad (4.4.1b)$$

where the radius r and the field components are non-dimensional. These fields correspond to an axial current profile given by

$$J_z = \frac{r^2(3-r^2)}{(1+r^2)^3} \quad (4.4.2)$$

which is hollow. The wave vector for the perturbation in this coordinate system is taken to be

$$\underline{k} = \frac{m}{r} \hat{e}_\theta - k_z \hat{e}_z \quad (4.4.3)$$

where $k_z = 2\pi/\lambda$ is the axial wave number and m is an integer. Then setting $\underline{k} \cdot \underline{B} = 0$ yields

$$\frac{2\pi B_{z0}}{\lambda} = \frac{m B_\theta(r)}{r} \quad (4.4.4)$$

We now let L be the axial length of the cylinder, and note that the axial wavelength is then $\lambda = L/n$ (n an integer), so that we arrive at

$$\mu(r_s) \equiv \frac{2\pi r_s B_{z0}}{L B_\theta(r_s)} = \frac{m}{n} \quad (4.4.5)$$

as the condition that must be satisfied for r_s to be the radius of a singular surface. The function μ is called

the pitch, and its generalization to toroidal geometry is called the safety factor (see Chapter I). For the fields (4.4.1), the pitch function is

$$\mu(r) = \mu_0 \left(\frac{1+r^2}{r} \right)^2 \quad (4.4.6)$$

where

$$\mu_0 = \frac{2\pi B z_0}{L} \quad (4.4.7)$$

and wherever (4.4.6) takes on rational values there exists a possible singular surface. This function is plotted in Figure (4.12), and we note that, for this example, two roots for a given mode (i.e., given values of m and n) are possible. If these roots lie close to one another, the modes at each singular surface may interact leading to an enhanced growth rate. This interaction may also lead to enhanced plasma transport.

Similar situations have been studied by Stix [35] in connection with disruptive instability in Tokamaks, and by Spicer [47] in a model of a solar flare. Here, we consider a simple Cartesian model with a form of $F = \underline{k} \cdot \underline{B}$ relevant to the double tearing mode. Our particular choice is

$$F = 1 - (1 + F(0)) \operatorname{sech} \left[\frac{y}{y_s} \operatorname{sech}^{-1} \left(\frac{1}{1+F(0)} \right) \right] \quad (4.4.8)$$

which has the property that $F = 0$ at $y = \pm y_s$, $F = F(0)$ at $y = 0$, and $F \rightarrow 1$ as $y \rightarrow \pm \infty$. We place conducting walls at

$\pm y_w$, and study the linear behavior of the mode by solving (4.2.1a,b) as an initial value problem, as described in section 4.2.

For the results presented here, we have chosen $F(0)$ such as to make $F'(y_s)$ (the shear) the same for all runs. The particular value chosen was $F'(y_s) = \pi/2$ to allow for easy comparison with the results presented in section 4.3. The conducting walls were placed at $y_w = \pm 5$ and we have set $\alpha = .5$. We use the Spitzer resistivity model. Growth rates were then obtained as functions of S and y_s , the separation of the singular surfaces. The results are summarized in Table 4.4. We find that for small separation ($y_s = .25$), $p = .26 S^{-.254}$, which is close to ideal MHD time scale, while for larger separation ($y_s = .7$), $p = .97 S^{-.47}$. These are to be compared with the single tearing mode results $p = .75 S^{-.542}$ as obtained numerically, and $p = S^{-.6}$ as derived in FKR. We see that the proximity of two singular surfaces can greatly enhance the growth rate of the tearing mode.

In Figure (4.13) we plot growth rate as a function of separation distance for various values of magnetic Reynolds number. We see that the growth rate peaks at smaller values of y_s for larger values of S . This is to be expected in light of the reduced size of the singular layer for higher S modes. In Figure (4.14) we plot the exponent z in $p \sim S^{-z}$ as a function of y_s . Recalling that $z' \rightarrow 0$ implies that

$\omega \sim 1/t_H$ (i.e., growth on the MHD time scale) we again see the enhanced growth rate for smaller separation distance. The linear behavior of the function $Z(y_s)$ as displayed in this figure cannot be continued for indefinitely large y_s , since for $y_s \rightarrow \infty$ we expect to regain the single layer results, $Z = .542$.

In Figure (4.15a-d) we display the linear eigenfunctions for various values of S and y_s . Note that the y -components of both the velocity and the magnetic field are large over the entire region between the singular surfaces. Thus the perturbation is not limited to the singular layer about each surface, but is more evenly distributed. This increased y -directed flow convects more magnetic field into the reconnection region of each singular surface, thus enhancing the growth rate.

We use the zero order equilibrium and eigenfunctions obtained from the linear code as initial conditions for the non-linear code and follow the behavior of the mode into the non-linear regime. We find that the magnetic islands on the adjacent singular surfaces grow toward one another, thus creating an extended region of y -directed magnetic field between the two surfaces. We have now created an effective "short circuit" for particles to travel from below the lower singular surface to above the upper singular surface (and vice versa). This could account for enhanced diffusion in the direction normal to the zero order field (i.e., the y -direction). This behavior of the flux surfaces is shown in

Figure (4.16). Note that this figure is an enlargement of the region near the singular surfaces, the y-axis being scaled between +1 and -1.

The corresponding velocity field is shown in Figure (4.17). We note the extended vortex motion in the region between the surfaces. Fluid ejected from an island on one singular surface is forced directly into the reconnection region of the other surface, and vice versa. This leads to stronger convection of magnetic field into the reconnection regions and could account for an increased rate of field annihilation.

In Figure (4.18) we plot the y-component of the magnetic field at a point half-way between the singular surfaces ($y = 0$) as a function of time. Note the initial exponential growth (at a rate in agreement with the linear results) followed by non-linear saturation. Figure (4.19) shows the kinetic energy in the entire plasma as a function of time, and we note that it is still growing (approximately exponentially) well after $B_y(0)$ has saturated. This could be due to the fact that reconnection is still taking place near each singular surface releasing magnetic energy into internal thermal and directed kinetic energy long after the field at the mid-plane has reached its maximum value. We find that after 14 Alfvén transit times (the approximate time at which $B_y(0)$ is completely quenched) 4.17% of the initial magnetic energy has been dissipated. Of that, 10.1% has gone into kinetic energy, and 89.9% into thermal energy.

GROWTH RATE $p = \omega t_H$

TABLE 4.1 $\eta = \cosh^2 \frac{\eta}{2} y$, $y_{wall} = 5$, $\alpha = .5$, $\beta = 1$

S	10^2	10^3	10^4	10^5
0.844 $S^{-.494}$	9.094×10^{-2}	2.917×10^{-2}	9.354×10^{-3}	3.0×10^{-3}
0.792 $S^{-.463}$	8.667×10^{-2}	2.988×10^{-2}	1.03×10^{-2}	3.835×10^{-3}
RIPPLE3	8.317×10^{-2}	3.244×10^{-2}	9.884×10^{-3}	2.791×10^{-3}
FKR, $S^{-.6}$	6.31×10^{-2}	1.58×10^{-2}	3.98×10^{-3}	1.0×10^{-3}
MHDG	7.838×10^{-2}	3.137×10^{-2}	8.86×10^{-3}	1.71×10^{-3}
0.735 $S^{-.473}$	8.307×10^{-2}	2.793×10^{-2}	9.39×10^{-3}	3.172×10^{-3}
1.28 $S^{-.559}$	9.391×10^{-2}	2.627×10^{-2}	7.348×10^{-3}	2.055×10^{-3}

TABLE 4.2 $\eta = T^{-3/2}$, $y_{wall} = 5$, $\alpha = .5$, $\beta = 1$

S	10^2	10^3	10^4	10^5
0.75 $S^{-.542}$	6.185×10^{-2}	1.776×10^{-2}	5.099×10^{-3}	1.464×10^{-3}
0.595 $S^{-.504}$	5.837×10^{-2}	1.828×10^{-2}	5.725×10^{-3}	1.797×10^{-3}
RIPPLE3	5.682×10^{-2}	1.929×10^{-2}	5.573×10^{-3}	1.342×10^{-3}
FKR, $S^{-.6}$	6.31×10^{-2}	1.58×10^{-2}	3.98×10^{-3}	1.0×10^{-3}
MHDG	5.591×10^{-2}	1.842×10^{-2}	4.48×10^{-3}	7.0×10^{-4}
0.734 $S^{-.548}$	5.881×10^{-2}	1.665×10^{-2}	4.712×10^{-3}	1.336×10^{-3}
1.23 $S^{-.632}$	6.691×10^{-2}	1.561×10^{-2}	3.641×10^{-3}	8.494×10^{-4}

SATURATION AMPLITUDE VS MAGNETIC REYNOLDS NUMBER

$$\alpha = .5, \beta_p = 1, \beta = .0625, B_z = 4, Y_W = 5$$

S	Numerical		Analytic
	$\Delta\phi_S$	$.86 S^{-.69}$	$1.8 S^{-.8}$
10^2	3.54×10^{-2}	3.55×10^{-2}	4.52×10^{-2}
3×10^2	1.63×10^{-2}	1.66×10^{-2}	1.88×10^{-2}
5×10^2	1.23×10^{-2}	1.16×10^{-2}	1.25×10^{-2}
7×10^2	8.89×10^{-3}	9.22×10^{-3}	9.56×10^{-3}

TABLE 4.3

TABLE 4.4

$$p = cS^{-2}$$

<u>Separation Distance y_s</u>	<u>Linear Growth Rate</u>
.25	$p = .26 S^{-.254}$
.4	$p = .48 S^{-.325}$
.5	$p = .64 S^{-.368}$
.6	$p = .81 S^{-.415}$
.7	$p = .97 S^{-.46}$
∞	$p = .75 S^{-.542}$

Fig. 4.1 Linear growth rate versus magnetic Reynolds number for the sheet pinch

Fig. 4.2 Sheet pinch eigenfunctions for $S = 10^2$

Fig. 4.2(continued). Sheet pinch eigenfunctions for $S = 10^2$

Fig. 4.3 Sheet pinch eigenfunctions for $S = 10^5$

Fig. 4.3 (continued) Sheet pinch eigenfunctions for $S = 10^5$

Fig. 4.4 Reconnected flux versus time for the sheet pinch

Fig. 4.5 Growth rate versus time for the sheet pinch

Fig. 4.6 Magnetic flux surfaces, $S = 10^2$, $t = 10t_H$

Fig. 4.7 Velocity field, $S = 10^2$, $t = 10t_H$

Fig. 4.8 Structure of a magnetic island

Fig. 4.9 Saturation study. Reconnected flux vs. time

Fig. 4.10 Saturation study. Growth rate vs. time

Fig. 4.11 Saturation amplitude vs. magnetic Reynolds number

Fig. 4.12 Pitch vs. radius for hollow current profile

Fig. 4.13 Linear growth rate p vs. y_s , the separation of the singular surfaces, for the double tearing mode

Fig 4.14 The exponent z in $p \sim S^{-z}$ vs. y_s .

Fig. 4.15 (a) and (b). Linear eigenfunctions of the double tearing mode for (a) $S = 10^2$, $y_s = .25$, (b) $S = 10^5$, $y_s = .25$

Fig. 4.15 (c) and (d). Linear eigenfunctions of the double tearing mode for (c) $S = 10^2$, $y_s = .7$, (d) $S = 10^5$, $y_s = .7$

Fig. 4.16 Flux surfaces in region between singular surfaces for the double tearing mode

Fig. 4.17 Velocity field corresponding to Fig. 4.16. Note that the singular surfaces are at $y = \pm .5$, and the conducting walls are at $y = \pm 5$.

Fig. 4.18 $B_y(0)$ vs. time

Fig. 4.19 Kinetic energy vs. time

CHAPTER V - CYLINDRICAL APPLICATIONS: THE TEARING MODE IN A REVERSED FIELD PINCH

5.1 Introduction

In this chapter we study the behavior of the tearing mode in cylindrical geometry. The purpose of this investigation is twofold: 1) to demonstrate the versatility of the orthogonal curvilinear formulation described in Chapter II, and 2) to extend our study of the tearing mode to geometries more typical of current controlled fusion experiments.

A recent study by Taylor [89] has dealt with the relaxation of toroidal plasma discharges to final quiescent states. These final configurations of a slightly resistive plasma are found to minimize the energy subject to the constraint that

$$K_0 = \int_{V_0} \underline{A} \cdot \underline{B} \, d\tau \quad (5.1.1)$$

be invariant during the discharge. Here, $\underline{A}$ is the vector potential, $\underline{B}$ is the magnetic flux density, and the integral is taken over the entire volume of the plasma. Then it is shown that these final states are characterized by the condition

$$\nabla \times \underline{B} = \mu \underline{B} \quad (5.1.2)$$

where μ is a constant. From Ampère's law we see that this is equivalent to requiring that the current density be

everywhere parallel to the magnetic field, i.e., the Lorentz force $\underline{J} \times \underline{B}$ vanishes at every point in the plasma. Such equilibrium configurations are termed force-free. In addition, there exist two operational windows in which the final configuration is stable against ideal MHD modes. These are the Tokamak regime, $\chi \lesssim a/R$, and the Reversed Field Pinch regime, $\chi \gtrsim 1$, where

$$\chi = \frac{2I}{aB_0} \quad (5.1.3)$$

is the pinch ratio, B_0 is the initial toroidal field, I is the induced current, " a " is the minor radius of the torus, and R is the major radius of the torus. The nomenclature for the regime $\chi \gtrsim 1$ arises from the fact that discharges in these experiments, such as in the ZETA device at Culham, show a reversal of the toroidal magnetic field near the outer regions of the plasma.

The Tokamak regime is necessarily limited to low beta operation ($\beta \sim (a/R)^2$), so that the total energy and the magnetic energy will be almost equal. In our non-linear code the pressure is computed as the difference between these quantities, i.e.,

$$p = (\gamma - 1)(u - B^2 - \rho v^2) \quad ,$$

so that in this regime this calculation would suffer severe roundoff errors. We thus restrict ourselves to high β ($\gtrsim 5\%$) and focus our attention on the Reversed Field Pinch.

In the cylindrical approximation, one solution to equation (5.1.2) is

$$B_r = 0 \quad (5.1.4a)$$

$$B_\theta = a_0 J_1(\mu r) \quad (5.1.4b)$$

$$B_z = a_0 J_0(\mu r) \quad (5.1.4c)$$

where a_0 is constant, J_0 and J_1 are the zero and first order Bessel functions, and the parameter μ is related to the pinch ratio by $\chi = \mu a/2$ [89]. This magnetic field configuration is referred to as the Lundquist [90], or force free Bessel function model. We note that for $\mu a > 2.4$ (i.e., the first root of the zero order Bessel function) the axial field reverses. This corresponds to a pinch ratio $\chi = 1.2$ which is consistent with the Reversed Field Pinch regime defined above. These fields are shown in figure (5.1).

The linear stability of this equilibrium against tearing modes has been extensively studied by Dibiase [60]. In this chapter we examine the non-linear properties of these modes. In section 5.2 we briefly describe the linear code used to generate the tearing mode eigenfunctions. In section 5.3 we discuss the axially symmetric case of $m = 0$. In section 5.4 we describe the reduction of the full three dimensional resistive MHD equations to two dimensions using the helical symmetry approximation, and in section 5.5 we discuss the numerical treatment of the origin ($r = 0$) in polar coordinates. Section 5.6 discusses the application of the non-linear

computer code to the $m = 1$ and $m = 2$ tearing modes. We find that the $m = 2$ mode saturates quickly, while the $m = 1$ mode is more dangerous in that it can continue to grow to large amplitude.

5.2 Linear Calculations

We use a linear, one-dimensional resistive MHD computer code [57,60] to generate initial conditions for the non-linear code. This code solves the linearized equations

$$\frac{\partial \underline{B}_1}{\partial t} = \nabla \times (\underline{V}_1 \times \underline{B}_0) - \frac{c^2}{4\pi} \nabla \times (\eta_0 \nabla \times \underline{B}_1 + \eta_1 \nabla \times \underline{B}_0) \quad (5.2.1a)$$

$$- \rho_0 \frac{\partial}{\partial t} (\nabla^2 \underline{V}_1) = \frac{1}{4\pi} \nabla \times \nabla \times (\underline{B}_0 \cdot \nabla \underline{B}_1 + \underline{B}_1 \cdot \nabla \underline{B}_0) \quad (5.2.1b)$$

as an initial value problem for the first order quantities $\underline{B}_1$ and $\underline{V}_1$ in a manner similar to that described in Chapter IV in connection with the sheet pinch calculations. Here the equations are expanded in cylindrical coordinates, and the z -components of $\underline{B}_1$ and $\underline{V}_1$ are eliminated by using

$\nabla \cdot \underline{B}_1 = \nabla \cdot \underline{V}_1 = 0$. Fourier analysis is then performed in the θ and z directions by assuming perturbations of the form

$$f(\underline{r}, t) = f_1(r, t) e^{i(m\theta + k_z z)} \quad (5.2.2)$$

Numerical solution of the resulting equations then yields exponentially growing eigenfunctions for given mode numbers m and k_z . This code, and another that includes viscosity, thermal conductivity, and compressibility [59,60], have

been used extensively to study the tearing mode in various cylindrical equilibria [60-63].

The radius of the singular surface, r_s , is defined as the root of the equation $\underline{k} \cdot \underline{B} = 0$. In cylindrical geometry this becomes

$$\frac{mB_\theta}{r} + k_z B_z = 0 \quad (5.2.3)$$

where m is the azimuthal mode number and k_z is the axial wave vector. If we think of the cylinder in some sense approximating a torus, then m and k_z can be identified as the poloidal and toroidal mode numbers, respectively. In that case the axial length of the cylinder should correspond to the major circumference of the torus, and if we require that only an integral number of the wavelengths can fit into this dimension, then (5.2.3) becomes

$$\frac{mB_\theta}{r} + \frac{nB_z}{R} = 0 \quad (5.2.4)$$

where n is an integer (the toroidal mode number) and R is the major radius of the torus. If distances are normalized to the minor radius "a", then a singular surface will occur when the equality

$$\frac{rB_z(r)}{AB_\theta(r)} = - \frac{m}{n} \quad (5.2.5)$$

is satisfied, where $A = R/a$ is the aspect ratio of the torus. Similar conditions were derived in Chapter I (see equation (1.1.5)), and in Chapter IV (see equation (4.4.5)). The

non-dimensional axial wave vector is then simply

$$k_z = \frac{n}{A} \quad (5.2.6)$$

In practice, equations (5.2.1) are cast in terms of dimensionless variables in a manner similar to that of the linear Cartesian code described in section 4.2. The non-dimensional quantities here are

$$\psi = B_{r1}/B_0, \quad \phi = iB_{\theta 1}/B_0$$

$$w = -ikt_R V_{r1}, \quad U = kt_R V_{\theta 1}$$

where $k = ((m/r_s)^2 + k_z^2)^{1/2}$ and t_R is the resistive diffusion time. Again, conversion must be made from this normalization to that used in the non-linear code. The appropriate formulas are

$$B_{\theta 1}(r, \theta) = -\phi(r) \sin m\theta \quad (5.2.7a)$$

$$B_{r1}(r, \theta) = \psi(r) \cos m\theta \quad (5.2.7b)$$

$$V_{\theta 1}(r, \theta) = \frac{U(r)}{\alpha S} \cos m\theta \quad (5.2.7c)$$

$$V_{r1}(r, \theta) = \frac{W(r)}{\alpha S} \sin m\theta \quad (5.2.7d)$$

where S is the magnetic Reynolds number,

$$\alpha = \left[\left(\frac{m}{r_s} \right)^2 + \left(\frac{n}{A} \right)^2 \right]^{1/2} \quad (5.2.8)$$

is the non-dimensional wave number, and r_s is the radius of the singular surface. The z-components of $\underline{B}_1$ and $\underline{V}_1$ are then found from $\nabla \cdot \underline{B}_1 = \nabla \cdot \underline{V}_1 = 0$. The result is

$$B_{z1}(r, \theta) = \frac{A}{mnr} \left[\frac{\partial}{\partial r} (rB_{r1}) - mB_{\theta1} \right] \cos m\theta \quad (5.2.9a)$$

$$V_{z1}(r, \theta) = - \frac{A}{mnr} \left[\frac{\partial}{\partial r} (rV_{r1}) + mV_{\theta1} \right] \sin m\theta \quad (5.2.9b)$$

These formulas obviously only hold for both m and n non-zero.

5.3 Axial Symmetry, the $m = 0$ Mode

We see from equation (5.2.2) that when the azimuthal mode number vanishes, the problem becomes axially symmetric and can be posed in the (r, z) plane. In this case, the location of the singular surface is determined by the vanishing of the axial magnetic field (see equation (5.2.3)). For computational purposes we take a coordinate system for which $x_1 = z$, $x_2 = r$, $x_3 = \theta$. The corresponding scale factors for use in equations (2.2.18) are $h_1 = 2\pi/\alpha$, $h_2 = 1$, $h_3 = r \equiv x_2$, where $\alpha = k_z a$ is the non-dimensional axial wave number.

We follow the mode into the non-linear regime for the case of $S = 96.65$, $\beta = .3164$. In figure (5.2) we plot the exponential growth rate as a function of time, and we note the non-linear saturation. Figure (5.3) shows the magnetic flux surfaces in the (z, r) plane after four Alfvén transit times. Note the well formed magnetic island.

5.4 Transformation to Helical Coordinates

Equations (5.2.1), as solved in the linear code, are equations in three space dimensions, with two of them (θ, z) Fourier analyzed. To describe the cylindrical perturbations

in our non-linear code, we therefore need to reduce the dimensionality of the problem from three to two. This can be done by employing a coordinate transformation based on the preservation of certain symmetries in the problem, and this will be discussed in this section.

For the cases we are considering the equilibrium magnetic fields have no radial component. The resulting field lines thus reside on nested cylindrical surfaces (called flux surfaces), each field line forming a helix of pitch rB_z/B_θ . This situation has been discussed in Chapter I, and we set the length of the cylinder to $2\pi R$, the major circumference of the torus we are approximating. We note that since B_θ and B_z are functions of r , the ratio B_z/B_θ will in general be a continuous function of radius. The pitch of the field lines will thus vary from one nested flux surface to the next, and in general the field lines will not close upon themselves after a finite number of turns around the major circumference of the torus (i.e., the number of transits of the axial length $2\pi R$) but rather will fill the flux surface ergodically.

We now consider a perturbation to this equilibrium characterized by a wave vector $\underline{k} = k_\theta \hat{a}_\theta + k_z \hat{a}_z$. This perturbation must be single valued (i.e., only integral numbers of wavelengths can fit in the θ and z directions) so that the components of $\underline{k}$ are

$$k_\theta = \frac{m}{r} \quad , \quad k_z = \frac{n}{R} \quad (5.4.1a,b)$$

where m and n are integers. In figures (5.2a-d) we sketch wavefronts of this perturbation on "unrolled" flux surfaces for various values of m and n . For clarity, we plot the perturbation ϕ as a function of θ and z next to the appropriate axis.

We recall that the tearing mode can develop about a singular surface defined as a root of the equation $\underline{k} \cdot \underline{B}_0 = 0$. This means that on the singular surface the wavefronts of the perturbation (i.e., $\phi = \text{constant}$ surfaces) must lie parallel to the zero order magnetic field, and the direction of the maximum gradient of ϕ is parallel to the wave vector $\underline{k}$. This direction is marked $\nabla\phi$ in figures (5.4). We know from Appendix A that the tearing mode is two dimensional in the (k, r) plane. Thus if we take our new coordinates to be r and the helical coordinate

$$\phi = m\theta + k_z z \quad (5.4.2)$$

which has the same pitch as the wavefronts (and as the magnetic field on the singular surface), we can reduce the problem from three space dimensions to two. We note that this is just the exponential factor in the Fourier decomposition (5.2.2). Such a transformation is clearly an approximation since it does not allow for the non-linear generation of modes with different helicity, i.e., the original helical symmetry is preserved.

To carry out the transformation, one first writes out the fully three dimensional resistive MHD equations in

cylindrical coordinates (r, θ, z) , and then makes the substitutions

$$\frac{\partial}{\partial \theta} \rightarrow m \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial z} \rightarrow k_z \frac{\partial}{\partial \phi} \quad (5.4.3a, b)$$

For example, a scalar conservation law of the form

$$\frac{\partial f}{\partial t} = - \nabla \cdot \underline{F}$$

becomes

$$\frac{\partial f}{\partial t} = - \frac{1}{r} \frac{\partial}{\partial r} (r F_r) - \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} \quad (5.4.4)$$

where

$$F_\phi = m F_\theta + r k_z F_z \quad (5.4.5)$$

is the helical flux of f , i.e., the flux through a helical ribbon with pitch equal to the pitch of the perturbation wavefronts. The result of applying the transformation to equations (3.1.15) is the set

$$\frac{\partial B_r}{\partial t} = - \frac{1}{r} \frac{\partial}{\partial \phi} (m E_z - r k_z E_\theta) \quad (5.4.6a)$$

$$\frac{\partial B_\theta}{\partial t} = \frac{\partial E_z}{\partial r} - k_z \frac{\partial E_r}{\partial \phi} \quad (5.4.6b)$$

$$\frac{\partial B_z}{\partial t} = - \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) + \frac{m}{r} \frac{\partial E_r}{\partial \phi} \quad (5.4.6c)$$

$$\frac{\partial (\rho v_r)}{\partial t} = - \frac{1}{r} \frac{\partial}{\partial r} (r T_{rr}) - \frac{1}{r} \frac{\partial}{\partial \phi} (m T_{r\theta} + r k_z T_{rz})$$

$$- \frac{1}{2} \frac{\partial}{\partial r} (p + B_r^2 + B_\theta^2 + B_z^2) + \frac{T_{\theta\theta}}{r}$$

(5.4.6d)

$$\begin{aligned} \frac{\partial(\rho v_\theta)}{\partial t} = & -\frac{1}{r} \frac{\partial}{\partial r} (r T_{r\theta}) - \frac{1}{r} \frac{\partial}{\partial \phi} (m T_{\theta\theta} + r k_z T_{z\theta}) \\ & - \frac{m}{2r} \frac{\partial}{\partial \phi} (p + B_r^2 + B_\theta^2 + B_z^2) - \frac{T_{r\theta}}{r} \end{aligned} \quad (5.4.6e)$$

$$\begin{aligned} \frac{\partial(\rho v_z)}{\partial t} = & -\frac{1}{r} \frac{\partial}{\partial r} (r T_{rz}) - \frac{1}{r} \frac{\partial}{\partial \phi} (m T_{z\theta} + r k_z T_{zz}) \\ & - \frac{1}{2} k_z \frac{\partial}{\partial \phi} (p + B_r^2 + B_\theta^2 + B_z^2) \end{aligned} \quad (5.4.6f)$$

$$\frac{\partial p}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) - \frac{1}{r} \frac{\partial}{\partial \phi} (m \rho v_\theta + r k_z \rho v_z) \quad (5.4.6g)$$

and

$$\frac{\partial u}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (r F_r) - \frac{1}{r} \frac{\partial}{\partial \phi} (m F_\theta + r k_z F_z) \quad , \quad (5.4.6h)$$

where

$$E_r = v_z B_\theta - v_\theta B_z + \frac{\eta}{S} \frac{1}{r} \frac{\partial}{\partial \phi} (m B_z - r k_z B_\theta) \quad (5.4.7a)$$

$$E_\theta = v_r B_z - v_z B_r + \frac{\eta}{S} \left(k_z \frac{\partial B_r}{\partial \phi} - \frac{\partial B_z}{\partial r} \right) \quad (5.4.7b)$$

$$E_z = v_\theta B_r - v_r B_\theta + \frac{\eta}{S} \left[\frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) - \frac{m}{r} \frac{\partial B_r}{\partial \phi} \right] \quad (5.4.7c)$$

are the components of the electric field,

$$T_{\alpha\beta} = \rho v_\alpha v_\beta - B_\alpha B_\beta \quad (5.4.8)$$

$(\alpha, \beta = r, \theta, z)$ are the components of the symmetric stress tensor, and

$$F_r = (u + p - B_r^2 + B_\theta^2 + B_z^2)v_r - 2B_r(B_\theta v_\theta + B_z v_z) \\ + \frac{2\eta}{S} \left[\left(\frac{mB_\theta}{r} + k_z B_z \right) \frac{\partial B_r}{\partial \phi} - \frac{B_\theta}{r} \frac{\partial}{\partial r} (rB_\theta) - \frac{1}{2} \frac{\partial}{\partial r} B_z^2 \right] \quad (5.4.9a)$$

$$F_\theta = (u + p + B_r^2 - B_\theta^2 + B_z^2)v_r - 2B_\theta(B_r v_r + B_z v_z) \\ + \frac{2\eta}{S} \left[\frac{B_r}{r} \frac{\partial}{\partial r} (rB_\theta) + k_z B_z \frac{\partial B_\theta}{\partial \phi} - \frac{m}{2r} \frac{\partial}{\partial \phi} (B_r^2 + B_z^2) \right] \quad (5.4.9b)$$

$$F_z = (u + p + B_r^2 + B_\theta^2 - B_z^2)v_z - 2B_z(B_r v_r + B_\theta v_\theta) \\ + \frac{2\eta}{S} \left[B_r \frac{\partial B_z}{\partial r} + \frac{mB_\theta}{r} \frac{\partial B_z}{\partial \phi} - \frac{1}{2} k_z \frac{\partial}{\partial \phi} (B_r^2 + B_\theta^2) \right] \quad (5.4.9c)$$

are the components of the energy flux. Note that by setting $k_z = 0$, $m = 1$ we recover the two dimensional equations derived in Chapter II with $x_1 = r$, $x_2 = \theta$, $x_3 = z$, $h_1 = r$, $h_2 = h_3 = 1$. Equations (5.4.6) are then solved numerically by methods previously described.

5.5 Numerical Treatment of the Origin

The problem of numerically advancing the solution to (5.4.6) at the origin of coordinates ($r = 0$) is a difficult one, for unless symmetry conditions exist there is no natural boundary condition to be imposed at this point. This is because it is a boundary of the computational domain only

by construction of the coordinate system. In this section we will present some special methods we have applied to this problem. For mode numbers where symmetry condition exist ($m \neq 1$) we have been fairly successful. However, the physically interesting case of $m = 1$ is characterized by gross motion of fluid across the origin, and here our results are more limited.

5.5.1 Scalar Quantities

Scalar quantities, such as ρ and u , must have unique values at the origin for any direction of approach, i.e., as we near the origin on any ray $\theta = \text{constant}$, these quantities must approach the same limiting value. In this coordinate system this is also the case for B_z and ρv_z . This comes about because the orientation of the unit vector in the z direction is independent of position. In this section we describe a scheme for advancing such quantities.

Consider the scalar conservation law

$$\frac{\partial u}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (r F_r) - \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} \quad (5.5.1)$$

We integrate this over a wedge-shaped region $0 \leq r \leq \Delta r/2$, $-\Delta\theta/2 \leq \theta \leq +\Delta\theta/2$ and ignore the θ -flux. Then we have

$$\frac{\partial}{\partial t} \int_{-\Delta\theta/2}^{\Delta\theta/2} d\theta \int_0^{\Delta r/2} u_{1,j} r dr = -\Delta\theta \frac{\Delta r}{2} F_{r3/2,j} \quad (5.5.2)$$

where $u_{1,j}$ is a (non-unique) value of u at the origin for the ray $\theta = \theta_j$. Approximating the integral as

$$\int_{-\Delta\theta/2}^{\Delta\theta/2} d\theta \int_0^{\Delta r/2} u_{1,j} r dr \approx \frac{1}{2} \left(\frac{\Delta r}{2}\right)^2 \Delta\theta u_{1,j} \quad (5.5.3)$$

we arrive at the equation

$$\frac{\partial u_{1,j}}{\partial t} = - \frac{4}{\Delta r} Fr_{3/2,j} \quad (5.5.4)$$

for advancing the point at the origin for the j^{th} ray. When spatial discretization is applied to the right-hand side, this equation becomes a boundary condition for the difference equations of the form discussed in Chapter III.

We know that u must be unique at the origin, but advancing in the above manner has made $u_{1,j}$ different for each j . Therefore, after we advance all these points, we redefine u_1 for all j by an average

$$\begin{aligned} \bar{u}_1 &= \frac{1}{\Delta v_0} \int_{\Delta v_0} u_1(\theta) dv \\ &= \frac{m}{2\pi} \int_0^{2\pi/m} u_1(\theta) d\theta \end{aligned} \quad (5.5.5)$$

where Δv_0 is the volume of the cylindrical cell surrounding the origin, and we have taken advantage of periodicity by allowing θ to range from zero to $2\pi/m$ ($m \neq 0$). On a uniform mesh with J points in the θ -direction, we then have

$$\bar{u}_1 = \frac{1}{J-2} \sum_{j=2}^{J-1} u_{1,j} \quad (5.5.6)$$

as the computed average quantity at the origin, which we take to be the proper unique value. In deriving (5.5.6) we have used

$$\Delta\theta = \frac{2\pi/m}{J-2} \quad (5.5.7)$$

so that we have let θ vary between $-\Delta\theta/2$ and $2\pi/m + \Delta\theta/2$ on the difference mesh.

To see that this scheme is conservative we write the conservation sum as

$$U = U_1 + U_{INT} \quad (5.5.8)$$

where

$$U_{INT} = \sum_{i=2}^I \sum_{j=2}^{J-1} u_{1,j} \Delta v_{ij} \quad (5.5.9)$$

is the sum over the interior of the mesh (including the boundary at $i = I$), and

$$U_1 = \bar{u}_1 \Delta v_0 \quad (5.5.10)$$

is the contribution from the cell at the origin. Then the rate of change of U is, from (5.5.6), (5.5.8), (5.5.9), and (5.5.10)

$$\frac{\partial U}{\partial t} = \frac{\Delta v_0}{J-2} \sum_{j=2}^{J-1} \frac{\partial u_{1,j}}{\partial t} + \sum_{i=2}^I \sum_{j=2}^{J-1} \frac{\partial u_{1,j}}{\partial t} \Delta v_{ij} \quad (5.5.11)$$

where

$$\Delta v_0 = \int_0^{2\pi/m} d\theta \int_0^{\Delta r/2} r dr = \frac{\pi (\Delta r)^2}{4m} \quad (5.5.12)$$

is the volume of the cell at the origin, and

$$\Delta v_{ij} = \frac{1}{2} r_i \Delta\theta (\Delta_+ r + \Delta_- r) \quad (5.5.13)$$

is the volume of an interior cell (on a variable radial and

uniform θ mesh). In performing the interior sum the usual cell to cell cancellation takes place and, using periodic conditions in θ , we are left with

$$\frac{\partial U_{INT}}{\partial t} = \frac{\Delta r \Delta \theta}{2} \sum_{j=2}^{J-1} Fr_{3/2,j} \quad (5.5.14)$$

while (5.5.4), (5.5.6), (5.5.7), and (5.5.12) can be combined to yield

$$\frac{\partial U_1}{\partial t} = - \frac{\Delta r \Delta \theta}{2} \sum_{j=2}^{J-1} Fr_{3/2,j} \quad (5.5.15)$$

We thus find that U is identically conserved on the difference mesh.

5.5.2 Symmetry Conditions, $m^2 \neq 1$

At the origin, we know that the x and y (Cartesian) components of any vector quantity must be uniquely defined, whereas the r and θ components are not necessarily unique (i.e., they can have different values on each ray approaching the origin). Let $v_{r1,j}$, $v_{\theta1,j}$ be the r and θ components of a vector $\underline{v}$ at the origin for each value of θ_j ($v_{r1,j} = v_{r1}(\theta_j)$, $v_{\theta1,j} = v_{\theta1}(\theta_j)$). We define the unique Cartesian components at the origin by the averaging procedure

$$\bar{v}_{x1} = \frac{m}{2\pi} \int_0^{2\pi/m} [v_{r1}(\theta) \cos\theta - v_{\theta1}(\theta) \sin\theta] d\theta \quad (5.5.16a)$$

$$\bar{v}_{y1} = \frac{m}{2\pi} \int_0^{2\pi/m} [v_{r1}(\theta) \sin\theta + v_{\theta1}(\theta) \cos\theta] d\theta \quad (5.5.16b)$$

But from Equations (5.2.7), one of the components varies like

$\sin m\theta$, while the other varies like $\cos m\theta$, so that

$$\bar{v}_{x1} \approx \int_0^{2\pi/m} \left[\begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix} \cos\theta - \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \sin\theta \right] d\theta \quad (5.5.17a)$$

$$\bar{v}_{y1} \approx \int_0^{2\pi/m} \left[\begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix} \sin\theta + \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \cos\theta \right] d\theta \quad (5.5.17b)$$

Both of these vanish identically for $|m| > 1$. Then from

$$v_{r1,j} = \bar{v}_{x1} \cos\theta_j + \bar{v}_{y1} \sin\theta_j \quad (5.5.18a)$$

$$v_{\theta 1,j} = -\bar{v}_{x1} \sin\theta_j + \bar{v}_{y1} \cos\theta_j \quad (5.5.18b)$$

we see that for these modes we have the symmetry condition that all vector components must vanish at the origin for each ray $\theta_j = \text{constant}$.

To summarize the results of the last two sections, we have the following boundary conditions at the origin for modes $|m| > 1$: a) Scalar quantities (hold for all m):

$$\frac{\partial \rho_{1,j}}{\partial t} = -\frac{4}{\Delta r} (\rho v_r)_{3/2,j} \quad (5.5.19a)$$

$$\frac{\partial u_{1,j}}{\partial t} = -\frac{4}{\Delta r} F_{r3/2,j} \quad (5.5.19b)$$

$$\frac{\partial B_{z1,j}}{\partial t} = -\frac{4}{\Delta r} E_{\theta 3/2,j} \quad (5.5.19c)$$

$$\frac{\partial (\rho v_z)_{1,j}}{\partial t} = -\frac{4}{\Delta r} T_{rz3/2,j} \quad (5.5.19d)$$

for each j , followed by the averaging procedure defined in section 5.4. Here F_r , E_θ , and T_{rz} are defined in equations (5.4.9a), (5.4.7b), and (5.4.8) respectively.

b) Components of vector quantities (hold for $|m| > 1$):

$$B_{r1,j} = B_{\theta1,j} = \rho v_{r1,j} = \rho v_{\theta1,j} = 0 \quad (5.5.20)$$

5.5.3 Conditions on Vector Components for $m = 1$

When $m = 1$, the integrals in equations (5.5.17) no longer vanish, so that this mode is characterized by gross motion (e.g., non-vanishing velocity) across the origin. Then there is no symmetry at the origin for all rays $\theta_j = \text{constant}$. There is, however, overall symmetry across an axis passing through the origin in the direction of the velocity there, and we can take advantage of this to derive boundary conditions on vector components at $r = 0$.

We place a Cartesian coordinate system over our cylindrical domain (with $x = 0$, $y = 0$ at $r = 0$) oriented in such a way that the x -axis corresponds to the ray $\theta = 0$. This is depicted in Figure (5.5), where we also show the velocity vector $\underline{V}_0$ at the origin, and a typical flow pattern away from the origin. We set up the problem in such a way that the total vector quantities are given by

$$B_r = B_{r1}(r) \cos m\theta \quad (5.5.20a)$$

$$B_\theta = B_{\theta0}(r) - B_{\theta1}(r) \sin m\theta \quad (5.5.20b)$$

$$v_r = v_{r1}(r) \sin m\theta \quad (5.5.20c)$$

$$v_\theta = v_{\theta1}(r) \cos m\theta \quad (5.5.20d)$$

Then along the y-axis ($\theta = \pm\pi/2$), we have

$$B_r \equiv B_y = 0 \quad (5.5.21a)$$

$$v_\theta \equiv v_x = 0 \quad (5.5.21b)$$

$$\frac{\partial B_\theta}{\partial \theta} \equiv \frac{\partial B_x}{\partial x} = 0 \quad (5.5.21c)$$

$$\frac{\partial v_r}{\partial \theta} \equiv \frac{\partial v_y}{\partial x} = 0 \quad (5.5.21d)$$

so that the y-axis defines a plane of symmetry for the problem. Then we also have the symmetry conditions on the scalar quantities

$$\left. \frac{\partial B_z}{\partial \theta} \right|_{\theta=\pm\pi/2} \equiv \left. \frac{\partial B_z}{\partial x} \right|_{x=0} = 0 \quad (5.5.22a)$$

$$\left. \frac{\partial \rho}{\partial \theta} \right|_{\theta=\pm\pi/2} \equiv \left. \frac{\partial \rho}{\partial x} \right|_{x=0} = 0 \quad (5.5.22b)$$

$$\left. \frac{\partial u}{\partial \theta} \right|_{\theta=\pm\pi/2} \equiv \left. \frac{\partial u}{\partial x} \right|_{x=0} = 0 \quad (5.5.22c)$$

and

$$\left. \rho v_z \right|_{\theta=\pm\pi/2} \equiv \left. \rho v_z \right|_{x=0} = 0 \quad (5.5.22d)$$

This last condition is required if we are to have no net Z-momentum in the problem.

To make use of these conditions, we set up the mesh in such a way that there is a ray corresponding to $\theta = 0$, and we sweep that ray first. On this ray we have $r \sim x$, $\theta \sim y$, so that the above formulas reduce to

$$B_{\theta} = \rho v_r = \rho v_z = 0 \quad (5.5.22a,b,c)$$

$$\frac{\partial B_r}{\partial r} = \frac{\partial B_z}{\partial r} = \frac{\partial (\rho v_{\theta})}{\partial r} = \frac{\partial \rho}{\partial r} = \frac{\partial u}{\partial r} = 0 \quad (5.5.22d-h)$$

for $\theta = 0$. These boundary conditions are applied to this ray only. The scalar quantities ρ and u at the origin obtained on this sweep are then taken to be the required unique values and are applied as fixed boundary conditions for the advancement of all other rays. The vector components on the other rays are then advanced using the fixed boundary conditions

$$B_{r1,j} = B_x \cos \theta_j \quad (5.5.23a)$$

$$B_{\theta 1,j} = -B_x \sin \theta_j \quad (5.5.23b)$$

$$(\rho v_r)_{1,j} = (\rho v_y) \sin \theta_j \quad (5.5.23c)$$

$$(\rho v_{\theta})_{1,j} = (\rho v_y) \cos \theta_j \quad (5.5.23d)$$

where B_x and ρv_y are the values of B_r and ρv_{θ} obtained from the advancement of the ray $\theta = 0$.

To summarize, then, we advance the origin for $m = 1$ modes by the following algorithm:

- a) Set up the mesh so that $\theta = 0$ (the x-axis) is a ray (with index $j = 2$, say).
- b) Advance this ray subject to the boundary conditions at the left-hand side of the mesh

$$B_{\theta 1,2} = (\rho v_r)_{1,2} = (\rho v_\tau)_{1,2} = 0$$

$$\left(\frac{\partial B_r}{\partial r}\right)_{1,2} = \left(\frac{\partial B_z}{\partial r}\right)_{1,2} = \left(\frac{\partial (\rho v_\theta)}{\partial r}\right)_{1,2} = \left(\frac{\partial \rho}{\partial r}\right)_{1,2} = \left(\frac{\partial u}{\partial r}\right)_{1,2} = 0$$

- c) Save the values at the origin ($i = 1$) obtained from this sweep as

$$B_{r1,2} \rightarrow B_{x1} \quad , \quad (\rho v_\theta)_{1,2} \rightarrow (\rho v_y)_1$$

- d) Advance all other rays ($j \neq 2$) subject to the boundary conditions

$$B_{r1,j} = B_{x1} \cos \theta_j \quad , \quad B_{\theta 1,j} = - B_{x1} \sin \theta_j$$

$$(\rho v_r)_{1,j} = (\rho v_y)_1 \sin \theta_j \quad , \quad (\rho v_\theta)_{1,j} = (\rho v_y)_1 \cos \theta_j$$

$$B_{z1,j} = B_{z1,2} \quad , \quad \rho_{1,j} = \rho_{1,2} \quad , \quad u_{1,j} = u_{1,2}$$

$$(\rho v_z)_{1,j} = 0$$

5.6 Results

We apply the methods described above to the study of the tearing mode in the Bessel function equilibrium given in Equations (5.1.4). The linear growth of the tearing mode for this model has been extensively studied by Dibiase [60], using the linear code described in section 5.2, and we use this code to generate eigenfunctions for our non-linear code. Due to the large amount of computer time required we have not performed an extensive parameter study of the non-linear behavior of the modes. Instead, we concentrate on the qualitative long-time behavior of solutions characterized

by the azimuthal mode numbers $m = 1$ and $m = 2$.

Since the equilibrium under consideration is force-free, and since the plasma is assumed to fill the cylinder with constant $\rho = 1$ initially, we are free to choose a pressure level arbitrarily. For these runs we have chosen this value to correspond to $\beta = 0.6$. Additional parameters to be specified are the magnetic Reynolds number S , the axial wave number k_z , the wall radius R_w , the radius of the singular surface r_s , and the non-dimensional wave number

$$\alpha = (k_z^2 + \frac{m^2}{2})^{\frac{1}{2}} \cdot r_s \quad (5.6.1)$$

For the cases presented below, we have taken $S = 100$ and have set $R_w = 5.5$. We choose an axial mode number $n = -1$, and an aspect ratio $R/a = 3$, so that the axial wave number is

$$|k_z| = |n|/(R/a) = 1/3 \quad (5.6.2)$$

The calculation of the growth rate is similar to that described for the Cartesian calculations except now we must define the reconnected flux along a singular surface that wraps itself helically around the cylinder. We define the helical flux to be

$$\begin{aligned} \psi(r, \theta) = & - \int_{(0,0)}^{(r,0)} (mB_\theta + r'k_zB_z)_{\theta=0} dr' \\ & + r \int_{(r,0)}^{(r,\theta)} B_r(r, \theta') d\theta' \end{aligned} \quad (5.6.3)$$

and calculate the growth rate as

$$p = \frac{1}{\Delta\psi_s} \frac{\partial \Delta\psi_s}{\partial t} \quad (5.6.4)$$

where

$$\Delta\psi_s = \max_{0 \leq \theta \leq \frac{2\pi}{m}} [\psi(r_s, \theta)] - \min_{0 \leq \theta \leq \frac{2\pi}{m}} [\psi(r_s, \theta)] \quad (5.6.5)$$

in the amplitude of the reconnected helical flux at the singular surface.

For the case $m = 2$, the singular surface is the root of

$$\frac{2J_1(r)}{r} - \frac{J_0(r)}{3} = 0$$

which occurs at $r_s = 3.389$, giving a value of $\alpha = 0.662$. We put 31 mesh uniformly spaced mesh points in the range $0 \leq \theta \leq \pi$, and put 51 mesh points in the range $0 \leq r \leq 5.5$. The radial grid is non-uniform, with $\Delta r = .055$ near the singular surface and increasing away from it. This is the same resolution of this region obtained in the linear code with 101 equally spaced radial mesh points.

In Figure (5.6a-f) we plot surfaces of constant helical flux (see equation (5.6.3)) at various times for the case $m = 2$. The same contour values are retained in each plot so that the evolution of a particular flux surface can be followed in time. Figure (5.6a) shows the initial conditions for the run. The perturbation is evident, but the magnetic

island is too small to be resolved. Figure (5.6b) shows the flux surfaces after 0.75 Alfvén transit times, and we see that previously unconnected field lines have now linked to form an island. In Figure (5.6c) we see that this connected flux surface is diffusing away, while two more field lines are on the verge of reconnecting, which is shown in Figure (5.6d) after 3.75 Alfvén transit times. In Figure (5.6e) we see that both reconnected flux contours are shrinking, and in Figure (5.6f) we see that, after 5 Alfvén transit times, the island has completely saturated with no further reconnection taking place.

This behavior is summarized in Figure (5.7) where we plot the exponential growth rate (see Equation (5.6.4)) as a function of time. The initial value is in agreement with the results of the linear code. This value then rises slightly before saturation takes place in a manner similar to that described and explained in Chapter IV. We note that this mode saturates within a few Alfvén transit times, with island growth ceasing shortly thereafter.

For the $m = 1$ mode, we find the singular surface to be located at $r_s = 3.139$, so that $\alpha = .438$, and Figure (5.8a-i) show the contours of constant helical flux as a function of time. We note that reconnection becomes apparent in Figure (5.8c) after 4.25 Alfvén transit times. This closed flux contour then disappears due to diffusion (Figures (5.8d-g)), but reconnection continues to take place, as is seen in subsequent figures. In Figure (5.9) we plot the reconnected

helical flux at the singular surface as a function of time and note the initial exponential growth (again at a rate in agreement with the linear results) followed by non-linear saturation. In Figure (5.10) we display the exponential growth rate as a function of time for this case, and we note the extended period of rapid growth.

We were unable to run this case to larger amplitude because of numerical problems encountered at the origin for large times. This may be due to the fact that, since we place a mesh point $r = 0$, the derivative boundary conditions described in Equations (5.5.22d-h) are necessarily only first order accurate. This generates errors in the numerical solution which are found to become unstable near the origin. This does not seem to affect the solution in the body of the plasma, so that we have some confidence in the saturation results presented above.

We conclude, then, that for the force-free Bessel function magnetic field model, the $m = 2$ mode saturates quickly, whereas the $m = 1$ mode is more dangerous in that it can continue to grow to large amplitude.

Fig. 5.1 Magnetic field profiles for the Bessel function model (BFM)

Fig. 5.2 Growth rate vs. time for the $m = 0$ mode

Fig. 5.3 Magnetic flux surfaces in the (z, r) plane for the $m = 0$ mode

Fig. 5.4(a) $m = 1$, $n = -1$, $B_z/B_\theta = k_\theta/k_z \sim R/r$

Fig. 5.4(b) $m = 2, n = -1, B_z/B_\theta = -k_\theta/k_z$

Fig. 5.4(c) $m = 1$, $n = -2$, $B_z/B_\theta = -k_\theta/k_z \sim R/2r$

Fig. 5.4(d) $m = 2, n = -2, B_z/B_\theta = -k_\theta/k_z \sim R/r$. Note that this case has the same pitch as that shown in Fig. 5.4(a)

Fig. 5.5 Symmetry for the $m = 1$ mode

Fig. 5.6(a) Helical flux surfaces for $m = 2$, $t = 0$

Fig. 5.6(b) Helical flux surfaces for $m = 2$, $t = 0.75$

Fig. 5.6(c) Helical flux surfaces for $m = 2$, $t = 3.0$

Fig. 5.6(d) Helical flux surfaces for $m = 2$, $t = 3.75$

Fig. 5.6(e) Helical flux surfaces for $m = 2$, $t = 4.25$

Fig. 5.6(f) Helical flux surfaces for $m = 2$, $t = 5.0$

Fig. 5.7 Growth rate vs. time for the $m = 2$ mode

Fig. 5.8(a) Helical flux surfaces for $m = 1, t = 0$

Fig. 5.8(b) Helical flux surfaces for $m = 1$, $t = 4.0$

Fig. 5.8(c) Helical flux surfaces for $m = 1$, $t = 4.25$

Fig. 5.8(d) Helical flux surfaces for $m = 1$, $t = 4.75$

Fig. 5.8(e) Helical flux surfaces for $m = 1$, $t = 5.25$

Fig. 5.8(f) Helical flux surfaces for $m = 1$, $t = 5.75$

Fig. 5.8(g) Helical flux surfaces for $m = 1$, $t = 6.25$

Fig. 5.8(h) Helical flux surfaces for $m = 1$, $t = 6.5$

Fig. 5.8(i) Helical flux surfaces for $m = 1$, $t = 7.0$

Fig. 5.9 Reconnected helical flux vs. time for the $m \approx 1$ mode

Fig. 5.10 Growth rate vs. time for the $m = 1$ mode

CHAPTER VI - SUMMARY AND CONCLUSIONS

6.1 Summary

We have developed a general model suitable to the study of the tearing mode in a variety of geometries and magnetic field configurations. This model consists of the eight non-linear partial differential equations of magnetohydrodynamics, including the effects of compressibility and resistivity. These equations have been cast in the form of a set of conservation laws, the conserved quantities being magnetic flux, momentum, mass, and total energy. We have made no assumptions about the ordering of various terms in the equations. Specifically, we have not assumed high aspect ratio, low β , etc., as has been the case in previous work [34].

In our model we assume that the relevant physical phenomena are describable in two space dimension and time, as is reasonable for the tearing mode [10]. Then to make the model applicable to many geometric configurations, we have expressed the equations in orthogonal, curvilinear coordinates. In this way, geometric effects can be accounted for by specifying the scale factors as functions of the independent variables.

We have solved the equations of our model numerically in order to follow the time evolution of various magnetoplasma configurations. We employ the method of finite differences on an Eulerian mesh. To this end, we have developed spatial discretization techniques which assure the conservation on

the difference mesh of the same quantities conserved by the differential equations. This has required the development of special boundary conditions on thermodynamic quantities based on these conservation laws. Since the tearing mode proceeds at a rate slower than the MHD time scale, we have used implicit methods for the temporal discretization. The particular technique chosen was the Alternating Direction Implicit (ADI) method. In this way we have not been limited to the Alfvén time scale, and we have avoided the effects of numerical dispersion and diffusion inherent in an explicit determination of the magnetic fields [77].

We have applied our model to several geometries. For the case of the sheet pinch in Cartesian coordinates, we have compared the results obtained from our numerical model with previous analytic [10] and numerical [57] work. We have started our calculations by specifying an equilibrium, and then perturbing it with tearing mode eigenfunctions obtained from a code which solves the linearized resistive MHD equations [57]. When the amplitude of the perturbation is made small enough, we have observed purely exponential growth at a rate in good agreement with previous results [10,57]. When the mode grows to larger amplitude, non-linear saturation is observed. We have performed a study of the saturation amplitude as a function of magnetic Reynolds number, S . This study has been limited to a low range of Reynolds numbers due to difficulty in numerically resolving the narrowing singular layer as this parameter increases.

A least squares fit of our results has given that the saturation amplitude, ψ_s , behaves like $\psi_s = 0.86 S^{-.69}$ as compared with the theoretical prediction [27] of $\psi_s = 1.85 S^{-.8}$.

We have also studied the linear and non-linear behavior of the double-tearing mode [35], which can occur when two singular surfaces lie in close proximity. These results were also obtained in Cartesian coordinates. We have found that the interaction between the singular surfaces leads to growth rates which are greatly enhanced over those of the single tearing mode. We have also allowed this mode to evolve non-linearly to large amplitude, where it results in magnetic field configurations which are consistent with enhanced cross-field particle and thermal transport.

We have also applied our code to several cases in cylindrical geometry, thus demonstrating the versatility of our general computational model. In particular, we have investigated the evolution of the tearing mode in a cylindrical Reversed Field Pinch. We have taken the equilibrium to be the force-free Bessel function model. For the case of the $m = 0$ mode, which possesses axial symmetry, we have used standard (r, z) cylindrical coordinates. For modes with $m > 0$, we have employed a transformation to helical coordinates which allows us to reduce the full three dimensional equations, which are required to describe these modes in this geometry, to two dimensions. The resulting equations preserve helical symmetry, i.e., modes with different helicity cannot be non-linearly generated. In

addition, we require special finite difference techniques to treat the singular origin $r = 0$. For cases in which there is no gross motion of fluid across the origin ($m \neq 1$), a modification of the conservative boundary conditions has been applied. For the more difficult case characterized by flow across the origin ($m = 1$), we have derived special boundary conditions based on certain planar symmetries in the problem. Using these techniques, we have followed the $m = 1$ and $m = 2$ modes into the non-linear regime, where we have observed that the $m = 2$ mode saturates quickly, where as the $m = 1$ mode continues to exponentiate for longer times.

6.2 Conclusions

For the single tearing mode in a sheet pinch, our results indicate that the effect of the instability is localized to a region near the singular surface, i.e., there is no overall disruption of the magnetic field configuration. We also conclude that, at least over the range of magnetic Reynolds numbers studied, the mode saturates when the width of the magnetic island becomes comparable to the width of the singular layer. This is in agreement with the theory of Rutherford [27].

For the double tearing mode, we conclude that the interaction between singular surfaces which lie in close proximity can lead to enhanced linear growth rates over those of the single tearing mode. In addition, the eigenfunctions are no longer localized to regions near each singular surface,

but are of greater extent. We also find that the non-linear evolution of this mode can lead to magnetic field configurations which could result in enhanced particle and thermal transport in a direction perpendicular to the zero order magnetic field. For this mode, we find that approximately ten percent of the released magnetic energy goes into directed kinetic energy, while ninety percent goes into internal thermal energy.

From our studies of the tearing mode in the Bessel function magnetic field model for a cylindrical Reversed Field Pinch, we conclude that the $m = 2$ mode saturates quickly, while the $m = 1$ mode is more dangerous in that it can continue to grow to large amplitude.

Finally, in light of the results that we have obtained in several different geometries, we conclude that our general MHD model is an extremely useful tool for studying the non-linear evolution of diverse magnetoplasma systems.

6.3 Future Work

In order to successfully simulate the operation of high beta experimental devices, more realistic geometries must be studied. In particular, the extension of the present work to a toroidal coordinate system is of current interest. This would allow for the study of the non-linear behavior of unstable modes which have no counterpart in cylindrical systems [56], as well as continued studies of the tearing mode. For these high beta systems, a general

model, such as the one developed in this thesis, will be most useful.

The extension of the present model to include tensor transport coefficients (including viscosity and thermal conductivity) would be of interest. Also, the inclusion of a more general form of Ohm's law (including the Hall term, for example) would be useful.

Although the linear behavior of the tearing mode is essentially two dimensional, the non-linear coupling of modes with different helicities can be studied only in three spatial dimensions, and codes are currently being developed to this end. One of these [75] is an extension of the model which assumes Tokamak ordering [29,34], which will be strictly applicable only to low beta systems. Another [73] is a model similar to the one reported here, and will be more generally useful in the analysis of high beta devices.

In the present study, the most difficult computational procedure was the treatment of the singular origin in cylindrical coordinates. Such a point will also occur in toroidal coordinates, and it is clear that a better understanding of means of numerically transporting both scalar and vector quantities across the axis is of utmost importance to achieving successful simulations. In addition, the effect of the failure to time center the mixed derivative is not clear. Any studies devoted to a better understanding of these computational problems would be most welcome.

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APPENDIX A: ANALYSIS OF THE TEARING MODE BASED ON THE MHD EQUATIONS

In Chapter I we presented a heuristic derivation of the growth rate, tearing width, and stability condition for the tearing mode. To verify that discussion, one must solve the problem more exactly by analytical means. In this appendix we outline the development of the dispersion relation from the MHD equations, as presented in FKR [10]. A discussion based on the kinetic equations can be found in Appendix B.

Under the assumptions that the fluid is incompressible and inviscid, the pertinent equations are the combination of Maxwell's equation and Ohm's law

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B} - \frac{c^2}{4\pi} \nabla \times \underline{B}) \quad (A1)$$

and the equation of motion

$$\rho \frac{d\underline{v}}{dt} = -\nabla p + \frac{1}{4\pi} (\nabla \times \underline{B}) \times \underline{B} \quad (A2)$$

where $d\underline{v}/dt = \partial \underline{v}/\partial t + \underline{v} \cdot \nabla \underline{v}$.

From (A1), the static equilibrium condition is

$$\nabla \times (\eta \nabla \times \underline{B}) = 0 \quad (A3)$$

We take the curl of (A2) to eliminate the pressure, linearize, and assume perturbations of the form

$$f_1(x, y, z, t) = f_1(y) e^{i(k_x x + k_z z) + \omega t}$$

Then equations (A1) and (A2) become

$$\omega \underline{B}_1 = \nabla \times (\underline{v}_1 \times \underline{B}_0) - \frac{c^2}{4\pi} \nabla \times [\eta_0 \nabla \times \underline{B}_1 + \eta_1 \nabla \times \underline{B}_0] \quad (A4a)$$

$$\omega \nabla \times (\rho_0 \underline{v}_1) = \frac{1}{4\pi} \nabla \times [(\underline{B}_0 \cdot \nabla) \underline{B}_1 + (\underline{B}_1 \cdot \nabla) \underline{B}_0] \quad (A4b)$$

Using $\nabla \cdot \underline{v}_1 = \nabla \cdot \underline{B}_1 = 0$, these equations can be written as two equations in the two unknowns B_{y1} and v_{y1} . In terms of the dimensionless variables

$$\psi = B_{y1}/B_0$$

$$W = -iv_{y1}k\tau_R$$

$$F = (k_x B_{x0} + k_z B_{z0})/kB_0 \quad k = (k_x^2 + k_z^2)^{1/2} \quad (A5)$$

$$\alpha = ka \quad \tau_R = \frac{4\pi a^2}{c^2 \eta_0} \quad \tau_H = \frac{a\sqrt{4\pi\rho_0}}{B_0}$$

$$S = \tau_R/\tau_H \quad P = \omega\tau_R$$

these are

$$\frac{\psi''}{\alpha^2} = \psi \left(1 + \frac{P}{\alpha^2}\right) + \frac{W}{\alpha^2} \left(\frac{F}{\eta} + \frac{\eta' F'}{\eta P}\right) \quad (A6a)$$

$$\frac{(\rho W')'}{\alpha^2} = W \left[\rho + \frac{FS^2}{P} \left(\frac{F}{\eta} + \frac{\eta' F'}{\eta P}\right) + \psi S^2 \left(\frac{F}{\eta} - \frac{F''}{P}\right)\right] \quad (A6b)$$

where ()' indicates differentiation with respect to $\mu = y/a$.

The non-dimensional version of (A3) is

$$\frac{\eta'}{\eta} F' = -F'' \quad (A7)$$

Equations (A6), together with appropriate boundary conditions, represent an eigenvalue problem for p . However, solution of this problem has only been achieved numerically [57,58] and to make progress analytically further approximations must be

made.

We first choose a normalization for resistivity such that $\eta F' = 1$. Then from (A6b) we find

$$\frac{p^2}{\alpha^2 S^2 F} [(\rho W')' - \alpha^2 W \rho] = (p\psi + WF)(pF' - \frac{F''}{F}) \quad (A8a)$$

$$= p\psi'' - p\psi(\alpha^2 + \frac{F''}{F}) \quad (A8b)$$

The last expression is obtained by using (A6a).

We now look for solutions where $S \rightarrow \infty$, i.e., the limit of small resistivity. We then see that the left hand side of (A8) is negligible except in a small region near $F = 0$. From (A5) we have that

$$F = (\underline{k} \cdot \underline{B}) / (kB_0)$$

so that we can use the ideal MHD equation

$$\psi'' - (\alpha^2 + \frac{F''}{F})\psi = 0 \quad (A9)$$

(i.e., the right hand side of (A8b)) everywhere except in a small region about the singular ($\underline{k} \cdot \underline{B} = 0$) surface. This equation is just the non-dimensional form of

$$\nabla \times (\underline{J} \times \underline{B}) = 0 \quad (A10)$$

which is the condition for marginal stability.

From (A8a) we also have the infinite conductivity equation

$$p\psi = -FW \quad (A11)$$

which is the non-dimensional form of (1.2.4) (with $\eta = 0$), and again tells us that finite ψ at $F = 0$ implies infinite W . We can thus obtain solutions to (A9) on either side of the singular surface, which in general cannot be joined at the singular surface without a discontinuity in the logarithmic derivative

$$\Delta' = \frac{1}{\psi} \frac{d\psi}{d\mu} \Big|_{\mu_S - \epsilon}^{\mu_S + \epsilon} \quad (A12)$$

where μ_S is a solution of $F = 0$. Inside the region $|\mu - \mu_S| < \epsilon$, the left hand side of (A8) does not vanish, and we must solve the full resistive equations. The eigenvalue equation for p is obtained by matching this "inner" solution to the outer infinite conductivity solutions.

An expression for Δ' can be obtained in terms of the Green's function for the operator in the outer region

$$\psi'' - \alpha^2 \psi = 0$$

with boundary conditions that ψ vanish as $|\mu| \rightarrow \infty$. Then for the case $\psi(\mu_S + \epsilon) = \psi(\mu_S - \epsilon)$ we get

$$\Delta' = -2\alpha - \frac{1}{\psi} \int_{-\infty}^{\infty} d\mu e^{-\alpha|\mu|} \psi \frac{F''}{F}$$

For $\alpha^2 \ll 1$, we expand ψ as

$$\psi = e^{-\alpha|\mu - \mu_S|} (\psi_0 + \alpha\psi_1 + \alpha^2\psi_2 + \dots)$$

Then, from (A9), we arrive at the recursion relation

$$-(\psi_n F' - \psi_n' F)' = \frac{2|\mu - \mu_S|}{(\mu - \mu_S)} F' \psi_{n-1}$$

and Δ' becomes

$$\Delta' = \frac{1}{\alpha} F'^2 \left(\frac{1}{F_{-\infty}^2} + \frac{1}{F_{\infty}^2} \right) \quad (\text{A13})$$

where $F_{\pm\infty}$ refer to the values of F as $\mu \rightarrow \pm\infty$. For the specific case

$$F = \tanh \mu \quad (\text{A14})$$

we have

$$\Delta' = 2 \left(\frac{1}{\alpha} - \alpha \right) \quad (\text{A15})$$

To obtain the solution in the inner region, we assume $\mu_S = 0$, that F' , F'' , η , and η' are constants, and write F as $F'\mu$. Then equations (A6) can be written as

$$\frac{d^2 \psi}{d\theta^2} - \epsilon^2 \alpha^2 \psi = \epsilon \Omega [4\psi + U(\theta + \delta_1)] \quad (\text{A16a})$$

$$\frac{d^2 U}{d\theta^2} + U(\Lambda - \frac{1}{4}\theta^2) = \psi(\theta - \delta) \quad (\text{A16b})$$

where

$$\theta = \frac{1}{\epsilon} \left(\mu + \frac{\eta'}{2p} \right) \quad (\text{A17a})$$

$$\epsilon = \left[\frac{p\eta\rho}{4\alpha^2 S^2 F'^2} \right]^{\frac{1}{2}} \quad (\text{A17b})$$

$$U = W \left(\frac{4\epsilon F'}{p} \right) \quad (\text{A17c})$$

$$\delta = \frac{1}{4\Omega} \left(\frac{F''}{F'} + \frac{\eta'}{2\eta} \right) \quad (\text{A17d})$$

$$\delta_1 = \frac{1}{8\Omega} \frac{\eta'}{\eta} \quad (\text{A17e})$$

$$\Lambda = \frac{\eta'^2}{16\epsilon^2 p^2} - \alpha^2 \epsilon^2 \quad (\text{A17f})$$

$$\Omega = \frac{pe}{4\eta} \quad (\text{A17g})$$

Note the similarity between (A17b) and (1.2.9).

The solution to (A16b) can be obtained in terms of Hermite polynomials, i.e.

$$U = \sum_{n=0}^{\infty} a_n H_n(\theta) \quad (\text{A18a})$$

where

$$a_n = \frac{1}{\Lambda - (n + \frac{1}{2})} \int_{-\infty}^{\infty} d\theta_1 H_n(\theta_1) \psi(\theta_1 - \delta) \quad (\text{A18b})$$

We now assume that, within the resistive layer, $\psi \approx$ constant, i.e., we require that $\Delta\psi/\psi \ll 1$ for $|\mu| < \epsilon_0$, where $\epsilon_0 > \epsilon$.

Writing $\Delta\psi \approx \Delta\psi' \epsilon_0$, we have the condition

$$\epsilon_0 |\Delta'| \ll 1 \quad (\text{A19})$$

for the constant ψ approximation to be valid. If this is true, then using (A16a) and (A18a,b) we obtain the following expression for Δ' :

$$\Delta' = 2^{7/2} \Omega \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m+1)} \left[\frac{\Lambda - 1/2}{\Lambda - (2m + 3/2)} - \frac{\delta \delta_1 / 4}{\Lambda - (2m + 1/2)} \right] \quad (\text{A20})$$

The eigenvalue equation is now realized by equating this expression to (A13), the value of Δ' as obtained from the

"outer" solution, and noting that Λ is a function of the growth rate p .

The tearing mode is characterized by the condition that $|\Lambda| \ll \frac{1}{2}$. Then equation (A20) becomes

$$\Lambda' = \Omega(12 + 13\delta\delta_1) > 0 \quad . \quad (A21)$$

When this is equated to the outer solution (A13), we obtain the growth rate

$$p = (F')^2 \left(\frac{2S\eta^{3/2}}{9\alpha\rho^{1/2}} \right)^{2/5} \left(\frac{1}{F_{-\infty}^2} + \frac{1}{F_{\infty}^2} \right)^{4/5} \\ \propto \alpha^{-2/5} S^{2/5} \quad (A22)$$

in agreement with the heuristic result, equation (1.2.20).

We finally note that we can obtain an energy principle for this problem [91] by multiplying (A9), the marginal stability condition for ideal MHD, by ψ and integrating over all μ , i.e.,

$$W = \int_{-\infty}^{\infty} d\mu \left[(\psi')^2 + \psi^2 \left(\alpha^2 + \frac{F''}{F} \right) \right] \quad . \quad (A23)$$

The integral across the singular surface is to be interpreted in the sense that

$$\int_{-\infty}^{\infty} = \lim_{\epsilon \rightarrow 0} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] \quad .$$

Integrating the last term by parts, we obtain

$$W = W_{\infty} - \lim_{\epsilon \rightarrow 0} \psi^2 \frac{F'}{F} \Big|_{-\epsilon}^{\epsilon} \quad (A24a)$$

where

$$W_{\infty} = \int_{-\infty}^{\infty} d\mu [(\psi')^2 + \alpha^2 \psi^2] \quad (\text{A24b})$$

is the infinite conductivity energy integral and is positive definite. We thus see that having ψ non-zero at the singular surface allows for lowering of the potential energy and hence to destabilization. The last term in (A24a) behaves like

$$\psi^2(\epsilon) - \psi^2(-\epsilon) \approx \psi(0)\psi' \Big|_{-\epsilon}^{\epsilon} = \psi^2(0)\Delta' \quad (\text{A25})$$

so that the condition for instability is

$$\Delta' > 0 \quad (\text{A26})$$

APPENDIX B: ANALYSIS OF THE TEARING MODE FROM THE KINETIC VIEWPOINT

In this Appendix we review work which demonstrates that kinetic theory, when applied to the tearing mode, yields results consistent with fluid theory.

Furth [12] has investigated the case where the field vanishes at the singular surface in the collisionless limit. The particular Vlasov equilibrium is

$$f_0 = \frac{n_0}{(2\pi)^{3/2} v_\perp^3 \delta \cosh^2(\frac{2z}{\delta})} \exp \{-[v_x^2 + v_z^2 + (v_y^2 - v_\perp^2)/2v^2]\} \quad (B1)$$

where

$$B_{x0} = 2v(2\pi n_0 m)^{1/2} \tanh(\frac{2z}{\delta})$$

$$\delta^2 = \frac{2}{\pi n_0 r_c} \left(\frac{v}{v_\perp}\right)^2$$

and

$$r_c = \frac{e^2}{mc^2}.$$

Here, the z-direction is normal to the current sheet, and the notation $(\)_\perp$ and $(\)_\parallel$ refer to components relative to the zero-order magnetic field. The form of the perturbation is

$$B_z = b e^{\omega t} \sin k_\parallel x \quad (B2a)$$

$$B_x = B_{x0} + \frac{1}{k} \frac{db}{dz} e^{\omega t} \cos k_\parallel x \quad (B2b)$$

$$E_y = \frac{\omega}{ck} b e^{\omega t} \cos k_\parallel x. \quad (B2c)$$

The resulting linearized Vlasov equation is

$$\omega f_1 + v_x \frac{\partial f_1}{\partial x} + v_z \frac{\partial f_1}{\partial z} + \frac{e}{mc} [B_{x0} \frac{\partial f_1}{\partial \phi} + \frac{1}{v} f_0 \{-bv_x v_z \sin k_{\parallel} x + \frac{db}{dz} \frac{v_x v_z}{k_{\parallel}} \cos k_{\parallel} z - \frac{bv_z}{k_{\parallel}} (v_y - v_z) \cos k_{\parallel} x\}] = 0 \quad (B3)$$

where $\phi = \tan^{-1}(v_y/v_z)$. An eigenvalue equation is obtained by equating the perturbed current density

$$J_y = \frac{e}{c} e^{i\omega t} \int_{-\infty}^{\infty} dv_x dv_y dv_z f_1 v_y \quad (B4)$$

as found from the solution of the Vlasov equation, to the perturbed current density as found from Ampere's law

$$J_y = -\frac{k_{\parallel}}{4\pi} (b - \frac{1}{k_{\parallel}^2} \frac{d^2 b}{dz^2}) \cos k_{\parallel} z \quad (B5)$$

The solution of (B3) is

$$f_1 = -\frac{bv_z}{B_{x0} r_g k_{\parallel} v} f_0 \cos k_{\parallel} x \quad (B6)$$

where $r_g = mvc/eB_0$ is the particle gyroradius. The result of equating (B4) and (B5) is that the marginal stability condition is

$$4\pi r_g n_0 \delta^2 \left(\frac{v_z}{v}\right)^2 < \lambda$$

where λ is an eigenvalue of the problem

$$\frac{d^2 b}{d\zeta^2} = b [(k_{\parallel} \delta)^2 - \frac{\lambda}{\cosh^2 2\zeta}] \quad (B7)$$

and $\zeta = z/\delta$, with the boundary conditions that b vanish at

$\zeta = w$ and $db/d\zeta = 0$ at $\zeta = 0$.

We now note that equation (B7) is identical to the marginal stability condition (A9) as obtained in the fluid theory, so that the fluid and kinetic approaches yield identical eigenvalue problems for this case. We therefore expect the fluid treatment to yield reliable answers. There is, however, a basic difference in the physics. In the fluid model, the perturbed electric field along the zero order current sheet is supported by resistivity, i.e., collisions. There is a resulting perturbation current which drives the perturbed field normal to the current sheet, and transfers energy into Joule heat and plasma kinetic energy. In the collisionless treatment just described, the electric field is supported by the inertia of the current carriers. Now, however, kinetic energy flows from the current carriers into the perturbed magnetic field, thus driving the instability.

There have been two recent attempts to deal with the tearing mode across the entire range of collisionality. Hazeltine, Dobrott, and Wang [48] used a variational solution of the guiding center kinetic equation with Fokker-Plank collision term to obtain a form of Ohms law which is uniformly valid for any ratio of wave frequency to collision frequency. However, this theory requires that the particle gyroradius be negligible, and this may clearly be violated in the tearing mode. Drake and Lee [13] dispensed with the guiding center approximation and obtained growth rates which are valid in sheared magnetic fields for all degrees of collisionality

regardless of the size of the gyroradius. The basic physics of this model will now be described.

Let the current sheet lie in the y-z plane, so that the transverse coordinate is x, and let the singular surface be at $x = 0$. Also, let there be a large z-component of the magnetic field (parallel to the current sheet) which is non-vanishing at the singular surface. Then considering perturbations of the form

$$f_1(x, y, t) = f_1(x) e^{i k_y y + i \omega t} \quad (B8)$$

we have

$$B_{x1} = i k_y A_{z1} \quad (B9a)$$

$$B_{y1} = - \frac{\partial A_{z1}}{\partial x} \quad (B9b)$$

Where A_1 is the perturbed vector potential. Then from Faraday's law, there is an electric field parallel to the current sheet

$$E_{z1} = - \frac{\omega}{c} A_{z1} \quad (B10)$$

which has a component parallel to the zero order field B_0

$$E_{\parallel 1} = E_{z1} \cos \theta(x) \quad (B11)$$

where $B_0 = B_{y0}(x) \hat{j} + B_{z0}(x) \hat{k}$ and

$$\theta(x) = \tan^{-1} \frac{B_{y0}(x)}{B_{z0}(x)}.$$

This parallel electric field will drive a large electron

current along $\underline{B}_0$ unless something prevents it, and we will find that the width of the singular layer about $x = 0$ (i.e., $\underline{k} \cdot \underline{B}_0 = 0$) is determined by the region in which this current can flow.

The current can be inhibited by either of two mechanisms. First, consider the electrons flowing along $\underline{B}_0$ but slightly away from the singular surface (i.e., $\theta(x) \neq 0$). There will then be a component of the wave vector $\underline{k}$ parallel to $\underline{B}_0$, which will cause $E_{||}$ to vary spatially along $\underline{B}_0$ (i.e., it will behave like $e^{ik_{||} s}$, where s measures displacement along $\underline{B}_0$). If the electrons flow along the field with effective velocity v_{eff} , they will feel a time dependent electric field with frequency equal to the Doppler frequency, i.e.,

$$\omega_d \approx k_{||} v_{eff} \quad (B12)$$

The electrons will thus feel an effective AC field which can limit their response, i.e., $J_{||}$ may be greatly reduced. This can happen if the electrons cannot be appreciably accelerated in one period of the time dependent (in the electron frame) electric field, so that the width of the layer will be determined by the value of x at which

$$\omega_d = k_{||} v_{eff} \approx \omega \quad (B13)$$

Secondly, one result of electron motion along $\underline{B}_0$ may be charge separation which could set up an electrostatic field effectively cancelling the induced field, i.e.

$$E_{\parallel \text{TOT}} = E_{\parallel \text{INDUCED}} + E_{\parallel \text{ELECTROSTATIC}}$$

$$= -\frac{\omega}{c} A_{z1} \cos \theta - ik_{\parallel} \phi_1 \quad (\text{B14})$$

where ϕ_1 is the potential due to charge separation. In this case the width of the singular layer is determined by that value of x at which $E_{\parallel \text{TOT}} \approx 0$.

Note that both the Doppler frequency and the electrostatic field increase with $k_{\parallel}$. We conclude that unless $k_{\parallel} \approx 0$, one of the criteria must be satisfied, thus restricting the region in which the instability can take place. From the geometry of the situation

$$k_{\parallel} = \frac{k_y B_{y0}}{B_0} \approx \frac{k_y x}{\ell_s} \quad (\text{B15})$$

where we have used $B_{y0} = B_0 x / \ell_s$ near $x = 0$. Then

$$x \approx \frac{\ell_s k_{\parallel}}{k_y} \quad (\text{B16})$$

and since $k_{\parallel} \approx 0$ within the singular region, and $\ell_s / k_y \sim 1$, we see that the width of the layer of perturbed current is quite small.

The perturbed field B_{y1} must suffer a change as we cross this narrow current layer. We estimate this change by integrating Ampere's law across the layer. The result is

$$B_{y1}(x) \Big|_{-\Delta}^{\Delta} \approx \frac{4\pi}{c} J_{z1}(0) \Delta \quad (\text{B17})$$

where Δ is the width of the layer. Then, using (B9b), we find that there is a discontinuity in the slope of the perturbed

z-component of magnetic potential of

$$\left. \frac{\partial A_{z1}}{\partial x} \right|_{-\Delta}^{\Delta} \approx - \frac{4\pi}{c} J_{z1}(0) \Delta \quad (B18)$$

We now assume that A_{z1} is constant across the singular layer. This is equivalent to the "constant ψ " approximation of FKR, which is described in Appendix A. Then we can measure the discontinuity in the slope of the vector potential as

$$\Delta' = \frac{1}{A_{z1}(0)} \left. \frac{\partial A_{z1}}{\partial x} \right|_{-\Delta}^{\Delta} \quad (B19)$$

and we note that Δ' scales as $1/a$, where "a" is the half-width of the current sheet. Then (B18) becomes

$$\Delta' A_{z1}(0) = - \frac{4\pi}{c} J_{z1}(0) \Delta \quad (B20)$$

which yields the dispersion relation once the term $J_{z1}(0)\Delta$ is related independently to $A_{z1}(0)$, ω , and k .

We now proceed to consider a range of collisionality. First, let the growth rate be much greater than the collision frequency, i.e.,

$$\omega \gg \nu_z \quad (B21)$$

This defines the collisionless regime. The z-component of the electron equation of motion is

$$m \frac{dv_{ez1}}{dt} = - eE_{z1}$$

or, using (B8) and (B10)

$$v_{ez1} = \frac{eA_{z1}}{mc} \quad (B22)$$

Then the perturbed current is related to the vector potential by

$$J_{z1} = - \frac{n_0 e^2 A_{z1}}{mc} \quad (B23)$$

We get Δ by using (B12), (B15), and setting $v_{\text{eff}} = v_{\text{the}}$, the electron thermal speed. The result is

$$\Delta = \frac{\omega l_s}{k_y v_{\text{the}}} \quad (B24)$$

Then using (B23) and (B24) in (B20), we find the collisionless growth rate to be

$$\omega_k = \frac{(\Delta' k_y a^2)}{(a/\delta)^2} \left(\frac{v_{\text{the}}}{l_s} \right) \quad (B25)$$

where $\delta = c/\omega_{pe}$ is the collisionless skin depth, and $\omega_{pe}^2 = 4\pi n_0 e^2/m$ is the square of the electron plasma frequency. Recalling that Δ' and k_y scale like $1/a$, we have the estimate

$$\omega_k = \left(\frac{v_{\text{the}}}{l_s} \right) \left(\frac{\delta}{a} \right)^2 \quad (B26)$$

which, of course, is independent of resistivity (collision frequency). The width of the singular layer is now obtained from (B24). We find

$$\Delta_k = \Delta' \delta^2 \approx \left(\frac{\delta}{a} \right) \delta \ll \delta \quad (B27)$$

so that the collisionless tearing layer is much smaller than the collisionless skin depth. For many cases the ion gyroradius exceeds the skin depth; then $\Delta_k < \rho_i$ also.

Next, consider the case when the growth rate is less than the collision frequency (but not much less), i.e.,

$$\omega \approx v_c \quad . \quad (B28)$$

In this semi-collisional regime the response of electrons is limited by collisions with ions, and we can define a resistivity. Thus, from Ohm's law and (B9)

$$J_{z1} = \frac{1}{\eta} E_{z1} \approx - \frac{\omega}{\eta c} A_{z1} \quad (B29)$$

which relates J_{z1} and A_{z1} .

Instead of streaming along the magnetic field at their normal velocities, the electrons are effectively diffusing along B_0 with diffusion coefficient

$$D = \frac{v_{the}^2}{v_c} \quad . \quad (B30)$$

We then define the effective Doppler frequency to be

$$\omega_d = k_{\parallel}^2 D = \frac{k_{\parallel}^2 v_{the}^2}{v_c} \quad . \quad (B31)$$

In this regime, as we move away from the singular surface, the Doppler frequency becomes much greater than the growth rate before electrostatic effects become important. Equating these two quantities and using (B15), we find that Δ and ω are related by

$$\Delta = (\omega v_c)^{1/2} \frac{k_s}{k_y v_{the}} \quad . \quad (B32)$$

Then using (B29) and (B32) in (B20), we find

$$\omega = \omega_k^{2/3} v_c^{1/3} \quad . \quad (B33)$$

To get this result, we have used the formula

$$\eta = \frac{mv_c}{ne^2} \quad (B34)$$

The tearing width is then obtained from (B32). The result is

$$\Delta = \Delta_k \left(\frac{v_c}{\omega_k} \right)^{2/3} \quad (B35)$$

We see that as the collision frequency surpasses the collisionless growth rate, the tearing layer broadens. This is because as collisions become important the effective Doppler frequency of the electrons decreases allowing more particles to receive a DC acceleration.

Finally, in the collisional regime, electron motion along B_0 is greatly inhibited due to the large collision frequency, and the effective Doppler frequency is sharply reduced. We then expect that the parallel electric field will be shorted out due to charge separation.

From the parallel component of the electron equation of motion and equation (B14), we find

$$v_{e\parallel} \approx - \frac{e}{mv_c} \left(- \frac{\omega A z_1}{c} - ik_{\parallel} \phi_1 \right). \quad (B36)$$

The resulting electron density perturbation is obtained from the linearized electron continuity equation

$$\frac{\partial n_{e1}}{\partial t} + v_{e\parallel} \frac{\partial n_{e0}}{\partial x_{\parallel}} = 0,$$

or

$$n_{e1} = \frac{ik_{\parallel} e E_{\parallel} n_{e0}}{m v_c \omega} \quad (B37)$$

Due to this density perturbation, there will be an ion polarization drift with velocity

$$\underline{v}_D = \frac{M}{e} \frac{c^2 \underline{\dot{E}}}{|B_0|^2}$$

where $\underline{E}$ is the electric field generated by the charge separation. The resulting ion density perturbation is found by substituting this expression in the ion continuity equation

$$\frac{\partial n_{i1}}{\partial t} + n_0 \frac{\partial v_D}{\partial x} = 0.$$

Using $E = -\nabla\phi$, we find

$$n_{i1} = \left(\frac{c^2}{4\pi e c_A^2} \right) \frac{\partial^2 \phi_1}{\partial x^2} \quad (B38)$$

where $c_A = B_0 / \sqrt{4\pi n_{i0} M}$ is the Alfvén velocity.

Charge neutrality requires that $n_{e1} \cong n_{i1}$. Then equating (B37) and (B38), we have that

$$\frac{\partial^2 \phi_1}{\partial x^2} = ik_{\parallel} \left(\frac{\Omega_e \Omega_i}{\omega v_c} \right) \left(-\frac{\omega A_{z1}}{c} - ik_{\parallel} \phi_1 \right) \quad (B39)$$

where Ω_e and Ω_i are the electron and ion gyrofrequencies. The width of the singular layer is determined when the right hand side of this equation is dominated by the static field.

Writing $\partial^2 \phi_1 / \partial x^2 \sim \phi_1 / \Delta^2$, we obtain

$$\Delta = \left(\frac{\Omega_e \Omega_i}{\omega v_c} \right)^{-1/2} \left(\frac{g_s}{k_y} \right)^{1/2}. \quad (B40)$$

Then combining (B20), (B29), and (B40) we arrive at the collisional growth rate

$$\omega = v_c^{3/5} \omega_k^{2/5} \left(\frac{\Delta_k}{\rho_i}\right)^{2/5} \quad (\text{B41a})$$

$$\approx \eta^{3/5} \quad (\text{B41b})$$

which is the same scaling obtained by FKR using the fluid model. From (B40), the size of the singular layer is

$$\Delta = \left(\frac{v_c}{\omega_k}\right)^{2/5} \rho_i^{2/5} \Delta_k^{3/5}.$$

We thus see that the kinetic treatment yields results which are consistent with the predictions of fluid theory, and we choose the latter model in our formulation of the problem.

APPENDIX C: FLUX-DIVERGENCE FORM OF MAXWELL'S EQUATIONS AND OHM'S LAW

We derive the flux divergence form of the pseudo-vector conservation law

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B} - \frac{\eta}{S} \nabla \times \underline{B}) \quad (C1)$$

which arises from the combination of Maxwell's equations and Ohm's law (see Chapter II).

We expand the operation $\nabla \times (\underline{v} \times \underline{B})$ by using the vector identity

$$\underline{A} \times (\underline{B} \times \underline{C}) = \underline{B} \underline{A} \cdot \underline{C} - \underline{C} \underline{A} \cdot \underline{B} \quad (C2)$$

and treating ∇ as a vector. The result is

$$\nabla \times (\underline{v} \times \underline{B}) = \overset{\curvearrowright}{\underline{v} \nabla \cdot \underline{B}} - \overset{\curvearrowright}{\underline{B} \nabla \cdot \underline{v}}$$

where the arrows are to remind us that ∇ must operate on both vectors in each term [92]. Thus

$$\nabla \times (\underline{v} \times \underline{B}) = \underline{v} \nabla \cdot \underline{B} + \underline{B} \cdot \nabla \underline{v} - \underline{B} \nabla \cdot \underline{v} - \underline{v} \cdot \nabla \underline{B} \quad (C3)$$

or

$$\nabla \times (\underline{v} \times \underline{B}) = \nabla \cdot (\underline{B} \underline{v} - \underline{v} \underline{B}) \quad (C4)$$

When we use $\nabla \cdot \underline{B} = 0$, the second term on the right hand side of equation (C1) becomes

$$\nabla \times (\eta \nabla \times \underline{B}) = - \eta \nabla^2 \underline{B} + \nabla \eta \times (\nabla \times \underline{B}) \quad (C5)$$

We expand the second term on the right hand side of (C5) in Cartesian tensor form, using (C2) and our "arrow" notation.

We have

$$\begin{aligned} (\nabla \eta \times \nabla \times \underline{B})_j &= \overbrace{\partial_j (\partial_i \eta) B_i} - \overbrace{B_j (\partial_i \eta) \partial_i} \\ &= (\partial_i \eta) (\partial_j B_i - \partial_i B_j) \end{aligned} \quad (C6)$$

where $\partial_i \equiv \partial/\partial x_i$, and we have invoked the summation convention.

A straightforward calculation yields

$$\partial_i (\eta \partial_j B_i - \eta \partial_i B_j) = (\partial_i \eta) (\partial_j B_i - \partial_i B_j) - \eta \partial_i \partial_i B_j \quad (C7)$$

Then transforming back to vector notation, and using (C7) in (C6), we arrive at

$$\nabla \eta \times (\nabla \times \underline{B}) = -\nabla \cdot [\eta (\nabla \underline{B} - \nabla \underline{B}^{\dagger})] + \eta \nabla^2 \underline{B} \quad (C8)$$

where $(\)^{\dagger}$ denotes the transpose of a tensor, i.e., $(A_{ij})^{\dagger} = A_{ji}$.

Equation (C5) then becomes

$$\nabla \times (\eta \nabla \times \underline{B}) = -\nabla \cdot [\eta (\nabla \underline{B} - \nabla \underline{B}^{\dagger})] \quad (C9)$$

so that equation (C1) can be written as

$$\frac{\partial \underline{B}}{\partial t} = \nabla \cdot [\underline{B} \underline{v} - \underline{v} \underline{B} + \frac{\eta}{S} (\nabla \underline{B} - \nabla \underline{B}^{\dagger})] \quad (C10)$$

in agreement with equation (2.2.17).