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Informal Research and
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GAMP-8842

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SYNTHESIS METHODS IN HEAT CONDUCTION

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U. S. Atomic Energy Commission
Contract No. AT(04-3)-167,
Project Agreement 17
Project No. 317.1801

20 January 1969

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ABSTRACT

This report presents the results of a study on the application of synthesis methods in heat conduction. The synthesis technique, which is a form of the integral methods used to obtain approximate solutions to differential equations, was developed with the objective of reducing computer time for calculating three-dimensional neutron flux distributions in reactor cores. In this regard it has been successful.

The essence of synthesis in obtaining, for example, the solution $T(x,y,z)$ of a differential equation is to approximate $T(x,y,z)$ as

$$T(x,y,z) \approx \sum_{i=1}^n H_i(x,y) Z_i(z)$$

where the number of terms n in the expansion is arbitrarily chosen. The functions $H_i(x,y)$ are determined by separate solutions of the two-dimensional differential equation in various z -regions and become input information for the assumed solution. The solutions $Z_i(z)$ are obtained by substituting the assumed form of $T(x,y,z)$ into the complete differential equation and performing weighted averages of this equation over the x,y plane. The result is a set of ordinary differential equations and/or algebraic equations to determine the $Z_i(z)$ functions, thus completing the solution.

To evaluate the potential use of the synthesis method in the area of heat conduction, the method was applied to a simple steady-state heat transfer problem and a simple transient problem. The numerical results for these specific problems are presented in Figs. 4 through 7, where the synthesized solutions are compared with finite difference solutions. For the steady state problem, the synthesized temperature distribution at internal points and along the homogeneous boundary agrees with the finite difference solution within 15%. At the nonhomogeneous boundary, the

disagreement between the two solutions is not as good. This disagreement results because the assumed finite expansion for the synthesized solution cannot match the imposed temperature distribution along the boundary. This difficulty with the nonhomogeneous boundary conditions has not occurred in the synthesis of neutron fluxes, where all the boundary conditions have been homogeneous. It is expected, however, that this difficulty can be eliminated or the inaccuracies they produce can be reduced, if the number of terms in the synthesized expansion is increased from the two used in the present analysis to approximately ten, which is commonly used in the neutron flux synthesis.

The application of the synthesis method to the transient heat transfer problem indicates that it can be used to extrapolate a spatial temperature distribution at time t_0 to future times ($t \gg t_0$) and also to steady state. Needed for the extrapolation are several spatial distributions at times $t_1 < t_2 < \dots t_0$, that could be evaluated by means of a finite difference calculation.

The overall conclusions from this study indicate that synthesis can be used to obtain solutions to heat conduction problems. It appears that it is not superior to a finite difference calculation for all types of problems. However, it may offer advantages for certain specific problems, for example, the extrapolation of transients. The two problems analyzed in this study and the assumed expansions that were used were relatively simple and a complete assessment of the synthesis method in heat conduction cannot be made. To obtain a better evaluation of the method requires the use of larger expansions in the assumed solution and the analysis of more complex problems. These problems should include nonhomogeneous boundary conditions, irregular geometries, geometries with several materials, temperature dependent properties, and radiation heat transfer. Since the synthesized solution is only approximate and its accuracy depends on the number of terms used in the assumed series expansion, future investigations must also consider this aspect.

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NOMENCLATURE

<u>SYMBOL</u>	<u>DEFINITION</u>
A	area, ft^2
a	width of the two-dimensional plate, ft. Also a coefficient in Eq. 2
b	height of the two-dimensional plate, ft
c	constant of integration. Also volumetric specific heat, Btu/hr-ft^3
C	constant defined by Eq. B12
d_i	coefficients in Eq. A16
D	differential operator = $d/\hat{dz}$, or d/dt , dimensionless
D_x, D_y, D_z	neutron diffusion coefficient
D_1, D_2	constants defined by Eqs. B13 and B14
f	function of x,y
h	heat transfer coefficient, $\text{Btu/hr-ft}^2\text{-}^\circ\text{F}$
H_i	two-dimensional temperature distribution, $^\circ\text{F}$
$\hat{H}_i$	nondimensional temperature = $(H_i - T_\infty)/T_\infty$, dimensionless
I_i	integrals defined in Eqs. A5 through A10, dimensionless
k	thermal conductivity, $\text{Btu/hr-ft-}^\circ\text{F}$
$\bar{k}_{ij}$	weighted averaged thermal parameter. Defined by Eq. 11b, $\text{Btu }^\circ\text{F/hr-ft}$
$\bar{K}_{ij}$	weighted averaged quantity. Defined by Eq. 11a, $\text{Btu-}^\circ\text{F/hr-ft}^3$
ℓ	parameter $(a/b)^2(\hat{T}_1 - \hat{T}_2)/\hat{Q}$, dimensionless
L	length of body normal to x,y plane, ft
L_i	differential operators defined by Eqs. A11 through A14, or Eqs. B3 through B6, dimensionless

Nomenclature (cont'd)

<u>SYMBOL</u>	<u>DEFINITION</u>
$\bar{L}$	combined operator. $L = L_4 L_1 - L_2 L_3$
n	limit of series expansion, dimensionless, and coordinate normal to the surface of a body, ft
Nu	Nusselt number = hb/k , dimensionless
q	surface heat transfer, Btu/hr
$\hat{q}$	nondimensional surface heat transfer = $q/ak T_\infty$, dimensionless
$\dot{Q}'''$	heat generation rate per unit volume, Btu/hr-ft ³
$\hat{Q}$	nondimensional heat generation rate = $\dot{Q}''' a^2 / k T_\infty$, dimensionless
$\bar{Q}_j$	weighted average heat generation parameter. Defined by Eq. 11c, Btu-°F/hr-ft ³
r_1, r_2	roots of Eq. 26, hr ⁻¹
S_p	neutron source coefficient - ft ⁻¹
t	time, minutes
t_f	indicates the time up to which the transient is computed by finite difference techniques, minutes
t_1	a time during the transient and less than t_f where the finite difference calculated spatial temperature distribution is set to $H_1(x,y)$, minutes
t_2	a time during the transient and less than t_f where the finite difference calculated spatial temperature distribution is set equal to $H_2(x,y)$, minutes
T	temperature, °F
$\hat{T}$	dimensionless temperature = $(T - T_\infty)/T_\infty$
W	weighting function
x	distance along x-axis, ft
X	function of x

Nomenclature (cont'd)

<u>SYMBOL</u>	<u>DEFINITION</u>
$\hat{x}$	dimensionless distance = x/a
y	distance along y-axis, ft
Y	function of y
z	distance along z-axis, ft
Z_1, Z_2	undetermined functions of z in steady-state synthesis, dimensionless
$\hat{z}$	dimensionless distance = z/b

Greek Symbols

α_{ij}	weighted averaged quantity. Defined by Eq. 24a, $\text{Btu-}^\circ\text{F/hr-ft}^3$
β_{ij}	weighted averaged quantity. Defined by Eq. 24b, $\text{Btu-}^\circ\text{F/ft}^3$
γ_i	weighted averaged quantity. Defined by Eq. 24c, $\text{Btu-}^\circ\text{F/hr-ft}^3$
θ_1, θ_2	undetermined functions of t in transient synthesis, dimensionless
ϕ	neutron flux, $\text{neutrons/ft}^2 \text{ sec}$
ϵ_a	neutron absorption coefficient - ft^{-1}

Superscripts

Bar ($\bar{}$)	weighted averaged quantity
Cap ($\hat{}$)	dimensionless quantity
Prime ($\prime$)	indicates a derivative with respect to distance, or a quantity per unit length

Subscripts

i	running index, $1, 2, \dots, n$
j	running index, $1, 2, \dots, n$
0	upper limit on a dimension
∞	refers to the ambient

ACKNOWLEDGEMENT

The author wishes to thank Dr. Robert Katz for his effort in initiating this research program and for the helpful suggestions offered throughout the investigation.

1. INTRODUCTION

In the analysis of conduction heat transfer problems, one of the most fruitful techniques is the use of a "code" or computer program based on the finite difference solution of the governing differential equation. This technique is very powerful since it can include variations in material properties and heat generation rates, internal coolants and voids, and many types of boundary conditions. As such, much work has gone into developing elaborate heat conduction codes, and they have found extensive use in industry.

In some instances, however, the heat transfer problem requires the use of many node points or long transient times to obtain a valid solution. The use of a computer program in these instances becomes quite costly. For this reason there have been attempts to develop approximate solutions which reduce the number of computations required for the solution. Among these are the integral and variational approaches. Here the form of the solution is assumed for at least one of the independent variables, thus reducing the number of independent variables for which a solution must be sought.

The technique of flux synthesis is a specific form of the general integral approach. It was developed with the objective of reducing the computer time for calculating three-dimensional neutron flux distributions in reactor cores, and it has been successful in this regard. Since the differential equation which governs heat conduction in a solid is similar to that which governs neutron diffusion, the same technique of flux synthesis was applied to problems in heat conduction.

This report describes briefly integral techniques in the general sense and then specifically the approach of flux synthesis in neutronics. With this background the application of synthesis to a steady state and a transient heat conduction problem is presented. The significant findings of this effort are summarized at the end of the report.

2. INTEGRAL TECHNIQUES

2.1 GENERAL INTEGRAL TECHNIQUES

One of the most common methods to obtain an approximate solution to a differential equation which describes a complex problem is by using integral techniques. In this approach the form of the solution is assumed for at least one of the independent variables. The complete solution, with part still unknown, is substituted into the governing differential equation, and an integration is performed with respect to the assumed independent variable over the region of interest. The result is a new differential equation containing less independent variables or simply an algebraic equation containing unknown coefficients.

As an example, consider the two-dimensional steady state conduction equation

$$\frac{\partial}{\partial x} \left(k(x,y) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k(x,y) \frac{\partial T}{\partial y} \right) + \dot{Q}'''(x,y) = 0 \quad (1)$$

which is to be solved in the two-dimensional region R of the x,y plane and having appropriate boundary conditions.

In the Ritz method the solution is assumed of the form

$$T(x,y) = \sum_{i=0}^n a_i f_i(x,y) \quad (2)$$

where the functions $f_i(x,y)$ are completely specified and the constant coefficients are unknown. Frequently, $f_i(x,y)$ is taken as a product of two polynomials $X_i(x)$ and $Y_i(y)$. Substituting Eq. 2 into the differential Eq. 1 and integrating over the x,y region yields one algebraic equation to evaluate the unknown coefficients a_0 through a_n .

a_0 through a_n .

Other equations for the unknown coefficients may be obtained by one of the following techniques:

1. Weighted Averages

Before integrating the differential equation over the region of interest, it is multiplied by the weighting function $W_i(x,y)$. By successive multiplication of different weighting functions, the required number of additional equations are obtained. Frequently, the weighting functions $W_i(x,y)$ are taken as the original functions $f_i(x,y)$, which is usually referred to as the Galerkin method.

2. Collocation

The original function, the differential equation, or its higher spatial derivatives are satisfied at a suitable number of discrete points, P_j . For example:

$$\sum_{i=0}^n a_i f_i(x,y)_{P_j} = T_{P_j}$$

or

$$\frac{\partial^m}{\partial x_k^m} \left(\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \dot{Q} \right)_{P_j} = 0$$

where $j = 1, 2, 3, \dots$

$m = 0, 1, 2, \dots$

x_k = appropriate spatial coordinate

3. Subdomain

Instead of performing the initial integration over the entire (x,y) region, separate integrations are performed over subdivisions of the entire region.

A second common way of selecting the assumed profile is the Kantorovich method or semi-direct method. Again a series solution is usually assumed. However, the unknown coefficients are now functions of one of the independent variables; namely,

$$T(x,y) = \sum_{i=0}^n \gamma_i(x) Y_i(y) \quad (3)$$

where $Y_i(y)$ is specified. Substituting Eq. 3 into the original differential equation and integrating with respect to y yields a differential equation for $\gamma_i(x)$. Additional differential equations may be obtained by either the weighted average or subdomain methods.

The integral approach using a first order Kantorovich profile (one unknown function) has long been used in boundary layer theory in convective heat transfer. Known as the von Karman-Polhausen integral technique, the velocity and temperature profiles are generally assumed of the form

$$u = u[y/\delta(x)]$$

$$T = T[y/\delta_T(x)]$$

where $\delta(x)$ and $\delta_T(x)$, the momentum and thermal boundary layer thicknesses, are the unknown functions or parameters. Integral techniques have been used to a lesser extent in conduction heat transfer problems. Most frequently these have been transient or propagation problems, where the single unknown parameter $\delta(t)$ represents the distance to which the transient has propagated. (1,3,4)

2.2 VARIATIONAL TECHNIQUES

The variational approach also falls in the category of approximate integral techniques, where a Ritz or Kantorovich profile may be assumed. However, the assumed solution is not substituted directly into the governing

differential equation and the integration performed in the usual manner. Rather, the governing differential equation is converted to a functional which is to be extremized by means of variational calculus. This functional involves an integration of the assumed solution over the region of interest. Finding the extremum of this functional generally yields a better fit of the assumed solution to the true solution of the original differential equation as compared with a straight integral approach.

3. SYNTHESIS METHODS IN REACTOR ANALYSES

An excellent survey of synthesis methods as applied to neutron diffusion is presented by Kaplan.⁽⁸⁾ Some of these developments are presented here for continuity in the application of the method to the area of heat transfer.

3.1 MOTIVATION FOR THE ORIGINAL METHOD OF FLUX SYNTHESIS

For illustration consider the following one-group, three-dimensional, neutron diffusion equation and associated homogeneous boundary conditions:

$$\frac{\partial}{\partial x} \left(D_x(x,y,z) \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_y(x,y,z) \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(D_z(x,y,z) \frac{\partial \phi}{\partial z} \right) \quad (4)$$

$$- \sum_a \phi + S_p \phi = 0$$

ϕ or $\partial \phi / \partial n = 0$ at the boundaries.

Since the diffusion coefficients D_x , D_y and D_z depend on the three space coordinates, the solution to this equation is not separable as a product solution. In many reactor cores the variation of the internal geometry and materials is generally slow in the vertical or z -direction. Herein lies the motivation for the original method of flux synthesis. By dividing the reactor core into several horizontal regions, it is assumed that the coefficients of the neutron diffusion equation do not vary with z within each region. The solution for $\phi(x,y,z)$ is then expressed as the product solution

$$\phi(x,y,z) \approx H_i(x,y) Z(z) \quad (5)$$

where $H_i(x,y)$ is the two-dimensional solution for the i^{th} horizontal region. See Fig. 1

The original problem is simplified to the extent that the three-dimensional solution is replaced by obtaining several $H_i(x,y)$ two-dimensional solutions and the single $Z(z)$ solution. When obtaining solutions numerically, the latter procedure usually requires less computer time than the former.

The equation to evaluate $Z(z)$ is obtained by substituting the assumed solution (5), with the known $H_i(x,y)$, into the original differential equation. Since the functions $H_i(x,y)$ differ for different ranges of z , the single differential equation applicable over the entire range of z becomes separated into several differential equations each of which is applicable within a single horizontal region. Each of these equations is integrated over the x,y plane yielding ordinary differential equations involving $Z(z)$ only. These latter equations actually represent one differential equation for $Z(z)$ which applies for the whole range of z .

It is important to note that the solution for $\phi(x,y,z)$, as given by Eq. 5, uses different $H_i(x,y)$ functions as z traverses from one region to another. For this reason $\phi(x_i, y_i, z)$ is inherently discontinuous as z crosses region boundaries (even though $Z(z)$ is continuous).

3.2 NEW METHOD OF FLUX SYNTHESIS

To eliminate the discontinuities in $\phi(x,y,z)$ of the early flux synthesis method, Kaplan⁽⁷⁾ suggests the solution for $\phi(x,y,z)$ be expressed as

$$\phi(x,y,z) \approx \sum_{i=1}^n H_i(x,y) Z_i(z) \quad (6)$$

where n is the number of horizontal regions into which the core is divided and $H_i(x,y)$ is generally taken as the x,y solution for the i^{th} horizontal region. Unlike the previously assumed solution where there is only one $Z(z)$ function, there now exist n of the $Z_i(z)$ functions. Each of these functions apply over the entire range of z . Thus, they become mixing or blending functions, and weigh their corresponding $H_i(x,y)$ functions to some

degree in all horizontal regions. With $\phi(x,y,z)$ given by Eq. 6 the solution is continuous with respect to z even across the boundaries of the horizontal regions.

The construction of the n equations which are necessary to evaluate the n of the $Z_i(z)$ functions may follow one of the three methods stated in Section 2.1; namely, weighted average, collocation and subdomain. Kaplan⁽⁸⁾ reports that the weighted average method with $W_i(x,y) = H_i(x,y)$ yields a good approximation to the true solution.

Kaplan⁽⁷⁾ also used variational techniques to synthesize a solution for a three-dimensional flux distribution. He found that the same governing equations result as those which are obtained when the standard integral technique is used provided that (1) the weighted average method is employed for constructing the n equations for the $Z_i(z)$'s and (2) the weighting functions are taken as $H_i^k(x,y)^{\text{adjoint}}$. (In the neutronics problem the spatial functions are double subscripted and form the elements of a matrix. The first subscript indicates the number of spatial functions used in the series expansion of Eq. 6. The second subscript is a result of evaluating a flux distribution for several neutron energy ranges. $H_i^k(x,y)^{\text{adjoint}}$ is the i,k element of the adjoint matrix).

A comparison of this new method of flux synthesis with the previously discussed integral approaches indicates that it is an integral method whereby a Kantorovich expansion is employed. The assumed portion of the solution is the $H_i(x,y)$ functions. The unknown parameter functions are the $Z_i(z)$ functions - which become the "synthesized" part of the solution.

4. APPLICATION OF SYNTHESIS METHODS TO STEADY STATE CONDUCTION HEAT TRANSFER

4.1 DEVELOPMENT OF THE GENERAL EQUATIONS FOR SPATIAL SYNTHESIS

Since flux synthesis is successful in the neutronics field and since there exists a similarity between the neutron diffusion equation and the conduction heat transfer equation, the technique of synthesis is applied to the thermal conductance problem. In this section the equations are developed for the temperature synthesis in one of the spatial directions for a steady state three-dimensional problem. The region of interest is assumed to be a regular parallelepiped similar to that shown in Fig. 1. It is further assumed that the heat transfer within the solid is by conduction only with directional conductivities $k_x(x,y,z)$, $k_y(x,y,z)$ and $k_z(x,y,z)$; the internal heat generation rate per unit volume is $\dot{Q}'''(x,y,z)$; and the boundary conditions are homogeneous. The governing differential equation and boundary conditions are then*

$$\frac{\partial}{\partial x} \left(k_x(x,y,z) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y(x,y,z) \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z(x,y,z) \frac{\partial T}{\partial z} \right) + \dot{Q}'''(x,y,z) = 0 \quad (7)$$

$$T_{\text{surf}} \text{ or } \left. \frac{\partial T}{\partial n} \right|_{\text{surf}} = 0$$

Synthesizing in the z -direction a solution is assumed of the form

$$T(x,y,z) \approx \sum_{i=1}^n H_i(x,y) Z_i(z) \quad (8)$$

As in the neutron flux synthesis development, the functions $H_i(x,y)$ are known and each satisfies the homogeneous boundary conditions. Generally, these functions are taken as the two-dimensional solutions for certain

* It is important to note that with directional conductivities k_x, k_y, k_z , the conduction equation has the form (7) only if the axes x, y, z are the principle axes. See Carslaw and Jaeger⁽⁶⁾ pp. 38-49.

z-regions. That is, for $z_a < z < z_b$ it is assumed that there exists no variation of properties or temperature with respect to z and $H_i(x,y)$ satisfies the differential equation and boundary conditions

$$\frac{\partial}{\partial x} \left(k_x(x,y) \frac{\partial H_i}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y(x,y) \frac{\partial H_i}{\partial y} \right) + \dot{Q}'''(x,y) = 0 \quad (9)$$

$$H_i \Big|_{\text{surf}} \quad \text{or} \quad \frac{\partial H_i}{\partial n} \Big|_{\text{surf}}$$

For all but the simplest problems the solutions for the $H_i(x,y)$'s will be obtained numerically by finite difference methods. The totality of the $H_i(x,y)$'s may include other functions that, a priori, would yield a good final temperature distribution. Here Kaplan^(7,8) suggests functions which bracket the expected temperature distribution.

When the $H_i(x,y)$ functions are chosen to satisfy the two-dimensional conduction equation, they represent the temperature within the solid. The $Z_i(z)$ functions are then merely dimensionless quantities which alter this temperature level.

Using the weighted average method to construct the equations to evaluate the n $Z_i(z)$ functions, one proceeds as follows. The assumed form (8) with known $H_i(x,y)$'s is substituted into Eq. 9, multiplied by $W(x,y)$, and averaged over the x,y plane. For $W(x,y) = W_j(x,y)$ the resulting differential equation is

$$\sum_{i=1}^n \left[\frac{d}{dz} \left(\bar{k}_{ij}(z) \frac{dZ_i}{dz} \right) + \bar{k}_{ij}(z) Z_i(z) \right] + \bar{Q}_j(z) = 0 \quad (10)$$

where

$$\bar{k}_{ij}(z) = \frac{1}{A_{x,y}} \int_{x,y} W_j(x,y) \left[\frac{\partial}{\partial x} \left(k_x(x,y,z) \frac{\partial H_i}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y(x,y,z) \frac{\partial H_i}{\partial y} \right) \right] dx dy \quad (11a)$$

$$\bar{k}_{ij}(z) = \frac{1}{A_{x,y}} \int_{x,y} W_j(x,y) k_z(x,y,z) H_i(x,y) dx dy \quad (11b)$$

$$\bar{Q}_j(z) = \frac{1}{A_{x,y}} \int_{x,y} W_j(x,y) \dot{Q}''(x,y,z) dx dy . \quad (11c)$$

(The derivation of Eq. 10 and other significant equations are given in the Appendix). Repeating this procedure for n different $W_j(x,y)$ functions yields a set of n, linear, second order, ordinary differential equations for the n $Z_i(z)$ functions.

To solve this set of differential equations for the $Z_i(z)$'s, requires evaluating the quantities $\bar{K}_{ij}(z)$, $\bar{k}_{ij}(z)$ and $\bar{Q}_j(z)$. Again, it is most likely that the integration involved would be done numerically. If the thermal conductivities or the heat generation depend on z, it is necessary to divide the z-direction into several segments over which these quantities are assumed to be independent of z. Values of $\bar{K}_{ij}$, $\bar{k}_{ij}$ and $\bar{Q}_j$ are then computed for each segment. Note that the division of the z-direction into segments at this point need not correspond to the sub-divisions for which the $H_i(x,y)$'s are evaluated.

It is important to mention that in Eq. 11a the term within the square brackets for $\bar{K}_{ij}$ equals $-\bar{Q}(x,y)$ when the function $H_i(x,y)$ satisfies the two-dimensional conduction Eq. 9.

In the formal solution of the set of the n differential equations for the $Z_i(z)$ functions. there exists 2n arbitrary constants (each equation is of second order). These constants are specified by satisfying the homogeneous boundary conditions at $z=0$ and $z=z_0$ for each of the $Z_i(z)$ functions.

4.2 DISCUSSION OF THE SYNTHESIS METHOD FOR GENERAL APPLICATIONS IN SPATIAL SYNTHESIS

The application of the above procedure to heat transfer problems in general leads to some difficulties. Boundary conditions present one of the

major difficulties. In the neutron flux problems the boundary conditions are usually homogeneous. Each $H_i(x,y)$ and $Z_i(z)$ can satisfy the homogeneous boundary conditions, and the synthesized solution will also satisfy these boundary conditions. In heat transfer problems the boundary conditions are frequently nonhomogeneous. If each $H_i(x,y)$ is made to satisfy certain nonhomogeneous conditions along the x,y boundaries, the synthesized solution according to Eq. 8 will generally not satisfy the same boundary conditions. In some instances the boundary conditions can be made homogeneous by a proper referencing of the temperature. In cases where there are several boundaries at different temperatures or one boundary with an arbitrary temperature distribution this is not possible. In these cases the nonhomogeneous boundary conditions along the x,y boundaries can partially be satisfied by using collocation. That is, at certain discrete points (x_o, y_o) the boundary conditions are satisfied. The more points that are employed, the more exact is the matching of the complete boundary condition.

Nonhomogeneous boundary conditions for the z boundaries can be satisfied in the following way if the temperature is specified along the boundary; namely, $T(x,y,z_o) = f(x,y)$. One of the $H_i(x,y)$ functions of the assumed solution (Eq. 6) is taken as $f(x,y)$. For example, $H_k(x,y) = f(x,y)$. By requiring that $Z_k(z_o) = 1.0$ and $Z_i(z_o) = 0$ for $i \neq k$, the synthesized solution then satisfies identically the boundary condition along $z=z_o$. If the boundary condition has a specified heat flux or heat transfer coefficient, the temperature distribution along the boundary is not known and this technique cannot be used. However, the results of the problem analyzed in Section 4.3 indicate that the synthesized temperature distributions along the boundary agrees fairly well with the true solution for these latter types of nonhomogeneous boundary conditions.

A second difficulty with the spatial synthesis of temperature occurs when the geometry is other than uniform. For example, consider the solid illustrated in Fig. 2. Since each Z_i extends over the entire range of z ($0 < z < z_o$), the H_i functions which extend to x_o, y_o at $z=0$ will also extend to these values at $z=z_o$. In this latter region the problem does not exist.

Yasinsky and Kaplan⁽⁹⁾ indicate problems with moderately irregular geometries can be handled by using different $H_i(x,y)$ functions as one moves through the different z -regions. In each z -region the appropriate $H_i(x,y)$ functions extend to the x,y boundaries of that region. This technique is similar to that used in the earliest methods of flux synthesis (Section 3.1), and it produces discontinuities at the z -region boundaries. However, in an example presented by Yasinsky and Kaplan, where a total of ten $H_i(x,y)$ functions are used for two z -regions differing in geometry, the discontinuities are nearly imperceptible.

Additional difficulties may also arise in the thermal problem because of the following:

1. Thermal properties that depend on temperature;
2. Internal gaps with radiation heat transfer that depends on the absolute temperature level;
3. Internal coolant channels.

If the $H_i(x,y)$ function satisfies the two-dimensional conduction equation and represents temperature in each of these cases, the quantities depending on the temperature level (thermal conductivity and radiation heat transfer) will have the correct values for the two-dimensional problem. When $H_i(x,y)$ is multiplied by $Z_i(z)$ and the temperature level is altered, the values used may no longer be correct.

4.3 ANALYSIS OF A STEADY-STATE HEAT TRANSFER PROBLEM

4.3.1 Development of Equations

The problem selected for analysis was a two-dimensional heat conduction problem with constant properties and uniform internal heat generation. The problem selected was kept simple so that the synthesized solution could be obtained analytically. That is, the $H_i(x)$'s and $Z_i(z)$'s were obtained by analytical integration of the governing differential equations and not numerical integration. Figure 3 illustrates the problem analyzed. The boundary conditions were intentionally mixed to evaluate various aspects of the method.

For the problem illustrated, the differential equation and boundary conditions are

$$k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial z^2} + \dot{Q}''' = 0 \quad (12)$$

$$T(x, 0) = T_\infty \quad (13a)$$

$$-k \frac{\partial T(x, b)}{\partial z} = h [T(x, b) - T_\infty] \quad (13b)$$

$$\frac{\partial T(0, z)}{\partial x} = 0 \quad (13c)$$

$$T(0, 0 < z < b/2) = T_2 \quad (13d)$$

$$T(0, b/2 < z < b) = T_1 \quad (13e)$$

Synthesizing in the z -direction, a solution in nondimensional quantities is assumed of the form

$$\hat{T}(\hat{x}, \hat{z}) = \hat{H}_1(\hat{x}) Z_1(\hat{z}) + \hat{H}_2(\hat{x}) Z_2(\hat{z}) \quad (14)$$

where

$$\hat{T} = (T - T_\infty)/T_\infty$$

$$\hat{H}_i = (H_i - T_\infty)/T_\infty$$

$$\hat{x} = x/a$$

$$\hat{z} = z/a$$

The solutions $\hat{H}_1(\hat{x})$ and $\hat{H}_2(\hat{x})$ are chosen to satisfy the one-dimensional conduction equation $d^2 H/dx^2 + \dot{Q}'''/k = 0$ and the boundary conditions in Regions 1 and 2, respectively. These solutions are

$$\hat{H}_1(\hat{x}) = \frac{\hat{Q}}{2} (1 - \hat{x}^2) + \hat{T}_1 \quad (15a)$$

$$\hat{H}_2(\hat{x}) = \frac{\hat{Q}}{2} (1 - \hat{x}^2) + \hat{T}_2 \quad (15b)$$

where $\hat{Q} = \dot{Q}''' a^2 / kT_\infty$

Since the boundary conditions for both $\hat{H}_1$ and $\hat{H}_2$ at $\hat{x} = 0$ are homogeneous ($\hat{H}_1'(0) = \hat{H}_2'(0) = 0$), the final solution for $T(x,z)$ also satisfies this homogeneous boundary condition. This is not the case at $\hat{x} = 1$. Here the boundary condition is satisfied only at two discrete points.

The construction of the two differential equations to evaluate Z_1 and Z_2 utilizes the weighted average method (Section 2.1) with the weighting functions chosen as $H_1(\hat{x})$ and $H_2(\hat{x})$. The solutions for Z_1 and Z_2 are:

$$Z_1 = 1 + c_1 + c_2 z + c_3 z^2 + c_4 z^3 \quad (16a)$$

$$Z_2 = -c_1 + 2(a/b)^2 (\hat{T}_1 - \hat{T}_2) c_3 / \hat{Q} \\ + \left[-c_2 + 6(a/b)^2 (\hat{T}_1 - \hat{T}_2) c_4 / \hat{Q} \right] z - c_3 z^2 - c_4 z^3 \quad (16b)$$

where c_1 through c_4 are the four arbitrary constants resulting from the general solution of the differential equations. The equations to specify these four constants are obtained by partially satisfying the nonhomogeneous boundary conditions as indicated:

1. Satisfy $T = T_\infty$ at $(x,z = 0)$ on the average;

$$\frac{1}{a} \int_0^a T(x,0) dx = T_\infty \quad (17a)$$

2. Satisfy $-k\partial T/\partial z = h(T - T_\infty)$ at $(x, z=b)$ on the average;

$$\frac{1}{a} \int_0^a \left[-k \frac{\partial T(x, b)}{\partial z} \right] dx = \frac{1}{a} \int_0^a h \left[T(x, b) - T_\infty \right] dx \quad (17b)$$

3. Satisfy $T = T_1$ at $x = a$ and the point $z = 3b/4$;

$$T(a, 3b/4) = T_1 \quad (17c)$$

4. Satisfy $T = T_2$ at $x = a$ and the point $z = b/4$;

$$T(a, b/4) = T_2. \quad (17d)$$

Note that the method suggested in Section 4.2 to satisfy temperature-specified boundary conditions along the z -boundaries is not used.

4.3.2 Numerical Results

In terms of nondimensional quantities, the temperature distribution for the preceding problem can be expressed functionally as

$$\hat{T}(\hat{x}, \hat{z}) = \hat{T}(\hat{x}, \hat{z}, a/b, \hat{T}_1, \hat{T}_2, \hat{Q}, Nu) . \quad (18)$$

Numerical results were obtained for four different cases. In each case the following parameters remained fixed at the values indicated:

$$\hat{T}_1 = 1.0; \quad \hat{T}_2 = 0.2; \quad Nu = 1.0 .$$

The four cases differ according to the specification of the remaining parameters as follows:

Case 1:	$b/a = 2; \quad \hat{Q} = 1$
Case 2:	$b/a = 2; \quad \hat{Q} = 10$
Case 3:	$b/a = 5; \quad \hat{Q} = 1$
Case 4:	$b/a = 5; \quad \hat{Q} = 10$

The selection of these cases was motivated by trying to make the heat flow in the z -direction comparable to that in the x -direction (Case 1 with low $\hat{Q}$ and low b/a) and this heat flow much less than that in the x -direction (Case 4 with high $\hat{Q}$ and high b/a).

The temperature distributions and surface heat flows from the synthesis method are compared with results from the Gulf General Atomic TAC finite difference heat conduction code. For the finite difference solution, a mesh size of $\Delta x/a = .08333$ and $\Delta z/b = .04166$ was used for the two cases with $b/a = 2$, and a mesh size of $\Delta x/a = .08333$ and $\Delta z/b = .03333$ was used for the two cases with $b/a = 5$. For Case 2 a solution was also obtained using a mesh size of one-half of this stated size. No difference in final temperature distributions was noted, indicating a small enough grid spacing was used.

The comparison between the temperature distributions for the two methods of solution is shown in Figs. 4 and 5 for $T_\infty = 100^\circ\text{F}$, $k = 10 \text{ Btu/hr-ft-}^\circ\text{F}$ and $a = 1 \text{ ft}$. (Only those results for Cases 1 and 4 are presented in these figures since the results for Cases 2 and 3 are similar to Cases 1 and 4, respectively. The results for Cases 2 and 3 are presented in Appendix C). The first set of curves in Figs. 4 and 5 (figure numbers with A) are plots of temperature versus $\hat{z}$ for different values of $\hat{x}$. The second set of curves (figure numbers with B) are plots of temperature versus $\hat{x}$ for different values of $\hat{z}$. As expected (refer to Section 4.2), the agreement between the two methods is not good along the non-homogeneous boundaries $\hat{z} = 0$ and $\hat{x} = 1$. The continuously varying distributions of $Z_1(z)$ and $Z_2(z)$ cannot approach the uniform boundary conditions at these points. As seen in Figs. 4 and 5, however, agreement within 15% is generally obtained along the homogeneous boundary $\hat{x} = 0$ and at internal points away from the edge $\hat{x} = 1$. In particular, at the lines $\hat{z} = 1/4$ and $\hat{z} = 3/4$, where the synthesized solution satisfies exactly the boundary conditions at both ends ($\hat{x} = 0$ and $\hat{x} = 1$) and where the true solution has the same shape as the assumed functions $H_1(\hat{x})$ and $H_2(\hat{x})$, the agreement is excellent (approximately 2%). Agreement within 15% is also obtained along

most of the nonhomogeneous boundary at $\hat{z} = 1$, where the finite, nonzero heat transfer coefficient applies. Along this boundary, the finite difference solution retains the general shape of a one-dimensional profile, unlike the boundaries $\hat{z} = 0$ and $\hat{x} = 1$ where the solution is forced to satisfy the given surface temperature.

A comparison between the surface heat flows is given in Table 1. At the nonhomogeneous boundaries $\hat{z}=0$ and $\hat{x}=1$ where the agreement between the temperature distributions is only fair, the agreement between the surface heat flows is also fair. On the other hand, at $\hat{z}=1$ there is good agreement between the surface heat flows as well as the temperature distributions. The sum of the surface heat flows is given in column 4, and the sum equals the total heat generated, which is given in column 5.

The distribution of Z_1 and Z_2 with respect to $\hat{z}$ is illustrated in Fig. 6 for all four cases.

In addition to using $\hat{H}_1(\hat{x})$ and $\hat{H}_2(\hat{x})$ as given by Eqs. 15a,b, numerical results were also obtained for Case 1 with $\hat{H}_2(\hat{x})$ taken as

$$\hat{H}_2(x) = \frac{\pi\hat{Q}}{2} \cos \frac{\pi\hat{x}}{2} + \hat{T}_2 \quad (19)$$

Equation 19 does not satisfy the one-dimensional conduction equation, but it does possess the shape of a one-dimensional profile and satisfies the boundary conditions at $\hat{x}=0$ and $\hat{x}=1$.

The temperature profiles for this case are presented in Fig. 4A,B and are labeled "Synthesis-2." They possess the same characteristics of the first synthesized solution; however, they do not agree with the numerical solution as well as the profiles of the first synthesized solutions.

Table 1

Surface Heat Flows - Btu/hr(For $a = 1$ ft., $T_{\infty} = 100^{\circ}\text{F}$, and $k = 10$ Btu/hr-ft- $^{\circ}\text{F}$)

	$0 < \hat{x} < 1; \hat{z} = 0$ ($h = \infty$)	$0 < \hat{x} < 1; \hat{z} = 1$ (h finite)	$\hat{x} = 1; 0 < \hat{z} < 1$ ($h = \infty$)	Sum of surface heat flows	Total heat generated
Case 1: Finite difference	1261	453	287	2001	2000
Synthesis	881	459	661	2001	
Case 2: Finite difference	6046	1501	12,451	19,998	20,000
Synthesis	5439	1482	13,080	20,001	
Case 3: Finite difference	1048	237	3717	5002	5000
Synthesis	400	237	4360	4997	
Case 4: Finite difference	5887	771	43,343	50,001	50,000
Synthesis	3443	688	45,800	49,931	

5. APPLICATION OF SYNTHESIS METHODS TO TRANSIENT HEAT TRANSFER PROBLEMS

5.1 INTRODUCTORY COMMENTS

The technique of synthesis can be applied to transient heat transfer problems, whereby the temperature is synthesized with respect to time. In this application the temperature distribution is assumed to be of the form

$$T(\vec{r}, t) \approx \sum_{i=1}^n H_i(\vec{r}) \theta_i(t) \quad (20)$$

where the spatial distributions $H_i(\vec{r})$ are known for some previous calculations and the time dependent functions $\theta_i(t)$ are to be determined. To obtain an accurate synthesized transient using Eq. 20 requires an expansion of more than one term. It does not appear possible that this method could be used to predict an entire transient, since the only spatial distribution available without performing transient calculations is the initial temperature distribution. The technique may be used, however, to extrapolate the spatial temperature distribution to future times ($t \gg t_f$) using several spatial distributions at times $t_1 < t_2 < \dots < t_f$ that have been evaluated from a finite difference calculation. By allowing the variable t in the functions $\theta_i(t)$ to go to infinity, the extrapolation can be extended to yield the steady state solution.

To illustrate the method of time dependent synthesis and to evaluate its application in transient analyses, the technique was applied to extrapolate the transient temperature distribution and to predict the steady state temperature distribution for the problem analyzed in Section 4.3. In the following section a brief development of the significant equations is given, and in Section 5.3 the numerical results are discussed.

5.2 DEVELOPMENT OF THE EQUATIONS FOR TRANSIENT SYNTHESIS

Consider a two-dimensional, transient heat transfer problem satisfying the partial differential equation

$$\frac{\partial}{\partial x} \left(k_x(x,y) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y(x,y) \frac{\partial T}{\partial y} \right) + \dot{Q}'''(x,y,t) = C(x,y) \frac{\partial T}{\partial t} \quad (21)$$

and arbitrary boundary conditions. Synthesizing with respect to the variable t , a second order expansion solution is assumed of the form

$$T(x,y,t) \approx H_1(x,y) \theta_1(t) + H_2(x,y) \theta_2(t) \quad (22)$$

The two spatial distributions $H_1(x,y)$ and $H_2(x,y)$ can be obtained by initially using finite difference techniques to compute the transient to some time t_f . The spatial temperature distributions at times $t_1 < t_2 < t_f$ can then be used as $H_1(x,y)$ and $H_2(x,y)$, respectively. Of course t_1 can be taken as zero, whereby $H_1(x,y)$ becomes the initial temperature distribution.

The evaluation of $\theta_1(t)$ and $\theta_2(t)$ can follow one of several of the methods cited in Section 2.1. Using the method of weighted averages and selecting the weighting functions as $H_1(x,y)$ and $H_2(x,y)$, the resulting differential equations for $\theta_1(t)$ and $\theta_2(t)$ are:

$$\beta_{11} \frac{d\theta_1}{dt} + \beta_{12} \frac{d\theta_2}{dt} - \alpha_{11}\theta_1 - \alpha_{12}\theta_2 = \gamma_1 \quad (23a)$$

$$\beta_{21} \frac{d\theta_1}{dt} + \beta_{22} \frac{d\theta_2}{dt} - \alpha_{21}\theta_1 - \alpha_{22}\theta_2 = \gamma_2 \quad (23b)$$

where

$$\alpha_{ij} = \frac{1}{A_{x,y}} \int_{x,y} H_i(x,y) \left[C(x,y) \frac{\partial H_j}{\partial t} - \dot{Q}'''(x,y,t_j) \right] dx dy \quad (24a)$$

$$\beta_{ij} = \frac{1}{A_{x,y}} \int_{x,y} H_i(x,y) C(x,y) H_j(x,y) dx dy \quad (24b)$$

$$\gamma_i = \frac{1}{A_{x,y}} \int_{x,y} H_i(x,y) \dot{Q}'''(x,y,t_i) dx dy \quad (24c)$$

It is important to note that Eq. 24a is valid only if $H_j(x,y)$ is taken from the early part of the transient. In this case, $H_j(x,y)$ is indeed a spatial distribution. However, it also satisfies Eq. 21, and the spatial derivatives may be replaced by $[C(x,y)\partial H_j/\partial t - \dot{Q}'''(x,y,t)]_{t_j}$ as is done in Eq. 24a.

Solving the set of differential equations (23a,b) yields the following general solutions for θ_1 and θ_2 :

$$\theta_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t} - \frac{\alpha_{22}\gamma_1 - \alpha_{12}\gamma_2}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \quad (25a)$$

$$\theta_2 = -\left[\frac{\beta_{11}r_1 - \alpha_{11}}{\beta_{12}r_1 - \alpha_{12}} \right] c_1 e^{r_1 t} - \left[\frac{\beta_{11}r_2 - \alpha_{11}}{\beta_{12}r_2 - \alpha_{12}} \right] c_2 e^{r_2 t} - \frac{\alpha_{11}\gamma_2 - \alpha_{21}\gamma_1}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \quad (25b)$$

In these equations c_1 and c_2 are arbitrary constants and r_1 and r_2 are the roots of the equation

$$(\beta_{11}\beta_{22} - \beta_{12}\beta_{21})r^2 + (\alpha_{12}\beta_{21} + \alpha_{21}\beta_{12} - \alpha_{11}\beta_{22} - \alpha_{22}\beta_{11})r + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) = 0 \quad (26)$$

An examination of Eqs. 25a,b indicates that both r_1 and r_2 should be real negative numbers since θ_1 and θ_2 have finite asymptotic values as time goes to infinity.

The two equations for evaluating c_1 and c_2 are obtained by specifying "initial conditions." Since the functions $H_1(x,y)$ and $H_2(x,y)$ are taken as the spatial temperature distributions calculated from the finite difference solution at times t_1 and t_2 , respectively, these initial conditions may be one of the following:

1. Require that the averaged synthesized temperature distribution equal the averaged actual temperature distribution at the two times t_1 and t_2 ; namely,

$$\int_{A_{x,y}} T(x,y,t_i) dx dy = \int_{A_{x,y}} H_i(x,y) dx dy \quad i = 1,2$$

2. Require that the averaged synthesized temperature distribution and the average of the derivative of the temperature distribution with respect to time equal the corresponding averages of the actual temperature distribution at either time t_1 or t_2 .
3. Require that the synthesized temperature distribution equals the actual temperature distribution at all points and at either time t_1 or t_2 . This is accomplished at time t_1 by setting $\theta_1(t_1) = 1$ and $\theta_2(t_1) = 0$, and at time t_2 by setting $\theta_1(t_2) = 0$ and $\theta_2(t_2) = 1$. (Refer to Eq. 22).

Combining the solutions for $\theta_1(t)$ and $\theta_2(t)$ with $H_1(x,y)$ and $H_2(x,y)$ according to Eq. 22 gives the complete synthesized solution. By allowing t to approach infinity, the steady-state distribution is obtained.

It is significant to mention that equations were also developed to evaluate $\theta_1(t)$ and $\theta_2(t)$ using variational techniques according to Section 8.11 of Ref. 1. The identical differential equations for θ_1 and θ_2 (Eqs. 23a,b) resulted. The same is also true for an n term expansion provided the weighting function $W_i(x,y)$ is taken as $H_i(x,y)$.

5.3 NUMERICAL RESULTS

The developments of the preceding section were used to extrapolate the transient and to calculate the steady-state temperature distribution for the simple two-dimensional problem analyzed in Section 4.3. The initial temperature distribution was taken uniformly at 100°F.

When using the synthesis method, some latitude exists for the selection of the two times t_1 and t_2 and for the finite difference approximation of the term $\partial H_i(x,y,t)/\partial t$ appearing in the integrand for α_{ij} . For this reason numerical results were evaluated for four cases which differ according to the selection of t_1 and t_2 . In each case the derivative $\partial H_i(x,y,t)/\partial t$ was approximated by the simple difference quotient

$$\frac{H_i(x,y,t_i + \Delta t_i/2) - H_i(x,y,t_i - \Delta t_i/2)}{\Delta t_i} ; i = 1,2$$

The times and time increments for each case are listed in Table 2. In addition, the arbitrary constants c_1 and c_2 of Eqs. 25a,b were evaluated by requiring that the synthesized solution equal the finite difference solution at all points and at time t_2 . Thus, the "initial conditions" become $\theta_1(t_2) = 0$ and $\theta_2(t_2) = 1$.

The transient of four node points from the synthesized and finite difference solutions are compared in Fig. 7 for Case 2 of the steady state synthesis where $b/a = 2$ and $\hat{Q} = 10$. The finite difference solution was calculated using the Gulf General Atomic code TAC. The curves from the synthesized solution are only for Cases 3 and 4 of the transient analysis. The lines with the double arrowhead in Fig. 7 indicate the time span over which the spatial distributions were taken from the finite difference solution to construct $\partial H_i(x,y,t)/\partial t$. The increment marks labeled t_1 and t_2 indicate the times for the selection of $H_1(x,y)$ and $H_2(x,y)$ from the finite difference solution. Since the synthesized solution is matched to the finite difference solution at time t_2 , the two solutions agree identically at this point.

For Cases 1 and 2 the exponential coefficients r_1 and r_2 are complex.* According to Eqs. 25a,b this does not give a physically realistic solution. An examination of the discriminant resulting from Eq. 26 indicates that for all four cases the discriminant is very nearly zero. For the two cases for which a solution is obtained, the discriminant is slightly greater than zero. For the two cases for which no solution is obtained, it is slightly less than zero. The reason for the negative discriminant in Cases 1 and 2 has not been examined, and will have to be treated in future investigations.

A comparison of the synthesized steady-state solutions (Cases 3 and 4) with the finite difference steady-state solution in Fig. 8 shows a maximum deviation of less than 2%.

* Case 2 was rerun by curve fitting a parabola through the temperature of each node at the times $t_i - \Delta t_i/2$, t_i , and $t_i + \Delta t_i/2$. The term $\partial H_i(x,y,t_i)/\partial t$ was then obtained by taking the derivative of the equation of the curve and evaluating it at time t_i . Still complex roots were obtained for r_1 and r_2 .

Table 2

Times t_1 and t_2 and Time Increments Δt_1 and Δt_2
(Minutes)

Case 1:	$t_1 = .4$ $t_2 = .8$	$\Delta t_1 = .4$ $\Delta t_2 = .4$
Case 2:	$t_1 = .8$ $t_2 = 1.5$	$\Delta t_1 = .4$ $\Delta t_2 = 1.0$
Case 3:	$t_1 = 1.5$ $t_2 = 3.0$	$\Delta t_1 = 1.0$ $\Delta t_2 = 2.0$
Case 4:	$t_1 = 3.0$ $t_2 = 6.0$	$\Delta t_1 = 2.0$ $\Delta t_2 = 4.0$

6. SUMMARY OF RESULTS AND CONCLUSIONS

The synthesis method has been developed to a considerable extent in the neutronics field, where it has been used to calculate three-dimensional flux distributions in reactor cores. The technique of synthesis has been examined with the objective of reducing the computation time in heat conduction problems. It has been found that this technique is a specific form of the general integral methods used to obtain approximate solutions to differential equations.

To evaluate the potential use of the synthesis method in the area of heat conduction, the method was applied to a simple steady-state heat transfer problem and a simple transient problem. In the steady-state problem the solution was synthesized in one of the spatial directions. In the non-steady-state problem the transient temperature distribution was synthesized.

For the two problems analyzed, the synthesized temperature distribution agrees quite well with the exact solution in certain instances. In the steady-state problem, there is good agreement at points where the boundary conditions are homogeneous and at points interior to the region. Difficulties are encountered, however, with nonhomogeneous boundary conditions, and agreement is not good at points where such boundary conditions occur. Better agreement is expected if the assumed solution contained more terms in the series expansion of the undetermined functions. For the present analysis only two terms were used; whereas, in the typical neutron flux synthesis ten terms are used. Sometimes heat conduction problems are solved only to find the temperature at a single point or, at least, in a region away from the boundaries. In these cases, the boundary conditions need only be satisfied approximately, and synthesis may be useful even if the boundary conditions are nonhomogeneous.

A second significant difficulty in the synthesis of spatial temperature distributions arises when the geometry is other than uniform. Yasinsky and Kaplan⁽⁹⁾ show that moderately irregular geometries can be handled by using different sets of trial functions in different z -regions.

The application of synthesis to transient heat transfer problems indicates that it can be used to extrapolate transient temperature distributions to future times and to steady state. For the particular transient problem analyzed, good agreement between the synthesized and the true solutions was obtained when the latter portion of the transient curve was synthesized. Difficulties were encountered in trying to get the proper exponential decay with respect to time when the early parts of transient were synthesized. Future efforts will have to investigate this difficulty.

As an overall appraisal, the method of synthesis appears to have application in obtaining approximate solutions to heat conduction problems. It is not certain at this time what its ultimate potential is. To obtain a better evaluation of this potential requires the analysis of more complex problems, both steady-state and transient. The steady-state problems should include nonhomogeneous boundary conditions and possibly internal coolant channels and internal gaps with radiation heat transfer. The series expansion of the unknown functions should consist of more than the two terms presently used. This will enable better matching of the nonhomogeneous boundary conditions and hopefully the prediction of the early part of the transient in nonsteady problems. The solutions to such complex problems will most likely be done numerically with the use of computers. This will require programs to evaluate the integrals for $K_{ij}(z)$, $\bar{k}_{ij}(z)$, α_{ij} , etc., and to solve the set of n simultaneous differential equations for the $Z_i(z)$'s or the $\theta_i(t)$'s. The final appraisal of the synthesis method must include a comparison of cost and machine time with the finite difference approach. This aspect has not been considered at all in this effort.

Fig. 1 - Division of reactor core into several horizontal regions

Fig. 2 - Nonuniform solid with respect to the z -direction

Uniform $\dot{Q}'''$ and k throughout

Fig. 3 - Two-dimensional problem for temperature synthesis

Fig. 4A - Spatial temperature distributions for Case 1: $T = f(\hat{z})$

Fig. 4B - Spatial temperature distribution for Case 1: $T = f(\hat{x})$

Fig. 5A - Spatial temperature distributions for Case 4 $T = f(\hat{z})$

Fig. 5B - Spatial temperature distributions for Case 4: $T = f(\hat{x})$

Fig. 6 - Distributions of z_1 and z_2

Fig. 7 - Transient temperatures of four node points

APPENDIX A: DERIVATION OF THE EQUATIONS FOR THE SYNTHESIS OF SPATIAL
TEMPERATURE DISTRIBUTIONS

A.1 Derivation of Eq. 10

Substituting the assumed temperature distribution

$$T(x,y,z) = \sum_{i=1}^n H_i(x,y) Z_i(z)$$

into the three-dimensional conduction equation (Eq. 9) yields

$$\sum_{i=1}^n \left\{ Z_i(z) \frac{\partial}{\partial x} \left(k_x(x,y,z) \frac{\partial H_i}{\partial x} \right) + Z_i(z) \frac{\partial}{\partial y} \left(k_y(x,y,z) \frac{\partial H_i}{\partial y} \right) + H_i(x,y) \frac{\partial}{\partial z} \left(k_z(x,y,z) \frac{dZ_i}{dz} \right) \right\} + \dot{Q}'''(x,y,z) = 0$$

Multiplying this equation by $W_j(x,y)$ and averaging each term over the x,y plane gives the equation

$$\begin{aligned} \sum_{i=1}^n \left\{ Z_i(z) \frac{1}{A_{x,y}} \int W_j(x,y) \left[\frac{\partial}{\partial x} \left(k_x(x,y,z) \frac{\partial H_i}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y(x,y,z) \frac{\partial H_i}{\partial y} \right) \right] dx dy \right. \\ \left. + \frac{d}{dz} \left[\frac{1}{A_{x,y}} \int W_j(x,y) k_z(x,y,z) H_i(x,y) dx dy \right] \frac{dZ_i}{dz} \right\} \\ + \frac{1}{A_{x,y}} \int W_j(x,y) \dot{Q}'''(x,y,z) dx dy = 0 \end{aligned} \quad (A1)$$

Defining the three integrals respectively as $\bar{K}_{ij}(z)$, $\bar{k}_{ij}(z)$, and $\bar{Q}_j(z)$ produces Eq. 10.

A.2 Construction of the two differential equations for evaluating $Z_1(z)$ and $Z_2(z)$

Combining Eqs. 14 and 15a,b, the assumed temperature profile becomes

$$\hat{T}(\hat{x}, \hat{z}) = \left[\frac{\hat{Q}}{2}(1 - \hat{x}^2) + \hat{T}_1 \right] Z_1(\hat{z}) + \left[\frac{\hat{Q}}{2}(1 - \hat{x}^2) + \hat{T}_2 \right] Z_2(\hat{z}) \quad (A2)$$

Nondimensionalizing Eq. 12 and substituting $\hat{T}(\hat{x}, \hat{z})$ from Eq. A2 yields

$$\begin{aligned} & -\hat{Q}Z_1 - \hat{Q}Z_2 + (a/b)^2 \left[\frac{\hat{Q}}{2}(1 - \hat{x}^2) + \hat{T}_1 \right] Z_1'' \\ & + (a/b)^2 \left[\frac{\hat{Q}}{2}(1 - \hat{x}^2) + \hat{T}_2 \right] Z_2'' + \hat{Q} = 0 \end{aligned} \quad (A3)$$

Multiplying each term by $\hat{H}_1(\hat{x})$ and integrating over $\hat{x}$ gives the first differential equation for $Z_1(z)$ and $Z_2(z)$:

$$-\hat{Q}I_1Z_1 - \hat{Q}I_1Z_2 + (a/b)^2 I_2Z_1'' + (a/b)^2 I_9Z_2'' + I_1\hat{Q} = 0 \quad (A4)$$

where

$$I_1 = \int_0^1 \hat{H}_1(\hat{x}) d\hat{x} = \hat{Q}/3 + \hat{T}_1 \quad (A5)$$

$$I_2 = \int_0^1 \hat{H}_1^2(\hat{x}) d\hat{x} = \frac{2}{15} \hat{Q}^2 + \frac{2}{3} \hat{Q}\hat{T}_1 + \hat{T}_1^2 \quad (A6)$$

$$I_9 = \int_0^1 \hat{H}_1(\hat{x}) \hat{H}_2(\hat{x}) d\hat{x} = \frac{2}{15} \hat{Q}^2 + \frac{\hat{Q}}{3}(\hat{T}_1 + \hat{T}_2) + \hat{T}_1\hat{T}_2 \quad (A7)$$

Multiplying Eq. A3 by $\hat{H}_2(\hat{x})$ and integrating over $\hat{x}$ yields the second differential equation:

$$-\hat{Q}I_5Z_1 - \hat{Q}I_5Z_2 + (a/b)^2 I_9Z_1'' + (a/b)^2 I_6Z_2'' + I_5\hat{Q} = 0 \quad (A8)$$

where

$$I_5 = \int_0^1 \hat{H}_2(\hat{x}) d\hat{x} = \hat{Q}/3 + \hat{T}_2 \quad (A9)$$

$$I_6 = \int_0^1 \hat{H}_2^2(\hat{x}) d\hat{x} = \frac{2}{15} \hat{Q}^2 + \frac{2}{3} \hat{Q}\hat{T}_2 + \hat{T}_2^2 \quad (A10)$$

A.3 Solution of the differential equations for Z_1 and Z_2

The solution of the differential equations, Eq. A4 and Eq. A8 to obtain Eqs. 16a,b follows the procedure outlined in Hildebrand⁽⁵⁾ pp. 21-26.

Defining the operators $D = d/d\hat{z}$ and

$$L_1 = \frac{(a/b)^2}{\hat{Q}} \frac{I_2}{I_1} D^2 - 1 \quad (A11)$$

$$L_2 = \frac{(a/b)^2}{\hat{Q}} \frac{I_9}{I_1} D^2 - 1 \quad (A12)$$

$$L_3 = \frac{(a/b)^2}{\hat{Q}} \frac{I_9}{I_5} D^2 - 1 \quad (A13)$$

$$L_4 = \frac{(a/b)^2}{\hat{Q}} \frac{I_6}{I_5} D^2 - 1 \quad (A14)$$

Equations A4 and A8 are written in the standard form

$$L_1 Z_1 + L_2 Z_2 = -1$$

$$L_3 L_1 + L_4 Z_2 = -1$$

From Eqs. 25a and 25b of Hildebrand⁽⁵⁾ page 21, both Z_1 and Z_2 (now denoted by Z) satisfy the fourth order differential equation

$$(L_4 L_1 - L_2 L_3) Z = 0$$

Substituting for the L operators, this equation becomes

$$\left\{ \frac{(a/b)^4}{\hat{Q}^2 I_1 I_5} (I_2 I_6 - I_9^2) D^4 - \left[\frac{(a/b)^2}{\hat{Q} I_1} (I_2 - I_9) + \frac{(a/b)^2}{\hat{Q} I_5} (I_6 - I_9) \right] \right\} Z = 0$$

After substituting for the integrals I_i from Eqs. A5-A10 and performing some algebraic manipulations, it is found that the term within the square brackets $\left[\right]$ is identically zero. The differential equation then reduces to

$$D^4 Z = 0$$

The solution to this equation is simply

$$Z = c_1 + c_2 z + c_3 z^2 + c_4 z^3$$

and the solutions Z_1 and Z_2 are expressed as

$$Z_1 = c_1 + c_2 z + c_3 z^2 + c_4 z^3 \quad (A15)$$

$$Z_2 = d_1 + d_2 z + d_3 z^2 + d_4 z^3 \quad (A16)$$

Since the original two differential equations are each second order equations, the number of arbitrary constants contained in the general solution is four. Thus, $d_1 \dots d_4$ of Eq. A16 are related to $c_1 \dots c_4$ of Eq. A15. These relationships are obtained by inserting the solutions A15 and A16 into the homogeneous form of one of the differential equations A4 or A8 and equating the coefficients of the powers of z ; i.e., z^0 , z^1 , and z^3 . This procedure yields

$$d_1 = -c_1 + 2(a/b)^2 \frac{(\hat{T}_1 - \hat{T}_2)}{\hat{Q}} c_3$$

$$d_2 = -c_2 + 6(a/b)^2 \frac{(\hat{T}_1 - \hat{T}_2)}{\hat{Q}} c_4$$

$$d_3 = -c_3$$

$$d_4 = -c_4$$

The homogeneous solutions for Z_1 and Z_2 then become

$$Z_{1h} = c_1 + c_2 z + c_3 z^2 + c_4 z^3$$

$$Z_{2h} = -c_1 + 2(a/b)^2 \frac{(\hat{T}_1 - \hat{T}_2)}{\hat{Q}} c_3 + \left[-c_2 + 6(a/b)^2 \frac{(\hat{T}_1 - \hat{T}_2)}{\hat{Q}} c_4 \right] z$$

$$-c_3 z^2 - c_4 z^3$$

Selecting the particular solutions as

$$Z_{1p} = 1$$

$$Z_{2p} = 0$$

yields the complete solutions as indicated by Eqs. 16a,b.

A.4 Equations for the Surface Heat Flows

The synthesized temperature is given as

$$\hat{T}(\hat{x}, \hat{z}) = \hat{H}_1(\hat{x})Z_1(\hat{z}) + \hat{H}_2(\hat{x})Z_2(\hat{z})$$

where:

$$\hat{H}_1(\hat{x}) = \hat{Q}/2 + \hat{T}_1 - (\hat{Q}/2)\hat{x}^2$$

$$\hat{H}_2(\hat{x}) = \hat{Q}/2 + \hat{T}_2 - (\hat{Q}/2)\hat{x}^2$$

$$Z_1 = 1 + c_1 + c_2 z + c_3 z^2 + c_4 z^3$$

$$Z_2 = (2\ell c_3 - c_1) + (6\ell c_4 - c_2)z - c_3 z^2 - c_4 z^3$$

$$\ell = (a/b)^2(\hat{T}_1 - \hat{T}_2)/\hat{Q}$$

A.4.1 Surface Heat Flow for $\hat{Z} = 0$

$$q_1 = -L \int_0^a k \frac{\partial T(x, 0)}{\partial z} dx$$

defining $\hat{q}_1 = q_1/akT_\infty$, one finds

$$\hat{q}_1 = -(a/b)(L/a) \int_0^1 [\hat{H}_1 Z_1'(0) + \hat{H}_2 Z_2'(0)] d\hat{x}$$

Substituting for $\hat{H}_1$ and $\hat{H}_2$ and performing the integration yields

$$\hat{q}_1 = -(a/b)(L/a) \left[(\hat{Q}/3 + \hat{T}_1) c_1 + (\hat{Q}/3 + \hat{T}_2)(6\ell c_4 - c_2) \right]$$

A.4.2 Surface Heat Flow for $\hat{z}=1$

$$q_2 = -L \int_0^a k \frac{\partial T(x, 1)}{\partial z} dx$$

Defining $\hat{q}_2 = q_2/akT_\infty$, yields

$$\hat{q}_2 = -(a/b)(L/a) \int_0^1 \left[\hat{H}_1 Z_1'(1) + \hat{H}_2 Z_2'(1) \right] d\hat{x}$$

and

$$\begin{aligned} \hat{q}_2 = -(a/b)(L/a) & \left[(\hat{Q}/3 + \hat{T}_1)(c_2 + 2c_3 + 3c_4) \right. \\ & \left. + (\hat{Q}/3 + \hat{T}_2)(6\ell c_4 - c_2 - 2c_3 - 3c_4) \right] \end{aligned}$$

A.4.3 Surface Heat Flow for $\hat{x}=1$

$$q_3 = -L \int_0^b k \frac{\partial T(1, z)}{\partial x} dz$$

with $\hat{q}_3 = q_3/akT_\infty$, one finds

$$\hat{q}_3 = -(b/a)(L/a) \int_0^1 (-\hat{Q} Z_1 - \hat{Q} Z_2) d\hat{z}$$

and

$$\hat{q}_3 = (b/a)(L/a) \hat{Q} (1 + 2\ell c_3 + 3\ell c_4)$$

APPENDIX B: DERIVATION OF THE EQUATIONS FOR THE SYNTHESIS OF TRANSIENT TEMPERATURE DISTRIBUTIONS

B.1 Construction of the two differential equations for evaluating $\theta_1(t)$ and $\theta_2(t)$

Substituting the assumed temperature distribution Eq. 22 into the governing differential equation

$$k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + \dot{Q}''' = c \frac{\partial T}{\partial t} \quad (B1)$$

yields the equation

$$k \left(\frac{\partial^2 H_1}{\partial x^2} + \frac{\partial^2 H_1}{\partial y^2} \right) \theta_1 + k \left(\frac{\partial^2 H_2}{\partial x^2} + \frac{\partial^2 H_2}{\partial y^2} \right) \theta_2 + \dot{Q}''' = c H_1 \frac{d\theta_1}{dt} + c H_2 \frac{d\theta_2}{dt} \quad (B2)$$

Utilizing the fact that H_1 and H_2 satisfy separately Eq. B1, the coefficients of θ_1 and θ_2 in Eq. B2 are replaced by the terms $(c\partial H_1/\partial t - \dot{Q}''')$ and $(c\partial H_2/\partial t - \dot{Q}''')$, respectively. Multiplying Eq. B2 successively by H_1 and H_2 and integrating each term over the x,y region yields the two differential equations Eqs. 23a,b.

B.2 Solution of the differential equations for θ_1 and θ_2

The solution of the differential equations Eqs. 23a,b follows the procedure outlined in Hildebrand⁽⁵⁾ pp. 21-26.

Defining the operators

$$D = d/dt$$

$$L_1 = \beta_{11}D - \alpha_{11} \quad (B3)$$

$$L_2 = \beta_{12}D - \alpha_{12} \quad (B4)$$

$$L_3 = \beta_{21}D - \alpha_{21} \quad (B5)$$

$$L_4 = \beta_{22}D - \alpha_{22} \quad (B6)$$

Equations 23a,b are written in the standard form

$$L_1 \theta_1 + L_2 \theta_2 = \gamma_1$$

$$L_3 \theta_1 + L_4 \theta_2 = \gamma_2$$

From Eqs. 25a,b of Hildebrand p. 21, the homogeneous portions of both θ_1 and θ_2 satisfy the second order differential equation

$$\bar{L} \theta_H = (L_4 L_1 - L_2 L_3) \theta_H = 0$$

where $\bar{L}$ is an operator consisting of L_1 through L_4 . Substituting for the operators L , this equation becomes

$$\begin{aligned} & \left[(\beta_{11} \beta_{22} - \beta_{12} \beta_{21}) D^2 - (\alpha_{11} \beta_{22} + \alpha_{22} \beta_{11} - \alpha_{12} \beta_{21} - \alpha_{21} \beta_{12}) D \right. \\ & \left. + (\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}) \right] \theta_H = 0 \end{aligned} \quad (B7)$$

The nonhomogeneous differential equations for θ_1 and θ_2 become, respectively,

$$\bar{L} \theta_1 = -(\alpha_{22} \gamma_1 - \alpha_{12} \gamma_2) \quad (B8)$$

$$\bar{L} \theta_2 = -(\alpha_{11} \gamma_2 - \alpha_{21} \gamma_1) \quad (B9)$$

The homogeneous solutions satisfying Eq. B7 are

$$\theta_{1H} = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad (B10)$$

$$\theta_{2H} = c_3 e^{r_1 t} + c_4 e^{r_2 t} \quad (B11)$$

where r_1 and r_2 are the roots of Eq. 26. Since the original set of differential equations consists of two, first order equations, only two of the constants in Eqs. B10 and B11 are independent. Two relationships between the constants are obtained by substituting the homogeneous solutions B10 and B11 into the homogeneous differential equation Eq. B7. By equating the coefficients of the terms $e^{r_1 t}$ and $e^{r_2 t}$ the following relations are obtained

$$c_3 = - \frac{\beta_{11} r_1 - \alpha_{11}}{\beta_{12} r_1 - \alpha_{12}} c_1$$

$$c_4 = - \frac{\beta_{11} r_2 - \alpha_{11}}{\beta_{12} r_2 - \alpha_{12}} c_2$$

The particular solutions to θ_1 and θ_2 are determined by the method of undetermined parameters, and they are:

$$\theta_{1P} = - \frac{\alpha_{22} \gamma_1 - \alpha_{12} \gamma_2}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}}$$

$$\theta_{2P} = - \frac{\alpha_{11} \gamma_2 - \alpha_{21} \gamma_1}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}}$$

B.3 Evaluation of the arbitrary coefficients c_1 and c_2

The two arbitrary constants appearing in Eqs. 25a,b are specified by requiring that the synthesized solution equal the finite difference solution at all points at time t_2 . This is accomplished by setting $\theta_1(t_2) = 0$ and $\theta_2(t_2) = 1.0$ (refer to Eq. 22). From Eqs. 25a,b the coefficients c_1 and c_2 become:

$$c_1 = \frac{1 + d_2 D_1/C - D_2/C}{d_1 - d_2} e^{-r_1 t_2}$$

$$c_2 = \frac{1 + d_1 D_1/C - D_2/C}{d_1 - d_2} e^{-r_2 t_2}$$

where

$$d_1 = - \frac{\beta_{11} r_1 - \alpha_{11}}{\beta_{12} r_1 - \alpha_{12}}$$

$$d_2 = - \frac{\beta_{11} r_2 - \alpha_{11}}{\beta_{12} r_2 - \alpha_{12}}$$

$$C = \alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} \quad (B12)$$

$$D_1 = - (\alpha_{22} \gamma_1 - \alpha_{12} \gamma_2) \quad (B13)$$

$$D_2 = - (\alpha_{11} \gamma_2 - \alpha_{21} \gamma_1) \quad (B14)$$

APPENDIX C: NUMERICAL RESULTS FOR CASES 2 and 3 OF THE SYNTHESIZED
STEADY-STATE TEMPERATURE DISTRIBUTION

The temperature distribution plots of T versus $\hat{x}$ and $\hat{z}$ for the synthesized spatial temperature distributions of Cases 2 and 3 are given in Figs. C1A,B and C2A,B. These figures correspond to Figs. 4A,B and 5A,B, which give the temperature distributions for Cases 1 and 4 of the spatial synthesis analysis.

Fig. C1-A - Spatial temperature distributions for Case 2: $T = f(\hat{z})$

Fig. C1-B - Spatial temperature distributions for Case 2: $T = f(\hat{x})$

Fig. C2-A - Spatial temperature distributions for Case 3 $T = f(\hat{z})$

Fig. C2-B - Spatial temperature distributions for Case 3: $T = f(\hat{x})$

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