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MASTER

High Resolution Difference Schemes
for compressible gas dynamics

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The new schemes to be presented here have grown out of an extensive com-
parison of a variety of difference methods. The new schemes represent an
attempt to combine the advantages and avoid the disadvantages of the schemes
which were compared — namely, the von-Neumann-Richtmyer scheme [1], Godunov's
scheme [2], MUSCL [3,4], and Glimm's scheme [5,6,7].

We list the advantages of the various schemes first. The principal advan-
tage of the von-Neumann-Richtmyer scheme is its use of a staggered grid.
Densities, internal energies, and hence pressures also are prescribed at zone
centers, while velocities are prescribed at zone interfaces. This grid struc-
ture is well suited to the Lagrangian equations of hydrodynamics, because it
allows narrow-based differences to be used to construct the necessary gradi-
ents. The result is that unusually high resolution of flow structure is
obtained in Lagrangian problems.

The advantage of Godunov's scheme is the clear physical picture upon which
it is based. Rather than replacing an infinite Taylor series by a truncated
one, this scheme replaces a physical system of complex structure by a simpler
one consisting of structureless zones. This simpler system is evolved exactly
for a time step, and then a similar replacement is made. Naturally, the
method rests on the assumption that a Taylor series can be truncated, but the
physical picture is always clear. To a physicist, this formulation has im-
mense appeal. By carrying the accuracy of the physical representation one
order higher than Godunov's scheme, MUSCL combines the advantage of the clear
physical picture with very high resolution of the flow structure.

Glimm's scheme differs from Godunov's scheme in one essential way. For
the variable values in its structureless zones it chooses those at some repre-
sentative point within each zone. This point has the same location within
each zone at a given time step, and it follows some well-distributed, pseudo-
random sequence from one time step to another. The most important effect of
this procedure is to give up exact conservation of mass, momentum, and energy
in an effort to force all flow discontinuities to zone boundaries, where they
can be treated exactly by the method. Because errors arising in the improper
treatment of discontinuities in the flow can severely contaminate a computa-
tion with a standard difference method, the treatment of discontinuities in
Glimm's method gives that scheme unequaled resolution of flow structure in
one-dimensional problems.

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When the above schemes are compared on a difficult two-dimensional flow
problem, their disadvantages are readily apparent. The staggered grid of the
von-Neumann-Richtmyer scheme, which is so convenient in Lagrangian calcula-
tions, is very badly suited to Eulerian calculations. Particularly difficult
to formulate is the conservation of total energy. An additional disadvantage
is the necessity to treat discontinuities as smooth flow regions with steep
gradients. This is done by adding in an artificial viscous pressure which
smears out the discontinuities over at least two zones. The main disadvantage
of Godunov's scheme is its relatively poor resolution of flow structure.
MUSCL has the highest resolution of these four schemes, but that resolution is
limited by an extrapolation procedure which is made at the beginning of each
time step. MUSCL uses as data a zone-centered average value and first deriva-
tive of each variable. From these, values of all variables at the zone inter-
faces must be constructed in order to compute fluxes of conserved quantities
during the time step. The extrapolation from the center of the zone to the
zone interface is responsible for most of the error in MUSCL. Finally, the
disadvantage of Glimm's scheme is that its very special properties in one-
dimensional problems are lost in two dimensions, and the scheme must be aban-
doned in favor of a much lower resolution method in the neighborhood of
discontinuities (see [7]). Because the principal advantage of Glimm's scheme
in 1-D flows was its treatment of discontinuities, the hybridization of the
scheme for 2-D problems results in a poorer scheme than either MUSCL or the
von-Neumann-Richtmyer scheme.

We have devised two new difference schemes which avoid all these disadvan-
tages and combine the advantages listed above. The key ingredients are: (1)
the approach of Godunov's method in replacing a complicated physical system
with a simpler one of a standard form, (2) the translation of this assumed
spatial structure inside zones into temporal structure at the interfaces by
solving Riemann's problem as in Godunov's scheme and using the characteristic
equations as in MUSCL, and a new ingredient (3) the use of both zone-averaged
values and interface values of variables in order to define a distribution of
each variable at every point which is continuous except at true flow discon-
tinuities and which conserves mass, momentum, and energy exactly. We have
devised a second-order method which uses a piecewise linear distribution for
each variable with kinks at zone centers and zone interfaces. In addition we
constructed a second-order method which uses a piecewise parabolic distribu-
tion for each variable with kinks only at the zone interfaces. In 1-D test
problems, both new schemes show at least twice the resolution of MUSCL, the
best of the four schemes discussed above. Only the piecewise parabolic scheme
has been run on 2-D problems. It preserves the high resolution of its 1-D
tests and is thus able to obtain a more accurate a flow description than MUSCL
while using only half as many zones in each dimension. The gains over the
other schemes discussed are still larger. The new algorithm is not yet opti-
mized, but it presently requires only 30% more computer time per zone per time
step than does MUSCL. The gain in time consumed to achieve a given accuracy
is thus a factor of 3 in 1-D. In 2-D, a more complicated operator splitting
algorithm doubles the time consumed, so that the gain is still only a factor
of 3. However, we expect that the new method can be speeded up considerably,
and we hope to do so in the near future.

In Fig. 1 all the schemes discussed above are compared using the example
of the flow of air (gamma is 1.4) through a duct containing a step. Initially
the flow is everywhere to the right at Mach 3, with $\rho = 1.4$, $p = 1$, $c = 1$.
The duct width is 1, its length is 3, and the step of height .2 is located a

distance of .6 from the entrance. All the results in Fig. 1 were obtained
with a uniform Cartesian grid with $\Delta x = \Delta y = .05$. At the exit a "flowout"
boundary condition is applied, but this is unimportant because the flow to the
right is always slightly supersonic there. The system is shown at time 4,
when a complicated system of shock reflections, rarefaction waves, and contact
discontinuities is present. This problem was used by Emery in 1969 [8] to
compare the methods of Lax, Rusanov, and Lax and Wendroff (Emery used a very
slightly different duct with a grid of nodal points with $\Delta x = \Delta y = 1/27$).
One of us also used this problem to demonstrate the MUSCL scheme described by
van Leer in [3]. In that article the relatively structureless steady flow in
the duct is displayed. This steady flow is attained at about time 12.

Because of the lack of space, we show only the contours of density at time
4. These are the most difficult to compute correctly, because of the weak
contact discontinuities which emanate from the two shock triple points associ-
ated with the two Mach reflections of the bow shock at time 4. In Emery's
article 8 , only pressure contours are shown, and it is likely that the weak
contact discontinuities in the flow were not resolved by any of the three
methods he compared. In Fig. 1a the results of Godunov's scheme are shown.
There is some indication of the Mach reflection at the upper wall. In Fig.
1b, the Glimm-Godunov hybrid scheme shows only some improvement over Fig. 1a
at the cost of introducing noise from the random choice feature of Glimm's
scheme. If the Mach reflection could be fully resolved (it is indeed resolved
with 4 times as many zones), the weak contact discontinuity would be quite
sharp. After an initial smearing by Godunov's method, Glimm's method
preserves the relatively narrow contact region.

A dramatic increase in resolution results from using a second-order accu-
rate scheme. The results in Fig. 1c were obtained with the BBC code [9].
This code uses a modified von-Neumann-Richtmyer scheme devised by DeBar [10]
for its Lagrangian step, and a MUSCL remap step on a staggered grid devised by
Woodward. To obtain the thin shocks shown here, the artifical viscosity was
set to zero in the Lagrangian step. This has resulted in a mild oscillation
behind each shock which is most evident when the pressure is plotted. When
the artificial viscosity is turned on, the shocks double in width and the flow
resolution is significantly degraded. Especially to one who considers the
staggered grid formulation both confusing and inconvenient, these results are
remarkably good. The contact discontinuity near the upper wall is spread over
2 to 3 zones, but it is clearly visible. Also the upper Mach stem is in the
correct position directly above the step, and it has the correct length (as
proven by a run on a more refined mesh of $\Delta x = \Delta y = .02$ which is shown in
Fig. 2). The MUSCL results shown in Fig. 1d are of comparable quality. They
are superior in that the post-shock oscillations of BBC are not present and
the contact discontinuity is more sharply defined. However, somewhat more
entropy is artificially produced as the flow rounds the corner of the step.
The result is a classic interaction of a shock with a boundary layer, which
produces the second, very weak reflected shock from the top of the step at $x =$
1.4. These two codes run at almost precisely the same speed — 2600 points
per sec per cycle on a CDC 7600 and 20000 pts/sec/cy on a Cray I. They both
make use of separate Lagrangian and remap steps in each 1-D pass.

In Fig. 1e we show the results of the new scheme which uses piecewise
parabolic interpolation. In 1-D this scheme can be made third-order accurate,
but because of its use of 1-D passes it can only be second-order accurate in
2-D. The scheme used here is thus made only second-order accurate in 1-D,

although several gestures toward higher order are included. The results of this new scheme are comparable in quality to those of Fig. 2, which were obtained with BBC using a much finer grid of $\Delta x = \Delta y = .02$. The resolution of the weak contact discontinuities from both Mach stems is particularly notable. A "monotonicity trick" has been used to constrain the interpolation parabolae so that the post-shock oscillations usually associated with high-order schemes are completely absent. In Fig. 3 we show results of the new scheme using a grid with $\Delta x = \Delta y = .1$. Evidently, even on this coarse grid the new scheme correctly resolves all the essential features of this complicated flow.

Finally, for comparison with the schemes discussed by Sod [11], we have shown in Fig. 4 results of the new piecewise linear scheme on his shock tube problem. The results in Fig. 4 use a grid of 50 zones rather than Sod's 100, and are more accurate than the results of any of the 12 schemes he compared. MUSCL results on this problem have been given by van Leer [3].
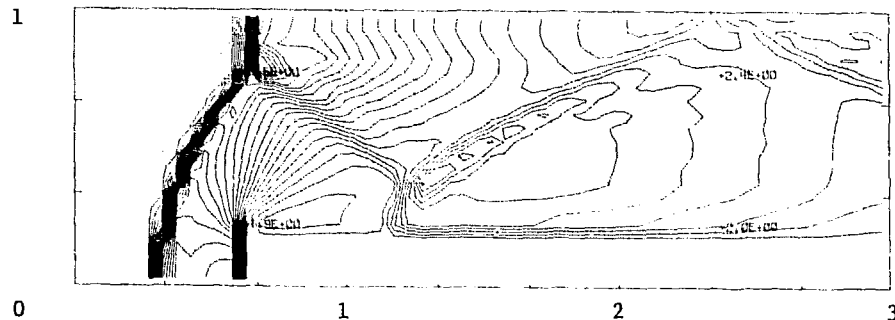
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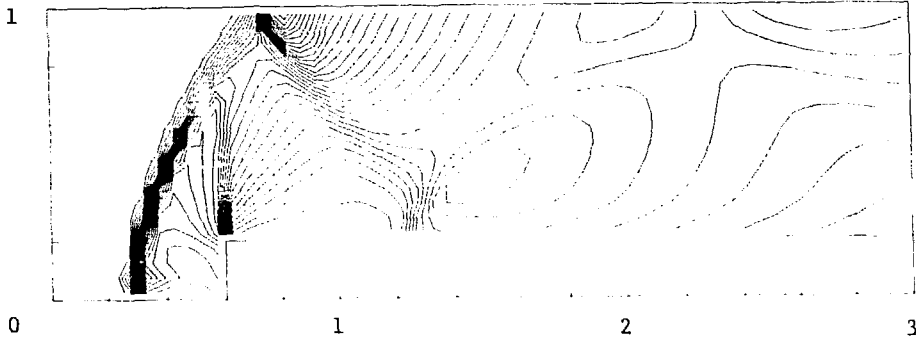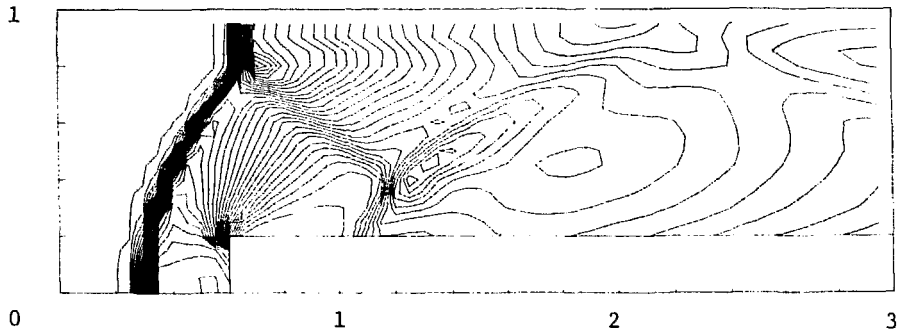
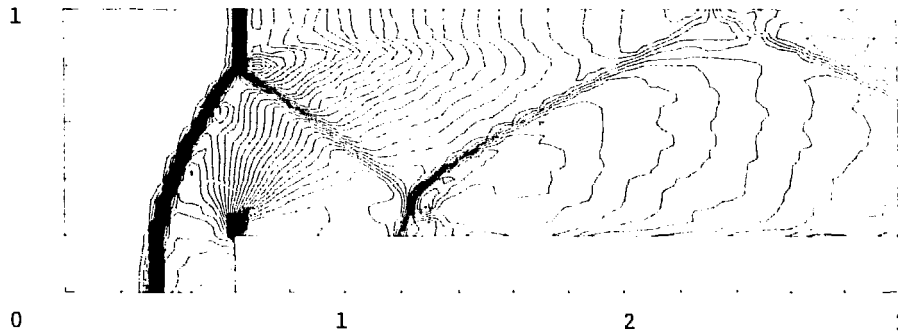1a)   Godunov's Method   29 density contours from 1.18 to 7.14
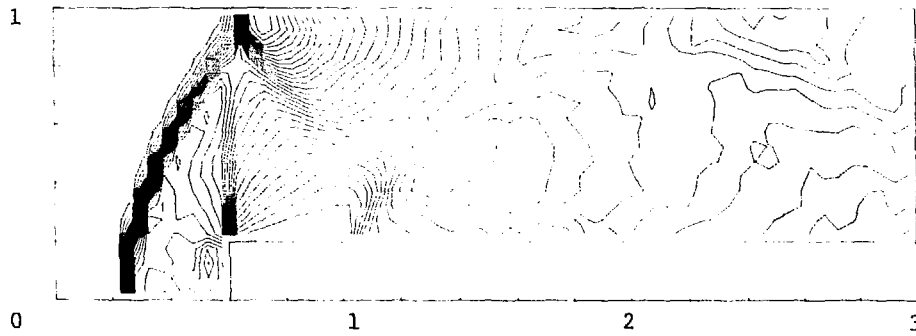
1b)   Glimm-Godunov Hybrid   29 density contours from 1.15 to 6.74

1c)   BBC   22 density contours from 1.00 to 6.25

1d)　MUSCL　　　　　　　　30 density contours from 0.85 to 6.34

1e)　Piecewise Parabolic Method　　30 density contours from 0.73
to 6.31

Fig. 1　Results of several difference schemes for the flow problem
described in the text.　All schemes use a series of 1-D
sweeps and Courant numbers of 0.8 or 0.9.　All use a uniform
grid of 20 x 60 zones.　The methods are:　(a) Godunov's method,
(b) Glimm-Godunov hybrid, (c) BBC, (d) MUSCL, (e) the new
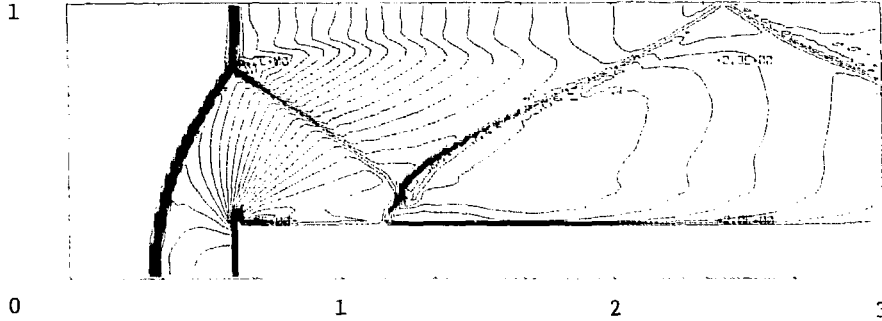piecewise parabolic scheme.

Fig. 2 BBC results for the same problem as in Fig. 1 but using a
finer, uniform grid of 50 x 150 zones.

23 density contours from 0.75 to 6.25 are shown as well as
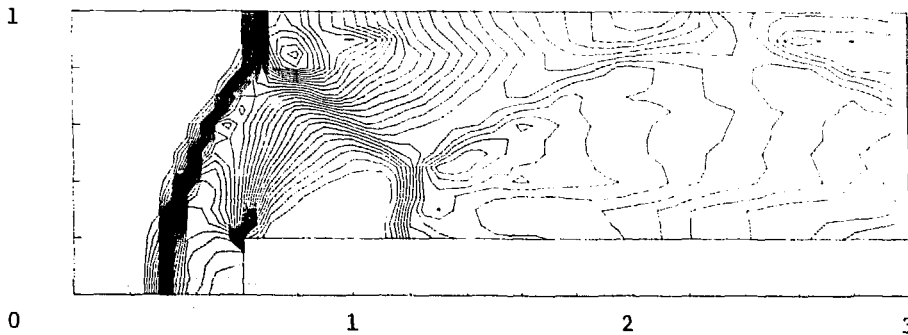contours at densities 0.5 and 0.6.

Fig. 3 Results of the new piecewise parabolic scheme for the same
problem as in Fig. 1 but using a coarser, uniform grid of
10 x 30 zones.

30 density contours from 1.03 to 6.09 are shown.
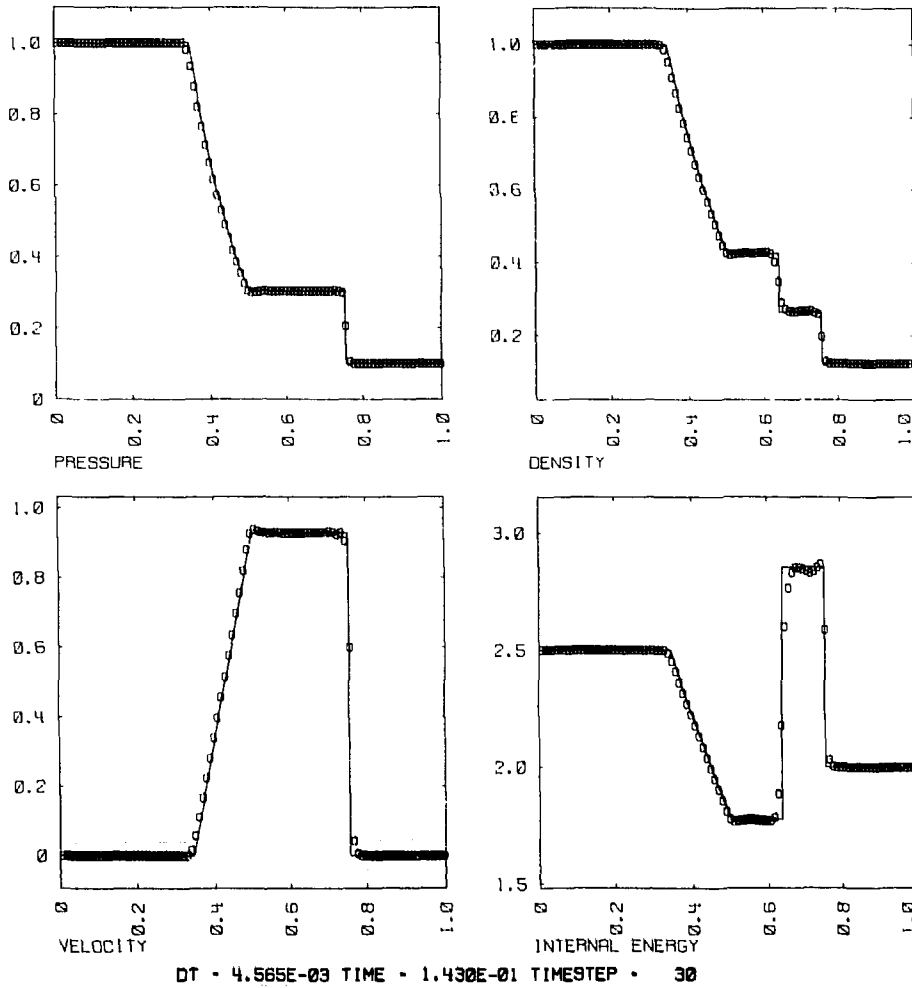
Fig. 4   Results of the new piecewise linear scheme for the shock tube
          problem studied by Sod. The solid line shows the exact
          solution, and the zone average and interface values for the
          grid of 50 zones are shown as circles.