A Monte Carlo Method of Solving Heat Conduction Problems

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MASTER

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A MONTE CARLO METHOD OF SOLVING HEAT
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ABSTRACT

A new approach in the use of Monte Carlo to solve heat conduction problems was developed using a transport equation approximation to the heat conduction equation. The method was shown to be applicable to the solution of multimedia problems in complex geometries with no inherent limitations as to the geometric complexity of problems which can be solved. Nuclear radiation transport and heat conduction problems can be calculated with the same computer code; and, in the case where the nuclear reactions are the heat source, the problem can be performed in a coupled mode with a single computer run.

The method was demonstrated for problems involving multimedia, internal sources, fixed temperature boundary conditions, convective boundary conditions, time-dependent sources, and for a coupled nuclear radiation-heat conduction problem. Comparisons were made to analytical solutions when they were available or could be generated.

The adjoint formulation was found to be the most suitable mode of solution for the class of problems which was considered and therefore the method appears to be more useful for calculating temperatures at specific points rather than temperature distributions.

CHAPTER I

INTRODUCTION

Monte Carlo methods are particularly useful in the solution of neutron and gamma-ray transport problems which involve complex geometries. Recent work¹ has suggested that the Monte Carlo approach might also be useful in the solution of certain classes of heat conduction problems.

In this paper a new approach to the use of Monte Carlo in the solution of heat conduction problems is developed. This method, which uses a transport equation approximation, will allow the solution of problems with as much geometric complexity as is necessary to adequately describe the physical model. The method is also completely compatible with Monte Carlo radiation transport programs. Thus, criticality, shielding, and heat transfer calculations can be performed with one geometric description and one computer code. Coupled radiation transport and heat conduction problems can also be performed in a single computer run where the heat generated by the nuclear radiation (for example, through fission reactions or by energy absorption) is the heat source for the heat transfer calculation.

There are many ways to solve the basic heat conduction problem both numerically and analytically. This method is not intended to replace existing methods. Numerical techniques such as finite difference or finite element will in general provide a much faster solution. The proposed method will, however, be able to solve certain problems for

which solutions were not previously feasible. Since the method uses the same logic as existing Monte Carlo radiation transport computer codes, it was not necessary to generate completely new computer programs for its implementation. The solution to the problems solved in this paper were performed using existing computer codes.

Other continuous Monte Carlo techniques for solving heat conduction problems have been reported,¹⁻¹⁰ and a summary of these methods is given in Reference 1. However, all published results using these methods have been limited to single medium problems. Discrete Monte Carlo approaches have also been used.¹¹⁻¹³ These methods are Monte Carlo methods which solve the finite difference equations and have not proven to be competitive with other finite difference techniques. Hybrid Monte Carlo approaches have also been developed,^{14, 15} but these require the use of both analog and digital systems and appear to be limited in their potential applications.

The equations of heat conduction and their relationship to the transport equation approximation in its solution are described in Chapter II. Chapter III provides a description of the application of Monte Carlo to the solution of the transport equation.

The validity of the method is established in Chapter IV where Monte Carlo solutions are compared to analytical results for a wide range of problems. Since a major area of interest is the demonstration of the applicability of the transport equation approximation to multimedia problems, and analytical solutions were available for only a very limited number of multimedia problems, Chapter IV also includes

the development of the analytical solutions for those problems for which solutions were not previously available. The verification of these solutions is given in Appendix A. A derivation of a bounded next event (flux at a point) estimator which was used in a Monte Carlo solution in Chapter IV is given in Appendix B.

Methods to correct for the differences between transport theory and diffusion theory as applied to heat conduction are developed in Chapter V.

A coupled gamma-ray, heat conduction problem is solved in Chapter VI as an example of the type of problem for which this method is applicable. A listing of the input data and output for this problem is given in Appendix C.

Conclusions and recommendations are given in Chapter VII.

CHAPTER II

TRANSPORT THEORY APPROXIMATION TO THE DIFFERENTIAL EQUATION OF HEAT CONDUCTION

The differential equation of heat conduction through a stationary, isotropic solid is given by:¹⁶

$$\nabla \cdot K(\vec{r}, T) \nabla T(\vec{r}, t) + S(\vec{r}, t) = \rho c \frac{\partial T}{\partial t}, \quad (2-1)$$

where

$K(\vec{r}, T)$ = thermal conductivity

ρ = density

c = specific heat

$S(\vec{r}, t)$ = internal heat source

If the thermal conductivity is not a function of position or temperature, this simplifies to:

$$K \nabla^2 T(\vec{r}, t) + S(\vec{r}, t) = \rho c \frac{\partial T}{\partial t}. \quad (2-2)$$

It is desired to solve the above differential equation subject to the following types of standard boundary conditions.

1. Boundary condition of the first kind:

The temperature is specified along a boundary surface.

$$T = f(\vec{r}_s, t), \quad (2-3)$$

where $\vec{r}_s$ is on a boundary surface.

2. Boundary condition of the second kind:

The normal derivative of the temperature is specified along a boundary surface.

$$\frac{\partial T}{\partial n} = f(\bar{r}_s, t) , \quad (2-4)$$

where $\bar{r}_s$ is on a boundary surface, and $\partial/\partial n$ represents differentiation along the outward normal to the surface.

3. Boundary condition of the third kind:

A combination of the temperature and normal derivative is specified at a boundary.

$$K \frac{\partial T}{\partial n} + hT = f(\bar{r}_s, t) , \quad (2-5)$$

where $\bar{r}_s$ is on a boundary surface, $\partial/\partial n$ represents differentiation along the outward normal to the surface, K is the thermal conductivity, and h is the convective heat transfer coefficient. This type of boundary condition is normally associated with forced convection and written as

$$K \frac{\partial T}{\partial n} = h(T_\infty - T) , \quad (2-6)$$

where T_∞ is the temperature of the surroundings.

4. Natural convection boundary condition:

$$K \frac{\partial T}{\partial n} = h(T_\infty - T)^{5/4} . \quad (2-7)$$

5. Thermal radiation boundary condition:

$$K \frac{\partial T}{\partial n} = \sigma \epsilon (T_{\infty}^4 - T^4) , \quad (2-8)$$

where σ is the Stefan-Boltzmann constant

ϵ is the emissivity of the surface.

Across interfaces between two different media

$$K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n} , \quad (2-9)$$

where K_1 and T_1 are near the boundary in the first medium and K_2

and T_2 are near the boundary in the second medium and

$T_1 = T_2$ at boundary.

The neutron diffusion equation can be written as¹⁷

$$D \nabla^2 \phi(\vec{r}, t) - \Sigma_a \phi(\vec{r}, t) + S(\vec{r}, t) = \frac{1}{v} \frac{\partial \phi}{\partial t} , \quad (2-10)$$

where ϕ is the neutron flux

D is the diffusion coefficient

Σ_a is the absorption cross section, and

v is the neutron's speed.

D is normally given by

$$D = \frac{1}{3(\Sigma - \bar{\mu} \Sigma_s)} , \quad (2-11)$$

where Σ is the total cross section,

$\bar{\mu}$ is the average cosine of the scattering angle, and

Σ_s is the scattering cross section.

If $\Sigma_a = 0$, then $\Sigma = \Sigma_s$ and

$$D = \frac{1}{3\Sigma(1-\bar{\mu})} . \quad (2-12)$$

If the scattering is isotropic, $\bar{\mu} = 0$, and

$$D = \frac{1}{3\Sigma} . \quad (2-13)$$

Therefore, under these condition, the diffusion equation becomes

$$\frac{1}{3\Sigma} \nabla^2 \phi + S(\vec{r}, t) = \frac{1}{v} \frac{\partial \phi}{\partial t} , \quad (2-14)$$

and by comparing the above to the heat conduction equation, the two equations are equivalent if

$$\Sigma = \frac{1}{3K} , \quad (2-15)$$

and

$$v = \frac{1}{\rho c} . \quad (2-16)$$

In neutron transport theory problems, the diffusion equation is frequently used as an approximation to the Boltzmann transport equation. Monte Carlo methods are available for solving the transport equation in problems with very complex geometries and these methods may then be used to approximate the solutions to the equivalent heat conduction problems.

Transport theory differs from diffusion theory when scattering is highly anisotropic or the angular distribution of particles (angular

flux) is highly anisotropic. Transport theory also differs from diffusion theory in the application of various boundary conditions. For example, diffusion theory allows the specification of a zero flux boundary condition whereas transport theory allows only a zero incoming flux boundary condition.

In trying to adjust neutron diffusion theory to approximate the transport equation, various methods of correction have been developed. For example, to account for anisotropic scattering the diffusion coefficient is modified by the average scattering angle and a zero flux boundary condition at an extrapolated distance¹⁸ from the boundary is used to simulate a zero incoming flux at the boundary. In optically thin regions (very few mean free paths thick), in the vicinity of strong absorbers or sources or when scattering is strongly anisotropic, the differences between the neutron diffusion approximation and transport theory may be significant and in these cases, the use of neutron diffusion theory is inappropriate. These differences are more easily corrected when transport theory is used to approximate diffusion theory. Diffusion theory is based on isotropic scattering and the scattering used in the transport solution can always be made isotropic. To correct for the anisotropic flux which transport theory will give in the vicinity of strong absorbers or sources or in optically thin regions, the actual cross sections used in the transport calculation can be scaled so that the regions of interest are no longer optically thin. An anisotropic flux will therefore exist only over a small region of the

problem and the transport theory approximation to the diffusion equation will be appropriate.

Scaling of cross sections is accomplished by letting $T = \beta\phi$ where β is a constant.

Then

$$K\beta\nabla^2\phi + S(\bar{r},t) = \beta\rho c \frac{\partial\phi}{\partial t} . \quad (2-17)$$

1. Boundary condition of first kind:

$$\phi = \frac{f(r_s,t)}{\beta} . \quad (2-18)$$

2. Boundary condition of second kind:

$$\frac{\partial\phi}{\partial n} = \frac{f(r_s,t)}{\beta} . \quad (2-19)$$

3. Boundary condition of third kind:

$$K\beta \frac{\partial\phi}{\partial n} + n\beta\phi = f(r_s,t) . \quad (2-20)$$

4. Natural convection boundary condition:

$$K\beta \frac{\partial\phi}{\partial n} = h\beta^{5/4} (\phi_\infty - \phi)^{5/4} . \quad (2-21)$$

5. Thermal radiation boundary condition:

$$K\beta \frac{\partial\phi}{\partial n} = \sigma\epsilon\beta^4 (\phi_\infty^4 - \phi^4) \quad (2-22)$$

across interface

$$\kappa_1 \beta \frac{\partial \phi_1}{\partial n} = \kappa_2 \beta \frac{\partial \phi_2}{\partial n} , \quad (2-23)$$

and

$$\beta \phi_1 = \beta \phi_2 . \quad (2-24)$$

Therefore, by letting $T = \beta \phi$ and comparing to the diffusion equation

$$\Sigma = \frac{1}{3K\beta} , \quad (2-25)$$

and

$$v = \frac{1}{\rho c \beta} . \quad (2-26)$$

The source doesn't change, $S = S(\vec{r}, t)$ and the above revised boundary conditions should be used.

The method of solution using the transport equation approximation is then to solve the Boltzmann transport equation for the flux ϕ under conditions of isotropic scattering and no absorption. The actual cross sections and velocities used in the calculation may be scaled using Equations (2-25) and (2-26) so as to make the differences between the transport equation solution and the diffusion equation solution as small as desired (with a penalty of increasing computation time). The boundary conditions used during the calculation are obtained from Equations (2-18) through (2-24). Since the diffusion equation is equivalent to the heat conduction equation under the above conditions,

the transport equation approximation of the solution to the heat conduction equation is then obtained from

$$T = \beta \phi . \quad (2-27)$$

The next chapter describes the application of Monte Carlo to the solution of the Boltzmann transport equation.

CHAPTER III

APPLICATION OF MONTE CARLO TO THE TRANSPORT EQUATION

As shown in the previous chapter, the solution of a heat conduction problem can be approximated by the solution of an appropriate transport problem. Once the appropriate transport problem has been defined, any of the methods which would normally be used in its solution can be used without additional justification. Monte Carlo methods have been developed for solving transport problems and the previously developed methodology is thereby appropriate. A mathematical treatment of the application of the Monte Carlo method to solve the Boltzmann transport equation is given in a book by Spanier and Gelbard,¹⁹ and a review of the Monte Carlo method as applied to particle transport is given by Carter and Cashwell.²⁰

Only a brief discussion of the application of Monte Carlo methods to the solution of the transport equation will be given here. The formulation given is based primarily on the article by Coveyou, Cain, and Yost.²¹

The integral form of the transport equation can be written as

$$\psi(x) = S(x) + \int_{\Omega} \psi(y) G(y, x) dy, \quad (3-1)$$

where $\psi(x)$ is the event density (collisions and leakage are considered events)

$S(x)$ is the first event source distribution function

$G(y, x)$ is the transition kernel (the probability of an event at x given an event at y), and

x, y are vectors in the space Ω of interest and may include position, direction, time, and energy dependence.

$\int_{\Omega} G(x, y) dy$ represents the nonabsorption probability at x .

The flux at any point in a medium is obtained from the event density by dividing by the cross section of the medium.

For nonmultiplying media

$$\int_{\Omega} G(x, y) dy \leq 1 \text{ for all } x \in \Omega. \quad (3-2)$$

For the case of heat transfer all media will be nonabsorbing and

$$\int_{\Omega} G(x, y) dy = \begin{cases} 1 & \text{for all } x \text{ at collision sites,} \\ 0 & \text{for all } x \text{ at leakage sites.} \end{cases} \quad (3-3)$$

The usual problem is to calculate an effect of interest λ , which is given by the functional

$$\lambda = \int_{\Omega} \psi(x) R(x) dx, \quad (3-4)$$

where $R(x)$ is the response function.

The adjoint formulation of the same problem can be written

$$\tilde{\psi}(x) = R(x) + \int_{\Omega} G(x, y) \tilde{\psi}(y) dy, \quad (3-5)$$

and

$$\lambda = \int_{\Omega} \tilde{\psi}(x) S(x) dx. \quad (3-6)$$

Note that in the adjoint formulation the integration of the transition kernel is with respect to the second variable instead of the first. The integral equation for ψ will be referred to as the forward formulation to differentiate it from the adjoint formulation.

Let

$$\psi(x) = \sum_{p=0}^{\infty} \psi_p(x) , \quad (3-7)$$

where

$$\psi_0(x) = S(x) , \quad (3-8)$$

and

$$\psi_{p+1}(x) = \int_{\Omega} \psi_p(y) G(y, x) dy . \quad (3-9)$$

Then

$$\int_{\Omega} \psi(y) G(y, x) dy = \int_{\Omega} \sum_{p=0}^{\infty} \psi_p(y) G(y, x) dy . \quad (3-10)$$

The summation, if it exists (existence is discussed in Chapter Three of Reference 19) is absolutely convergent since $\psi_p(y) \geq 0$ and $G(y, x) \geq 0$ and therefore summation and integration can be interchanged giving

$$\int_{\Omega} \psi(y) G(y, x) dy = \sum_{p=0}^{\infty} \int_{\Omega} \psi_p(y) G(y, x) dy = \sum_{p=0}^{\infty} \psi_{p+1}(x) = \sum_{p=1}^{\infty} \psi_p(x) . \quad (3-11)$$

Therefore, adding $S(x)$ (or $\psi_0(x)$) to both sides gives

$$S(x) + \int_{\Omega} \psi(y) G(y, x) dy = \psi(x) , \quad (3-12)$$

and $\psi(x)$ as defined by Equation (3-7) satisfies the integral form of the transport equation.

Rewriting gives:

$$\begin{aligned}
 \psi(x_1) = & S(x_1) + \int_{\Omega} S(x_2)G(x_2, x_1)dx_2 \\
 & + \int_{\Omega} G(x_3, x_1) \int_{\Omega} S(x_2)G(x_2, x_3)dx_2 dx_3 \\
 & + \int_{\Omega} G(x_4, x_1) \int_{\Omega} G(x_3, x_4) \int_{\Omega} S(x_2)G(x_2, x_3)dx_2 dx_3 dx_4 \\
 & + \dots \dots \dots
 \end{aligned} \tag{3-13}$$

And λ is given by

$$\lambda = \int_{\Omega} \psi(x_1)R(x_1)dx_1 . \tag{3-14}$$

Or

$$\begin{aligned}
 \lambda = & \int_{\Omega} S(x_1)R(x_1)dx_1 + \iint_{\Omega\Omega} R(x_1)S(x_2)G(x_2, x_1)dx_2 dx_1 \\
 & + \iint_{\Omega\Omega} R(x_1)G(x_3, x_1) \int_{\Omega} S(x_2)G(x_2, x_3)dx_2 dx_3 dx_1 \\
 & + \iint_{\Omega\Omega} R(x_1)G(x_4, x_1) \int_{\Omega} G(x_3, x_4) \int_{\Omega} S(x_2)G(x_2, x_3)dx_2 dx_3 dx_4 dx_1 \\
 & + \dots \dots \dots
 \end{aligned} \tag{3-15}$$

Or, interchanging dummy integration variables,

$$\begin{aligned}
 \lambda = & \int_{\Omega} S(x_1)R(x_1)dx_1 + \iint_{\Omega\Omega} R(x_2)S(x_1)G(x_1, x_2)dx_1 dx_2 \\
 & + \iint_{\Omega\Omega} R(x_3)G(x_2, x_3) \int_{\Omega} S(x_1)G(x_1, x_2)dx_1 dx_2 dx_3 \\
 & + \iint_{\Omega\Omega} R(x_4)G(x_3, x_4) \int_{\Omega} G(x_2, x_3) \int_{\Omega} S(x_1)G(x_1, x_2)dx_1 dx_2 dx_3 dx_4 \\
 & + \dots \dots \dots
 \end{aligned} \tag{3-16}$$

The above formulation may be used to construct a Monte Carlo random walk. 1) A position x_1 is picked from $S(x)$ and $R(x_1)$ is scored. This is an estimate of the first integral. 2) Starting at x_1 , a new position x_2 is picked using $G(x_1, y)$ and a value w_1 is assigned by

$$w_1 = \int_{\Omega} G(x_1, y) dy . \quad (3-17)$$

The product $w_1 \cdot R(x_2)$ is then scored as the estimate of the second integral. 3) Starting at x_2 a new position x_3 is picked using $G(x_2, y)$ and a value w_2 is assigned by

$$w_2 = w_1 \int_{\Omega} G(x_2, y) dy . \quad (3-18)$$

The product $w_2 \cdot R(x_3)$ is then scored as the estimate of the third integral, 4)

This process is continued until an external boundary is reached or until the value of w is so small that additional scores of $w \cdot R(x)$ would be negligible. In this case Russian roulette may be used to determine if the random walk is to be continued.

The random walk which comprises the series of points $x_1, x_2, \dots$ in Ω will be called a history.

The cumulative score from the history is then an estimate of λ . If a large number of histories are performed, the mean will be a reasonable estimate of λ . The variance of the estimations should also be computed to provide an indication of the statistical uncertainty of the estimation.

Similarly, the adjoint formulation of the solution for λ can be written

$$\lambda = \int_{\Omega} R(x_1) S(x_1) dx_1 + \iint_{\Omega\Omega} S(x_2) R(x_1) G(x_2, x_1) dx_1 dx_2 + \dots \quad (3-19)$$

Note that this is identical to the forward formulation with the order of integration interchanged.

The Monte Carlo solution of the adjoint formulation is accomplished in a similar manner except that x_1 is chosen from $R(x)$ and x_2 is chosen from $G(x, x_1)$, i.e., given a collision at x_1 , $G(x, x_1)$ is the probability that it was preceded by a collision at x .

A number of techniques have been developed to reduce the variance of the estimate of λ for given computational times. Some of the more common techniques include source biasing, Russian roulette, splitting, and exponential transform (path length stretching). Standard techniques are also available for obtaining an estimate of λ from the Monte Carlo histories. Although only a few of these techniques are used in the following sections in the solution of the demonstration problems, all of the techniques which have been developed for the Monte Carlo solution of the transport equation are available for use.

CHAPTER IV

COMPARISONS OF MONTE CARLO TRANSPORT SOLUTIONS TO ANALYTICAL SOLUTIONS

In this chapter Monte Carlo solutions are formulated for a number of multimedia problems and compared against analytical solutions. Since analytical solutions were not available in the literature for multimedia problems involving more than one dimension,²² the development of the analytical solutions for these problems is also presented.

I. TWO MEDIA SPHERE

Problem 1: To calculate the temperature at several points within a one-dimensional, two-media sphere (Figure 1).

Assume:

1. A point source of strength s at the center
2. K_1 is the thermal conductivity for $0 \leq r \leq a$
3. K_2 is the thermal conductivity for $a \leq r \leq b$
4. No contact resistance between media
5. $T(r=b) = 80^\circ\text{C}$
6. steady state.

The analytical solution can be obtained from Carslaw and Jaeger.²²

$$T(r) = \frac{s}{4\pi K_1 r} + \frac{s[b(K_1 - K_2) - aK_1]}{4\pi K_1 K_2 ab} + T_b, \quad 0 \leq r \leq a. \quad (4-1)$$

$$T(r) = \frac{s}{4\pi K_2 r} - \frac{s}{4\pi K_2 b} + T_b, \quad a \leq r \leq b. \quad (4-2)$$

Figure 1. Two-media sphere with point source at center.

The parameters for the problem are:

$$K_1 = .3333 \text{ cal/sec cm}^\circ\text{C}$$

$$K_2 = .1667 \text{ cal/sec cm}^\circ\text{C}$$

$$a = 5 \text{ cm}$$

$$b = 10 \text{ cm}$$

$$s = 238.8 \text{ cal/sec (1000 watts).}$$

This gives:

$$T(r) = \begin{cases} \frac{57.0}{r} + 80.0 & 0 \leq r \leq 5 \\ \frac{114.0}{r} + 68.6 & 5 \leq r \leq 10 \end{cases}, \quad (4-3)$$

where the temperature is given in $^\circ\text{C}$.

The Monte Carlo calculations were performed using the MORSE-SGC Monte Carlo neutron and gamma-ray transport code,²³ which provides a Monte Carlo solution to either the forward or adjoint Boltzmann transport equation.

The parameters used in the transport equation solution (using the conversion in Chapter II) were

$$1. \quad \Sigma_1 = \frac{1}{3K_1} = 1 \text{ cm}^{-1}$$

$$2. \quad \Sigma_2 = \frac{1}{3K_2} = 2 \text{ cm}^{-1}$$

$$3. \quad a = 5 \text{ cm}$$

$$4. \quad b = 10 \text{ cm}$$

$$5. \quad s = 238.8/\text{sec}$$

and the flux ϕ has units of $\text{cm}^{-2} \text{sec}^{-1}$. The flux was calculated at 1 cm radial intervals in the sphere using a boundary crossing estimator. The solution was obtained using the forward formulation. The results of these calculations are given in Table 1 along with the analytical results. (The estimate of the standard deviation of the Monte Carlo solution is also given). The Monte Carlo results were obtained using 4.5 min of CPU time on the IBM 369/91.

To more accurately approximate the diffusion equation by the transport equation, an extrapolation distance correction could have been made to account for edge effects. The extrapolation distance d is given by:

$$d = \frac{.71}{\Sigma_2} = .355 \text{ cm.} \quad (4-4)$$

This distance would be subtracted from the radius giving an outer radius of 19.645 cm. The Monte Carlo solution as performed corresponds to the solution of a problem with outer radius of 20.355 cm, if the extrapolation distance correction is incorporated.

The analytical solution for a composite sphere with an outer radius of 20.355 cm is included in the table to show the effects of including the correction. The Monte Carlo solution to the transport equation compares very well to the analytical solution once this correction is made, except in the vicinity of the point source.

TABLE 1
TEMPERATURE COMPARISONS FOR A TWO-MEDIA SPHERE
WITH A POINT SOURCE

Radius (cm)	Analytic Temperature °C	Analytic Temperature with Extrapolation Distance Correction °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
1.0	137.0	137.4	142	1.8
2.0	108.5	108.9	108	0.70
3.0	99.00	99.39	98.6	0.54
4.0	94.25	94.64	93.6	0.44
5.0	91.40	91.79	91.2	0.26
6.0	87.6	87.99	87.8	0.17
7.0	84.89	85.28	85.2	0.14
8.0	82.85	83.24	83.2	0.083
9.0	81.27	81.66	81.7	0.047
10.0	80.00	80.39	80.3	0.0004

Note: All numbers calculated with Monte Carlo in this and subsequent tables will be given with an estimation of the standard deviation of the numbers.

II. TWO-MEDIA RPP GREEN'S FUNCTION

The next series of problems requires analytical solutions for three-dimensional, two-media rectangular parallelepipeds. The first step in generating the solutions to these problems is to derive an appropriate Green's function (Figure 2).

From Carslaw and Jaeger,²² the one-medium Green's function for a rectangular parallelepiped (RPP) $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, with zero surface temperature and a point source at (x', y', z', τ) is

$$T = \frac{8}{abc} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sin \frac{l\pi x}{a} \sin \frac{l\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} \right. \\ \left. e^{-K\pi^2(t-\tau) \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \right]. \quad (4-5)$$

To obtain the time independent Green's function integrate $\int_0^t d\tau$ and take $\lim t \rightarrow \infty$, to get

$$T = \frac{8}{abcK\pi^2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \sin \frac{l\pi x}{a} \sin \frac{l\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c}. \quad (4-6)$$

Also from Carslaw and Jaeger,²² if the RPP face at $x=0$ is maintained at $\phi(y, z, t)$, other faces at zero, with initial conditions $T(x, y, z, 0) = 0$

$$T = \frac{8K\pi}{a^2bc} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[l \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \int_0^t d\tau \int_0^c \int_0^b \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} \phi(y', z', \tau) \right. \\ \left. e^{-K\pi^2(t-\tau) \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} dy' dz' \right]. \quad (4-7)$$

Figure 2. Two-media RPP with point source in medium one.

Let $\phi(y, z, t) = \phi(y, z)$ and take $\lim t \rightarrow \infty$ to get the steady-state solution

$$T = \frac{4}{bc} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{2l\pi}{l^2\pi^2 + a^2\pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \int_0^b \int_0^c \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\}. \quad (4-8)$$

But

$$\sum_{l=1}^{\infty} \frac{2l\pi}{l^2\pi^2 + a^2\pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{l\pi x}{a} = \frac{\sinh \left[\pi(a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}. \quad (4-9)$$

Therefore

$$T = \frac{4}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \left[\pi(a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \int_0^b \int_0^c \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\}. \quad (4-10)$$

To obtain a Green's function for a composite, two-media region, first assume an RPP $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$, $K=K_2$, $\phi(y, z)$ at $x=0$, other faces at zero, then

$$T_2 = \frac{4}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \left[\pi(a+x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \int_0^c \int_0^b \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\} \quad (4-11)$$

Now assume RPP $0 \leq x \leq z$, $0 \leq y \leq b$, $0 \leq z \leq c$, $K = K_1$, $\phi(y, z)$ at $x = 0$, other faces zero, and point source at (x', y', z') , then

$$T_1 = \frac{4}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \left[\pi(a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \cdot \int_0^c \int_0^b \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\} + \frac{8}{K_1 \pi^2 abc} \\ \cdot \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \sin \frac{\ell\pi x}{a} \sin \frac{\ell\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} \right\} \quad (4-12)$$

For continuity of heat flow at $x = 0$

$$K_2 \left. \frac{\partial T_2}{\partial x} \right|_{x=0} = K_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=0} \quad (4-13)$$

From Equation (4-11)

$$\left. \frac{\partial T_2}{\partial x} \right|_{x=0} = \frac{4\pi}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \cdot \int_0^c \int_0^b \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\}. \quad (4-14)$$

From Equation (4-12)

$$\left. \frac{\partial T_1}{\partial x} \right|_{x=0} = - \frac{4\pi}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi z}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \cdot \int_0^c \int_0^b \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\} + \frac{4}{K_1 bc} \\ \cdot \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{2\ell\pi}{\ell^2\pi^2 + a^2\pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{\ell\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c}. \quad (4-15)$$

Using

$$\sum_{\ell=1}^{\infty} \frac{2\ell\pi}{\ell^2\pi^2 + a^2\pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{\ell\pi x'}{a} = \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}, \quad (4-16)$$

and setting

$$K_2 \left. \frac{\partial T_2}{\partial x} \right|_{x=0} = K_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=0}, \quad (4-17)$$

gives

$$\begin{aligned} & \frac{4\pi}{bc} (K_1 + K_2) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ & \left. \int_0^c \int_0^b \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\} \\ & = \frac{4\pi}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c}. \end{aligned} \quad (4-18)$$

Assuming termwise equality gives

$$\begin{aligned} & \int_0^c \int_0^b \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \\ & = \frac{1}{\pi (K_1 + K_2) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c}. \end{aligned} \quad (4-19)$$

Using this in Equations (4-11) and (4-12) gives the solution as

$$T_2 = \frac{4}{bc(K_1 + K_2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \left[\pi(a+x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. - \frac{1}{\left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} \right\}, \quad (4-20)$$

and

$$T_1 = \frac{4}{\pi bc(K_1 + K_2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{\left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \frac{\sinh \left[\pi(a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. - \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} \right\} + \frac{8}{K_1 \pi^2 abc} \\ \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \sin \frac{\ell\pi x}{a} \sin \frac{\ell\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c}. \quad (4-21)$$

To show that these equations reduce to the one-media equations when $K_1 = K_2$, consider an RPP $0 \leq x \leq 2a$, $0 \leq y \leq b$, $0 \leq z \leq c$, with zero surface temperature and a point source at (x', y', z') . The steady-state solution (Equation (4-6)) is then

$$T = \frac{4}{K\pi^2 abc} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\frac{l^2}{4a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \sin \frac{l\pi x}{2a} \sin \frac{l\pi x'}{2a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c}, \quad (4-22)$$

but

$$\begin{aligned} & \sum_{l=1}^{\infty} \frac{1}{\frac{l^2}{4a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \sin \frac{l\pi x}{2a} \sin \frac{l\pi x'}{2a} \\ &= \frac{\pi a}{2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \frac{\sinh \left[\pi x \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] \sinh \left[\pi (2a - x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] \cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}; \end{aligned} \quad (4-23)$$

therefore,

$$T = \frac{2}{K\pi bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{\left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \frac{\sinh \left[\pi \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] \sinh \left[\pi (2a - x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] \cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \cdot \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} \right\}, \quad (4-24)$$

which is Equation (4-20) with $K_1 = K_2 = K$, and translated by a .

III. TWO-MEDIA RPP WITH SINUSOIDAL INTERNAL SOURCE

Problem 2: To calculate the temperature distribution within a three-dimensional, two-media rectangular parallelepiped (Figure 3).

Assume:

1. K_1 is the thermal conductivity for $0 \leq x \leq a$, $0 \leq y \leq b$,
 $0 \leq z \leq c$
2. K_2 is the thermal conductivity for $-a \leq x \leq 0$, $0 \leq y \leq b$,
 $0 \leq z \leq c$
3. No contact resistance between media
4. All external surface temperatures are zero
5. Steady state
6. $S = S_0 \sin \pi x/a \sin \pi y/b \sin \pi z/c$ for $0 \leq x \leq a$, $0 \leq y \leq b$,
 $0 \leq z \leq c$

The analytical solution for this problem can be obtained using the previously generated Green's function, Equations (4-20) and (4-21).

Denoting the Green's function solution by $T_g(x, y, z, x', y', z')$ where x', y', z' is the source point, the temperature distribution due to a distributed source $S(x, y, z)$ is then given by

$$T(x, y, z) = \iiint T_g(x, y, z, x', y', z') S(x', y', z') dx' dy' dz' \quad . \quad (4-25)$$

The solution for the Green's function in medium two (no source) was

Figure 3. Two-media RPP with distributed source in medium one.

$$T_{2g} = \frac{4}{\pi bc (K_1 + K_2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \left[\pi(a+x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \frac{1}{\left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \right. \\ \left. + \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} \right\} . \quad (4-26)$$

Using the following integrals:

$$\int_0^b \sin \frac{\pi y'}{b} \sin \frac{m\pi y'}{b} dy' = \begin{cases} b/2, & m = 1, \\ 0, & m \neq 1, \end{cases} \quad (4-27)$$

$$\int_0^a \sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] \sin \frac{\pi x'}{a} dx' \\ = \sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] \frac{a}{\pi + a^2 \pi \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)}, \quad (4-28)$$

gives

$$T_2(x, y, z) = \frac{S_0 a}{\pi^2} \frac{1}{(K_1 + K_2)} \frac{1}{1 + a^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right)} \\ \cdot \frac{1}{\left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}} \frac{\sinh \left[\pi(a-x) \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]}{\cosh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} . \quad (4-29)$$

The solution for the Green's function in medium one was

$$T_{1g}(x,y,z,x',y',z') = T_{2g}(-x,y,z,x',y',z') + \frac{8}{K_1 \pi^2 abc} \cdot \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \cdot \sin \frac{l\pi x}{a} \sin \frac{l\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} \right\} \cdot (4-30)$$

Integrating Equation (4-30) gives

$$T_1(x,y,z) = T_2(-x,y,z) + \frac{S_0}{\pi^2 K_1} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \cdot (4-31)$$

To verify the solution, it is necessary that

$$\nabla^2 T_2(x,y,z) = 0, \quad (4-32)$$

$$\nabla^2 T_1(x,y,z) = -\frac{S(x,y,z)}{K_1}, \quad (4-33)$$

$$-K_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=0} = -K_2 \left. \frac{\partial T_2}{\partial x} \right|_{x=0}, \quad (4-34)$$

and $T(x,y,z) = 0$ for x, y, z on outer boundaries. The verification of this solution is shown in Appendix A.

The parameters for the problem are:

1. $K_1 = .1667 \text{ cal/sec cm}^\circ\text{C}$
2. $K_2 = .3333 \text{ cal/sec cm}^\circ\text{C}$

$$3. \quad a = 1 \text{ cm}$$

$$4. \quad b = 1 \text{ cm}$$

$$5. \quad c = 1 \text{ cm}$$

$$6. \quad S_0 = 238.8 (\pi/2)^3 \text{ cal/sec cm}^3$$

with an external surface temperature of 80°C.

This gives:

$$T_2(x,y,z) = \frac{238.8\pi}{12\sqrt{2}} \frac{\sinh[\sqrt{2} \pi(1+x)]}{\cosh[\sqrt{2} \pi]} \sin\pi y \sin\pi z + 80, \quad (4-35)$$

for $-1 \leq x \leq 0$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, and

$$\begin{aligned} T_1(x,y,z) = & \frac{238.8\pi}{12\sqrt{2}} \frac{\sinh[\sqrt{2} \pi(1-x)]}{\cosh[\sqrt{2} \pi]} \sin y \sin z \\ & + \frac{238.8\pi}{4} \sin\pi x \sin\pi y \sin\pi z + 80, \end{aligned} \quad (4-36)$$

for $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$.

The source parameter S_0 is chosen to give a source of 1000 watts (238.8 cal/sec) when the source is integrated over the source volume.

The Monte Carlo solution was attempted using the forward formulation and a next-event flux at a point estimator. The parameters for the transport equation solution were

$$1. \quad \Sigma_1 = \frac{1}{3K_1} = 2 \text{ cm}^{-1}$$

$$2. \quad \Sigma_2 = \frac{1}{3K_2} = 1 \text{ cm}^{-1}$$

$$3. \quad a = b = c = 1 \text{ cm}$$

and the source S is given by

$$S = 238.8 \left(\frac{\pi}{2} \right)^3 \sin \pi x \sin \pi y \sin \pi z, \quad (4-37)$$

for $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, and zero elsewhere.

Since these parameters define a problem which is only one mean free path thick in one dimension, scaling was necessary. A scaling factor of 10 was used. This gives:

$$\Sigma_1 = 20 \text{ cm}^{-1}, \quad (4-38)$$

and

$$\Sigma_2 = 10 \text{ cm}^{-1}, \quad (4-39)$$

and the correct temperature estimate T is given by

$$T = \frac{\phi}{10}, \quad (4-40)$$

where ϕ is the flux in units of $\text{cm}^{-2} \text{ sec}^{-1}$ which is found using the scaled parameters. The results of performing a one-hour Monte Carlo calculation on the IBM 360/195 are compared to the analytical results in Table 2. The temperature was calculated at .1 cm intervals along the $y = .5 \text{ cm}$, $z = .5 \text{ cm}$ centerline.

Although the results are not unreasonable, the standard deviations are high for a one-hour run on the IBM 360/195. Since the standard deviations vary significantly from point to point, as for example, 1.52 C° to $.34 \text{ C}^\circ$ to 2.97 C° at $x = -.7$, $-.6$, and $-.5$, the possibility existed that the fluctuations and poor statistics were the result of using an unbounded estimator.

TABLE 2

TEMPERATURE COMPARISONS FOR A TWO-MEDIA RPP WITH A SINUSOIDAL
INTERNAL SOURCE USING THE FORWARD FORMULATION

x (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
-1.0	80.000	80.21	.080
- .9	80.478	80.96	.38
- .8	81.046	81.96	.55
- .7	81.832	84.30	1.52
- .6	82.985	82.39	.34
- .5	84.728	87.40	2.97
- .4	87.451	90.27	1.72
- .3	91.653	95.0	2.57
- .2	98.149	106.3	3.93
- .1	108.417	113.4	3.29
0.0	124.178	132.5	7.82
.1	166.207	192.	19.0
.2	208.713	195.	12.7
.3	243.817	238.	18.8
.4	266.025	264.	15.8
.5	272.473	276.	19.3
.6	261.488	254.	15.2
.7	233.548	226.	16.0
.8	191.042	195.	10.7
.9	138.506	146.9	12.0
1.0	80.000	91.5	1.37

Methods have been developed which produce a bounded next event flux at a point estimate,²⁴⁻²⁶ but these schemes are usually difficult to implement and require a modification of the random walk procedure. To eliminate these problems, a new bounded flux at a point estimator was developed.

For a monoenergetic flux, with isotropic scattering, the flux at a point $r = 0$, is given by

$$\phi(r=0) = \int \Sigma_s \phi_{\text{avg}}(r) 4\pi r^2 \frac{e^{-\Sigma_t r}}{4\pi r^2} dr + \int S(r) 4\pi r^2 \frac{e^{-\Sigma_t r}}{4\pi r^2} dr \quad (4-41)$$

where

$$\phi_{\text{avg}}(r) = \frac{\int \int_A \phi dA}{\int \int_A dA} \quad (4-42)$$

and A is the surface area at r . A Monte Carlo estimate of Equation (4-41) is obtained using each collision as an estimate of $\Sigma_s \phi_{\text{avg}}(r) 4\pi r^2$ and scoring $e^{-\Sigma_t r}/4\pi r^2$ at each collision. The source term may be added on analytically or $e^{-\Sigma_t r}/4\pi r^2$ is scored at each source location chosen.

The large or unbounded estimates occur from collisions very near the point at which the flux is being estimated. In this case the $e^{-\Sigma_t r}/4\pi r^2$ contribution is very large. (Note, however, that the total integral is bounded since the r^2 terms cancel.)

To eliminate this problem, consider the contributions to the flux estimator from collisions which occur within a sphere of radius 'a' about the point

$$C = \int_0^a \Sigma_s \phi_{avg}(r) 4\pi r^2 \frac{e^{-\Sigma_t r}}{4\pi r^2} dr \quad . \quad (4-43)$$

If $\phi_{avg}(r)$ is known, then a 'd' can be calculated such that

$$\int_0^a \Sigma_s \phi_{avg}(r) 4\pi r^2 \frac{e^{-\Sigma_t r}}{4\pi r^2} dr = \int_0^a \Sigma_s \phi_{avg}(r) 4\pi r^2 \frac{e^{-\Sigma_t r}}{4\pi d^2} dr \quad (4-44)$$

where $e^{-\Sigma_t r}/4\pi d^2$ is now scored at each collision which occurs within radius 'a'. This estimate is bounded and gives the correct estimate. But, the calculation of the correct 'd' depends upon knowing $\phi_{avg}(r)$. For spheres with a radius 'a' up to a mean free path, 'd' is fairly independent of the flux shape within the sphere. Calculations of 'd' are performed in Appendix B for a flux which varies linearly across the sphere and for a flux which varies exponentially across the sphere.

Using these results, the calculation of the temperature at a point was performed using several different size spheres to bound the estimator. The results of these calculations are shown in Table 3. These calculations are of the temperature at (.1, .5, .5), and were made using 4.5 min of CPU time on the IBM 360/91. The analytical result for this point is 166.207°C.

These calculations demonstrate that the poor statistics were not primarily the result of using an unbounded estimator. No collisions

TABLE 3
TEMPERATURES AT A POINT USING A BOUNDED NEXT EVENT ESTIMATOR

Bounding Radius (mean free paths)	Total Response °C	Estimated Standard Deviation C*	Response in Bounding Sphere °C	Collisions* in Bounding Sphere
0.0	160.7	17.7	0.0	0
0.1	160.7	17.7	0.0	0
0.25	176.0	26.4	34.4	3
0.50	167.9	14.9	38.0	15
1.00	159.8	9.6	46.3	86
2.00	167.2	7.9	74.0	870

*There was a total of 151,387 collisions in the problem.

occurred within .1 mean free path of the point detector and only 870 out of 151387 collisions occurred within two mean free paths. Note, however, that the collisions within two mean free paths contribute approximately 85% of the answer due to collisions (80 C° of the answer is due to the external boundary condition) and that collisions within one mean free path contributes over 50% of the answer due to collisions. The poor statistics are a result then of not having a sufficient number of collisions in the area of interest. These results suggest that the adjoint formulation would be more appropriate for this problem.

In the adjoint formulation the forward response becomes the adjoint source (all particles are started at the point of interest) and the forward source becomes the adjoint response (a collision density estimator was used to score the source at each collision site). With these changes a Monte Carlo calculation of the temperature at (.1, .5, .5) was made using 4.5 min of CPU time on the IBM 360/91. The answer obtained was 175.0°C with an estimated standard deviation of 1.52 C°. The standard deviation is approximately a factor of 10 lower than the forward calculation using the same computing time. A reduction of the standard deviation by a factor of 10 is roughly equivalent to a factor of 100 in reduction of computing time. The disadvantage of the adjoint calculation is that a separate computer run must be made for each point at which a temperature calculation is to be made.

Adjoint calculations were made for the temperatures at four points along the $y = .5$, $z = .5$ centerline. The results are shown in Table 4. Each calculation represents 4.5 min of IBM 360/91 CPU time.

TABLE 4

TEMPERATURE COMPARISONS FOR A TWO-MEDIA RPP WITH A SINUSOIDAL
INTERNAL SOURCE USING THE ADJOINT FORMULATION

x (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
- .7	81.83	82.75	.18
- .5	84.73	86.50	.31
.1	166.2	175.0	1.52
.5	272.5	284.3	3.94

The Monte Carlo results are high. The magnitude of the error is that expected due to the edge effects (not including an extrapolation distance correction). The effects of not including this correction factor and ways of minimizing this error are discussed in more detail in two-media right circular cylinder problems later in the section, in which case deterministic computer codes could be used to calculate the transport theory solution and therefore eliminate the uncertainty associated with the Monte Carlo results. One method to reduce the error associated with the edge effects is to increase the scale factor used in the calculations. The calculations of the temperature at (.1, .5, .5) were performed using a scale factor of 40. The calculation represents 20 min of CPU time on the IBM 360/195. The Monte Carlo result was 171.0°C with an estimated standard deviation of 2.05 C°. An even better answer would be obtained if a larger scale factor (requiring more CPU time) was used.

IV. TWO-MEDIA RPP WITH UNIFORM INTERNAL SOURCE

Problem 3: To calculate the temperature distribution in a three-dimensional, two media-rectangular parallelepiped as in the previous problem except with a source distribution of

$$S(x,y,z) = 2384.8 \text{ cal/sec cm}^3, \quad (4-45)$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, and zero elsewhere.

The analytical solution is obtained using the two-media RPP Green's function.

Using the following integrals

$$\int_0^b \sin \frac{m\pi y}{b} dy = \begin{cases} \frac{2b}{m\pi}, & m \text{ odd}, \\ 0, & m \text{ even}, \end{cases} \quad (4-46)$$

and

$$\int_0^a \sinh \left[\pi(a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] dx = \frac{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - 1}{\pi \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}, \quad (4-47)$$

gives

$$T_2(x, y, z) = \frac{16S_0}{\pi^4(K_1 + K_2)} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left\{ \frac{1}{mn \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \frac{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - 1}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \right. \\ \left. \cdot \frac{\sinh \left[\pi(a+x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right\}, \quad (4-48)$$

and

$$T_1(x, y, z) = T_2(-x, y, z) + \frac{64S_0}{K_1 \pi^5} \sum_{\ell \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left\{ \frac{1}{\ell mn \left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \right. \\ \left. \cdot \sin \frac{\ell\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right\}. \quad (4-49)$$

This solution is verified in Appendix A.

This solution still requires the evaluation of an infinite series to obtain numerical results. However, the series is absolutely convergent and converges very quickly.

The Monte Carlo solution was performed using the adjoint formulation. A scaling factor of 10 was used. Adjoint calculations were made for the temperature at four points along the $y = .5$, $z = .5$ centerline. The results are shown in Table 5. Each Monte Carlo calculation represents 4.5 min of IBM 360/91 CPU time.

V. TWO-MEDIA RPP WITH GIVEN SURFACE TEMPERATURE DISTRIBUTION

Problem 4: To calculate the temperature distribution in a three-dimensional, two media rectangular parallelepiped as in the previous problem except:

1. No internal sources
2. $T(x=a) = \sin \pi y/b \sin \pi z/c$.

The analytical solution is obtained using the two-media RPP Green's function. Denoting the Green's function solution by $T_g(x,y,z,x',y',z')$ where (x',y',z') is the source point, the temperature distribution due to a surface temperature $\phi(x,y,z)$ is given by

$$T(x',y',z') = K \iint_S \frac{\partial T_g(x,y,z,x',y',z')}{\partial n} \phi(x,y,z) dx dy dz, \quad (4-50)$$

where $\partial/\partial n$ denotes differentiation with respect to the x,y,z variables along the inward normal to the surface.

TABLE 5
 TEMPERATURE COMPARISONS FOR A TWO-MEDIA RPP WITH
 A UNIFORM INTERNAL SOURCE

x (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation °C
- .7	81.58	82.5	.12
- .5	84.07	86.0	.21
.1	136.67	145.2	.78
.5	164.61	174.8	.86

Performing the indicated integration using the following integral:

$$\int_0^b \sin \frac{\pi y}{b} \sin \frac{m\pi y}{b} dy = \begin{cases} b/2, & m=1, \\ 0, & m \neq 1, \end{cases} \quad (4-51)$$

and the following Fourier expansion:

$$\frac{2}{\pi} \sum_{\ell=1}^{\infty} \frac{\ell(-1)^\ell \sin \frac{\ell\pi x}{a}}{\ell^2 + a^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right)} = - \frac{\sinh \pi x \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}}, \quad (4-52)$$

gives

$$T_2(x, y, z) = \frac{K_1}{(K_1 + K_2)} \frac{\sinh \left[\pi(a+x) \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}}{\sinh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \cosh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]}, \quad (4-53)$$

for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$

and

$$T_1(x, y, z) = T_2(-x, y, z) + \frac{\sinh \pi x \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (4-54)$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, and $0 \leq z \leq c$.

This solution is verified in Appendix A. For numerical results a temperature of 80°C was used on all external boundaries except for the face at $x = 1$ which was maintained at $T(x = 1) = 2200 \sin \pi y \sin \pi z + 80$.

The Monte Carlo solutions were performed using the adjoint formulation. The external temperature was scored for each particle which escaped. A scaling factor of 10 was used. Adjoint calculations were made for the temperatures at four points along the $y = .5$, $z = .5$ centerline. The results of the Monte Carlo solutions are compared to the analytical results in Table 6. Each Monte Carlo calculation was made using 4.5 min of CPU time on the IBM 360/91.

VI. THREE-MEDIA RPP GREEN'S FUNCTION

To solve the next problem, it is first necessary to derive a Green's function for a three-media rectangular parallelepiped (Figure 4).

The same geometry as used in the two-media rectangular parallelepiped Green's function will be retained with an additional medium having a conductive coefficient K_3 added. This gives K_2 for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$, K_1 for $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, and K_3 for $a \leq x \leq 2a$, $0 \leq y \leq b$, and $0 \leq z \leq c$.

The Green's function will be calculated for a point source in the medium with conductive coefficient K_1 . Assume that the temperature at the interface at $x = 0$ is given by $\phi(y,z)$ and the temperature at the interface at $x = a$ is given by $\psi(y,z)$. Assuming no contact resistance at media interfaces and that all external boundaries are at zero temperatures, the temperatures in the three media can then be obtained from the single medium Green's function and are given by:

TABLE 6
TEMPERATURE COMPARISONS FOR A TWO-MEDIA RPP WITH A
GIVEN SURFACE TEMPERATURE DISTRIBUTION

x (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
- .7	80.716	80.94	.25
- .5	81.849	82.86	.51
.1	114.826	117.3	3.17
.5	317.666	312.5	8.13

Figure 4. Three-media RPP with point source in medium one.

$$T_2(x, y, z) = \frac{4}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \left[\pi(a+x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \cdot \int_0^c \int_0^b \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\}, \quad (4-55)$$

for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$

$$T_3(x, y, z) = \frac{4}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \left[\pi(2a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \cdot \int_0^c \int_0^b \psi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\}, \quad (4-56)$$

for $a \leq x \leq 2a$, $0 \leq y \leq b$, $0 \leq z \leq c$,

and

$$T_1(x, y, z) = \frac{4}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \left[\pi(a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right. \\ \left. \cdot \int_0^c \int_0^b \phi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\} \\ + \frac{4}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \pi x \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right\}$$

$$\begin{aligned}
& \cdot \int_0^c \int_0^b \psi(y'z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \Bigg\} + \frac{8}{K_1 \pi^2 abc} \\
& \cdot \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \sin \frac{l\pi x}{a} \sin \frac{l\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} ,
\end{aligned} \tag{4-57}$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$,

where $\phi(y,z)$ and $\psi(y,z)$ are still to be determined. At the interface at $x = 0$ continuity of heat flow gives

$$K_2 \left. \frac{\partial T_2}{\partial x} \right|_{x=0} = K_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=0} . \tag{4-58}$$

Solving for $\left. \frac{\partial T_2}{\partial x} \right|_{x=0}$ gives

$$\begin{aligned}
\left. \frac{\partial T_2}{\partial x} \right|_{x=0} &= \frac{4\pi}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \\
&\cdot \int_0^c \int_0^b \phi(y'z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \Bigg\} .
\end{aligned} \tag{4-59}$$

For simplicity of notation the integral term will be denoted $F(m,n)$.

Solving for $\left. \frac{\partial T_1}{\partial x} \right|_{x=0}$ gives

$$\begin{aligned}
\left. \frac{\partial T_1}{\partial x} \right|_{x=0} = & - \frac{4\pi}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} F(m,n) \\
& + \frac{4\pi}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \right. \\
& \cdot \left. \int_0^c \int_0^b \psi(y', z') \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} dy' dz' \right\} + \frac{4\pi}{K_1 bc} \\
& \cdot \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{2l\pi}{l^2 \pi^2 + a^2 \pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{l\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c},
\end{aligned} \tag{4-60}$$

where the remaining integral term will be denoted $G(m,n)$.

Using

$$\sum_{l=1}^{\infty} \frac{2l\pi}{l^2 \pi^2 + a^2 \pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{l\pi z'}{a} = \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}, \tag{4-61}$$

and setting $K_2 \left. \frac{\partial T_2}{\partial x} \right|_{x=0} = K_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=0}$ gives

$$\begin{aligned}
& \frac{4\pi}{bc}(K_1 + K_2) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} F(m, n) \\
&= \frac{4\pi K_1}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} G(m, n) + \frac{4}{bc} \\
&\cdot \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} . \quad (4-62)
\end{aligned}$$

For termwise equality, which follows from the orthogonality of sine functions,

$$\begin{aligned}
& \pi(K_1 + K_2) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} F(m, n) \\
&= \pi K_1 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} G(m, n) + \sinh \pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} . \\
&\hspace{15em} (4-63)
\end{aligned}$$

For continuity of heat flow at the interface at $x = a$

$$K_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=a} = K_3 \left. \frac{\partial T_3}{\partial x} \right|_{x=a} . \quad (4-64)$$

Solving for $\left. \frac{\partial T_3}{\partial x} \right|_{x=a}$ gives

$$\left. \frac{\partial T_3}{\partial x} \right|_{x=a} = \frac{4\pi}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} G(m,n) . \quad (4-65)$$

Solving for $\left. \frac{\partial T_1}{\partial x} \right|_{x=a}$ gives

$$\begin{aligned} \left. \frac{\partial T_1}{\partial x} \right|_{x=a} = & - \frac{4\pi}{bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} F(m,n) + \frac{4\pi}{bc} \\ & \cdot \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} G(m,n) + \frac{4}{K_1 bc} \\ & \cdot \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{2\ell\pi(-1)^\ell}{\ell^2 \pi^2 + a^2 \pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{\ell\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} . \end{aligned} \quad (4-66)$$

Using

$$\sum_{\ell=1}^{\infty} \frac{2\ell\pi(-1)^\ell}{\ell^2 \pi^2 + a^2 \pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{\ell\pi x}{a} = - \frac{\sinh \pi x \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} , \quad (4-67)$$

and setting $K_3 \left. \frac{\partial T_3}{\partial x} \right|_{x=a} = K_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=a}$ gives

$$\begin{aligned}
 & - \frac{4\pi}{bc} (K_1 + K_3) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} G(m,n) \\
 & = - \frac{4\pi}{bc} K_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \frac{\sin \frac{m\pi y}{b} \sin \frac{n\pi x}{c}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} F(m,n) - \frac{4}{bc} \\
 & \quad \cdot \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sinh \pi x' \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} . \quad (4-68)
 \end{aligned}$$

For termwise equality

$$\begin{aligned}
 & \pi (K_1 + K_3) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} G(m,n) \\
 & = \pi K_1 \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} F(m,n) + \sinh \pi x' \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c} . \quad (4-69)
 \end{aligned}$$

This gives two equations for $F(m,n)$ and $G(m,n)$. Solving these gives

$$F(m, n) = \frac{\sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c}}{\pi \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \cdot \left\{ \frac{(K_1 + K_3) \cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{(K_1 + K_3)(K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - K_1^2} + \frac{K_1 \sinh \pi x' \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{(K_1 + K_3)(K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - K_1^2} \right\}, \quad (4-70)$$

and

$$G(m, n) = \frac{\sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c}}{\pi \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}} \cdot \left\{ \frac{(K_1 + K_2) \cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sinh \pi x' \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{(K_1 + K_3)(K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - K_1^2} + \frac{K_1 \sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{(K_1 + K_3)(K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - K_1^2} \right\}. \quad (4-71)$$

Note that

$$\lim_{K_3 \rightarrow \infty} F(m, n) = \frac{\sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\pi (K_1 + K_2) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}, \quad (4-72)$$

and

$$\lim_{K_3 \rightarrow \infty} G(m,n) = 0, \quad (4-73)$$

which gives the two-media RPP Green's function derived previously.

Using $F(m,n)$ and $G(m,n)$ then gives

$$T_2(x,y,z) = \frac{4}{\pi bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\sin \frac{m\pi y'}{b} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} \sinh \left[\pi(a+x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \right. \\ \cdot \left. \frac{\left\{ (K_1 + K_3) \cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] \right\}}{(K_1 + K_3)(K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - K_1^2} \right. \\ \left. + \frac{K_1 \sinh \pi x' \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{(K_1 + K_3)(K_1 + K_2) \cosh^2 \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} - K_1^2} \right\}, \quad (4-74)$$

for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$,

$$\begin{aligned}
T_3(x, y, z) = \frac{4}{\pi bc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} & \left\{ \frac{\sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c} \sinh \left[\pi(2a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \right. \\
& - \frac{\left((K_1 + K_3) \cosh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \sinh \pi x' \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right)}{\left((K_1 + K_3) (K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - K_1^2 \right)} \\
& \left. + \frac{K_1 \sinh \left[\pi(a-x') \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\left((K_1 + K_3) (K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] - K_1^2 \right)} \right\}, \quad (4-75)
\end{aligned}$$

for $a \leq x \leq 2a$, $0 \leq y \leq b$, $0 \leq z \leq c$,

and

$$\begin{aligned}
T_1(x, y, z) = T_2(-x, y, z) + T_3(2a-x, y, z) + \frac{8}{K_1 \pi^2 abc} \\
\cdot \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \sin \frac{\ell\pi x}{a} \sin \frac{\ell\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi z'}{c}, \quad (4-76)
\end{aligned}$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

VII. THREE-MEDIA RPP WITH SINUSOIDAL INTERNAL SOURCE

Problem 5: To calculate the temperature distribution in a three-media, rectangular parallelepiped with no contact resistance between media and with a fixed, uniform temperature on all external boundaries (Figure 5).

Assume:

1. K_1 is the thermal conductivity for $0 \leq x \leq a$, $0 \leq y \leq b$,
 $0 \leq z \leq c$
2. K_2 is the thermal conductivity for $-a \leq x \leq 0$, $0 \leq y \leq b$,
 $0 \leq z \leq c$
3. K_3 is the thermal conductivity for $a \leq x \leq 2a$, $0 \leq y \leq b$,
 $0 \leq z \leq c$
4. Steady state
5. $S = S_0 \sin \pi x/a \sin \pi y/b \sin \pi z/c$ for $0 \leq x \leq a$, $0 \leq y \leq b$,
 $0 \leq z \leq c$

The analytical solution for this problem can be obtained using the previously generated Green's function.

$$T(x,y,z) = \iiint_V T_g(x,y,z,x',y',z') S(x',y',z') dx' dy' dz' \quad (4-77)$$

Using

$$\int_0^b \sin \frac{\pi y}{b} \sin \frac{m\pi y}{b} dy = \begin{cases} b/2, & m=1, \\ 0, & m \neq 1, \end{cases} \quad (4-78)$$

and

Figure 5. Three-media RPP with distributed source in medium one.

$$\int_0^a \sinh \left[\pi(a-x) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right] \sin \frac{\pi x}{a} dx = \frac{a \sinh \pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}}{\pi + a^2 \pi \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)}, \quad (4-79)$$

gives

$$\begin{aligned} T_2(x, y, z) = & \frac{S_0 a}{\pi^2} \frac{1}{\left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}} \cdot \frac{1}{1+a^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right)} \\ & \cdot \frac{(K_1 + K_3) \cosh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] + K_1}{(K_1 + K_3) (K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] - K_1^2} \\ & \cdot \sinh \left[\pi(a+x) \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \end{aligned} \quad (4-80)$$

for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$,

$$\begin{aligned} T_3(x, y, z) = & \frac{S_0 a}{\pi^2} \left\{ \frac{1}{\left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}} \cdot \frac{1}{1+a^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right)} \right. \\ & \cdot \frac{(K_1 + K_3) \cosh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] + K_1}{(K_1 + K_3) (K_1 + K_2) \cosh^2 \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] - K_1^2} \end{aligned}$$

$$\left. \sinh \left[\pi(2a-x) \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \right\}, \quad (4-81)$$

for $a \leq x \leq 2a$, $0 \leq y \leq b$, $0 \leq z \leq c$,

and

$$T_1(x,y,z) = T_2(-x,y,z) + T_3(2a-x,y,z) + \frac{S_0}{\pi^2 K_1} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (4-82)$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

This solution is verified in Appendix A. The parameters for the problem are:

1. $K_1 = .1667 \text{ cal/sec cm}^\circ\text{C}$
2. $K_2 = .3333 \text{ cal/sec cm}^\circ\text{C}$
3. $K_3 = .3333 \text{ cal/sec cm}^\circ\text{C}$
4. $a = 1 \text{ cm}$
5. $b = 1 \text{ cm}$
6. $c = 1 \text{ cm}$
7. $S_0 = 238.8 (\pi/2)^3 \text{ cal/sec cm}^3$,

with an external surface temperature of 80°C .

The Monte Carlo solution was performed using the adjoint formulation. A scaling factor of 10 was used. Adjoint calculations were made for the temperature at four points along the $y = .5$, $z = .5$

centerline. Each Monte Carlo calculation represents 4.5 min of IBM 360/91 CPU time. The results are shown in Table 7.

VIII. TWO-MEDIA RCC GREEN'S FUNCTIONS

The next series of problems requires analytical solutions for two-media right circular cylinders. The first step in generating the solutions to these problems is to derive an appropriate Green's function (Figure 6).

From Carslaw and Jaeger,²² the one-medium Green's function for a right circular cylinder (RCC) $0 \leq z \leq l$, $0 \leq \theta \leq \pi$, $0 \leq r \leq a$ with zero surface temperature and a unit instantaneous point source at $(z', \theta', r', t = 0)$ is

$$T = \frac{2}{\pi a^2 l} \sum_{m=1}^{\infty} \left[e^{-K m^2 \pi^2 z z' / l^2} \sin \frac{m \pi z}{l} \sin \frac{m \pi z'}{l} \right. \\ \left. \cdot \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \sum_{\alpha} e^{-K \alpha^2 t} \frac{J_n(\alpha r) J_n(\alpha r')}{[J_n'(\alpha a)]^2} \right], \quad (4-83)$$

where α are positive roots of

$$J_n(\alpha a) = 0. \quad (4-84)$$

For a source with symmetry about the z axis, the integration over θ can be performed using

TABLE 7
 TEMPERATURE COMPARISONS FOR A THREE-MEDIA RPP WITH
 A SINUSOIDAL INTERNAL SOURCE

x (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
- .7	81.849	82.6	.16
- .5	84.89	86.7	.32
.1	166.9	176.7	1.93
.5	276.9	290.6	2.30

Figure 6. Two-media RCC with a ring source in medium one.

$$\int_0^{2\pi} \cos n(\theta - \theta') d\theta = \begin{cases} 2\pi, & n=0, \\ 0, & n \neq 0, \end{cases} \quad (4-85)$$

to give

$$T = \frac{4}{a^2 \ell} \sum_{m=1}^{\infty} e^{-Km^2 \pi^2 t / \ell^2} \sin \frac{m\pi z}{\ell} \sin \frac{m\pi z'}{\ell} \sum_{\alpha} e^{-K\alpha t} \frac{J_0(\alpha r) J_0(\alpha r')}{[J_0^2(\alpha a)]^2}, \quad (4-86)$$

where α are positive roots of $J_0(\alpha a) = 0$, or, since $J_0'(x) = -J_1(x)$

$$T = \frac{4}{a^2 \ell} \sum_{m=1}^{\infty} e^{-Km^2 \pi^2 t / \ell^2} \sin \frac{m\pi z}{\ell} \sin \frac{m\pi z'}{\ell} \sum_{n=1}^{\infty} e^{-K\alpha_n t} \frac{J_0(\alpha_n r) J_0(\alpha_n r')}{J_1^2(\alpha_n a)}, \quad (4-87)$$

where α_n are positive roots of $J_0(\alpha_n a) = 0$.

For steady state the solution is given by $\lim_{t \rightarrow \infty} \int_0^t T(r, z, t - \tau) d\tau$.

Using

$$\int_0^t e^{-K(m^2 \pi^2 / \ell^2 + \alpha_n^2)(t - \tau)} d\tau = \frac{1}{K(m^2 \pi^2 / \ell^2 + \alpha_n^2)} (1 - e^{-K(m^2 \pi^2 / \ell^2 + \alpha_n^2)t}), \quad (4-88)$$

taking $\lim_{t \rightarrow \infty}$ gives $\frac{1}{K(\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2)}$

or, for a steady-state source symmetric about the z axis

$$T_g = \frac{4}{a^2 \ell} \sum_{m=1}^{\infty} \sin \frac{m\pi z}{\ell} \sin \frac{m\pi z'}{\ell} \sum_{n=1}^{\infty} \frac{1}{K(\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2)} \frac{J_0(\alpha_n r) J_0(\alpha_n r')}{J_1^2(\alpha_n a)}. \quad (4-89)$$

If the RCC face is maintained at $f(r)$, other faces at zero, the temperature is given by

$$T = K \int_0^a r' f(r') \left. \frac{\partial T}{\partial z'} \right|_{z'=0} dr' . \quad (4-90)$$

This gives

$$T = \frac{4\pi}{a^2 \ell^2} \sum_{m=1}^{\infty} m \sin \frac{m\pi z}{\ell} \sum_{n=1}^{\infty} \frac{1}{\left[\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right]} \frac{J_0(\alpha_n r)}{J_1^2(\alpha_n a)} \int_0^a r' f(r') J_0(\alpha_n r') dr' . \quad (4-91)$$

Substituting

$$\sum_{m=1}^{\infty} \frac{2m\pi}{\ell^2} \frac{1}{\left[\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right]} \sin \frac{m\pi z}{\ell} = \frac{\sinh \alpha_n (\ell - z)}{\sinh \alpha_n \ell} \quad (4-92)$$

into Equation (4-91) gives

$$T = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{J_1^2(\alpha_n a)} \frac{\sinh \alpha_n (\ell - z)}{\sinh \alpha_n \ell} \int_0^a r' f(r') J_0(\alpha_n r') dr' . \quad (4-93)$$

To obtain a Green's function for a composite two-media region, first assume an RCC $-\ell \leq z \leq 0$, $0 \leq r \leq a$, $K = K_2$ and $f(r)$ at $z = 0$, with the other faces at zero, then

$$T_2 = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{J_1^2(\alpha_n a)} \frac{\sinh \alpha_n (\ell + z)}{\sinh \alpha_n \ell} \int_0^a r' f(r') J_0(\alpha_n r') dr' . \quad (4-94)$$

Now assume an RCC $0 \leq z \leq \ell$, $0 \leq r \leq a$, $K = K_1$, and $f(r)$ at $z = 0$ with the other faces at zero and a symmetric ring source at z' , r' , then

$$T_1 = T_2(r, -z) + \frac{4}{a^2 \ell} \sum_{m=1}^{\infty} \left[\sin \frac{m\pi z}{\ell} \sin \frac{m\pi z'}{\ell} \right. \\ \left. \cdot \sum_{n=1}^{\infty} \frac{1}{K_1 \left(\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right)} \frac{J_0(\alpha_n r) J_0(\alpha_n r')}{J_1^2(\alpha_n a)} \right]. \quad (4-95)$$

For continuity of heat flow at $z = 0$

$$K_2 \left. \frac{\partial T_2}{\partial z} \right|_{z=0} = K_1 \left. \frac{\partial T_1}{\partial z} \right|_{z=0}. \quad (4-96)$$

From Equation (4-94)

$$\left. \frac{\partial T_2}{\partial z} \right|_{z=0} = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{J_1^2(\alpha_n a)} \frac{\alpha_n \cosh \alpha_n \ell}{\sinh \alpha_n \ell} \int_0^a r' f(r') J_0(\alpha_n r') dr'. \quad (4-97)$$

From Equation (4-95)

$$\left. \frac{\partial T_1}{\partial z} \right|_{z=0} = - \left. \frac{\partial T_2}{\partial z} \right|_{z=0} + \frac{4\pi}{a^2 \ell^2} \sum_{m=1}^{\infty} m \sin \frac{m\pi z'}{\ell} \sum_{n=1}^{\infty} \frac{1}{K_1 \left(\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right)} \frac{J_0(\alpha_n r) J_0(\alpha_n r')}{J_1^2(\alpha_n a)}. \quad (4-98)$$

Using

$$\sum_{m=1}^{\infty} \frac{2\pi m}{\ell^2} \frac{1}{\left[\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right]} \sin \frac{m\pi z}{\ell} = \frac{\sinh(\ell-z)\alpha_n}{\sinh \alpha_n \ell}, \quad (4-99)$$

and setting

$$K_2 \left. \frac{\partial T_2}{\partial z} \right|_{z=0} = K_1 \left. \frac{\partial T_1}{\partial z} \right|_{z=0}, \quad (4-100)$$

gives

$$\begin{aligned} \frac{2}{a^2} (K_1 + K_2) \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{J_1^2(\alpha_n a)} \alpha_n \frac{\cosh \alpha_n \ell}{\sinh \alpha_n \ell} \int_0^a r' f(r') J_0(\alpha_n r') dr' \\ = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r) J_0(\alpha_n r')}{J_1^2(\alpha_n a)} \frac{\sinh(\ell - z') \alpha_n}{\sinh \alpha_n \ell}. \end{aligned} \quad (4-101)$$

Assuming termwise equality gives

$$\int_0^a r' f(r') J_0(\alpha_n r') dr' = \frac{1}{K_1 + K_2} \frac{\sinh \alpha_n (\ell - z')}{\alpha_n \cosh \alpha_n \ell} J_0(\alpha_n r'). \quad (4-102)$$

Substituting Equation (4-102) into Equations (4-94) and (4-95) gives the solution as

$$T_2(r, z, r', z') = \frac{2}{a^2 (K_1 + K_2)} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r) J_0(\alpha_n r')}{\alpha_n \sinh \alpha_n \ell \cosh \alpha_n \ell} \frac{\sinh \alpha_n (\ell + z) \sinh \alpha_n (\ell - z')}{\cosh \alpha_n \ell}, \quad (4-103)$$

and

$$T_1(r, z, r', z') = T_2(r, -z, r', z')$$

$$+ \frac{4}{a^2 \ell} \sum_{m=1}^{\infty} \sin \frac{m\pi z}{\ell} \sin \frac{m\pi z'}{\ell} \sum_{n=1}^{\infty} \frac{1}{K_1 \left(\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right)} \frac{J_0(\alpha_n r) J_0(\alpha_n r')}{J_1^2(\alpha_n a)}. \quad (4-104)$$

With the form of the Green's function solution $T_g(r, z, r', z')$, if $S(r, z, \theta)$ is any arbitrary source in medium one symmetric about z , i.e., $S(r, z, \theta) = S(r, z)$, then the temperature is given by

$$T = \int_{z'} \int_{r'} S(r', z') T_g(r, z, r', z') r' dr' dz' . \quad (4-105)$$

IX. TWO-MEDIA RCC WITH FUNCTIONAL INTERNAL SOURCE

Problem 6: To calculate the temperature distribution in a two-media, right circular cylinder (Figure 7).

Assume:

1. K_1 is the thermal conductivity for $0 \leq z \leq \ell$, $0 \leq r \leq a$
2. K_2 is the thermal conductivity for $-\ell \leq z \leq 0$, $0 \leq r \leq a$
3. Steady state
4. No contact resistance between media
5. All external surface temperatures are zero
6. $S = S_0 J_0(\alpha_1 r) \sin \pi z / \ell$, $0 \leq r \leq a$, $0 \leq z \leq \ell$.

The analytical solution for this problem can be obtained using the previously generated Green's function.

Using

$$\int_0^a r J_0(\alpha_n r) J_0(\alpha_m r) dr = \begin{cases} \frac{a^2}{2} J_1^2(\alpha_n a) & , \quad n=m , \\ 0 & , \quad n \neq m , \end{cases} \quad (4-106)$$

$$\int_0^\ell \sin \frac{m\pi z}{\ell} \sin \frac{\pi z}{\ell} dz = \begin{cases} \ell/2 & , \quad m=1 , \\ 0 & , \quad m \neq 1 , \end{cases} \quad (4-107)$$

DISTRIBUTED SOURCE IN MEDIUM ONE

Figure 7. Two-media RCC with a distributed source in medium one.

and

$$\int_0^{\ell} \sinh \alpha_n (\ell - z) \sin \pi z dz = \frac{\pi \ell}{\pi^2 + \ell^2 \alpha_n^2} \sinh \alpha_n \ell, \quad (4-108)$$

gives

$$T_2(r, z) = \frac{S_0 \pi \ell}{\pi^2 + \ell^2 \alpha_1^2} \frac{J_0(\alpha_1 r)}{\alpha_1 (K_1 + K_2)} \frac{\sinh \alpha_1 (\ell + z)}{\cosh \alpha_1 \ell}, \quad (4-109)$$

for $0 \leq r \leq a$, $-\ell \leq z \leq 0$,

and

$$T_1(r, z) = T_2(r, -z) + \frac{S_0}{K_1 \left(\alpha_1^2 + \frac{\pi^2}{\ell^2} \right)} J_0(\alpha_1 r) \sin \frac{\pi z}{\ell}, \quad (4-110)$$

for $0 \leq r \leq a$, $0 \leq z \leq \ell$.

This solution is verified in Appendix A.

The parameters for the problem are

1. $K_1 = .1667 \text{ cal/sec cm}^\circ\text{C}$
2. $K_2 = .3333 \text{ cal/sec cm}^\circ\text{C}$
3. $a = .5 \text{ cm}$
4. $\ell = 1 \text{ cm}$
5. $S = 238.8 (4.632259) J_0(r\alpha_1) \sin \pi z \text{ cal/sec cm}^2 \begin{cases} 0 \leq r \leq .5 \\ 0 \leq z \leq 1 \end{cases}$

with an external surface temperature of 80°C . The constant

4.632259 is the normalization factor for a unit source.

The Monte Carlo solution was performed using the adjoint formulation and a scaling factor of 10. Adjoint calculations were made for the temperature at four points along the $r = 0$ centerline. Each Monte Carlo calculation required 4.5 min of IBM 360/91 CPU time. The results are shown in Table 8.

The use of the scale factor to allow the transport equation to more closely approximate the diffusion equation was studied for this problem. Calculations were made using the DOT²⁷ discrete ordinates radiation transport code. These results are shown in Table 9. (The DOT diffusion theory calculations include an extrapolation distance correction to approximate transport theory.) As can be seen, the transport equation approximation can be made as close as desired by increasing the scale factor used in the calculations. However, as the scale factor is increased, the time involved in performing the calculation increases also. For example, a calculation of the temperature at $x = -.7$ cm was calculated by Monte Carlo using a scale factor of 40. The results obtained were 81.44°C with an estimated standard deviation of $.38^{\circ}\text{C}$. This run required 4.5 min of IBM 360/91 CPU time.

X. TWO-MEDIA RCC WITH UNIFORM INTERNAL SOURCE

Problem 7: To calculate the temperature distribution in a two-media, right circular cylinder as in the previous problem except with a source distribution of

TABLE 8
TEMPERATURE COMPARISONS FOR A TWO-MEDIA RCC
WITH A FUNCTIONAL INTERNAL SOURCE

x (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
- .7	81.426	82.69	.21
- .5	83.921	86.19	.35
.1	169.209	180.0	2.16
.5	285.031	301.9	2.63

TABLE 9
TEMPERATURE COMPARISONS FOR DOT RUNS WITH
DIFFERENT SCALING FACTORS

z Scale Factor (cm)	Analytic °C	DOT Diffusion 1000 °C	DOT Diffusion 100 °C	DOT Transport 40 °C	DOT Transport 20 °C	DOT Transport 10 °C
- .9	80.356	80.361	80.396	80.418	80.554	80.857
- .8	80.798	80.805	80.860	80.876	81.089	81.543
- .7	81.426	81.438	81.516	81.545	81.855	82.500
- .6	82.388	82.412	82.531	82.579	83.009	83.916
- .5	83.92	83.94	84.11	84.20	84.82	85.99
- .4	86.38	86.40	86.62	86.78	87.62	89.19
- .3	90.34	90.36	90.65	90.94	92.01	94.02
- .2	96.72	96.76	97.12	97.60	98.96	101.37
- .1	107.0	107.0	107.5	108.2	109.9	113.0
0.0	123.7	123.9	124.4	125.8	127.5	131.8
.1	169.2	168.8	169.6	171.7	174.6	180.3
.2	214.9	214.4	215.4	218.3	221.1	228.3
.3	253.1	252.4	253.6	257.2	260.3	268.4
.4	277.7	277.0	278.2	282.5	285.6	294.4
.5	285.0	284.4	285.6	289.9	293.2	302.3
.6	273.7	273.0	274.4	278.4	281.5	290.9
.7	244.1	243.6	245.0	248.6	251.9	260.8
.8	198.9	198.7	200.1	202.7	206.6	215.4
.9	142.6	142.6	144.0	146.1	150.9	159.5

$$S(x, y, z) = S_0, \quad (4-111)$$

for $0 \leq r \leq a$, $0 \leq z \leq \ell$.

The analytical solution is obtained using the two-media, right circular cylinder Green's function.

Using the following integrals:

$$\int_0^a r J_0(\alpha_n r) dr = \frac{a}{\alpha_n} J_1(\alpha_n a), \quad (4-112)$$

and

$$\int_0^\ell \sinh \alpha_n (\ell - z) dz = \frac{\cosh \alpha_n \ell - 1}{\alpha_n}, \quad (4-113)$$

gives

$$T_2(r, z) = \frac{2 S_0}{a(K_1 + K_2)} \sum_{n=1}^{\infty} \frac{\cosh \alpha_n \ell - 1}{\alpha_n^3 J_1(\alpha_n a)} \frac{J_0(\alpha_n r)}{\sinh \alpha_n \ell} \frac{\sinh \alpha_n (\ell + z)}{\cosh \alpha_n \ell}, \quad (4-114)$$

for $0 \leq r \leq a$, $-\ell \leq z \leq 0$,

and

$$T_1(r, z) = T_2(r, -z) + \frac{8 S_0}{a \pi} \sum_{m \text{ odd}}^{\infty} \frac{1}{m} \sin \frac{m \pi z}{\ell} \sum_{n=1}^{\infty} \frac{1}{K_1 \alpha_n \left(\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right)} \frac{J_0(\alpha_n r)}{J_1(\alpha_n a)}, \quad (4-115)$$

for $0 \leq r \leq a$, $0 \leq z \leq \ell$.

This solution is verified in Appendix A.

For a unit source, $S_0 = 1/\pi a^2 \ell$.

This solution still requires the evaluation of an infinite series to obtain numerical results. The series is absolutely convergent, however, and converges very quickly. A source of

$$S_0 = \frac{238.8}{\pi a^2 \ell} \text{ cal/sec ,} \quad (4-116)$$

with an external boundary temperature of 80°C was used for the numerical calculations.

The Monte Carlo solution was performed using the adjoint formulation and a scaling factor of 10. Adjoint calculations were made at four points along the $r = 0$ axis. Each Monte Carlo calculation required 4.5 min of IBM 360/91 CPU time. The results of these calculations are shown in Table 10.

XI. TWO-MEDIA RCC WITH GIVEN SURFACE TEMPERATURE DISTRIBUTION

Problem 8: To calculate the temperature distribution in a two-media right circular cylinder as in the previous problem except:

1. No internal sources
2. $T(z=\ell) = J_0(\alpha, r)$.

The analytical solution is obtained using the two-media, RCC Green's function. Denoting the Green's function solution by $T_g(r, z, r', z')$ for a ring source at (r', z') , the temperature distribution due to a surface temperature $\phi(r, z)$ is given by

TABLE 10
TEMPERATURE COMPARISONS FOR A TWO-MEDIA RCC
WITH A UNIFORM INTERNAL SOURCE

z (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
- .7	80.716	80.94	.25
- .5	81.849	82.86	.51
.1	114.826	117.3	3.17
.5	317.666	312.5	8.13

$$T(r', z') = K \int_r \int_z \frac{\partial T_g(r, z, r', z')}{\partial n} \phi(r, z) r dr dz, \quad (4-117)$$

where $\partial/\partial n$ denotes differentiation with respect to the r, z variables along the inward normal to the surface over which the integration is taken. Note that the integration over ϕ is already incorporated into the Green's function.

Performing the indicated integration using the following integral:

$$\int_0^a r J_0(\alpha_n r) J_0(\alpha_m r) dr = \begin{cases} \frac{a^2}{2} [J_1(\alpha_n a)]^2, & n=m, \\ 0, & n \neq m, \end{cases} \quad (4-118)$$

and the Fourier expansion

$$\frac{2}{\pi} \sum_{m=1}^{\infty} \frac{m(-1)^m \sin \frac{m\pi z}{\ell}}{m^2 + \frac{\ell^2 \alpha_1^2}{\pi^2}} = - \frac{\sinh \alpha_1 z}{\sinh \alpha_1 \ell} \quad (4-119)$$

gives

$$T_2(r, z) = \frac{K_1}{K_1 + K_2} J_0(\alpha, r) \frac{\sinh \alpha_1 (\ell + z)}{\sinh \alpha_1 \ell \cosh \alpha_1 \ell}, \quad (4-120)$$

for $0 \leq r \leq a$, $-\ell \leq z < a$,

and

$$T_1(r, z) = T_2(r, -z) + J_0(\alpha, r) \frac{\sinh \alpha_1 z}{\sinh \alpha_1 \ell}, \quad (4-121)$$

for $0 \leq r \leq a$, $0 \leq z \leq \ell$.

where α_1 is the first zero of $J_0(\alpha_n a)$. This solution is verified in Appendix A.

A temperature of 80°C used on all external boundaries except for the face at $z = 1$ which was maintained at $T(z = 1) = 2200 J_0(\alpha, r) + 80$. The Monte Carlo solutions were performed using the adjoint formulation and a scaling factor of 10. Adjoint calculations were made for the temperatures at four points along the $r = 0$ axis. The results of the Monte Carlo solutions compared to the analytical results in Table 11. Each Monte Carlo calculation was made using 4.5 min of CPU time on the IBM 360/91.

XII. TWO-MEDIA RCC WITH FUNCTIONAL INTERNAL SOURCE AND GIVEN SURFACE TEMPERATURE DISTRIBUTION

Problem 9: To calculate the temperature distribution in a two media right circular cylinder with both an internal source and a given external temperature distribution.

For convenience, the parameters to be used for this problem are those which will give the same analytic solution as problem 6. This is accomplished by setting the outer radius of the right circular cylinder at $a/2$ and using the solution for problem 6 at $r = a/2$ as the given external temperature distribution with all other parameters remaining the same.

The Monte Carlo solution was performed using the adjoint formulation and a scale factor of 10. Adjoint calculations were made at two points along the $r = 0$ axis. At $z = -.7$ the analytical solution

TABLE 11
 TEMPERATURE COMPARISONS FOR A TWO-MEDIA RCC WITH
 A GIVEN SURFACE TEMPERATURE DISTRIBUTION

z (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
- .7	80.389	80.66	.21
- .5	81.070	81.75	.37
.1	105.31	108.2	2.71
.5	278.08	279.2	6.98

is 81.426°C. The Monte Carlo solution using 4.2 min of IBM 360/91 CPU time was 81.91°C with an estimated standard deviation of .03 C°. At $z = .5$ the analytical solution is 285.031°C. The Monte Carlo solution using 4.5 min of IBM 360/91 CPU time was 302.66°C with an estimated standard deviation of .82 C°. The Monte Carlo solution with a scale factor of .40 using 4.2 min of IBM 360/91 CPU time was 287.76°C with an estimated standard deviation of 4.0 C°.

XIII. SPHERE WITH TIME DEPENDENT SOURCE

Problem 10: To calculate the time dependent temperature distribution in a one medium sphere with a uniform, time dependent source.

Assume:

1. A sphere of radius a
2. K is the thermal conductivity for $0 \leq r \leq a$
3. δ is the diffusivity for $0 \leq r \leq a$
4. Surface and initial temperature of 0°C
5. A uniform source of $S = S_0 e^{-\lambda t}$ for $t > 0$.

The analytical solution for this problem can be obtained from Carslaw and Jaeger.²²

$$T(r,t) = \frac{S_0 \delta}{K \lambda} e^{-\lambda t} \frac{a \sin r(\lambda/\delta)^{1/2}}{r \sin a(\lambda/\delta)^{1/2}} - 1$$

$$+ \frac{2a^3 S_0}{r \pi^3 K} \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n^2 - \lambda a^2 / \delta \pi^2)} \sin \frac{n \pi r}{a} e^{-\delta n^2 \pi^2 t / a^2} \quad (4-122)$$

For numerical results, the parameters used were:

1. $S_0 = 23.88 \text{ cal/sec cm}^3$
2. $K = .48 \text{ cal/sec cm}^\circ\text{C}$
3. $a = 20 \text{ cm}$
4. $\lambda = .01/\text{sec}$
5. $c = .22 \text{ cal/g}^\circ\text{C}$
6. $\rho = 2.7 \text{ g/cm}^3$

The temperature was calculated at various times for the center of the sphere both analytically and using Monte Carlo. The results of these calculations are shown in Table 12. The Monte Carlo calculations were made using the adjoint formulation. No scaling was used. For a time dependent calculation a velocity of $1/\rho c$ is used during the tracking. An extrapolation distance correction was made for the Monte Carlo calculation. The Monte Carlo results were obtained with 2.5 min of IBM 360/91 CPU time. An advantage of the adjoint formulation for problems with time dependent sources and boundary conditions is that the solution can be generated for many different times with the same calculation. For example, the above 2.5 min run performed the calculation for 59 different times.

XIV. SPHERE WITH TIME DEPENDENT SOURCE AND CONVECTIVE BOUNDARY CONDITION

Problem 11: To calculate the temperature distribution in a one-medium sphere with a convective boundary condition.

TABLE 12
 TEMPERATURE COMPARISONS FOR A SPHERE WITH
 A TIME DEPENDENT SOURCE

t (sec)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
1	40.09	41.	1.8
5	196.53	199.	3.6
10	383.48	389.	6.2
50	1435.92	1444.	17.3
70	1603.05	1610.	29.0
80	1619.85	1621.	34.0
90	1605.10	1598.	38.4
100	1566.83	1546.	40.2
110	1511.61	1483.	43.0
150	1210.86	1164.	47.7
200	830.41	787.	45.6
500	48.355	47.	.5

Assume:

1. A sphere of radius a
2. K is the thermal conductivity for $0 \leq r \leq a$
3. δ is the diffusivity for $0 \leq r \leq a$
4. Initial temperature is 0°C
5. A uniform source of S_0 for $t > 0$
6. Convection at $r = a$ into a medium at 0°C
7. h is the coefficient of convection transfer

The analytical solution can be obtained from Carslaw and Jaeger.²²

The temperature for $r = 0$ is given by

$$T(r=0,t) = \frac{S_0 a^2}{6K} + \frac{S_0 a}{3h} - \frac{2hS_0 a^2}{K^2} \sum_{n=1}^{\infty} \frac{e^{-\delta \alpha_n^2 t}}{\alpha_n^2 a^2 \alpha_n^2 + \frac{ah}{K} \left(\frac{ah}{K} - 1 \right) \sin a \alpha_n} \quad (4-123)$$

where α_n are the roots of $a\alpha \cot a\alpha = 1 - ah/K$. This series converges very quickly when evaluated numerically.

To perform the Monte Carlo calculation, it is necessary to develop a method for applying the convective boundary condition to transport theory. The convective boundary condition for heat conduction problems can be written:

$$K_1 \frac{\partial T}{\partial n} = h (T_\infty - T) \quad . \quad (4-124)$$

This can be rewritten as

$$K_1 \frac{\partial T}{\partial n} = h \Delta n \left(\frac{T_\infty - T}{\Delta n} \right) \quad . \quad (4-125)$$

If it is assumed that T_{∞} is the temperature at a distance Δn away from the surface, then $h\Delta n$ has the form of a conduction coefficient and $(T_{\infty}-T)/\Delta n$ represents the change in temperature per unit distance. Let $K_2 = h\Delta n$. Then this boundary condition could be approximated in transport theory by adding a layer Δn thick onto the boundary with cross section Σ given by

$$\Sigma = \frac{1}{3K_2} = \frac{1}{3h\Delta n} . \quad (4-126)$$

The thickness of this layer in mean-free-paths is given by

$$\Sigma\Delta n = \frac{\Delta n}{3K_2} = \frac{1}{3h} . \quad (4-127)$$

The thickness in mean-free-paths does not depend upon the Δn used in the approximation. If the limit as Δn approaches 0 is taken, then it is apparent that the convective boundary condition is equivalent to a physically infinitely thin layer of material which is $1/3h$ mean-free-paths thick. This type of boundary condition can be treated as an albedo problem. Letting β be the albedo (the ratio of the current returning from the boundary layer to the current entering the boundary layer) and $\delta = 1/3h$ be the thickness of the boundary layer in mean-free-paths, a simple diffusion theory solution (based on Fick's law) for the albedo gives

$$\beta = \frac{1 - \frac{2}{3\delta}}{1 + \frac{2}{3\delta}} . \quad (4-128)$$

The parameters for the problem are

1. $K = .5 \text{ cal/sec cm}^\circ\text{C}$
2. $h = .050 \text{ cal/sec cm}^2\text{C}$
3. $\rho c = .5^\circ\text{C/cal cm}^3$
4. $S_0 = 1 \text{ cal/cm}^3 \text{ sec}$
5. $r = 10 \text{ cm}$

The Monte Carlo solution was performed using the adjoint formulation. A scaling factor was not used. The temperature was calculated at the center of the sphere as a function of time. Using the adjoint formulation, the temperatures for all time desired can be made with the same computer run. The results are shown in Table 13. The Monte Carlo calculation required 4.2 min of IBM 360/91 CPU time.

The above problems have shown that the method is applicable to multimedia problems in a variety of geometries. The adjoint formulation appears to be the most suitable for most problems.

The major source of error in most of the problems was not due to statistics but to the use of the transport equation approximation. In the next chapter the differences between the approximate solution and the exact solution are examined to determine if corrections can be made.

TABLE 13
 TEMPERATURE COMPARISONS FOR A SPHERE WITH A TIME
 DEPENDENT SOURCE AND CONVECTIVE
 BOUNDARY CONDITION

t (sec)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
1	2.000	2.05	.074
2	4.000	4.08	.11
3	6.000	6.15	.15
4	8.000	8.18	.18
5	9.996	10.08	.17
10	19.775	19.57	.23
20	37.039	36.17	.33
30	50.775	49.49	.59
40	61.535	60.08	.78
50	69.945	68.79	.89
100	91.248	91.46	1.55
150	97.451	98.63	1.87
200	99.258	100.99	2.02

Note: The analytic steady solution as $t \rightarrow \infty$ is 100°C.

CHAPTER V

CORRECTIONS TO THE TRANSPORT EQUATION APPROXIMATION

In the problems solved in Chapter IV the major source of inaccuracy was the approximation associated with using radiation particle transport theory to solve heat conduction problems. In some of the problems the inaccuracy was reduced by the use of an extrapolation distance correction as suggested by neutron diffusion theory. Problem 11, however, which involved a convective boundary condition, had no obvious inaccuracies associated with its solution. These results suggested that the transport equation approximation should be examined to determine if additional refinements could be made.

The differences between transport theory and diffusion theory can be identified as two distinct types: 1) the differences in the description of the heat flow process, and 2) the differences in the boundary conditions which can be used.

The transport description of the heat flow process will be different from that provided by the heat conduction equation when the angular flux calculated by the transport equation is highly anisotropic. This condition will occur at points which are within several mean free paths of boundaries or nonuniformly distributed sources and can be regarded as a localized effect. If an estimate of the temperature is desired at points within these regions, it will often

be possible to scale up the cross sections used in the transport calculation so that the point of interest is no longer close to a boundary or strong source when the distance is measured in mean free paths. An alternate approach is also possible if the anisotropic angular flux has azimuthal symmetry about a given direction. In this case a localized one-dimensional calculation can be employed on the basis of the following analogy. The one-dimensional transport equation for the angular flux in a pure scattering medium with isotropic scattering can be written as

$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \Sigma \psi(x, \mu) = \frac{\Sigma}{2} \int_{-1}^1 \psi(x, \mu') d\mu' + \frac{1}{2} S(x) \quad (5-1)$$

where $\psi(x, \mu)$ is the angular flux

μ is the cosine of angle between the angular flux and the x-axis

Σ is the scattering cross section, and

$S(x)$ is an isotropic source.

Defining

$$\psi_L(x) = \int_{-1}^1 \mu^L \psi(x, \mu) d\mu, \quad (5-2)$$

and integrating Equation (5-1) with respect to μ gives

$$\frac{d\psi_L(x)}{dx} = S(x). \quad (5-3)$$

Multiplying Equation (5-1) by μ and integrating with respect to μ gives

$$\frac{d\psi_2(x)}{dx} + \Sigma\psi_1(x) = 0. \quad (5-4)$$

Taking the derivative of Equation (5-4) with respect to x and combining with Equation (5-3) gives

$$\frac{1}{\Sigma} \frac{d^2\psi_2(x)}{dx^2} + S(x) = 0. \quad (5-5)$$

Note that this is the one dimensional diffusion equation. Therefore, the second moment of the solution of the transport equation is a solution of the diffusion equation for one dimensional problems with isotropic scattering and isotropic sources regardless of the anisotropic shape of the angular flux. This approach should therefore be useful in making calculations of temperatures in the vicinity of boundaries or nonuniformly distributed sources.

The second kind of inaccuracy is caused by the differences between the boundary conditions which are associated with the transport equation and those which can be used with the diffusion equation. These effects are not localized and represent the major source of inaccuracy in most of the problems solved in Chapter IV.

As was shown in problem 11, the convective boundary condition can be represented by an equivalent albedo condition in transport theory; and results using this boundary condition are accurate. However, the fixed temperature boundary condition of heat conduction (which is

equivalent to the neutron diffusion theory zero flux condition) does not have an equivalent boundary condition in transport theory. The most common method used is to treat the zero flux diffusion theory boundary condition as being represented by a zero incoming flux condition in the transport equation. To account for the differences in the boundary conditions, the diffusion equation flux is assumed to be zero at an extrapolated distance beyond the boundary at which the zero incoming angular flux condition is imposed. The extrapolation distance used (.71 mean free paths) is determined from neutron transport theory. This correction was made in several of the problems in Chapter IV and resulted in excellent agreement with the analytical solutions. However, this type of correction is not always practical for problems with complex shapes or with several different types of materials on the external boundaries. The effect of the different boundary conditions can still be minimized in cases where an extrapolation distance correction is not practical by scaling up the cross sections used in the transport calculation (with a resultant penalty of increased computer running time).

A better boundary condition for the transport solution to simulate the diffusion theory zero flux condition is obtained by considering the limit of the convective boundary condition as the thickness of the convective layer (in mean free paths) goes to zero. This limit is equivalent to the fixed temperature boundary condition. Taking this limit in Equation (4-128) (pg. 87) gives a transport condition of an albedo of minus one on the boundary. Fractional negative albedo

conditions can be used in transport solutions but a negative albedo of minus one on all external boundaries is nonconvergent. To test the negative albedo boundary condition, various negative albedo conditions were applied at the boundary of the Milne problem and solution obtained with ANISN.²⁸ The Milne problem is a one dimensional infinite half-space problem (material from 0 to ∞) with a unit source at infinity. The albedo condition was imposed at the boundary $x = 0$. The results of the ANISN calculations for the total flux are given in Figure 8 for albedo conditions of 0 and -.4, and are compared to the diffusion theory solution with a zero flux boundary condition. Based on these calculations, if the change associated with going from a vacuum boundary condition (zero incoming angular flux) to a -.4 albedo condition is multiplied by 1.89 and added to the vacuum boundary transport solution, the results are the same as the diffusion theory solution at points which are not close to the boundary. The results are also very good even at points close to the boundary. An albedo of -.4 was chosen because it gave a significant correction (over 50% of the difference between the vacuum boundary transport theory solution as compared to the zero flux diffusion theory solution) and is still small enough so as to not prohibitively increase the running time for problem solutions. Test problems with a -.4 albedo condition ran 1.5 to 2.0 times as long as those with a vacuum boundary condition. This type of boundary condition is easily implemented in Monte Carlo transport codes and can be implemented so as to give the solution with a vacuum boundary condition and the negative albedo boundary condition in the same run.

Figure 8. Diffusion theory vs transport theory with different boundary conditions.

The negative albedo boundary condition was used to recalculate problem 2. The results of these calculations are shown in Table 14. The Monte Carlo solution was performed using the adjoint formulation and a scaling factor of 10. With the exception of the negative albedo boundary condition, the transport theory results were performed in the same manner as the calculations in Table 4 (p. 42). Since the estimated standard deviation for the results at $x = -.7$ cm were larger than desired to make an adequate evaluation of the effect of the negative albedo boundary condition, a second calculation was performed using 19 min of IBM 360/91 CPU time. The results of this calculation was a Monte Carlo estimate of 81.86°C with an estimated standard deviation of $.18^{\circ}\text{C}$ as compared to the analytical results of 81.83°C .

Using the above correction factor, the major source of error is now the statistical error associated with the Monte Carlo process.

TABLE 14
 TEMPERATURE COMPARISONS FOR PROBLEM TWO WITH
 THE NEGATIVE ALBEDO CORRECTION FACTOR

x (cm)	Analytic Temperature °C	Monte Carlo Temperature °C	Estimated Standard Deviation C°
- .7	81.83	81.79	.31
- .5	84.73	84.79	.62
.1	166.2	169.5	2.6
.5	272.5	271.6	3.3

CHAPTER VI

MONTE CARLO SOLUTION OF A COUPLED GAMMA RADIATION AND HEAT CONDUCTION PROBLEM

One of the major advantages of this method of solution of the heat conduction equation is that it will allow the solution of coupled nuclear radiation transport and heat conduction problems where nuclear radiation interactions with matter are the heat source for the heat conduction problem. Another important advantage is the capability of solving these problems in complex geometries. In this section the Monte Carlo method is applied to the solution of a coupled gamma radiation and heat conduction problem.

Problem 12: To calculate the temperature at several points in a sphere when the heat source is an incident parallel beam of gamma radiation (Figure 9).

Assumptions and Parameters:

1. Outer boundary of the sphere is maintained at a constant temperature.
2. Radius of sphere is 10 cm.
3. $K = .3333 \text{ cal/cm sec } ^\circ\text{C}.$
4. Σ_t for gammas (all energies) = $.1 \text{ cm}^{-1}.$
5. Σ_a for gammas = $.03 \text{ cm}^{-1}.$
6. $1 \text{ cal/cm}^2\text{sec}$ energy in the incident gamma radiation.
7. Steady-state condition.
8. Isotropic scattering of gamma radiation.

Figure 9. One-medium sphere with incident gamma beam.

The values chosen are typical of those for a metal such as aluminum.

To provide a check on the Monte Carlo solution, the average temperature as a function of radius of the sphere is estimated using previously available techniques. For calculating the average gamma flux as a function of radius, Kalos²⁹ has shown that the incident parallel beam of ϕ_0 particles/cm²-sec can be replaced by an isotropic flux of ϕ_0 particles/cm²-sec over the entire surface of the sphere. Using this transformation, the average gamma flux as a function of radius was solved using the discrete ordinates code ANISN²⁸ and the average energy deposition was calculated. With an incident gamma radiation beam of 1 cal/cm²-sec, the radius averaged gamma flux was fairly uniform in the sphere varying from a low of .685 cal/cm²-sec at the center to a high of .848 cal/cm²-sec at the edge. The volume averaged flux was .773 cal/cm²-sec. The average heat source in the sphere is then given by $\phi_{avg} \Sigma_a$ and is .0232 cal/cm³-sec. Assuming that this heat source is uniform across the sphere, an estimation of the temperature as a function of radius can then be made analytically.

The Monte Carlo calculation of the coupled gamma radiation and heat conduction problem was performed using MORSE-SGC. The logic used for performing a coupled gamma radiation and heat conduction problem with MORSE-SGC is the same as that used for solving a coupled neutron-gamma problem. For the forward formulation of the problem, a gamma starting location is chosen from the parallel beam distribution and the gamma particle is tracked until it escapes or until the weight of the

particle has been reduced by nonabsorption weighting at collision sites and its survival determined by Russian roulette. At each collision site energy is absorbed and represents a heat source. This heat source is treated as a secondary particle. In the actual calculation a secondary particle was generated with a probability of .1 at each collision site with particle weighting to maintain a fair game. The secondary particles (the heat source) are then tracked to determine temperatures.

As was expected, an adjoint formulation of the problem was necessary to obtain good statistics. (The adjoint calculation obtained fractional standard deviations of less than 1% in the same time that the forward calculation obtained fractional standard deviations of over 30%). A next event estimator was used in the adjoint formulation at each gamma collision site (the secondary particles in the adjoint formulation) and the probability of the gamma having reached this collision site from the incident beam was scored. By scoring to different detectors, where for each detector a different location of the incoming beam was used, the temperature at a number of points could be calculated with the same computer run. Each of these points will have the same radius but are at locations which have the same relative locations to the incoming beams as was used during the estimator process.

Comparisons of the analytical estimation of the average temperature as a function of radius and the Monte Carlo calculation of the temperature at various locations in the sphere are shown in Tables 15-19 and Figures 10-14. An adjoint Monte Carlo calculation was performed for each of the seven radii shown in the figures and

TABLE 15
TEMPERATURES FOR $\theta=0^\circ$ COMPARED TO ANALYTICAL ESTIMATE

r (cm)	Analytical Estimate °C	Monte Carlo Calculation °C	Estimated Standard Deviation C°
0	1.16	1.05	.021
1.0	1.15	1.13	.026
2.0	1.11	1.13	.024
3.0	1.056	1.15	.023
4.0	.974	1.15	.033
5.0	.871	1.03	.023
7.5	.508	.640	.019
9.0	.220	.312	.019

Note: The Monte Carlo temperatures are calculated along a radius which makes an angle θ with the direction towards the incident gamma radiation.

TABLE 16
TEMPERATURES FOR $\theta=45^\circ$ COMPARED TO ANALYTICAL ESTIMATE

r (cm)	Analytical Estimate °C	Monte Carlo Calculation °C	Estimated Standard Deviation C°
0	1.16	1.05	.021
1.0	1.15	1.12	.024
2.0	1.11	1.14	.023
3.0	1.056	1.11	.023
4.0	.974	1.12	.032
5.0	.871	.988	.021
7.5	.508	.611	.018
9.0	.220	.297	.018

Note: The Monte Carlo temperatures are calculated along a radius which makes an angle θ with the direction towards the incident gamma radiation.

TABLE 17
TEMPERATURES FOR $\theta=90^\circ$ COMPARED TO ANALYTICAL ESTIMATE

r (cm)	Analytical Estimate $^\circ\text{C}$	Monte Carlo Calculation $^\circ\text{C}$	Estimated Standard Deviation $^\circ\text{C}$
0	1.16	1.05	.021
1.0	1.15	1.09	.025
2.0	1.11	1.07	.021
3.0	1.056	1.00	.024
4.0	.974	.987	.031
5.0	.871	.853	.020
7.5	.508	.497	.015
9.0	.220	.238	.016

Note: The Monte Carlo temperatures are calculated along a radius which makes an angle θ with the direction towards the incident gamma radiation.

TABLE 18
TEMPERATURES FOR $\theta=135^\circ$ COMPARED TO ANALYTICAL ESTIMATE

r (cm)	Analytical Estimate $^\circ\text{C}$	Monte Carlo Calculation $^\circ\text{C}$	Estimated Standard Deviation $^\circ\text{C}$
0	1.16	1.05	.021
1.0	1.15	1.05	.025
2.0	1.11	.993	.021
3.0	1.056	.904	.024
4.0	.974	.835	.027
5.0	.871	.697	.020
7.5	.508	.348	.013
9.0	.220	.154	.012

Note: The Monte Carlo temperatures are calculated along a radius which makes an angle θ with the direction towards the incident gamma radiation.

TABLE 19
TEMPERATURES FOR $\theta=180^\circ$ COMPARED TO ANALYTICAL ESTIMATE

r (cm)	Analytical Estimate °C	Monte Carlo Calculation °C	Estimated Standard Deviation C°
0	1.16	1.05	.021
1.0	1.15	1.04	.026
2.0	1.11	.968	.022
3.0	1.056	.867	.024
4.0	.974	.760	.025
5.0	.871	.630	.023
7.5	.508	.289	.012
9.0	.220	.127	.011

Note: The Monte Carlo temperatures are calculated along a radius which makes an angle θ with the direction towards the incident gamma radiation.

Figure 10. Temperatures for $\theta = 0^\circ$ compared to analytical estimates.

Figure 11. Temperatures for $\theta = 45^\circ$ compared to analytical estimates.

Figure 12. Temperatures for $\theta = 90^\circ$ compared to analytical estimates.

Figure 13. Temperatures for $\theta = 135^\circ$ compared to analytical estimates.

Figure 14. Temperatures for $\theta = 180^\circ$ compared to analytical estimates.

tables. Each Monte Carlo calculation required 4.2 min of CPU time on the IBM 360/91. A listing of the input to MORSE-SGC and LAVA,³⁰ and the relevant output is given in Appendix C for the calculation at $r = 2.0$ cm. LAVA was used to create the necessary cross sections for MORSE-SGC.

The analytical estimate of the average temperature at a given radius differs from the Monte Carlo calculation of the same quantity because the analytical results are based on the assumption that the heat generation within the sphere is uniform. This is a fairly good approximation but is not exact. Since the heat generation is higher near the edges than at the center, the correct temperature at the center is lower than the analytical estimate.

The Monte Carlo results along various radii show the variation of the temperature from the average over the sphere. As is expected intuitively, the hottest point in the sphere is not at the center but is several centimeters off center in the direction towards the incident beam of gamma radiation.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

The transport theory approximation has been shown to be appropriate for use in solving many types of heat conduction problems. As was demonstrated in the problems, the transport approximation is not limited to Monte Carlo methods. However, since most other techniques can be used to solve the heat conduction equation directly, the approximation is not necessary with these other methods.

As is typical with current Monte Carlo methods, other methods of problem solutions appear to be preferable when they may be used. However, there are many heat conduction problems for which the solution is not feasible using other methods. The application of the Monte Carlo method to the transport equation approximation will provide solutions to many of these problems. There are no inherent limitations as to the geometric complexity of the heat conduction problems which can be solved using this method. It will also simplify, or make possible, the solution of coupled problems which involve nuclear reactions as the heat source for a heat conduction problem.

The Monte Carlo solution of the transport equation approximation is more suitable for calculating temperatures at individual points within a system rather than calculating complete temperature distributions. The adjoint formulation appears to be the most suitable mode of solution for most problems. The method is not applicable to problems where the thermal conductivity is temperature dependent.

The advantages and disadvantages of the Monte Carlo solution of the transport equation approximation are illustrated by the following comparison to the surface density walk and floating random walk techniques:

1. There are no published results on the use of either the surface density walk or floating random walk in multimedia problems.

2. The tracking algorithm used by the floating random walk is difficult since it requires the determination of the nearest boundary point. Both the transport equation approximation and the surface density walk use a simpler method which requires the finding of the intersect point between a surface and a given line.

3. The surface density walk tracks directly to surface points, with the number of tracks for a given walk being determined by the number of terms in the truncated Fredholm series being evaluated. The floating random walk tracks over large distances until the walk approaches a boundary in which case the distances become smaller and smaller. An approximation must then be used to reach the boundary. The transport equation approximation uses relatively small steps throughout the walk with the step size becoming smaller as a scale factor is increased to make the transport equation approximation better.

4. The surface density walk and the transport equation approximation may be used with or readily adapted to existing Monte Carlo radiation transport codes.

5. All three methods may be used to solve heat conduction problems with internal heat sources, given surface temperature distributions and time dependence.

6. The floating random walk and the transport equation approximation may be used to solve problems with convective boundary conditions.

7. The transport equation approximation may be used to solve coupled nuclear radiation and heat conduction problems.

The Monte Carlo solution of the transport equation approximation has been shown to be useful in the solution of many types of heat conduction problems. However, the following areas still require further research.

1. Contact resistance between media.

This type of problem should be solvable by the use of an appropriate albedo medium at the contact surface to transmit or reflect a fixed ratio of particles striking the surface. This technique still requires demonstration, however.

2. Other boundary conditions.

In Chapter II, many different types of standard boundary conditions are listed for heat conduction problems. Only the fixed surface temperature distribution and the convective boundary condition problems were solved. Further research is needed to determine effective means of handling the remaining types of boundary conditions.

3. Biasing techniques.

All of the standard biasing techniques available for use in solving transport problems with Monte Carlo are applicable to this method. Further research is still needed, however, to determine how these techniques may be effectively applied in the use of transport problems which approximate heat conduction problems.

4. Coupling with deterministic codes.

The method should also be useful when used in conjunction with deterministic codes to perform perturbation calculation or to couple the Monte Carlo calculation to deterministic codes as in the adjoint difference method.³¹ Further research is needed to establish effective and convenient methods to accomplish the coupling with heat conduction codes.

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APPENDIXES

APPENDIX A

VERIFICATION OF ANALYTICAL SOLUTIONS

Once an analytical solution has been developed for a heat conduction problem, the solution may be verified by showing that it satisfies the heat conduction equation and the boundary conditions. Many of the problems presented in Chapter IV involved the derivation of an appropriate Green's function to obtain the analytical solution. Since a large number of manipulations (which could introduce errors) were involved in the development of the solutions, a verification of the more difficult solutions is given in this appendix.

The analytical solution to problem 2 is

$$T_2(x, y, z) = \frac{S_o a}{\pi^2} \cdot \frac{1}{K_1 + K_2} \cdot \frac{1}{1 + a^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right)} \cdot \frac{1}{\left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}} \cdot \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \\ \cdot \frac{\sinh \pi \left[(a + x) \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]}{\cosh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]} \quad (A-1)$$

for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$, and

$$T_1(x, y, z) = T_2(-x, y, z) + \frac{S_o}{\pi^2 K_1} \cdot \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \cdot \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \quad (A-2)$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, and $0 \leq z \leq c$.

Since the problem is steady state, the solution must satisfy the following:

$$(1) \quad \nabla^2 T = - \frac{S(x,y,z)}{K} , \quad (A-3)$$

where

$$S(x,y,z) = S_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} , \quad (A-4)$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, zero elsewhere,

$$(2) \quad T(x,y,z) = 0 , \quad (A-5)$$

for all (x,y,z) on outer boundaries, and

$$(3) \quad K_1 \frac{\partial T_1}{\partial x} = K_2 \frac{\partial T_2}{\partial x} , \quad (A-6)$$

at the interface between the two media.

By inspection it is obvious that $T(x,y,z) = 0$ at outer boundaries.

To evaluate $\nabla^2 T_2(x,y,z)$, let

$$T_2(x,y,z) = \alpha \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \sinh \left[\pi (a + x) \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] . \quad (A-7)$$

Then

$$\frac{\partial^2 T_2}{\partial y^2} = -\alpha \left(\frac{\pi}{b} \right)^2 \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \sinh \left[\pi (a + x) \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] , \quad (A-8)$$

$$\frac{\partial^2 T_2}{\partial z^2} = -\alpha \left(\frac{\pi}{c}\right)^2 \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \sinh\left[\pi(a+x)\left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right], \quad (\text{A-9})$$

and

$$\frac{\partial^2 T_2}{\partial x^2} = \alpha \pi^2 \left(\frac{1}{b^2} + \frac{1}{c^2}\right) \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \sinh\left[\pi(a+x)\left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right]. \quad (\text{A-10})$$

Therefore ,

$$\nabla^2 T_2(x,y,z) = \frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} + \frac{\partial^2 T_2}{\partial z^2} = 0. \quad (\text{A-11})$$

To evaluate $\nabla^2 T_1(x,y,z)$, it is only necessary to evaluate the second term of $T_1(x,y,z)$, which will be denoted $T_s(x,y,z)$, since $\nabla^2 T_2(-x,y,z) = 0$. This gives

$$T_s(x,y,z) = \frac{S_0}{\pi^2 K_1} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}. \quad (\text{A-12})$$

This gives

$$\frac{\partial^2 T_s}{\partial x^2} = -\left(\frac{\pi}{a}\right)^2 \frac{S_0}{\pi^2 K_1} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (\text{A-13})$$

$$\frac{\partial^2 T_s}{\partial y^2} = -\left(\frac{\pi}{b}\right)^2 \frac{S_0}{\pi^2 K_1} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (\text{A-14})$$

and

$$\frac{\partial^2 T_s}{\partial z^2} = -\left(\frac{\pi}{c}\right)^2 \frac{S_0}{\pi^2 K_1} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} . \quad (\text{A-15})$$

Therefore,

$$\nabla^2 T_s(x, y, z) = - \frac{S_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}}{K_1} , \quad (\text{A-16})$$

and

$$\nabla^2 T_1(x, y, z) = - \frac{S(x, y, z)}{K_1} . \quad (\text{A-17})$$

The final condition will be checked by evaluating $K_1 \frac{\partial T_1}{\partial x} \Big|_{x=0}$ and $K_2 \frac{\partial T_2}{\partial x} \Big|_{x=0}$ and setting them equal to each other to see if the equality is verified. Performing these operations gives

$$\begin{aligned} K_2 \alpha \cosh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \pi \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} = \\ - K_1 \alpha \cosh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \pi \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \\ + \frac{S_0}{\pi^2} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \left(\frac{\pi}{a} \right) \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} , \end{aligned} \quad (\text{A-18})$$

where

$$\begin{aligned} \alpha = \frac{S_0 a}{\pi^2} \cdot \frac{1}{K_1 + K_2} \cdot \frac{1}{1 + a^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right)} \cdot \frac{1}{\left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2}} \\ \cdot \frac{1}{\cosh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]} , \end{aligned} \quad (\text{A-19})$$

or

$$\frac{S_0 a}{\pi} \frac{1}{1 + a^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right)} = \frac{S_0}{a\pi} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}, \quad (\text{A-20})$$

which checks. Therefore the analytical solution to problem 2 is verified.

The analytical solution to problem 3 is

$$T_2(x, y, z) = \frac{16 S_0}{\pi^4 (K_1 + K_2)} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left(\frac{1}{mn \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \left[\frac{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} - 1 \right]}{\sinh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \right. \right. \\ \left. \left. \cdot \frac{\sinh \left[\pi \left(a + x \right) \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]}{\cosh \left[\pi a \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2} \right]} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right] \right), \quad (\text{A-21})$$

for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$, and

$$T_1(x, y, z) = T_2(-x, y, z) + \frac{64 S_0}{K_1 \pi^5} \sum_{\ell \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left(\frac{1}{\ell mn \left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \right. \\ \left. \cdot \sin \frac{\ell\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right) \quad (\text{A-22})$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, and $0 \leq z \leq c$.

The solution must satisfy the following:

$$(1) \nabla^2 T = - \frac{S(x,y,z)}{K}, \quad (A-23)$$

where

$$S(x,y,z) = S_0, \quad (A-24)$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, zero elsewhere ,

$$(2) T(x,y,z) = 0, \quad (A-25)$$

for all (x,y,z) on outer boundaries, and

$$(3) K_1 \frac{\partial T_1}{\partial x} = K_2 \frac{\partial T_2}{\partial x}, \quad (A-26)$$

at the interface between the two media.

By inspection it is obvious that $T(x,y,z) = 0$ on outer boundaries.

To evaluate $\nabla^2 T_2(x,y,z)$ let

$$\alpha_{mn} = \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{1/2}. \quad (A-27)$$

Then

$$\begin{aligned} \frac{\partial^2 T_2}{\partial x^2} = & \frac{16 S_0}{\pi^2 (K_1 + K_2)} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left(\frac{1}{m n \alpha_{mn}^2} \left(\frac{\cosh(a \pi \alpha_{mn}) - 1}{\sinh(a \pi \alpha_{mn})} \right) \right. \\ & \left. \cdot \frac{\sinh[\pi(a+x)\alpha_{mn}]}{\cosh(a \pi \alpha_{mn})} \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c} \right), \end{aligned} \quad (A-28)$$

$$\begin{aligned} \frac{\partial^2 T_2}{\partial y^2} = & \frac{-16 S_0}{\pi^2 (K_1 + K_2)} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left(\frac{m}{n b^2 \alpha_{mn}^2} \left(\frac{\cosh(a \pi \alpha_{mn}) - 1}{\sinh(a \pi \alpha_{mn})} \right) \right. \\ & \left. \cdot \frac{\sinh[\pi(a+x)\alpha_{mn}]}{\cosh(a \pi \alpha_{mn})} \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c} \right), \end{aligned} \quad (A-29)$$

and

$$\frac{\partial^2 T_2}{\partial z^2} = \frac{-16 S_0}{\pi^2 (K_1 + K_2)} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left[\frac{n}{mc^2 \alpha_{mn}^2} \left(\frac{\cosh(a\pi\alpha_{mn}) - 1}{\sinh(a\pi\alpha_{mn})} \right) \right. \\ \left. \cdot \frac{\sinh[\pi(a+x)\alpha_{mn}]}{\cosh(a\pi\alpha_{mn})} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right]. \quad (\text{A-30})$$

Using the definition of α_{mn} gives

$$\frac{1}{mn} - \frac{m}{nb^2 \alpha_{mn}^2} - \frac{n}{mc^2 \alpha_{mn}^2} = 0, \quad (\text{A-31})$$

and

$$\nabla^2 T_2(x, y, z) = 0. \quad (\text{A-32})$$

To evaluate $\nabla^2 T_1(x, y, z)$, it is only necessary to evaluate the second term of $T_1(x, y, z)$, which will be denoted $T_s(x, y, z)$, since $\nabla^2 T_2(-x, y, z) = 0$.

This gives

$$T_s(x, y, z) = \frac{64 S_0}{K_1 \pi^5} \sum_{\ell \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left[\frac{1}{\ell mn \left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \right. \\ \left. \cdot \sin \frac{\ell\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \right]. \quad (\text{A-33})$$

This gives

$$\frac{\partial^2 T_s}{\partial x^2} = - \frac{64 S_0}{K_1 \pi^3} \sum_{\ell \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \left(\frac{\ell^2/a^2}{\ell^2 m^2 n^2 \left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \right. \\ \left. \cdot \sin \frac{\ell \pi x}{a} \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c} \right), \quad (\text{A-34})$$

and similar expressions for $\frac{\partial^2 T_s}{\partial y^2}$ and $\frac{\partial^2 T_s}{\partial z^2}$. Combining these gives

$$\nabla^2 T_s = - \frac{64 S_0}{K_1 \pi^3} \sum_{\ell \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \frac{1}{\ell m n} \sin \frac{\ell \pi x}{a} \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c}. \quad (\text{A-35})$$

To evaluate this expression, consider a Fourier expansion of the number "1" over the region $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$. This gives

$$1 = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{\ell m n} \sin \frac{\ell \pi x}{a} \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c}, \quad (\text{A-36})$$

where

$$A_{\ell m n} = \frac{8}{abc} \int_0^a \int_0^b \int_0^c \sin \frac{\ell \pi x}{a} \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c} dz dy dx. \quad (\text{A-37})$$

Using

$$\int_0^a \sin \frac{\ell \pi x}{a} dx = \begin{cases} \frac{2a}{\ell \pi} & \ell \text{ odd} \\ 0 & \ell \text{ even} \end{cases}, \quad (\text{A-38})$$

gives

$$A_{\ell m n} = \begin{cases} \frac{64}{\ell m n \pi^3} & \ell, m, n \text{ odd} \\ 0 & \text{otherwise} \end{cases}. \quad (\text{A-39})$$

Therefore

$$\nabla^2 T_s = - \frac{S_o}{K_1} . \quad (A-40)$$

The final condition will be checked by evaluating $K_1 \frac{\partial T_1}{\partial x} \Big|_{x=0}$ and $K_2 \frac{\partial T_2}{\partial x} \Big|_{x=0}$ and setting them equal to each other to see if the equality is verified. Performing these operations gives

$$\begin{aligned} & \frac{16 K_2}{\pi^3 (K_1 + K_2)} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \frac{1}{mn \alpha_{mn}} \left(\frac{\cosh(a \pi \alpha_{mn}) - 1}{\sinh(a \pi \alpha_{mn})} \right) \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c} \\ &= \frac{-16 K_1}{\pi^3 (K_1 + K_2)} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \frac{1}{mn \alpha_{mn}} \left(\frac{\cosh(a \pi \alpha_{mn}) - 1}{\sinh(a \pi \alpha_{mn})} \right) \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c} \\ &+ \frac{64}{a \pi^4} \sum_{\ell \text{ odd}}^{\infty} \sum_{m \text{ odd}}^{\infty} \sum_{n \text{ odd}}^{\infty} \frac{1}{mn \left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \sin \frac{m \pi y}{b} \sin \frac{n \pi z}{c} . \quad (A-41) \end{aligned}$$

To verify this equation, it is necessary to verify that

$$\frac{\pi}{4\beta} \frac{\cosh \pi \beta - 1}{\sinh \pi \beta} = \sum_{\ell \text{ odd}}^{\infty} \frac{1}{\ell^2 + \beta^2} , \quad (A-42)$$

where

$$\beta = a \alpha_{mn} . \quad (A-43)$$

To verify Eq. (A-42), consider a Fourier cosine expansion of $\frac{\cosh \beta(\pi-x)}{\beta \sinh \beta \pi}$. Using

$$\int_0^{\pi} \frac{\cosh \beta(\pi-x)}{\beta \sinh \beta\pi} \cos \ell x dx = \frac{1}{\ell^2 + \beta^2} , \quad \text{A-44)}$$

gives

$$\frac{\cosh \beta(\pi-x)}{\beta \sinh \beta\pi} = \frac{1}{\pi\beta^2} + \sum_{\ell=1}^{\infty} \frac{2}{\pi} \frac{1}{\ell^2 + \beta^2} \cos \ell x . \quad \text{(A-45)}$$

Setting $x = 0$ gives

$$\frac{\cosh \beta\pi}{\beta \sinh \beta\pi} = \frac{1}{\pi\beta^2} + \sum_{\ell=1}^{\infty} \frac{2}{\pi} \frac{1}{\ell^2 + \beta^2} . \quad \text{(A-46)}$$

Setting $x = \pi$ gives

$$\frac{1}{\beta \sinh \beta\pi} = \frac{1}{\pi\beta^2} - \sum_{\ell=1}^{\infty} \frac{2}{\pi} \frac{(-1)^{\ell}}{\ell^2 + \beta^2} . \quad \text{(A-47)}$$

Subtracting Eq. (A-47) from Eq. (A-46) and multiplying by $\frac{\pi}{4}$ gives

$$\frac{\pi}{4\beta} \frac{\cosh \pi\beta - 1}{\sinh \pi\beta} = \sum_{\ell \text{ odd}}^{\infty} \frac{1}{\ell^2 + \beta^2} , \quad \text{(A-48)}$$

which verifies the equality. Therefore the analytical solution to problem 3 is verified.

The analytical solution to problem 4 is

$$T_2(x,y,z) = \frac{K_1}{K_1 + K_2} \frac{\sin[\pi(a+x)(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}}{\sinh[\pi a(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] \cosh[\pi a(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}]} \quad \text{(A-49)}$$

for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$, and

$$T_1(x, y, z) = T_2(-x, y, z) + \frac{\sinh[\pi x(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}]}{\sinh[\pi a(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}]} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \quad (A-50)$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, and $0 \leq z \leq c$.

The solution must satisfy the following:

$$(1) \quad \nabla^2 T = 0, \quad (A-51)$$

$$(2) \quad T(x, y, z) = \begin{cases} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, & \text{on surface at } x = a, \\ 0, & \text{on all other external boundaries} \end{cases} \quad (A-52)$$

and

$$(3) \quad K_1 \frac{\partial T_1}{\partial x} = \frac{\partial T_2}{\partial x}, \quad (A-53)$$

at the interface between the two media.

By inspection it is obvious that the external boundary conditions as given in Eq. (A-52) are met. To evaluate $\nabla^2 T_2(x, y, z)$ let

$$T_2(x, y, z) = \alpha \sinh[\pi(a + x)(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (A-54)$$

then from Eqs. (A-7) and (A-11)

$$\nabla^2 T_2(x, y, z) = 0. \quad (A-55)$$

To evaluate $\nabla^2 T_1(x, y, z)$, it is only necessary to evaluate the second term of $T_1(x, y, z)$ which will be denoted $T_s(x, y, z)$. This gives

$$T_s(x, y, z) = \beta \sinh[\pi x(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}. \quad (A-56)$$

Therefore

$$\frac{\partial^2 T_s}{\partial x^2} = \beta \pi^2 \left(\frac{1}{b^2} + \frac{1}{c^2} \right) \sinh \left[\pi x \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (\text{A-57})$$

$$\frac{\partial^2 T_s}{\partial y^2} = -\beta \frac{\pi^2}{b^2} \sinh \left[\pi x \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (\text{A-58})$$

and

$$\frac{\partial^2 T_s}{\partial z^2} = -\beta \frac{\pi^2}{c^2} \sinh \left[\pi x \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}. \quad (\text{A-59})$$

This gives

$$\nabla^2 T_1(x, y, z) = 0. \quad (\text{A-60})$$

The final condition will be checked by evaluating $K_1 \frac{\partial T_1}{\partial x} \Big|_{x=0}$ and $K_2 \frac{\partial T_2}{\partial x} \Big|_{x=0}$ and setting them equal to each other to see if the equality is verified. Performing these operations gives

$$\begin{aligned} \frac{K_2 K_1}{K_1 + K_2} \frac{\pi \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}}{\sinh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]} = \\ - \frac{K_1^2}{K_1 + K_2} \frac{\pi \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}}{\sinh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]} \\ + \frac{K_1}{K_1 + K_2} \frac{\pi \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}}{\sinh \left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{1/2} \right]}, \end{aligned} \quad (\text{A-61})$$

which checks. Therefore the analytical solution to problem 4 is verified.

The analytical solution to problem 5 is

$$T_2(x, y, z) = \frac{S_o a}{\pi^2} \frac{1}{(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}} \frac{1}{1+a^2(\frac{1}{b^2} + \frac{1}{c^2})} \frac{(K_1+K_3)\cosh[\pi a(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] + K_1}{(K_1+K_3)(K_1+K_2)\cosh^2[\pi a(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] - K_1^2} \cdot \sinh[\pi(a+x)(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (A-62)$$

for $-a \leq x \leq 0$, $0 \leq y \leq b$, $0 \leq z \leq c$,

$$T_3(x, y, z) = \frac{S_o a}{\pi^2} \frac{1}{(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}} \frac{1}{1+a^2(\frac{1}{b^2} + \frac{1}{c^2})} \frac{(K_1+K_2)\cosh[\pi a(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] + K_1}{(K_1+K_3)(K_1+K_2)\cosh^2[\pi a(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] - K_1^2} \cdot \sinh[\pi(2a-x)(\frac{1}{b^2} + \frac{1}{c^2})^{1/2}] \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (A-63)$$

for $a \leq x \leq 2a$, $0 \leq y \leq b$, $0 \leq z \leq c$,

and

$$T_1(x,y,z) = T_2(-x,y,z) + T_3(2a-x,y,z) + \frac{S_0}{\pi^2 K_1} \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}, \quad (\text{A-64})$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, and $0 \leq z \leq c$.

The solution must satisfy the following:

$$(1) \quad \nabla^2 T = - \frac{S(x,y,z)}{K}, \quad (\text{A-65})$$

where

$$S(x,y,z) = S_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \quad (\text{A-66})$$

for $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, and zero elsewhere,

$$(2) \quad T(x,y,z) = 0, \quad (\text{A-67})$$

for all (x,y,z) on outer boundaries,

$$(3) \quad K_1 \frac{\partial T_1}{\partial x} = K_2 \frac{\partial T_2}{\partial x}, \quad (\text{A-68})$$

at the $x = 0$ interface, and

$$(4) \quad K_1 \frac{\partial T_1}{\partial x} = K_3 \frac{\partial T_3}{\partial x}, \quad (\text{A-68})$$

at the $x = a$ interface.

By inspection it is obvious that $T(x,y,z) = 0$ on the outer boundaries. From the verification of problem 2 it follows that Eq. (A-65) is also verified. The condition at the $x = 0$ interface will be checked by evaluating $K_1 \frac{\partial T_1}{\partial x} \Big|_{x=0}$ and $K_2 \frac{\partial T_2}{\partial x} \Big|_{x=0}$ and setting them

equal to each other to see if the equality is verified. Performing these operations gives

$$\begin{aligned}
 & \frac{(K_1+K_2) \cosh\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] \left\{ (K_1+K_3) \cosh\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] + K_1 \right\}}{(K_1+K_3)(K_1+K_2) \cosh^2\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] - K_1^2} \\
 &= \frac{K_1(K_1+K_2) \cosh\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] + K_1^2}{(K_1+K_3)(K_1+K_2) \cosh^2\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] - K_1^2} + \frac{1}{2}, \quad (A-70)
 \end{aligned}$$

which checks. The final condition at the $x = a$ interface will be checked by evaluating $K_1 \frac{\partial T_1}{\partial x} \Big|_{x=a}$ and $K_3 \frac{\partial T_3}{\partial x} \Big|_{x=a}$ and setting them equal to each other to see if the equality is verified. Performing these operations gives

$$\begin{aligned}
 & \frac{(K_1+K_3) \cosh\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] \left\{ (K_1+K_2) \cosh\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] + K_1 \right\}}{(K_1+K_3)(K_1+K_2) \cosh^2\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] - K_1^2} \\
 &= \frac{K_1(K_1+K_3) \cosh\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] + K_1^2}{(K_1+K_3)(K_1+K_2) \cosh^2\left[\pi a \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}\right] - K_1^2} + 1, \quad (A-71)
 \end{aligned}$$

which checks. Therefore the analytical solution to problem 5 is verified.

The analytical solution to problem 6 is

$$T_2(r, z) = \frac{S_0 \pi l}{\pi^2 + l^2 \alpha_1^2} \frac{J_0(\alpha_1 r)}{\alpha_1 (K_1 + K_2)} \frac{\sinh \alpha_1 (l + z)}{\cosh \alpha_1 l}, \quad (\text{A-72})$$

for $0 \leq r \leq a$, $-l \leq z \leq 0$,

and

$$T_1(r, z) = T_2(r, -z) + \frac{S_0}{K_1 (\alpha_1^2 + \frac{\pi^2}{l^2})} J_0(\alpha_1 r) \sin \frac{\pi z}{l}, \quad (\text{A-73})$$

for $0 \leq r \leq a$, and $0 \leq z \leq l$.

The solution must satisfy the following:

$$(1) \quad \nabla^2 T = - \frac{S(r, z)}{K}, \quad (\text{A-74})$$

where

$$S(r, z) = S_0 J_0(\alpha_1 r) \sin \frac{\pi z}{l}, \quad (\text{A-75})$$

for $0 \leq r \leq a$, $0 \leq z \leq l$, and zero elsewhere,

$$(2) \quad T(r, z) = 0, \quad (\text{A-76})$$

for all (r, z) on outer boundaries, and

$$(3) \quad K_1 \frac{\partial T_1}{\partial z} = K_2 \frac{\partial T_2}{\partial z}, \quad (\text{A-77})$$

at the interface between the two media.

By inspection and from the definition of α_1 as the first zero of $J_0(\alpha a)$, it is obvious that $T(r, z) = 0$ on the outer boundaries.

For cylindrical geometries with angular symmetry

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} . \quad (\text{A-78})$$

Using

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial J_0(\alpha_1 r)}{\partial r} \right) = -\alpha_1^2 J_0(\alpha_1 r) , \quad (\text{A-79})$$

and

$$\frac{\partial^2}{\partial z^2} \sinh \alpha_1 (\ell + z) = \alpha_1^2 \sinh \alpha_1 (\ell + z) , \quad (\text{A-80})$$

gives

$$\nabla^2 T_2(r, z) = 0 . \quad (\text{A-81})$$

To evaluate $\nabla^2 T_1(r, z)$, it is only necessary to evaluate the second term of $T_1(r, z)$, which will be denoted $T_s(r, z)$. This gives

$$\nabla^2 T_s(r, z) = - \frac{S_0 J_0(\alpha_1 r) \sin \pi_\ell z}{K_1} . \quad (\text{A-82})$$

Therefore

$$\nabla^2 T_1(r, z) = - \frac{S(r, z)}{K_1} . \quad (\text{A-83})$$

The final condition will be checked by evaluating $K_1 \frac{\partial T_1}{\partial z} \Big|_{z=0}$ and $K_2 \frac{\partial T_2}{\partial z} \Big|_{z=0}$ and setting them equal to each other to see if the equality is verified. Performing these operations gives

$$\frac{S_0 \pi \ell}{\pi^2 + \ell^2 \alpha_1^2} \frac{K_2}{K_1 + K_2} J_0(\alpha_1 r) = - \frac{S_0 \pi \ell}{\pi^2 + \ell^2 \alpha_1^2} \frac{K_1}{K_1 + K_2} J_0(\alpha_1 r) + \frac{S_0 \pi}{\ell(\alpha_1^2 + \frac{\pi^2}{\ell^2})} J_0(\alpha_1 r), \quad (\text{A-84})$$

which checks. Therefore the analytical solution to problem 6 is verified.

The analytical solution to problem 7 is

$$T_2(r, z) = \frac{2S_0}{a(K_1 + K_2)} \sum_{n=1}^{\infty} \left(\frac{\cosh \alpha_n \ell - 1}{\alpha_n^3 J_1(\alpha_n a)} \frac{J_0(\alpha_n r) \sinh \alpha_n (\ell + z)}{\sinh \alpha_n \ell \cosh \alpha_n \ell} \right) \quad (\text{A-85})$$

for $0 \leq r \leq a$, $-\ell \leq z \leq 0$

and

$$T_1(r, z) = T_2(r, -z) + \frac{8S_0}{a\pi} \sum_{m \text{ odd}} \frac{1}{m} \sin \frac{m\pi z}{\ell} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{K_1 \alpha_n (\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2) J_1(\alpha_n a)} \quad (\text{A-86})$$

for $0 \leq r \leq a$, and $0 \leq z \leq \ell$.

The solution must satisfy the following

$$(1) \quad \nabla^2 T = - \frac{S(r, z)}{K} \quad (\text{A-87})$$

where

$$S(r, z) = S_0 \quad (\text{A-88})$$

for $0 \leq r \leq a$, $0 \leq z \leq \ell$, and zero elsewhere

$$(2) \quad T(r, z) = 0 \quad (A-89)$$

for all (r, z) on outer boundaries, and

$$(3) \quad K_1 \frac{\partial T_1}{\partial z} = K_2 \frac{\partial T_2}{\partial z} \quad (A-90)$$

at the interface between the two media.

By inspection and from the definition of the α_n as the zeroes of $J_0(\alpha a)$, it is obvious that $T(r, z) = 0$ on outer boundaries. Using

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial J_0(\alpha_n r)}{\partial r} \right) = -\alpha_n^2 J_0(\alpha_n r) \quad (A-91)$$

and

$$\frac{\partial^2 \sinh \alpha_n (\ell + z)}{\partial z^2} = \alpha_n^2 \sinh \alpha_n (\ell + z) \quad (A-92)$$

gives

$$\nabla^2 T_2(r, z) = 0. \quad (A-93)$$

To evaluate $\nabla^2 T_1(r, z)$, it is only necessary to evaluate the second term of $T_1(r, z)$ which will be denoted $T_s(r, z)$. Using

$$\frac{\partial^2}{\partial z^2} \sin \frac{m\pi z}{\ell} = -\frac{m^2 \pi^2}{\ell^2} \sin \frac{m\pi z}{\ell} \quad (A-94)$$

gives

$$\nabla^2 T_s(r, z) = -\frac{8S_0}{a^2} \sum_{m \text{ odd}} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{m K_1 \alpha_n J_1(\alpha_n a)} \sin \frac{m\pi z}{\ell}. \quad (A-95)$$

To evaluate this, consider a series expansion of the number "1" over the region $0 \leq r \leq a$, $0 \leq z \leq \ell$. This is given by

$$1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi z}{l} J_0(\alpha_n r) , \quad (\text{A-96})$$

where

$$A_{mn} = \frac{4}{a^2 l J_1^2(\alpha_n a)} \int_0^a \int_0^l \sin \frac{m\pi z}{l} J_0(\alpha_n r) r dz dr . \quad (\text{A-97})$$

Using

$$\int_0^l \sin \frac{m\pi z}{l} dz = \begin{cases} \frac{2l}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases} \quad (\text{A-98})$$

and

$$\int_0^a r J_0(\alpha_n r) dr = \frac{a}{\alpha_n} J_1(\alpha_n a) \quad (\text{A-99})$$

gives

$$A_{mn} = \frac{8}{m\pi a J_1(\alpha_n a)} . \quad (\text{A-100})$$

Therefore

$$\nabla^2 T_s(r, z) = - \frac{S_0}{K_1} . \quad (\text{A-101})$$

The final condition will be checked by evaluating $K_1 \frac{\partial T_1}{\partial z} \Big|_{z=0}$ and $K_2 \frac{\partial T_2}{\partial z} \Big|_{z=0}$ and setting them equal to each other to see if the equality is verified. Performing these operations gives

$$\begin{aligned}
& \frac{2S_0 K_2}{a(K_1 + K_2)} \sum_{n=1}^{\infty} \frac{\cosh \alpha_n \ell - 1}{\alpha_n^2 J_1(\alpha_n a)} \frac{J_0(\alpha_n r)}{\sinh \alpha_n \ell} \\
&= - \frac{2S_0 K_1}{a(K_1 + K_2)} \sum_{n=1}^{\infty} \frac{\cosh \alpha_n \ell - 1}{\alpha_n^2 J_1(\alpha_n a)} \frac{J_0(\alpha_n r)}{\sinh \alpha_n \ell} \\
&+ \frac{8S_0}{a\ell} \sum_{m \text{ odd}}^{\infty} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{\alpha_n \left(\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right) J_1(\alpha_n a)}, \tag{A-102}
\end{aligned}$$

or

$$\begin{aligned}
& \frac{\ell}{4} \sum_{n=1}^{\infty} \frac{\cosh \alpha_n \ell - 1}{\alpha_n^2 J_1(\alpha_n a)} \frac{J_0(\alpha_n r)}{\sinh \alpha_n \ell} \\
&= \sum_{m \text{ odd}}^{\infty} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{\alpha_n \left(\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2 \right) J_1(\alpha_n a)}. \tag{A-103}
\end{aligned}$$

To verify this equality, it is necessary to verify that

$$\frac{\ell}{4\alpha_n} \frac{\cosh \alpha_n \ell - 1}{\sinh \alpha_n \ell} = \sum_{m \text{ odd}}^{\infty} \frac{1}{\frac{m^2 \pi^2}{\ell^2} + \alpha_n^2} \tag{A-104}$$

or, letting $\beta = \frac{\ell \alpha_n}{\pi}$,

$$\frac{\pi}{4\beta} \left(\frac{\cosh \pi\beta - 1}{\sinh \pi\beta} \right) = \sum_{m \text{ odd}}^{\infty} \frac{1}{m^2 + \beta^2}. \tag{A-105}$$

But, this is precisely Eq. (A-42) which was verified in checking the

solution to problem 3. Therefore the analytical solution to problem 7 is verified.

The analytical solution to problem 8 is

$$T_2(r, z) = \frac{K_1}{K_1 + K_2} J_0(\alpha_1 r) \frac{\sinh \alpha_1 (\ell + z)}{\sinh \alpha_1 \ell \cosh \alpha_1 \ell} \quad (\text{A-106})$$

for $0 \leq r \leq a$, $-\ell \leq z \leq 0$,

and

$$T_1(r, z) = T_2(r, -z) + J_0(\alpha_1 r) \frac{\sinh \alpha_1 z}{\sinh \alpha_1 \ell} \quad (\text{A-107})$$

for $0 \leq r \leq a$, and $0 \leq z \leq \ell$.

The solution must satisfy the following:

$$(1) \quad \nabla^2 T = 0 \quad (\text{A-108})$$

$$(2) \quad T(r, z) = \begin{cases} J_0(\alpha_1 r), & \text{on the surface at } z = \ell \\ 0, & \text{on all other external boundaries} \end{cases} \quad (\text{A-109})$$

and

$$(3) \quad K_1 \frac{\partial T_1}{\partial z} = K_2 \frac{\partial T_2}{\partial z}, \quad (\text{A-110})$$

at the interface between the two media.

By inspection it is obvious that the external boundary conditions as given in Eq. (A-109) are met. Using

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} J_0(\alpha_1 r) \right) = -\alpha_1^2 J_0(\alpha_1 r), \quad (\text{A-111})$$

and

$$\frac{\partial^2}{\partial z^2} \sinh \alpha_1 (\ell + z) = \alpha_1^2 \sinh \alpha_1 (\ell + z) , \quad (\text{A-112})$$

gives

$$\nabla^2 T = 0 . \quad (\text{A-113})$$

The final condition will be checked by evaluating $K_1 \left. \frac{\partial T_1}{\partial z} \right|_{z=0}$ and $K_2 \left. \frac{\partial T_2}{\partial z} \right|_{z=0}$ and setting them equal to each other to see if the equality is verified. Performing these operations gives

$$\frac{K_1 K_2}{K_1 + K_2} \alpha_1 \frac{J_0(\alpha_1 r)}{\sinh \alpha_1 \ell} = - \frac{K_1^2}{K_1 + K_2} \alpha_1 \frac{J_0(\alpha_1 r)}{\sinh \alpha_1 \ell} + K_1 \alpha_1 \frac{J_0(\alpha_1 r)}{\sinh \alpha_1 \ell} , \quad (\text{A-114})$$

which checks. Therefore the analytical solution to problem 8 is verified.

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APPENDIX B

DERIVATION OF A BOUNDED NEXT EVENT ESTIMATOR

The following derivation will be made assuming a monoenergetic flux and isotropic scattering. These assumptions are made for simplicity of notation only. The results are not changed if the flux is assumed to be energy dependent and if the scattering is anisotropic.

The flux at a point $r=0$ can be represented by

$$\phi(r=0) = \int_r \Sigma_s \phi_{\text{avg}}(r) 4\pi r^2 \frac{e^{-\Sigma r}}{4\pi r^2} dr + \int_r S(r) 4\pi r^2 \frac{e^{-\Sigma r}}{4\pi r^2} dr \quad (\text{B-1})$$

where

$$\phi_{\text{avg}}(r) = \frac{\int \int_A \phi dA}{\int \int_A dA} \quad (\text{B-2})$$

and A is the surface area at r .

The standard Monte Carlo unbounded next event estimator uses each collision as an estimate of $\Sigma_t \phi_{\text{avg}}(r) 4\pi r^2$ and scores $\frac{\Sigma_s e^{-\Sigma r}}{\Sigma_t 4\pi r^2}$ at each collision site. The estimator is considered unbounded since the score is unbounded as r approaches zero.

This problem can be eliminated by scoring all collisions within a sphere of radius " a " by $\frac{e^{-\Sigma r}}{4\pi r^2}$ where d is calculated from

$$\int_0^a \Sigma_s \phi_{\text{avg}}(r) 4\pi r^2 \frac{e^{-\Sigma r}}{4\pi d^2} dr = \int_0^a \Sigma_s \phi_{\text{avg}}(r) 4\pi r^2 \frac{e^{-\Sigma r}}{4\pi r^2} dr \quad (\text{B-3})$$

This calculation depends upon the flux shape assumed for $\phi(r)$. In this appendix "d" will be calculated as a function of "a" for (1) a flat flux and (2) an exponentially decreasing flux.

Assume $\phi(r) = \phi_0$, a constant. Then $\phi_{avg}(r) = \phi_0$ and

$$d^2 = \frac{\int_0^a r^2 e^{-\Sigma r} dr}{\int_0^a e^{-\Sigma r} dr} . \quad (B-4)$$

Evaluating the above integrals gives

$$d^2 = a^2 \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda} (1 + \frac{2}{\lambda})}{1 - e^{-\lambda}} \right) \quad (B-5)$$

where $\lambda = \Sigma a$ is the radius of the bounding sphere in mean free paths.

This can also be written in series form as

$$d^2 = a^2 \frac{\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{(n+3)!} \lambda^n}{\sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{(n+1)!}} . \quad (B-6)$$

Next assume $\phi = \phi_0 e^{-\Sigma x}$. $\phi_{avg}(r)$ is then given by

$$\phi_{avg}(r) = \frac{1}{4\pi r^2} \int_A \phi_0 e^{-\Sigma x} dA \quad (B-7)$$

where A is the area of the sphere with radius r and $r = (x^2 + y^2 + z^2)^{1/2}$.

Evaluating this integral gives

$$\phi_{\text{avg}}(r) = \phi_0 \frac{\sinh(\Sigma r)}{\Sigma r}, \quad (\text{B-8})$$

and

$$d^2 = \frac{\int_0^a r^2 \frac{\sinh(\Sigma r)}{\Sigma r} e^{-\Sigma r} dr}{\int_0^a \frac{\sinh(\Sigma r)}{\Sigma r} e^{-\Sigma r} dr}. \quad (\text{B-9})$$

Evaluating these integrals gives

$$d^2 = a^2 \left[\frac{\frac{1}{2} - \frac{1}{c^2} + \frac{e^{-c}}{c^2} (1+c)}{\text{Ei}(c) + \ln(c) + \gamma} \right] \quad (\text{B-10})$$

where $c = 2\Sigma a$,

γ is Euler's constant, and

$\text{Ei}(c)$ is the exponential integral function of c .

This can also be written in series form as:

$$d^2 = a^2 \frac{\sum_{n=0}^{\infty} (-1)^n \frac{(n+2)c^n}{(n+3)!}}{\sum_{n=0}^{\infty} (-1)^n \frac{c^n}{(n+1)(n+1)!}}. \quad (\text{B-11})$$

Table 20 gives d^2/a^2 versus Σ_a for both assumptions, i.e., $\phi=\phi_0$ and $\phi=\phi_0 e^{-\Sigma x}$.

Note that for bounding spheres with radii up to one mean path in length the flux shape used has very little effect on d^2 , and therefore on the score used in the estimator. In estimating the flux at a point

which is not near a strong point source or an external boundary, the above method should therefore be applicable for bounding spheres up to one mean free path in radius since the flux shape will usually be somewhere between a constant and exponentially decaying.

TABLE 20
 d^2/a^2 VERSUS Σa

Σa (mfp)	d^2/a^2	
	for $\phi = \phi_0$	for $\phi = \phi_0 e^{-\Sigma x}$
.1	.3250	.3252
.25	.3127	.3136
.50	.2925	.2960
1.00	.2541	.2665

This procedure does not require a modification of the random walk procedure and may be implemented very easily into Monte Carlo programs which use next event estimators.

APPENDIX C

LISTING OF COMPUTER INPUT DATA AND SELECTED OUTPUT

The input to MORSE-SGC and its output are given on the following pages for the coupled gamma radiation and heat conduction problem solved in Chapter VI.

The listings of the analysis routines are given on the next three pages. The input data to LAVA to generate the cross sections and to MORSE-SGC is given on the following page, and the remaining pages in this Appendix are MORSE-SGC output.

```

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINEMT=40,SIZE=0000K,
SOURCE=EBSDIC,NOLIST,NOUSECK,LOAD,MAP,NOEDIT,VJID,NOXREF
134 0002 SUBROUTINE HELCOL(D,XD,YD,ZD,VEL,EXTRG,EXTRD)
C
C SUBROUTINE HELCOL IS CALLED AFTER EACH COLLISION DURING
C THE RANDOM WALK
C
134 0003 COMMON/INPUT/IAJH,NSTRT,NMST,NIFS,NOUT,NCOLTP,ISTAT,NSP,T,
2 NMILL,NPAST,NOLEAK,IESIAS,NKCALC,NORMP,
3 MEDIA,NMIX,MEDALS,NXREG,NP1STP,NNGA,NNGA,NNGTP,
4 NGOPT,IGOPT,NDIN,NDIG,NCOLP,NECT,MAXGP,IRDSG,ISTR,
5 IFMU,IMOM,IPRIN,IPUN,ITAPE,JXTAPE,IOGR,IGOPT,
6 ISOUR,NGPFS,ISIAS,NSOUR,
7 NO,MNE,NE,NT,MA,NRESP,MEX,MEND,IFLAG(20),
8 TMAX,TCUT,UTSTRT,AGSTRT,XSTRT,YSTRT,ZSTRT,
9 UIMP,VIMP,WIMP,NXPM,NMISTR,NMISM,NBINO(36),NCOL,S(13),
10 NG,IFTG,IGG,NNUC,IOI,NRP,NLM,NEM,NGPS,TITLE(20),DATE(6),
11 JFTG,NFTG,LFTJ
134 0004 COMMON/POINT/LPP1,LPP2,LPP3,LPP4,LPP5,LPP6,LPP7,LPP8,LPP9,
1 LPP10,LPP11,LPP12,LPP13,LPP14,LPP15,LPP16,LPP17,LPP18,LPP19,
2 LPP20,LPP21,LPP22,LPP23,LPP24,LAP1,LAP2,LAP3,LAP4,LAP5,
3 LAP6,LAP7,LSP1,LSP2,LSP3,
4 LPP25,LPP26,LPP27,LPP28,LPP29,LPP30,LPP31,LPP32,
5 LX1,LX2,LX3,LX4,LT1,LT9
134 0005 COMMON/NUTRON/NAME,NAMEX,IG,IGO,NMED,MEDCL,DNREG,J,V,W,UDLD,VJLD
1 WOLD,X,Y,Z,XOLD,YOLD,ZOLD,WATE,OLDOT,WTBC,ELZNT,B,ZN,AGE,OLDAGE
134 0006 COMMON/APOLL/OPF,OPF,DEADOT(S),ETA,ETATH,ETAUSD,XTRA(10),
1 ITER,S,ITIME,ITSTR,MAXTIM,MGPREG,NALB,NOEAO(S),NEGNN,NGEON,
2 NLAST,NMEX,NMGP,NMTG,
3 NPSEL(13),NSIGL,NXTRA(10)
134 0007 COMMON/ODET/LOCASP,LOCND,LOCIB,LOCCO,LOCY,LOCJD,LOCSD,
1 LOCQE,LOCOT,LOCQE,LOCQAE,LMAX,EFIRST,EGTOP
134 0008 COMMON/GOMLOC/ KMA,KPPD,KLCR,KND,KIOR,KRIZ,KRCZ,KMIZ,KMCZ,
1 KKR1,KKR2,KNSR,KVUL,NADD,LDATA,LFMA,LPPD,NJMR,INTRJ,NJMR,NIA
134 0009 INTEGER EXTRD
134 0010 DIMENSION D(1),XD(1),YD(1),ZD(1),VEL(1),EXTRG(NMTG,1),
1 EXTRD(ND,1)
134 0011 ENTRY SGAM(D,XD,YD,ZD)
C
C ENTRY SGAM IS CALLED WHEN A SECONDARY PARTICLE IS GENERATED
C
134 0012 IF(IG.NE.2)RETURN
C
C ONLY SECONDARY PARTICLES (THE GAMMAS ARE IN GROUP 2 IN THIS
C ADJOINT FORMULATION) ARE SCORED
C
134 0014 DO 10 I=1,ND
C
C DETECTOR LOCATIONS ARE USED TO INDICATE THE DIRECTION
C TOWARD THE INCOMING BEAM. THE DIFFERENT DIRECTIONS ARE
C CHOSEN TO GIVE DIFFERENT ORIENTATIONS WITH RESPECT TO THE
C STARTING LOCATION
C
134 0015 ME=X+XD(I)
134 0016 VE=Y+YD(I)

```

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```

157 0017 ZE=Z+Z0(1)
157 0018 SC=X0(1)*Z+Y0(1)*Z+Z0(1)*Z
157 0019 OS=SORI(SO)
157 0020 NOTURNMED
157 0021 XI=Z
157 0022 ZI=Z
157 0023 MARK=1
157 0024 CALL EUC(DI,D(LFPI),NMTG,MEDIA,D(KMIZ),MARK,X,Y,Z,XE,
157 0025 VE,ZE,DS,IG,AR,1,ME,DUM,PLZNT,MREG)
157 0026 X=Z1
157 0027 Y=Z1
157 0028 Z=Z1
157 0029 IF(ARC.LT.-99.160 TO 5
157 0030 CON=DATE*EXP(ARC)/12.56637
157 0031 IF(ABS(CON).LT.1E-40)GO TO 5
157 0032 IL=2
157 0033 COSI=1.
157 0034 W=0
157 0035 IF(MEX.LE.-01)M=1
157 0036 IF(MEX.LE.0)CON=1.8)*CON
157 0037 CALL FLUXSY(1,IL,CON,AGED,COSI,M,D(LOC,H*39ME),DI,JCOF),
157 0038 1 D(LOC),D(LOC,TMO*MT),D(LOC),D(LOC),D(LOC),D(LOC),D(LOC),
157 0039 2 D(LOC),D(LOC),D(LOC),D(LOC),D(LOC),D(LOC),D(LOC),
157 0040 5 CONTINUE
157 0041 IF CONTINUE
157 0042 IF RETURN
157 0043 IF ENO
157 0044 IF ENO
157 0045 IF ENO

```


[illegible]

ANPA MODULE LOG AND LIST OF CARD INPUT DATA

MODULE LAVA HAS BEEN CALLED

088 5 E
188 1 1 3 4 5 1 1 0 17
288 1 388 0 488 1 788 2.0 1.0 888 2.0 1.0 27
2 5 0 1 DUMMY FOR HEAT WITH GAMMA ADJOINT
0. 0. 1. 1. C.
0.03 0.0 .1 .1 .07 .03

MODULE LAVA IS FINISHED.
CPU TIME .63 (SECONDS). 170'S 156 USER CODE J300 SYSTEM CODE 000

MODULE MGRSE HAS BEEN CALLED

TEST OF GAMMA HEATING

188 0 100 200 100 1 0 0 0 1 0 0 0 0
288 1 1 1000 1 0
388 1 1 0 42 1 0 2 1 52 4 32
488 1 0 0 0
588 5 0 0 0 0 1 1 1
72 1 1 F0

GAMMA HEATING TEST

42
SPH 32 10
SPH 32 1.E+5
SPH 32 3.E+5
END
241 1
242 2 -1
243 3 -2
END
1 1 1
1 1000 0
688 4.2 0 1. 0 6.0 0. 0. 0. 0. 0.
888 1.5 1.5 .05 .05 .1 .1 .5 .5
1188 F.1
1288 1 388 1 1488 1.
27

1988 1.E+5 22 1.E+4 1.E+4 0 0 1.E+5 0 -1.E+4 1.E+4 0
-1.E+5 22
2088 F12.56637
37

MODULE MGRSE IS FINISHED.
CPU TIME 258.49 (SECONDS). 170'S 1717 USER CODE J300 SYSTEM CODE 000

THURSDAY, MAY 19, 1977

TEST OF GAMMA HEATING

1 8 ARRAY

IAJN	ADJOINT INDICATOR	0	NSPLT	SPLITTING INDICATOR	0
PJRT	NUMBER OF PARTICLES PER BATCH	100	NRILL	RUSSIAN ROULETTE INDICATOR	1
NRJST	MAX NUMBER OF PARTICLES ALLOWED	200	NRJST	EXPONENTIAL TRANSFORM INDICATOR	0
NRIS	NUMBER OF BATCHES	100	NRLEAK	NON-LEAKAGE INDICATOR	0
NRJUT	NUMBER OF SETS OF BATCHES	1	NRBIAS	ENERGY BIAS INDICATOR	0
NRCLT	COLLISION TAPE INDICATOR	0	NRKCALC	BATCH USED IN K-CALCULATION	0
ISLAT	INDICATOR TO STORE LEGRENGRE COEFF	0	NRNMF	NORMALIZATION INDICATOR	0

2 8 ARRAY

MEDIA	NUMBER OF CROSS SECTION MEDIA	1	MRREG	NUMBER OF REGIONS	1
MMIX	NUMBER OF MIXING OPERATIONS	1	MRJSTP	FISSION INDICATOR	0
MEDA-B	ALBEDO INDICATOR	1000			

3 8 ARRAY

NRGA	NUMBER OF N-GPS TO ANALYZE	1	ISTR	PRINT CROSS SECTIONS AS STORED	0
NRGA	NUMBER OF G-GPS TO ANALYZE	1	IPMU	PRINT RESULTS OF MU CALC	0
NRGTD	COMPLETELY COUPLED INDICATOR	0	IMON	PRINT MOMENTS	0
NRDSN	NUMBER OF N DOWNSCATTERS	0	IPRIN	PRINT ANGLES AND PROB	0
NRDSG	NUMBER OF G DOWNSCATTERS	0	IPUN	PRINT RAD COEF RESULTS	0
NRDEF	NUMBER OF COEFFICIENTS	1	ITXTAPE	INPUT LOGICAL TAPE UNIT	4
NRDCT	NUMBER OF DISCRETE ANGLES	0	JXTAPE	OUTPUT LOGICAL TAPE UNIT	0
NRXGP	LAST GP FOR RR-SPLITTING, OR XFORM	2	IOBR	OUR LOGICAL TAPE UNIT	0
NRDSG	PRINT CROSS SECTIONS AS READ	1	IGOPT	LAST OUR GROUP	0
NRDGT	EXTRA NEUTRON GROUP OPTION	0	IGOPT	EXTRA GAMMA GROUP OPTION	0

THURSDAY, MAY 19, 1977

TEST OF GAMMA HEATING

4 8 ARRAY

ISSUR SOURCE ENERGY GROUP
 NGPFS NUMBER OF SOURCE GROUPS

ISSBIAS SOURCE ENERGY BIAS INDICATOR 0
 NSOURN FIXED SOURCE INDICATOR 0

5 8 ARRAY

ND NUMBER OF DETECTORS
 NNE NUMBER OF NEUTRON ENERGY BINS
 NE NUMBER OF ENERGY BINS
 NT NUMBER OF TIME BINS

NA NUMBER OF ANGLE BINS 0
 NRESP NUMBER OF RESPONSE FUNCTIONS 1
 NEX NO OF EXTRA MNGA+NGCA SIZE ARRAYS 1
 NERND NUMBER OF EXTRA NO SIZE ARRAYS 1

THE FOLLOWING ANALYSIS SUBROUTINES WILL BE CALLED

SUBROUTINE NRUN VERSION 1
 SUBROUTINE NSATCH VERSION 1
 SUBROUTINE STATCH VERSION 1
 SUBROUTINE SCAM VERSION 1
 SUBROUTINE RELCOL VERSION 1

THURSDAY, MAY 19, 1977

TEST OF GAMMA HEATING

6 * ARRAY

TMAX	MAXIMUM 01 CPU TIME	0.0200E 01	YSTRY	SOURCE Y-COORDINATE	0.0
TCUT	AGE PARTICLES RETIRED	0.0	ZSTRY	SOURCE Z-COORDINATE	0.0
WTSTAT	SOURCE PARTICLE WEIGHT	0.1000E 01	UIMP	SOURCE DIRECTION COSINE	0.0
AGSTRY	SOURCE AGE	0.0	VIMP	SOURCE DIRECTION COSINE	0.0
XSTRY	SOURCE X-COORDINATE	0.0000E 01	#IMP	SOURCE DIRECTION COSINE	0.0

7 * ARRAY

GROUP	1	NEUTRON VELOCITY	0.1504E 07
-------	---	------------------	------------

8 * ARRAY

GROUP	REGION	SPLITTING WT	RR WEIGHT	SURVIVING #T	STRETCH PARAMETERS
1	1	0.1500E 01	0.5000E-01	0.1200E 00	0.5000E 00
2	1	0.1500E 01	0.5000E-01	0.1200E 00	0.5000E 00

11 * ARRAY

GROUP	REGION		
1	0.1000E 00		
128.138 AND 14 * ARRAYS			
MEDIUM	1 ELEMENT	DENSITY	0.1000E 01

GAMMA HEATING TEST

IWOPT = 0

IDBG = 0

SPH	1	0.0	0.0	0.0	BODY DATA
SPH	2	0.0	0.0	0.0	0.10000000 02
SPH	3	0.0	0.0	0.0	0.20000000 06
SPH	3	0.0	0.0	0.0	0.30000000 06
NUMBER OF BODIES 3					
LENGTH OF FPD-ARRAY 18					

	INPUT ZONE	CODE ZONE	BODY NUMBERS	INPUT ZONE DATA
ZN1	1	1	1	
ZN2	2	2	2	-1
ZN3	3	3	3	-2
NUMBER OF INPUT ZONES 3				
NUMBER OF CODE ZONES 3				
LENGTH OF INTEGER ARRAY 74				

CODE ZONE	INPUT ZONE	ZONE DATA LOC.	NO. OF BODIES	REGION NO.	MEDIA NO.
1	1	22	1	1	1
2	2	27	2	1	1000
3	3	36	2	1	0

I	KR1(I)	KR2(I)
1	1	1
2	2	2
3	3	3

MORSE REGION IN INPUT ZONE(I) ARRAY (MRIZ(I),I=1, 3)

1	1	1
---	---	---

MORSE MEDIA IN INPUT ZONE(I) ARRAY (MMIZ(I),I=1, 3)

1	1000	0
---	------	---

TEST OF GAMMA HEATING

THURSDAY, MAY 19, 1977

DETECTOR	X	Y	Z	RAD	TO
1	0.1000E 00	0.0	0.0	3.9999E 05	0.3335E-05
2	0.1000E 05	0.1000E 05	0.0	0.1414E 05	0.4715E-05
3	0.0	0.1000E 06	0.0	0.1000E 06	0.3336E-05
4	-0.1000E 05	0.1000E 05	0.0	0.1415E 05	0.4719E-05
5	-0.1000E 06	0.0	0.0	0.1000E 06	0.3336E-05

RESPONSE FUNCTIONS

GROUP RESP(1)
 1 0.1257E 02
 2 0.1257E 02

TIME REQUIRED FOR INPUT WAS LESS THAN ONE SECOND.

***START BATCH 1 RANDOM=00071AFD49BD

SOURCE DATA

YOU ARE USING THE DEFAULT VERSION OF SOURCE WHICH SETS RATE TO DDF AND PROVIDES AN ENERGY IG.
 WT= 1.0000E 02 UAVE= -3.7939E-02 VAVE= 4.0694E-02 WAVE= -4.8966E-02 AGEAVE= 0.0
 IAVE= 1.00 XAVE= 9.0000E 00 YAVE= 0.0 ZAVE= 0.0

YOU ARE USING THE DEFAULT VERSION OF GTMED WHICH ASSUMES THAT THE GEOMETRY AND XSEC MEDIA ARE THE SAME

NUMBER OF COLLISIONS OF TYPE NCOLL

SOURCE SPL IT(D)	FISHN	GAMGEN	REALCOLL	ALBEDO	BDRYX	ESCAPE	E-CUT TIMEKILL R	KILL R R	SURV	GAMLOST
100	0	0	676	6958	884	0	0	332	180	0

TIME REQUIRED FOR THE PRECEDING BATCH WAS 8 SECONDS.

***START BATCH 2 RANDOM=282704AD4EF5

SOURCE DATA

WT= 1.0000E 02 UAVE= 7.1318E-03 VAVE= -3.4683E-02 WAVE= 1.6255E-02 AGEAVE= 0.0
 IAVE= 1.00 XAVE= 9.0000E 00 YAVE= 0.0 ZAVE= 0.0

NUMBER OF COLLISIONS OF TYPE NCOLL

SOURCE SPL IT(D)	FISHN	GAMGEN	REALCOLL	ALBEDO	BDRYX	ESCAPE	E-CUT TIMEKILL R	KILL R R	SURV	GAMLOST
100	0	0	579	5964	755	0	0	370	188	0

TIME REQUIRED FOR THE PRECEDING BATCH WAS 7 SECONDS.

***START BATCH 3 RANDOM=56D954594045

SOURCE DATA

WT= 1.0000E 02 UAVE= 7.2519E-02 VAVE= 5.7363E-02 WAVE= 4.3730E-03 AGEAVE= 0.0
 IAVE= 1.00 XAVE= 9.0000E 00 YAVE= 0.0 ZAVE= 0.0

NUMBER OF COLLISIONS OF TYPE NCOLL

SOURCE SPL IT(D)	FISHN	GAMGEN	REALCOLL	ALBEDO	BDRYX	ESCAPE	E-CUT TIMEKILL R	KILL R R	SURV	GAMLOST
100	0	0	632	6899	833	0	0	348	197	0

TIME REQUIRED FOR THE PRECEDING BATCH WAS 8 SECONDS.

***START BATCH 4 RANDOM=EF23D29500D1

SOURCE DATA

WT= 1.0000E 02 UAVE= 9.4811E-03 VAVE= 3.6692E-03 WAVE= -1.9085E-02 AGEAVE= 0.0
 IAVE= 1.00 XAVE= 9.0000E 00 YAVE= 0.0 ZAVE= 0.0

NUMBER OF COLLISIONS OF TYPE NCOLL

RUN TERMINATED BY EXECUTION TIME LIMIT
THIS CASE WAS RUN ON THURSDAY, MAY 19, 1977

DETECTOR	RESPONSES (DETECTOR)		TOTAL RESPONSE	TOTAL RESPONSE
	UNCOLL	PSU		
1	-2.1193E-01	-0.04397	3.1312E-01	0.06032
2	-2.0136E-01	-0.04503	2.0742E-01	0.06137
3	-1.6255E-01	-0.04127	2.1469E-01	0.06496
4	-1.4868E-01	-0.04127	1.5376E-01	0.08059
5	-9.0686E-02	-0.07379	1.2609E-01	0.08848