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SEPTEMBER 1980

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BY

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**PLASMA PHYSICS
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Two-Dimensional Modeling of the Formation of Spheromak Configurations

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A reduced set of two-dimensional MHD equations have been derived describing the axisymmetric time evolution of a MHD stable plasma evolving slowly due to resistive diffusion and changing boundary conditions. The equations are restricted to low β but allow changing topology. They are integrated in time to demonstrate a possible spheromak formation method. External circuit equations are integrated simultaneously with the plasma equations to determine the electromagnetic boundary conditions self consistently. The effects of a finite conductivity vacuum chamber are included.

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I. INTRODUCTION

The spheromak, a closed magnetic field line toroidal confinement configuration with null external toroidal field, has recently received considerable theoretical attention.¹⁻³ Much of this interest stems from the fact that a spheromak fusion reactor would enjoy certain advantages over a tokamak reactor.¹ Since most of the confining magnetic field is produced by currents in the plasma itself, the "engineering" beta value (with respect to externally imposed magnetic fields) would be large. Also, structural shielding and blanket requirements would be greatly eased since the configuration requires no structural linkage down the symmetry axis.

Theoretical studies indicate that once a spheromak equilibrium is established, stability to ideal and resistive MHD modes exists over a wide range of parameter space provided there is a conducting shell in the vicinity of the plasma.^{2,3} Before the spheromak reactor concept can be taken seriously, however, it must be demonstrated that it is possible to produce a spheromak configuration in a geometry without toroidal field coil linkage in a quasistatic manner that is applicable to parameters in reactor regimes.

An experiment designed to demonstrate such a formation method will soon take place.⁴ It is the intent of this paper to theoretically demonstrate the formation of such an equilibrium by numerically integrating the MHD equations in a realistic geometry with experimentally relevant boundary conditions. We take advantage of the two time scales implied by the large magnetic Reynolds number to simplify the dynamical MHD equations. A companion paper⁵ also presents numerical solutions demonstrating spheromak formation, but does so without utilizing reduced equations.

In the next section we modify the two-dimensional one fluid MHD equations

to obtain a reduced set of equations that can be integrated accurately numerically. This ordering is justified in Sec. III where the reduced equations are interpreted using asymptotic ordering and multiple time scale formalism. The imposition of realistic boundary conditions is essential in modeling a real experiment. In this case, this requires the simultaneous solution of a system of electrical circuit equations, discussed in Sec. IV. Results of the numerical solution of the reduced equations are presented in Sec. V where we illustrate the formation of a spheromak configuration.

II. THE REDUCED COMPRESSIBLE MHD EQUATIONS

The one fluid resistive MHD equations are

$$\frac{\partial}{\partial t} \vec{m} + \nabla \cdot \left(\frac{1}{\rho} \vec{m} \vec{m} \right) + \nabla p = \vec{J} \times \vec{B} \quad , \quad (1)$$

$$\frac{3}{2} \frac{\partial}{\partial t} p + \nabla \cdot \left(\vec{q} + \frac{5}{2} \frac{p}{\rho} \vec{m} \right) = \vec{J} \cdot \vec{E} + S_e \quad , \quad (2)$$

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \vec{m} = S_p \quad , \quad (3)$$

$$\frac{\partial}{\partial t} \vec{B} = -\nabla \times \vec{E} \quad , \quad (4)$$

$$\vec{E} + \frac{1}{\rho} \vec{m} \times \vec{B} = \eta \vec{J} \quad , \quad (5)$$

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} \quad , \quad (6)$$

$$T = \frac{M_i}{2\rho k_B} p \quad . \quad (7)$$

Here p is the fluid pressure, ρ is the mass density, $\vec{q}$ is the heat flux vector, and T is the fluid temperature. In an axisymmetric geometry with symmetry angle ϕ , the magnetic field $\vec{B}$ can be written in terms of the poloidal flux per radian ψ and the toroidal field function g ,

$$\vec{B} = \nabla\phi \times \nabla\psi + g\nabla\phi \quad . \quad (8)$$

The momentum density $\vec{m}$ can be written in terms of a stream function A , a toroidal component ω , and a potential Ω ,

$$\vec{m} = \nabla\phi \times \nabla A + \omega\nabla\phi + \nabla\Omega \quad . \quad (9)$$

As discussed in Sec. III, we are looking for solutions with slowly changing boundary conditions and small mass flow so that the ordering

$$\frac{\mu_0 m^2 / \rho}{B^2} \sim \epsilon^2 \ll 1 \quad (10a)$$

is appropriate. If in addition we restrict consideration to low β plasmas, the ordering

$$\frac{\mu_0 p}{B^2} \sim \epsilon^2 \quad (10b)$$

leads to considerable simplification. The reduced equations are then obtained by using Eq. (10) to justify dropping the convective derivative and pressure gradient terms in Eq. (1) and adding a viscous term (see Sec. III). Equations (2) and (7) are no longer needed to obtain a closed set of equations, and Eq. (1) becomes simply

$$\frac{\partial \vec{m}}{\partial t} - \nu \nabla^2 \vec{m} = \vec{J} \times \vec{B} \quad (1)'$$

Substitution of Eqs. (8) and (9) into Eqs. (1)', (3), (4), (5), (6), taking the curl and divergence of Eq. (1)', and performing the necessary vector manipulations yields the set of scalar evolution equations solved numerically,

$$\frac{\partial}{\partial t} \nabla^2 \Omega + \nabla \cdot \left(-\frac{\Delta^* \psi}{\mu_0 x^2} \nabla \psi + \frac{g}{\mu_0 x^2} \nabla g - \nu \nabla (\nabla^2 \Omega) \right) = 0 \quad , \quad (11)$$

$$\frac{\partial}{\partial t} \Delta^* A + x^2 \nabla \cdot \left(\frac{\Delta^* \psi}{\mu_0 x^2} \nabla \psi \times \nabla \phi + \frac{g}{\mu_0 x^2} \nabla g \times \nabla \phi - \frac{\nu}{x^2} \nabla (\Delta^* A) \right) = 0 \quad , \quad (12)$$

$$\frac{\partial}{\partial t} \omega + \frac{1}{\mu_0} (\nabla \phi \times \nabla g) \cdot \nabla \psi - \nu \Delta^* \omega = 0 \quad , \quad (13)$$

$$\frac{\partial}{\partial t} \rho + \nabla^2 \Omega = S_\rho \quad , \quad (14)$$

$$\frac{\partial \psi}{\partial t} + \frac{1}{\rho} (\nabla \phi \times \nabla A \cdot \nabla \psi + \nabla \Omega \cdot \nabla \psi) = \frac{\eta}{\mu_0} \Delta^* \psi \quad , \quad (15)$$

$$\frac{\partial g}{\partial t} + x^2 \nabla \cdot \left(-\frac{g}{2\rho x} (\nabla \phi \times \nabla A + \nabla \Omega) - \frac{\omega}{2\rho x} \nabla \phi \times \nabla \psi \right) = \frac{x^2}{\mu_0} \nabla \cdot \frac{\eta}{x^2} \nabla g \quad (16)$$

Here, $\Delta^* \equiv x^2 \nabla \cdot x^{-2} \nabla$ is the familiar toroidal elliptic operator and $x = |\nabla \phi|^{-1}$ measures the distance from the symmetry axis.

The set of Eqs. (11) through (16) offer several advantages over more primitive forms. By taking the divergence and curl of the momentum equation, the shear Alfvén wave motion has been removed from the velocity potential variable Ω and the fast compressible wave motion has been removed from the variables A and ω . As discussed in Ref. 6, this partial isolation of the different MHD characteristics leads to dramatic improvements in the accuracy and the stability of the numerical solutions.

A physically interesting subset of Eqs. (11) through (16) is that obtained by dropping Eq. (11) and setting $\Omega = 0$ in Eqs. (14) through (16). This is the familiar incompressible approximation. The fact that the incompressible equations are a subset of our full set of equations can be exploited in developing and debugging a numerical solution method for the full set of equations.

III. PHYSICAL INTERPRETATION

In this section we justify the replacement of the momentum Eq. (1) with the reduced momentum Eq. (1)'. The reduced Eqs. (11) through (16) expressed in vector form are

$$\frac{\partial \vec{m}}{\partial t} - \nu \nabla^2 \vec{m} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} \quad , \quad (17a)$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\frac{1}{\rho} \vec{m} \times \vec{B} \right) - \nabla \times \frac{\eta}{\mu_0} \nabla \times \vec{B} \quad , \quad (17b)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{m} = 0 \quad . \quad (17c)$$

The Eqs. (17) contain phenomena occurring on two distinct time scales, the ideal time scale

$$t_0 \sim \tau_h \sim \frac{a}{B} (\mu_0 \rho_0)^{1/2} \quad , \quad (18a)$$

and the resistive time scale

$$t_1 \sim \tau_R \sim \frac{a^2 \mu_0}{\eta} \sim \epsilon^{-1} t_0 \quad , \quad (18b)$$

where

$$\epsilon = \frac{\eta}{aB} \left(\frac{\rho_0}{\mu_0} \right)^{1/2} \ll 1 \quad . \quad (19)$$

For boundary conditions changing on the (slow) t_1 time scale, we look for an ordering that describes a configuration which is always near ($\sim \epsilon$) equilibrium but in which the equilibrium evolves on the t_1 time scale. Such a description is given formally by the multiple time scale method⁷ with the ordering

$$\eta = \epsilon \eta_1 \quad , \quad (20a)$$

$$\vec{B} = \vec{B}_0(t_1) + \epsilon \vec{B}_1(t_0, t_1) + \dots \quad , \quad (20b)$$

$$\rho = \rho_0(t_1) + \epsilon \rho_1(t_0, t_1) + \dots, \quad (20c)$$

$$\vec{m} = \epsilon \vec{m}_1(t_0, t_1) + \dots, \quad (20d)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \dots, \quad (20e)$$

where t_0 and t_1 are treated as independent variables.

Substitution of Eq. (20) into Eq. (17a) yields to order ϵ^0

$$(\nabla \times \vec{B}_0) \times \vec{B}_0 = 0, \quad (21)$$

and to order ϵ^1

$$\frac{\partial \vec{m}_1}{\partial t_0} - \nu \nabla^2 \vec{m}_1 = \frac{1}{\mu_0} [(\nabla \times \vec{B}_1) \times \vec{B}_0 + (\nabla \times \vec{B}_0) \times \vec{B}_1]. \quad (22)$$

Thus, the zero-order magnetic field is in equilibrium without zero order mass flow, and the first-order mass flow evolves on the fast time scale, being driven by the small departure of $\vec{B}$ from a force free field.

Substitution of Eq. (20) into Eq. (17b) gives, to order ϵ ,

$$\frac{\partial \vec{B}_0}{\partial t_1} + \frac{\partial \vec{B}_1}{\partial t_0} = \nabla \times \left(\frac{\vec{m}_1}{\rho_0} \times \vec{B}_0 \right) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \vec{B}_0 \right). \quad (23)$$

Averaging Eq. (23) over the t_0 time scale and subtracting that average from Eq. (23) gives an equation for the slow time scale evolution of the force free field $\vec{B}_0$,

$$\frac{\partial \vec{B}_0}{\partial t_1} = \nabla \times \left(\frac{\langle \vec{m}_1 \rangle}{\rho_0} \times \vec{B}_0 \right) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \vec{B}_0 \right) , \quad (24)$$

where $\langle \vec{m}_1 \rangle(t_1) \equiv \frac{1}{\tau} \int_0^\tau \vec{m}_1(t_0, t_1) dt_0$ denotes the average of $\vec{m}_1$ over the t_0 time scale, and an equation for the fast time scale evolution of $\vec{B}_1$,

$$\frac{\partial \vec{B}_1}{\partial t_0} = \nabla \times \left(\frac{(\vec{m}_1 - \langle \vec{m}_1 \rangle)}{\rho_0} \times \vec{B}_0 \right) . \quad (25)$$

Note that we used the fact that $\langle \vec{B}_1 \rangle(t_1) = 0$, which is implied by the two time scale ordering of Eq. (20).

Similar operations on the density evolution of Eq. (17c) yield separate evolution equations for ρ_0 and for ρ_1 ,

$$\frac{\partial \rho_0}{\partial t_1} + \nabla \cdot \langle \vec{m}_1 \rangle = 0 , \quad (26)$$

$$\frac{\partial \rho_1}{\partial t_0} + \nabla \cdot (\vec{m}_1 - \langle \vec{m}_1 \rangle) = 0 . \quad (27)$$

Thus, the zero-order density component evolves slowly due to the time average of $\vec{m}_1$, while the first order density component exhibits small amplitude fast time scale oscillations about zero.

A. Fast Time Scale Physics

A description of the fast time scale physics is given by Eqs. (22), (25), and (27). Taking the t_0 derivative of Eq. (22) and substituting in from Eq. (25) gives a dispersion relation for the fast time scale mass flow,

$$\frac{\partial^2 \vec{m}_1}{\partial t_0^2} - \nu \nabla^2 \frac{\partial \vec{m}_1}{\partial t_0} = \frac{1}{\mu_0} \left[\nabla \times \left[\nabla \times \left(\frac{\vec{m}_1}{\rho_0} \times \vec{B}_0 \right) \right] \times \vec{B}_0 \right]$$

$$+ (\nabla \times \vec{B}_0) \times \left[\nabla \times \left(\frac{\vec{m}_1}{\rho_0} \times \vec{B}_0 \right) \right] + \vec{R}(t_1) \quad (28)$$

With $\nu = 0$, Eq. (28) is just the zero pressure ideal MHD equation of motion, containing both the transverse and fast longitudinal wave branches. Since there are no unstable modes in two dimensions, the fast time scale behavior is simply that of small amplitude oscillations about the slowly varying equilibrium fields. With $\nu > 0$, these oscillations are damped. Setting $\nu = 2aB_0(\mu_0\rho_0)^{-1/2}$ makes the longest wavelength modes critically damped, decaying on the τ_A time scale.

B. Slow Time Scale Physics

A description of the slow time scale physics is given by Eqs. (21), (24), and (26). Taking the t_1 derivative of Eq. (21) and substituting in from Eq. (24) yields the constraint equation determining the slowly varying momentum density $\langle \vec{m}_1 \rangle(t_1)$,

$$\begin{aligned} & \left[\nabla \times \left(\frac{\langle \vec{m}_1 \rangle}{\rho_0} \times \vec{B}_0 - \frac{\eta}{\mu_0} \nabla \times \vec{B}_0 \right) \right] \times \vec{B}_0 + (\nabla \times \vec{B}_0) \\ & \times \left[\nabla \times \left(\frac{\langle \vec{m}_1 \rangle}{\rho_0} \times \vec{B}_0 - \frac{\eta}{\mu_0} \nabla \times \vec{B}_0 \right) \right] = 0 \quad (29) \end{aligned}$$

Equation (29) is the well known constraint equation for the velocity field associated with resistive diffusion. Other forms of this equation have been derived by Taylor,⁸ Grad,⁹ Pao,¹⁰ Hirshman and Jardin.¹¹ The fundamental point to note here is that the resistive component of the velocity $\langle \vec{m}_1 \rangle(t_1)$ is not determined from a time advancement equation as is $\vec{m}_1(t_0, t_1)$. It is determined instead from the constraint condition that the system evolve

through a series of near equilibrium states $\vec{B}_0(t_i)$ satisfying the equilibrium Eq. (21).

IV. BOUNDARY CONDITIONS

The time advancement Eqs. (11) through (16) are being applied to study the time evolution of the plasma in the experimental device shown in Fig. 1. The plasma occupies the region between the thin conducting metallic liner and the thick conducting vacuum chamber. Outside the vacuum chamber are equilibrium field (EF) coils which produce a nearly uniform steady state vertical field. Inside the metallic liner are two independent sets of current carrying windings. The toroidal windings are connected to the poloidal field (PF) capacitor banks, and the poloidal windings are connected to the toroidal field (TF) capacitor banks. In this section we discuss the boundary conditions at the interface between the plasma and the adjacent conductors.

Magnetic boundary conditions require information about the flux functions ψ and g at the plasma boundary. These functions are related to the electric field from Eqs. (4) and (8), thus

$$\frac{\partial \psi}{\partial t} = x^2 \nabla \phi \cdot \vec{E} \quad , \quad (30)$$

$$\frac{\partial g}{\partial t} = x^2 \nabla \cdot (\nabla \phi \times \vec{E}) \quad . \quad (31)$$

The boundary conditions on the velocity or mass flow variables are obtained from taking the vector product of the magnetic field $\vec{B}$ with Ohm's law, Eq. (5),

$$\vec{E} \times \vec{B} + \frac{1}{\rho} (\vec{m} \times \vec{B}) \times \vec{B} = \eta \vec{J} \times \vec{B} = 0 \quad , \quad (32)$$

where Eq. (i)' and the orderings (20a) and (20d) have been used in the second equality. We have the freedom to require that the mass flow parallel to $\vec{B}$ vanish at the boundary (no slip), thus Eq. (32) takes the form

$$\frac{1}{\rho} \vec{m} = \frac{\vec{E} \times \vec{B}}{B^2} \quad . \quad (33)$$

The normal component of the incompressible part of the velocity field must vanish at the boundary, yielding

$$A = 0 \quad . \quad (34)$$

The irrotational parts of the velocity field can now be determined using Eqs. (9), (33), and (34),

$$\omega = \rho \kappa^2 \frac{\nabla \phi \cdot \vec{E} \times \vec{B}}{B^2} \quad , \quad (35)$$

$$\nabla \Omega = \rho \frac{\vec{E} \times \vec{B}}{B^2} - \omega \nabla \phi \quad . \quad (36)$$

It is evident from Eqs. (30), (31), (34), (35), and (36) that the boundary conditions are determined once the tangential electric field at the boundary is known. This is the natural boundary condition since the tangential electric field must be continuous across the interface. The electric field is zero in the thick conducting vacuum vessel walls since the associated skin time is long compared to the lifetime of the plasma. The

determination of the electric field in the thin conducting metallic liner is the subject of the next two subsections.

A. The Toroidal Field Circuit Current

Figure 2 details the components which enter into the determination of the poloidal electric field at the boundary of the thin metallic liner and the plasma. The poloidal circuit current I_{TF} passes through the TF capacitor banks C_{TF} , resistance R_{TF} , and makes m loops around the flux core. Summing the voltage drops around this circuit gives

$$-m \oint_{a_c} \vec{E} \cdot d\vec{l} = R_{TF} I_{TF} + \frac{Q_{TF}}{C_{TF}} \quad (37)$$

By Faraday's law, the toroidal field function g , Eq. (8), must be uniform inside the radius a_c with a value g_1 which obeys the time advancement equation

$$\oint_{a_c} \vec{E} \cdot d\vec{l} = L_c \frac{d}{dt} (2\pi g_1) \quad (38)$$

where by direct computation $L_c = R_o - (R_o^2 - a_c^2)^{1/2}$.

If we approximate the metallic liner by N conducting shells, each with radius a_j carrying a current I_j , then by similar considerations the toroidal field function between current sheets I_j and I_{j+1} , g_j , satisfies the time advancement equation

$$\oint_j \vec{E} \cdot d\vec{l} - \oint_{j+1} \vec{E} \cdot d\vec{l} = (L_j - L_{j+1}) \frac{d}{dt} (2\pi g_j) \quad (39)$$

where $L_j = R_o - (R_o^2 - a_j^2)^{1/2}$. If the liner material has resistivity η_L and each current sheet has thickness Δa_j , then the line integral of the electric

field around each radius must satisfy

$$\oint_j \vec{E} \cdot d\vec{l} + R_j I_j = 0 \quad , \quad (40)$$

where $R_j = \eta_L a_j / R_O \Delta a_j$. Also, these currents are related to the toroidal field function just inside the plasma, g_b , by Ampere's law,

$$2\pi g_j = 2\pi g_b + \mu_O (I_1 + I_2 + \dots + I_j) \quad . \quad (41)$$

The final equation needed to self consistently determine the boundary poloidal electric field is obtained from Eqs. (16) and (31), which can be written

$$\nabla^2 \frac{\partial g}{\partial t} = \nabla \cdot \vec{R} \quad (42)$$

where

$$\vec{R} = \nabla \phi \times \vec{E} \quad (43a)$$

on the boundary and

$$\vec{R} \equiv \vec{R}_P = \frac{-g}{\rho x^2} (\nabla \phi \times \nabla A + \nabla \Omega) - \frac{\omega}{\rho x^2} \nabla \phi \times \nabla \psi + \frac{\eta}{\mu_O x^2} \nabla g \quad (43b)$$

in the plasma. Integrating Eq. (42) over the first row of finite difference zones in the plasma gives a time advancement equation for g_b ,

$$S \frac{\partial}{\partial t} g_b = - \oint_{a_1} \vec{E} \cdot d\vec{l} + K \quad . \quad (44)$$

Here

$$S = \int dA \frac{1}{x} \quad , \quad (45)$$

$$K = \oint x^2 d\vec{l} \cdot \nabla \phi \times \vec{R}_p \quad , \quad (46)$$

with the domain of integration in Eq. (45) being the annulus encompassing the first row of plasma finite difference zones and the line integral in Eq. (46) the outside boundary of that annulus.

Finally, combining Eqs. (37) through (41) with Eq. (44) gives a coupled set of equations describing the evolution of the poloidal currents in the circuit and the metallic liner,

$$\mu_0 \frac{dI_1}{dt} = - \left(\frac{1}{(L_1 - L_2)} + \frac{2\pi}{S} \right) R_1 I_1 + \frac{1}{(L_1 - L_2)} R_2 I_2 - \frac{2\pi K}{S} \quad , \quad (47a)$$

$$\mu_0 \frac{dI_j}{dt} = \frac{R_{j+1}}{(L_j - L_{j+1})} I_{j+1} + \frac{R_{j-1}}{(L_{j-1} - L_j)} I_{j-1}$$

$$- \left(\frac{R_j}{(L_j - L_{j+1})} + \frac{R_j}{(L_{j-1} - L_j)} \right) I_j \quad ; \quad j = 2, N-1 \quad , \quad (47b)$$

$$\mu_0 \frac{dI_N}{dt} = \frac{1}{m(L_N - L_C)} \left(R_{TF} I_{TF} + \frac{Q_{TF}}{C_{TF}} \right) + \frac{R_{N-1}}{(L_{N-1} - L_N)} I_{N-1}$$

$$- \left(\frac{R_N}{(L_N - L_C)} + \frac{R_N}{(L_{N-1} - L_N)} \right) I_N \quad , \quad (47c)$$

$$\mu_0 \frac{dI_{TF}}{dt} = \frac{R_N}{m(L_N - L_C)} I_N - \frac{L_N}{m^2 L_C (L_N - L_C)} (R_{TF} I_{TF} + \frac{Q_{TF}}{C_{TF}}) \quad (47d)$$

$$\frac{dQ_{TF}}{dt} = I_{TF} \quad (47e)$$

B. The Poloidal Field Circuit Current

The PF circuit current J_{PF} and the toroidal electric field at the boundary of the thin metallic liner and the plasma are determined in a manner analogous to that detailed for the TF circuit. Figure 3 illustrates the geometry for this calculation. The PF circuit current flows in a conductor of radius a_d while the liner is divided into N conducting shells, each with radius a_j carrying a toroidal current J_j . The current I_{PF} obeys the circuit equation

$$n(2\pi x E_\phi)|_{a_d} = R_{PF} J_{PF} + \frac{Q_{PF}}{C_{PF}} \quad (48)$$

where the toroidal electric field E_ϕ is evaluated at radius a_d .

If we denote by i the (continuous) grid index which numbers the grid lines that go around the liner, and by j the grid index which numbers the grid lines that go out from the liner, the time derivative of Ampere's law, Eq. (6), together with Eqs. (5), (8), and (30) give the relation

$$\begin{aligned} & -L_T^{-1} (2\pi x E_\phi)|_{a_1} + K_T \\ & = \mu_0 \frac{d}{dt} (J_1 + J_2 + \dots + J_N + n J_{PF}) \quad (49) \end{aligned}$$

Here

$$2\pi L_T^{-1} \equiv \oint_{a_N} \frac{dl}{x} (\hat{n} \cdot \nabla l) \quad , \quad (50)$$

$$K_T \equiv \oint_{a_e} \frac{dl}{\rho x} (\hat{n} \cdot \nabla l) (-\nabla \phi \times \nabla A \cdot \nabla \psi - \nabla \Omega \cdot \nabla \psi + \frac{n}{\mu_0} \Delta^* \psi) \quad , \quad (51)$$

with $\hat{n}$ being the unit normal to the liner, and the contours of integration in Eqs. (50) and (51) being the inside and outside surfaces of the first row of encircling computational zones, respectively.

Ohm's law applied to each of the N toroidal liner elements gives

$$2\pi x E_\phi|_{a_j} = r_j J_j \quad , \quad (52)$$

where $r_j = \eta_L R_0 / a_j \Delta a_j$. Application of Faraday's and Ampere's law to the interval between any two liner currents gives

$$\begin{aligned} & -(2\pi x E_\phi)|_{a_j} + (2\pi x E_\phi)|_{a_{j+1}} \\ & = \mu_0 R_0 \ln\left(\frac{a_{j+1}}{a_j}\right) \frac{d}{dt} (J_j + J_{j+1} + \dots + J_N + n J_{PF}) \quad . \end{aligned} \quad (53)$$

Finally, combining Eqs. (48) through (53) gives a coupled set of equations describing the evolution of the toroidal currents in the circuit and the metallic liner,

$$\begin{aligned} \mu_0 \frac{d}{dt} J_1 = & - \left(L_T^{-1} + \frac{1}{R_0 \ln(a_1/a_2)} \right) r_1 J_1 \\ & + \frac{1}{R_0 \ln(a_1/a_2)} r_2 J_2 + K_T \quad , \end{aligned} \quad (54a)$$

$$\begin{aligned} \mu_0 \frac{d}{dt} J_j &= \frac{1}{R_0 \ln(a_{j-1}/a_j)} (-r_j J_j + r_{j-1} J_{j-1}) \\ &\quad - \frac{1}{R_0 \ln(a_j/a_{j+1})} (-r_{j+1} J_{j+1} + r_j J_j) ; j = 2, N-1 \quad , \quad (54b) \end{aligned}$$

$$\begin{aligned} \mu_0 \frac{d}{dt} J_N &= \frac{1}{R_0 \ln(a_{N-1}/a_N)} (-r_N J_N + r_{N-1} J_{N-1}) \\ &\quad - \frac{1}{R_0 \ln(a_N/a_c)} \left[\frac{1}{n} (R_{PF} J_{PF} + \frac{Q_{PF}}{C_{PF}}) + r_N J_N \right] \quad , \quad (54c) \end{aligned}$$

$$\mu_0 \frac{d}{dt} J_{PF} = \frac{1}{n R_0 \ln(a_N/a_c)} \left[r_N J_N - \frac{1}{n} (R_{PF} J_{PF} + \frac{Q_{PF}}{C_{PF}}) \right] \quad , \quad (54d)$$

$$\frac{d}{dt} Q_{PF} = J_{PF} \quad . \quad (54e)$$

V. COMPUTED RESULTS

We present here the results of a two dimensional nonlinear initial value computer simulation which integrates in time the plasma Eqs. (11) through (16) together with the circuit Eqs. (47) and (54). The partial differential equations were differenced using the Leap-Frog¹² method on the convective and hyperbolic terms and the DuFort-Frankiel¹³ method on the diffusive or parabolic terms. A fast elliptic solver¹⁴ was used to invert the elliptic operators to obtain Ω and A from $\nabla^2 \Omega$ and $\Delta^* A$.

The thin metallic liner has a thickness of 0.025 cm, a major radius of 1 meter, a minor radius of 18.0 cm and a resistivity of $\eta_L = 12 \times 10^{-7}$ ohm-meters. The TF circuit has a minor radius of $a_C = 12.0$ cm and has 15 turns. The PF circuit has a minor radius of $a_d = 3.0$ cm and has 1 turn. The TF circuit has a capacitor $C_{TF} = 2320 \mu\text{f}$ charged to 16.3 kV and has a resistance $R_{TF} = 0.772 \text{ m}\Omega$. The PF circuit has a capacitor $C_{PF} = 1870 \mu\text{f}$, initially uncharged, and has a resistance $R_{PF} = 0.340 \text{ m}\Omega$.

The plasma resistivity is computed using the anomalous resistivity algorithm of Chocura,¹⁵

$$v = c\omega_{pi} [1 - \exp(-|v_d|/fV_s)] ,$$

$$\eta = \text{Max}[vM_e/n_0 e^2, \eta_{\text{spitzer}}(20\text{eV})] ,$$

where the ion acoustic speed $V_s = (\gamma kT_e/M_i)^{1/2}$, the drift velocity $V_d = j/n_0 e$, and the coefficients $n_0 = 10^{14} \text{ cm}^{-3}$, $c = 0.5$ and $f = 5.0$. This algorithm gives a classical resistivity when the ratio of the electron drift velocity to the ion acoustic speed is small, but an empirically enhanced value when this ratio is large. This same formula has been used in reversed field pinch simulations to give good agreement with experimentally measured profiles.¹⁶

Figure 4 shows the computed currents in the circuits, liner and plasma as a function of the time t . The EF and PF currents have been brought up slowly so that they have the values -240 kA and 500 kA respectively at time $t = 0$ when the plasma is preionized. At $t = 0$, the TF capacitor banks are discharged initiating a negative current in the TF circuit. Simultaneously

the PF circuit is closed, causing the current to decrease by charging the PF capacitors. The currents in the TF and PF circuits are crowbarred and allowed to decay resistively at time $t = 100 \mu\text{sec}$ and $t = 140 \mu\text{sec}$ respectively, the times when the voltage drop across the corresponding capacitors is zero. In response to the changing circuit currents, poloidal and toroidal currents are induced into the liner and plasma as indicated in the figure.

Figure 5(a) shows the computational grid used in the calculation. A nonorthogonal grid was utilized so that cell boundaries align with physical boundaries allowing accurate computation of boundary conditions. Figure 5(b) shows the poloidal magnetic flux distribution at time $t = 0$. Only the contours on the upper half plane are plotted since the device is symmetric about the midplane. The lower part of Fig. 5(b) shows the poloidal flux on the midplane plotted against the major radius R .

Figures 6 and 7 show the distribution of the poloidal magnetic flux, the toroidal current, the toroidal magnetic field, and the poloidal current at time $t = 75 \mu\text{sec}$. Comparison of Fig. 6(a) with Fig. 5(b) shows that after $75 \mu\text{sec}$, the value of the poloidal flux ψ at the flux core has been increased from $\psi_c = -.080$ to $\psi_c = -.063$. The increase of ψ_c , through magnetic induction, has caused a local minimum to form between the symmetry axis and the flux core. This local minimum is to become the magnetic axis of the spheromak.

Figure 6(b) shows the toroidal current density in the plasma, J_ϕ , at time $t = 75 \mu\text{sec}$. At this time the plasma current is distributed around the flux core, but shows a preferential inward expansion toward the symmetry axis. This preferential inward expansion is dictated by the "external field" produced by the currents in the EF coils shown in Fig. 1.

Figure 7(a) shows the toroidal field function $g = xR_{\perp}^2$ at times

$t = 75 \mu\text{sec}$, and Fig. 7(b) shows vectors indicating the relative magnitude and direction of the associated plasma poloidal current $\vec{j}_p = \nabla g \times \nabla \phi$. The poloidal current is induced into the plasma by the changing current in the PF circuit. The evolution of this current is governed by both resistive diffusion, and by the equilibrium equation which dictates that the poloidal current vectors lie in surfaces of constant poloidal flux. Since the poloidal flux surfaces are moving preferentially inward, the equilibrium constraint forces the toroidal field function contours to expand inward as well.

Figures 8 and 9 show the fields and currents at time $t = 250 \mu\text{sec}$, when the spheromak configuration is fully formed. The magnetic axis is now located at $R = 0.53 \text{ m}$. From Fig. 8(a) it is seen that enough volt-seconds have been supplied by the PF circuit to raise the value of the poloidal flux on the flux core, ψ_c , to the zero value, the same as that on the symmetry axis. All of the poloidal flux surfaces between the symmetry axis and the flux core have negative values of ψ , corresponding to closed magnetic surfaces that do not encircle the flux core.

Fig. 8(b) shows the toroidal current density at time $t = 250 \mu\text{sec}$. Inward expansion has continued so that nearly all of the toroidal plasma current is now located in the confined region of closed magnetic field lines between the symmetry axis and the flux core. From Fig. 4(b), it is seen that the total toroidal current in the plasma is now about 500 kA, the same as the initial toroidal current in the PF circuit. This current has effectively been transferred from the flux core into the plasma by magnetic induction.

Figure 9(a) shows the toroidal magnetic field function $g = xB_T$ at time $t = 250 \mu\text{sec}$ and Fig. 9(b) shows the associated poloidal current vectors. Essentially all of the toroidal flux has been "captured" in the closed magnetic field line region. From Fig. 9(b) we see that in contrast to Fig.

7(b), the poloidal current paths encircle the magnetic axis but no longer encircle the flux core. Thus poloidal current, as well as toroidal current, has effectively been inductively transferred from the flux core into the final spheromak configuration.

This calculation was halted at time $t = 250 \mu\text{sec}$. The formation phase is complete at this time and the configuration is expected to decay due to a resistive diffusion in a quasi static manner. It is reasonable to expect that, as in tokamaks, the resistivity will behave classically during this quasi-static decay phase and will take on the Spitzer value determined by the plasma temperature. The self consistent determination of this temperature lies outside the scope of our model, but reasonable estimates of radiative and transport losses suggest a central temperature of several hundred electron volts which would imply a lifetime of several milli-seconds.

Finally we comment about the sensitivity of the computational results to the number of mesh points used, the plasma density ρ_0 , the artificial viscosity ν and the assumed plasma temperature T_e , used in calculating the plasma resistivity $\eta(T_e)$. Comparison runs were made with twice the number of zones and ten times the density and viscosity. The location of the magnetic axis and the total amounts of poloidal and toroidal currents induced in the plasma differed by only a few percent compared to the run described in Figs. 4 through 11. When the assumed plasma temperature is increased, so that the resistivity is decreased, the formation process described above is made more efficient in that after $250 \mu\text{sec}$, not as much of the poloidal and toroidal current in the plasma has decayed due to classical field diffusion.

VI. DISCUSSION

The individual terms of Poyntings theorem are given in Fig. 10 which depicts the power balance as a function of time. The internal energy and inertial terms are negligible in the power balance and have been omitted from the figure. Initially, the toroidal electric field from the PF circuit is taking energy from the plasma volume and the poloidal field energy is decreasing from its $t = 0$ value. At the same time, the poloidal electric field from the TF circuit is putting energy into the plasma volume and the toroidal magnetic field energy is increasing from zero. Eventually, a quasi-static state is reached where the Poynting flux contributions to the energy balance vanish, and the ohmic losses balance the decreasing energy stored in both components of the magnetic field.

At early times, $t \lesssim 30 \mu\text{sec}$, the two components of the Poynting vector $\vec{E} \times \vec{B}$ integrated around the flux core tend to cancel; i.e.,

$$\int d\vec{A} \cdot \vec{E} \times \vec{B} = \int d\vec{A} \cdot \vec{E}_T \times \vec{B}_P + \int d\vec{A} \cdot \vec{E}_P \times \vec{B}_T \sim 0 \quad ,$$

so that from Eq. (33), the normal component of the velocity at the plasma-liner boundary is approximately zero. This design prevents the piling up of plasma around the flux core, or conversely the formation of a vacuum region.

A schematic illustration of the spheromak formation process is given in Fig. 11. Figure 11(a) shows three poloidal flux surfaces $\psi_1 > \psi_2 > \psi_3$ at an early stage in the formation. The induction of a poloidal current around the flux core and its subsequent outward diffusion cause toroidal flux to exist in the volume between surfaces ψ_1 and ψ_2 , and between surfaces ψ_2 and ψ_3 . The toroidal electric field set up by the decreasing PF current in the flux core

causes the value of the poloidal flux ψ_c on the flux core to increase according to Eq. (30). Figures 11(a) through (d) illustrate the geometry as $\psi_c = \psi_3$, $\psi_c = \psi_2$, $\psi_c = \psi_1$, and $\psi_c > \psi_1$. Note that all of the toroidal flux in the initial configuration 11(a) has been captured in the final spheromak configuration 11(d).

The formation method of Fig. 11 is to be compared with the previously proposed¹ formation method utilizing pinching coils, depicted in Fig. 12. In the pinching method, the value of the poloidal flux on the flux core, ψ_c , is held fixed while pinching coils are activated to "pinch off" a piece of the plasma to form a spheromak configuration. This method has the disadvantage of requiring an additional set of coils and is intrinsically less efficient. Only half¹⁷ of the toroidal flux which exists in the initial configuration of Fig. 12(a) can end up in the final spheromak configuration of Fig. 12(d), while in the present method, essentially all of the toroidal flux present in Fig. 11(a) can be captured in the spheromak configuration of Fig. 11(d).

VII. SUMMARY

The accurate integration of the multidimensional magnetohydrodynamic equations on the resistive time scale in a low β plasma is a problem that continues to challenge researchers in the field. Methods have previously been presented⁸⁻¹¹ in which the inertial terms in the momentum equation are completely dropped and the plasma transport equations are integrated with the constraint imposed that the plasma pass through a series of equilibrium states. This approach has been implemented successfully on some problems but can become cumbersome when modeling a geometry such as presented here

involving changing magnetic topology.

The present approach exploits the fundamental differences between phenomena occurring on the two MHD time scales. The fast (ideal) time scale velocity is physically determined from a time advancement equation, the momentum Eq. (1), while the slow (resistive) time scale velocity is determined from the Ohm's law Eq. (5), together with the constraint that the plasma evolve through a series of near equilibrium states. In particular, the inertial term in the momentum equation plays no roll in the determination of the slow time scale velocity other than that of keeping the plasma near an equilibrium configuration. If we are interested only in the slow resistive time scale evolution, we are therefore free to replace the true inertial term with a "fictitious" inertial term having the same fast time scale asymptotic behavior. This reasoning led to the introduction of the model momentum Eq. (1)', which in turn yielded the computationally convenient Eqs. (11) through (13).

The equations derived here have been applied to demonstrate a method for the formation of spheromak plasmas. This scenario is novel in that it utilizes magnetic induction to establish both toroidal and poloidal currents in the plasma, and that it occurs in a quasi-equilibrium fashion. The investigation of the three dimensional stability properties of this formation process will be presented in a future publication.

ACKNOWLEDGMENTS

We gratefully acknowledge extensive collaboration with Drs. M. Okabayashi, H. P. Furth, M. Yamada, J. Sinnis, and G. Sheffield, and useful discussions with Drs. C. K. Chu, H. C. Lui, and A. Aydemir.

This work was supported by the United States Department of Energy, contract # DE-AC02-76-CH03073.

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FIGURE CAPTIONS

Fig. 1. Electromagnetic components of the S-1 spheromak experiment.

802247

Fig. 2. Details of the toroidal field (TF) circuit which determines the poloidal electric field at the plasma-liner boundary.

802223

Fig. 3. Details of the poloidal field (PF) circuit which determines the toroidal electric field at the plasma-liner boundary.

802220

Fig. 4. Time histories of the currents in the circuits, liner, and plasma.

802261

Fig. 5. (a) Computational grid.

802253

Fig. 5. (b) Poloidal flux contours at time $t = 0$. Graph shows midplane values.

802255

Fig. 6. (a) Poloidal flux at time $t = 75 \mu\text{sec}$.

802256

Fig. 6. (b) Toroidal current at $t = 75 \mu\text{sec}$.

802257

Fig. 7. (a) Toroidal field strength at time $t = 75 \mu\text{sec}$.

802259

Fig. 7. (b) Poloidal current vectors at $t = 75 \mu\text{sec}$.

802252

Fig. 8. (a) Poloidal flux at time $t = 250 \mu\text{sec}$.

802258

- Fig. 8. (b) Toroidal current at time $t = 250 \mu\text{sec}$. 802260
- Fig. 9. (a) Toroidal field strength at time $t = 250 \mu\text{sec}$. 802254
- Fig. 9. (b) Poloidal current at time $t = 250 \mu\text{sec}$. 802251
- Fig. 10. Power flow in spheromak formation calculation. Surface integrals are over plasma-conductor interface. Volume integrals are over plasma volume. 802250
- Fig. 11. As the value of the poloidal flux ψ_c on the flux core is increased, a spheromak configuration is formed, capturing the toroidal flux. 802245
- Fig. 12. In a previously proposed method of formation, the value of the poloidal flux ψ_c on the flux core is held fixed while pinching coils are activated to "pinch off" a piece of the plasma. 802246

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802260

$t = 250 \mu\text{sec}$

Fig. 9. (a) Toroidal field strength at time $t = 250 \mu\text{sec}$.

802254

Fig. 9. (b) Poloidal current at time $t = 250 \mu\text{sec}$.

802251

Fig. 10. Power flow in spheromak formation calculation. Surface integrals are over plasma-conductor interface. Volume integrals are over plasma volume.

802250

Fig. 11. As the value of the poloidal flux ψ_c on the flux core is increased, a spheromak configuration is formed, capturing the toroidal flux.

802245

(a) Pinching Coil

(c)

(b)

(d)

Fig. 12. In a previously proposed method of formation, the value of the poloidal flux ψ_c on the flux core is held fixed while pinching coils are activated to "pinch off" a piece of the plasma.

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