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UCRL-50400
Vol. 5, Pl. 8

**An Integrated System for Production of
Neutronics and Photonics Calculational Constants
Volume 5, Part B**

**RELATIVISTIC TRANSFORMATIONS BETWEEN
CENTER-OF-MASS AND LABORATORY SYSTEMS
FOR TWO-BODY NUCLEAR REACTIONS**

R. J. Howerton

January 16, 1978

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FOREWORD

The UCRL-50400 series describes an integrated and computer-oriented system for the production and application of neutronics and photonics calculational constants.

The system must supply reliable up-to-date data, select specific types of data on request, provide output in a variety of forms (ultimately in the form of input to other computer codes), and function rapidly and efficiently. The system has now been developed to a point where we are achieving these goals.

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RELATIVISTIC TRANSFORMATIONS BETWEEN CENTER-OF-MASS AND LABORATORY SYSTEMS FOR TWO-BODY NUCLEAR REACTIONS

ABSTRACT

We describe a technique for transforming cross sections and secondary-particle energies between center-of-mass and laboratory systems for two-body nuclear reactions. We also review the derivations of the relativistic transformations needed for this technique.

INTRODUCTION

Differential cross sections for nuclear reactions are measured in the laboratory coordinate system (LAB), but, to facilitate comparison with theory, two-body reactions with defined Q-values are usually reported in the center-of-mass system (COM). Compilations of experimental data of this type also usually present the data in the COM system, as do libraries of evaluated data. On the other hand, users of evaluated data often require values of the average cosine of the scattering angle of a secondary particle weighted by the differential cross section and/or the average energy of a secondary particle weighted by the differential cross section, both in the LAB system. Likewise, processing codes that produce calculational constants from the data of nuclear reactions often use differential cross sections that are expressed in the LAB system.

Because some incident-particle energies require relativistic transformations for adequate accuracy, the transformation equations for differential cross sections and energies developed here are relativistic. Our objective is to provide a straightforward derivation of the pertinent equations with sufficient discussion of special problems to enable the reader to construct a computer code that will carry out transformations in both directions. First we present a short background that reviews the basic formulas associated with relativity, then we develop the transformation equations, and finally we present a summary of the equations needed for the calculations.

BACKGROUND

As a refresher, we develop the following equations from basic assumptions of the theory of relativity. For any moving particle, P, the total energy is

$$W = m_0 c^2 / (1 - v^2/c^2)^{1/2} ,$$

where

m_0 = particle mass,
 c = velocity of light,
 v = particle velocity.

Its mass equivalent is $m = m_0 \gamma^2$, so its kinetic energy is

$$E = W - m . \quad (1)$$

Let $v/c = \beta$ and let $\gamma = 1/(1-\beta^2)^{1/2}$. Then $\gamma = (E + m)/m = W/m$, which gives

$$\beta = (1 - m^2/W^2)^{1/2}$$

and

$$E = m(\gamma - 1)$$

Likewise from relativity theory, the momentum of a particle is

$$p' = m_0 / (1 - \beta^2)^{1/2}$$

More conveniently, to keep a consistent set of units,

$$p = p' \cdot c = m_0 c (\beta \cdot c) / (1 - \beta^2)^{1/2}$$

$$p = m\beta / (1 - \beta^2)^{1/2} = m\beta\gamma = W\beta \quad (2)$$

$$p = W(1 - m^2/W^2)^{1/2} = (W^2 - m^2)^{1/2} = [(E + m)^2 - m^2]^{1/2} \quad (3)$$

$$p = (E^2 + 2mE)^{1/2}$$

and the velocity of the particle is

$$v = \frac{p'}{m_0\gamma} = \frac{p}{m_0c\gamma} = \frac{p(1 - v^2/c^2)^{1/2}}{m_0c}$$

$$v = \frac{p \cdot c}{m + E}$$

$$\beta = \frac{(E^2 + 2mE)^{1/2}}{E + m}$$

GENERAL DERIVATIONS

A particle, P_1 , with mass m_1 and kinetic energy E_1 in the LAB system interacts with a particle, P_2 , with mass m_2 and zero kinetic energy in the LAB system when P_2 is in its ground state. The interaction results in two particles, P_3 and P_4 , having masses m_3 and m_4 and kinetic energies E_3 and E_4 in the LAB system ($\bar{E}_3$ and $\bar{E}_4$ in the COM system). P_3 , the lighter of the emergent particles, is in its ground state, whereas P_4 may be in either its ground or excited state, E_x . P_3 makes an angle ϕ in the LAB system (θ in the COM system) with the initial direction of P_1 .

While all values of θ ($0 \leq \theta \leq \pi$) are always possible, for endoergic reactions there is an energy range between the threshold energy and a critical energy E_c .

$$E_c = \frac{(m_3 + m_4 + E_x - m_1 - m_2) (m_1 + m_2 - m_3 + m_4 + E_x)}{2 (m_2 - m_3)}$$

where ϕ may take on values only between 0 and $\phi_{\max}$. (All masses are represented in equivalent energies; e.g., mass of the neutron = $1.008665 \text{ amu} \times 931.141 \text{ MeV/amu} = 939.209337 \text{ MeV}$.) The three cases of interest are illustrated in Figs. 1, 2, and 3.

Fig. 1. Velocity diagram for the case in which the velocity of the center of mass is greater than the velocity of P_3 in the center-of-mass system.

Fig. 2. Velocity diagram for the case in which the velocity of the center of mass is equal to the velocity of P_3 in the center-of-mass system.

Fig. 3. Velocity diagram for the case in which the velocity of the center of mass is less than the velocity of P_3 in the center-of-mass system.

Superscripted bars designate LAB variables, and v_{com} is the velocity of the center of mass in the LAB system. In Fig. 1, $\bar{v}_3 < v_{com}$, and a maximum angle ϕ_{max} limits the range of the LAB angle. Note that, for values of $\phi < \phi_{max}$, v_3 is double valued. In Fig. 2, $\phi_{max} = \pi/2$, a condition that corresponds to $\bar{v}_3 = v_{com}$. In Fig. 3, $\bar{v}_3 > v_{com}$, and ϕ ranges over all values from 0 to π .

For all three figures the following Lorentz transformations apply to spatial and time coordinates in the COM and LAB systems:

$$\left. \begin{aligned} \bar{x}_3 &= \frac{x_3 - v_{com} \cdot t}{[1 - (v_{com}/c)^2]^{1/2}} \\ x_3 &= \frac{\bar{x}_3 + v_{com} \cdot t}{[1 - (v_{com}/c)^2]^{1/2}} \end{aligned} \right\} \quad (4)$$

$$\left. \begin{aligned} \bar{t} &= \frac{t - v_{com} \cdot x_3/c^2}{[1 - (v_{com}/c)^2]^{1/2}} \\ t &= \frac{\bar{t} + v_{com} \cdot \bar{x}_3/c^2}{[1 - (v_{com}/c)^2]^{1/2}} \end{aligned} \right\} \quad (5)$$

$$\bar{y}_3 = y_3 \quad (6)$$

$$\left. \begin{aligned} x_3 &= v_3 t \cos \phi \\ y_3 &= v_3 t \sin \phi \end{aligned} \right\} \quad (7)$$

$$\left. \begin{aligned} \bar{x}_3 &= \bar{v}_3 \bar{t} \cos \theta \\ \bar{y}_3 &= \bar{v}_3 \bar{t} \sin \theta \end{aligned} \right\} \quad (8)$$

Now let $v_{com}/c = B$ and let $\bar{v}_3/c = \beta_3$. Then, using Eqs. (4-8),

$$x_3 = v_3 t \cos \phi = \frac{\bar{x}_3 + v_{com} \bar{t}}{(1 - B^2)^{1/2}},$$

from which

$$x_3 = \frac{\bar{v}_3 \bar{t} \cos \theta + v_{com} \cdot \bar{t}}{(1 - B^2)^{1/2}}.$$

Now we divide by t , substitute for t in terms of $\bar{t}$,

$$\frac{x_3}{t} = v_{3x} = \frac{\bar{v}_3 \bar{t} \cos \theta + v_{com} \cdot \bar{t}}{\bar{t} + v_{com} \cdot \bar{x}_3/c^2},$$

and divide both numerator and denominator of the right side by $\bar{t}$,

$$v_{3x} = \frac{\bar{v}_3 \cos \theta + v_{com}}{1 + v_{com} \cdot \bar{x}_3/c^2 \bar{t}} = \frac{\bar{v}_3 \cos \theta + v_{com}}{1 + B\bar{\beta}_3 \cos \theta}. \quad (9)$$

Similarly,

$$v_{3y} = v_3 \sin \theta = \frac{y}{t} = \frac{\bar{y}}{\bar{t}}$$

$$v_{3y} = \frac{\bar{v}_3 \bar{t} \sin \theta (1 - B^2)^{1/2}}{\bar{t} + v_{com} \cdot \bar{x}_3/c^2} = \frac{\bar{v}_3 \sin \theta (1 - B^2)^{1/2}}{1 + B\bar{\beta}_3 \cos \theta}, \quad (10)$$

and, since

$$\tan \phi = v_{3y}/v_{3x},$$

$$\tan \phi = \frac{\sin \theta (1 - B^2)^{1/2}}{\cos \theta + v_{com}/\bar{v}_3} = \frac{\sin \theta (1 - B^2)^{1/2}}{\cos \theta + B/\bar{\beta}_3}.$$

Let $B/\bar{\beta}_3 = K_3$ and let $\Gamma = \frac{1}{(1 - B^2)^{1/2}}$. Then

$$\tan \phi = \frac{\sin \theta}{\Gamma(\cos \theta + K_3)} \quad (11)$$

and

$$\frac{d \tan \phi}{d\theta} = \frac{1 + K_3 \cos \theta}{\Gamma^2(\cos \theta + K_3)^2}.$$

This expression vanishes at $\cos \theta = -1/K_3$, so ϕ has a maximum value for that condition. If $K_3 < 1$, the condition can never be reached and there is no limitation on ϕ . If the critical value $-1/K_3$ is substituted for $\cos \theta$ in Eq. (11),

$$\tan (\phi_{\max}) = \frac{1}{\Gamma(K_3^2 - 1)^{1/2}} \quad (12)$$

when $K_3^2 > 1$,

which corresponds to the condition shown in Fig. 1.

Now we invoke conservation of energy and momentum so that

$$m_1 + E_1 + m_2 = m_3 + E_3 + m_4 + E_4.$$

For convenience and because m_4 is represented in energy units, we shall assume that the mass of particle P_4 includes the excitation value E_x , so that

$$m_1 + E_1 + m_2 = m_3 + E_3 + m_4 + E_4. \quad (13)$$

In the LAB system

$$p_1 + p_2 = p_3 + p_4,$$

but $p_2 = 0$, since we assume that particle P_2 is at rest (see Fig. 4).

Fig. 4. Momentum magnitude diagram.

$$p_1 = p_3 \cos \phi_3 + p_4 \cos \phi_4$$

$$0 = p_3 \sin \phi_3 + p_4 \sin \phi_4 \quad (14)$$

The reaction energy, Q , is defined as

$$Q = m_1 + m_2 - m_3 - m_4,$$

or, using Eq. (13),

$$Q = E_3 + E_4 - E_1. \quad (15)$$

Manipulation of Eq. (14) yields

$$p_4^2 = p_1^2 + p_3^2 - 2 p_1 p_3 \cos \phi_3, \quad (16)$$

and, from Eqs. (2) and (15),

$$p_4^2 = 2 m_4 E_4 + E_4^2$$

$$E_4 = Q + E_1 - E_3.$$

so that

$$p_4^2 = 2 m_4(Q + E_1 - E_3) + (Q + E_1 - E_3)^2 \quad (17)$$

Direct substitution of Eqs. (17) and (2) into Eq. (16) gives

$$Q^2 + 2Q(m_4 + E_1 - E_3) + 2E_1(m_4 - m_1 - E_3) - 2E_3(m_4 + m_3) + 2 [(E_1^2 + 2m_1E_1)(E_3^2 + 2m_3E_3)]^{1/2} \cos \phi_3 = 0$$

An analogous manipulation of Eqs. (14) and (15) gives an expression involving $\cos \phi_4$ and E_4 .

Energy and Momentum Transformations

Since the following discussion deals with P_3 , the subscript will be dropped and the angle of P_3 will be (as before) ϕ in the LAB system and θ in the COM system.

Since

$$\beta_3^2 = \frac{v_3^2}{c^2} = \frac{v_{3x}^2 + v_{3y}^2}{c^2}$$

and using Eqs. (9) and (10),

$$\beta_3^2 = \frac{\bar{\beta}_3^2 + B^2 + 2B\bar{\beta}_3 \cos \theta - \bar{\beta}_3^2 B^2 \sin^2 \theta}{(1 + \bar{\beta}_3 B \cos \theta)^2}$$

Then

$$\gamma_3 = 1/(1 - \beta_3^2)^{1/2}$$

$$\gamma_3 = \frac{1 + \bar{\beta}_3 B \cos \theta}{(1 + \bar{\beta}_3^2 B^2 - \bar{\beta}_3^2 B^2 \sin^2 \theta)^{1/2}} = \frac{1 + \bar{\beta}_3 B \cos \theta}{[(1 - \bar{\beta}_3^2)(1 - B^2)]^{1/2}}$$

$$\gamma_3 = \Gamma \bar{\gamma}_3 (1 + \bar{\beta}_3 B \cos \theta) \quad (18)$$

Similarly,

$$\bar{\gamma}_3 = \gamma_3 \Gamma (1 - \beta_3 B \cos \theta)$$

From Eq. (2),

$$\bar{p}_3 = \bar{W}_3 \bar{\beta}_3, \quad p_3 = W_3 \beta_3, \quad (19)$$

and from Eqs. (2) and (18),

$$\gamma_3 m_3 \gamma_3 = m_3 \bar{\gamma}_3 \Gamma (1 + \bar{\beta}_3 B \cos \theta)$$

$$W_3 = \bar{W}_3 \Gamma (1 + \bar{\beta}_3 B \cos \theta)$$

$$W_3 = \Gamma (\bar{W}_3 + \bar{p}_3 B \cos \theta). \quad (20)$$

Similarly,

$$\bar{W}_3 = m_3 \bar{\gamma}_3 = W_3 \Gamma (1 - \beta_3 B \cos \theta)$$

$$\bar{W}_3 = \Gamma (W_3 - p_3 B \cos \theta).$$

Now, using Eqs. (2), (9), and (19), we can develop the momentum transformations,

$$\begin{aligned} p_{3x} &= m_3 \frac{v_{3x}}{c} \gamma_3 = \frac{m_3}{c} \frac{\bar{v}_{3x} + v_{com}}{1 + \bar{\beta}_3 B \cos \theta} \bar{\gamma}_3 \Gamma (1 + \bar{\beta}_3 B \cos \theta) \\ &= \frac{m_3 \bar{v}_{3x}}{c} \bar{\gamma}_3 (1 + v_{com}/\bar{v}_{3x}) \Gamma = \bar{p}_{3x} \Gamma (1 + v_{com}/\bar{v}_{3x}). \end{aligned}$$

But $\bar{v}_{3x} = \bar{v}_3 \cos \theta$ and $v_{com}/\bar{v}_3 = B/\bar{\beta}_3$, so

$$p_{3x} = \bar{p}_{3x} \Gamma (1 + B/\bar{\beta}_3 \cos \theta).$$

And, since $\bar{p}_{3x} = \bar{p}_3 \cos \theta$,

$$p_{3x} = \Gamma (\bar{p}_{3x} + \bar{W}_3 B),$$

which can be written

$$p_3 \cos \phi = \bar{p}_3 \Gamma (\cos \theta + K_3), \quad (21)$$

since $K_3 = B/\bar{\beta}_3$ and $\bar{W}_3 = \bar{p}_3/\bar{\beta}_3$. In the y direction, using Eq. (13),

$$\begin{aligned}
 p_{3y} &= m_3 v_{3y} \gamma_3 = m_3 \frac{v_{3y}(1 - B^2)^{1/2}}{1 + \beta_3 B \cos \theta} \cdot \gamma_3 \Gamma(1 + \beta_3 B \cos \theta) \\
 &= m_3 v_{3y} \bar{\gamma}_3 = \bar{p}_{3y} .
 \end{aligned}
 \tag{22}$$

As expected, $p_3 \sin \phi = \bar{p}_3 \sin \theta$. From Eqs. (21) and (22),

$$p_3^2 = \bar{p}_3^2 [\Gamma^2(\cos \theta + K_3)^2 + \sin^2 \theta] . \tag{23}$$

To obtain the value of B that expresses the velocity of the center of mass, we start with the total energy of P_1 and P_2 ,

$$W_T = W_1 + W_2.$$

Then, using the equation for P_1 and P_2 analogous to Eq. (20), we get

$$W_T = \Gamma(\bar{W}_1 + \bar{p}_1 B \cos \theta_1 + \bar{W}_2 + \bar{p}_2 B \cos \theta_2) .$$

Since by definition of the COM system

$$\bar{p}_1 \cos \theta_1 + \bar{p}_2 \cos \theta_2 = 0 ,$$

$$W_T = \Gamma(\bar{W}_1 + \bar{W}_2) = \Gamma \bar{W}_T .$$

If the x-components of the momentum are written

$$\begin{aligned}
 p_1 \cos \phi_1 + p_2 \cos \phi_2 &= \Gamma(\bar{p}_1 \cos \theta_1 + \bar{W}_1 B + \bar{p}_2 \cos \theta_2 + \bar{W}_2 B) \\
 &= \Gamma(\bar{W}_1 B + \bar{W}_2 B) = \Gamma B \bar{W}_T ,
 \end{aligned}$$

then

$$B = \frac{p_1 \cos \phi_1 + p_2 \cos \phi_2}{W_T} .$$

But since we assume that P_2 is at rest and that the path of P_1 is along the x-axis,

$$B = \frac{p_1}{W_T} = \frac{(E_1^2 + 2m_1 E_1)^{1/2}}{E_1 + m_1 + m_2} .$$

Once again, from the definition of the center-of-mass system,

$$\bar{p}_3 = \bar{p}_4 \quad ,$$

and since, from Eq. (3), $p^2 = W^2 - m^2$,

$$\bar{W}_3^2 - m_3^2 = \bar{W}_4^2 - m_4^2 \quad . \quad (24)$$

Using the relations

$$W_T = \Gamma \bar{W}_T \quad \text{and} \quad \bar{W}_T = \bar{W}_3 + \bar{W}_4 \quad ,$$

we can write Eq. (24)

$$(\bar{W}_3 + \bar{W}_4)(\bar{W}_3 - \bar{W}_4) = m_3^2 - m_4^2$$

$$\bar{W}_3 - \bar{W}_4 = (m_3^2 - m_4^2)/\bar{W}_T$$

$$\bar{W}_3 - \bar{W}_4 + \bar{W}_3 + \bar{W}_4 = (m_3^2 - m_4^2)/\bar{W}_T + \bar{W}_T \quad ,$$

so that

$$\bar{W}_3 = \frac{\bar{W}_T^2 + m_3^2 - m_4^2}{2\bar{W}_T} \quad ,$$

or

$$\bar{W}_3 = \frac{W_T^2 + \Gamma^2(m_3^2 - m_4^2)}{2\Gamma W_T} \quad .$$

By energy conservation,

$$W_1 + W_2 = W_3 + W_4 = W_T$$

$$W_T = E_1 + m_1 + m_2 \quad .$$

Cross Sections

Next, we consider cross sections in the LAB and COM systems.

$$\sigma(\theta) d\Omega = \sigma(\phi) d\Omega'$$

$$\sigma(\theta) d \cos \theta = \sigma(\phi) d \cos \phi$$

$$\sigma(\theta)/\sigma(\phi) = \frac{\sin \phi}{\sin \theta} \frac{d\phi}{d\theta}$$

From Eq. (11),

$$\tan \phi = \frac{\sin \theta}{\Gamma(\cos \theta + K_3)}$$

$$d \tan \phi = d \left[\frac{\sin \theta}{\Gamma(\cos \theta + K_3)} \right]$$

$$\sec^2 \phi d\phi = \frac{1 + K_3 \cos \theta}{\Gamma^2(\cos \theta + K_3)^2} d\theta$$

$$\frac{d\phi}{d\theta} = \frac{1 + K_3 \cos \theta}{\Gamma^2(\cos \theta + K_3)^2} \left/ \left[\frac{\sin^2 \theta}{\Gamma^2(\cos \theta + K_3)^2} - 1 \right] \right.$$

$$\frac{d\phi}{d\theta} = \frac{1 + K_3 \cos \theta}{\sin^2 \theta - \Gamma^2(\cos \theta + K_3)^2}$$

From Eqs. (22) and (23),

$$\frac{\sin \phi}{\sin \theta} = \frac{\bar{P}_3}{P_3} = [\sin^2 \theta + \Gamma^2(\cos \theta + K_3)^2]^{-1/2},$$

so that

$$\frac{\sigma(\theta)}{\sigma(\phi)} = \frac{1 + K_3 \cos \theta}{[\sin^2 \theta - \Gamma^2(\cos \theta + K_3)^2]^{3/2}}$$

or

$$\sigma(\theta) = \sigma(\phi) \frac{1 + K_3 \cos \theta}{[\sin^2 \theta - \Gamma^2(\cos \theta + K_3)^2]^{3/2}}$$

$$\sigma(\phi) = \sigma(\theta) \frac{[\sin^2 \theta - \Gamma^2(\cos \theta + K_3)^2]^{3/2}}{(1 + K_3 \cos \theta)}$$

(25)

If we solve Eq. (11),

$$\cos \theta = \frac{-\sin^2 \phi \Gamma^2 K_3 \pm \cos \phi [\Gamma^2 \sin^2 \phi (1 - K_3^2) + \cos^2 \phi]^{1/2}}{\Gamma^2 \sin^2 \phi + \cos^2 \phi} \quad (26)$$

$$\cos \phi = \frac{\Gamma(\cos \theta + K_3)}{[\sin^2 \theta + \Gamma^2(\cos \theta + K_3)^2]^{1/2}} \quad (27)$$

In Eq. (26) the two signs in front of the radical are used when $K_3^2 > 1$ [see Eq. (12)] to obtain the two values of $\cos \theta$ corresponding to a single value of $\cos \phi$, as illustrated in Fig. 1. When we use the two values of $\cos \theta$ with Eq. (25), we obtain two different values, $\sigma_1(\cos \phi)$ and $\sigma_2(\cos \phi)$. If an angular distribution $\sigma(\cos \theta)$ is given over the range $\pi \geq \theta \geq 0$, we obtain two angular distributions, $\sigma_1(\cos \phi)$ and $\sigma_2(\cos \phi)$, over the range $\phi_{\max} > \phi \geq 0$, as illustrated in Fig. 5. In Eq. (25), the value of $\sigma(\phi)$ is undefined if $\cos \theta = -1/k_3$, which is just the condition for defining $\phi_{\max}$. Thus, $\sigma_{1,2}(\phi)$ are asymptotic to the line $\phi = \phi_{\max}$.

If $K_3^2 < 1$, only the positive sign is used with Eq. (26) and $\sigma(\cos \phi)$ is single valued.

Fig. 5. Cross section $\sigma_1(\cos \theta)$ and $\sigma_2(\cos \theta)$ for the reaction ${}^3\text{H}(p,n){}^3\text{He}$ at 1.04 MeV.

SUMMARY OF EQUATIONS NEEDED FOR CALCULATION

To calculate the cross sections and secondary energies, the masses m_1 , m_2 , m_3 , and m_4 and the excitation value E_x of P_4 must be given. The masses must be expressed in energy units, and m_4 for the calculation will be the sum of the input ground-state mass and the excitation energy; i.e., $m_4 = m_{4gs} + E_x$.

The energy (E_1) of particle P_1 and the differential cross section in the COM system at a value of $\cos \theta$ are required if the desired quantity is the differential cross section in the LAB system at $\cos \phi$. Similarly, E_1 , $\cos \phi$, and $d\sigma(\cos \phi)$ must be given if the desired quantities are $\cos \theta$ and $d\sigma(\cos \theta)$. Usually this calculation is done for a complete angular distribution at a given energy, so a useful check for energies at which $K_3^2 < 1$ is

$$\int_{-1}^1 d\sigma(\cos \theta) \dot{\omega}(\cos \theta) = \int_{-1}^1 d\sigma(\cos \phi) d \cos \phi \quad (28)$$

For incident energies at which $K_3^2 > 1$,

$$\int_{-1}^1 d\sigma(\cos \theta) d \cos \theta = \int_{\cos(\phi_{\max})}^1 d \sigma_1(\cos \phi) d \cos \phi + \int_{\cos(\phi_{\max})}^1 d \sigma_2(\cos \phi) d \cos \phi. \quad (29)$$

If $K_3^2 = 1$, the situation is that illustrated in Fig. 3, and $d\sigma_2(\cos \phi) = 0$ for the range $\pi/2 > \phi > 0$. The quantities needed for the calculations are

$$W_T = E_1 + m_1 + m_2 \quad (30)$$

$$B = \frac{(E_1^2 + 2m_1 E_1)^{1/2}}{W_T} \quad (31)$$

$$\Gamma = \frac{1}{(1 - B^2)^{1/2}} \quad (32)$$

$$\bar{W}_3 = \frac{W_T^2 + \Gamma^2(m_3^2 - m_4^2)}{2\Gamma W_T} \quad (33)$$

$$\bar{\beta}_3 = \left(1 - \frac{m_3^2}{\bar{W}_3^2}\right)^{1/2} \quad (34)$$

$$K_3 = \frac{B}{\bar{\beta}_3} \quad (35)$$

If we represent the cross section in the COM system by a set of values $[\sigma_j(\cos \theta_j), \cos \theta_j]$ and an interpolation method, we assume a set of values for $\cos \phi_j$ sufficiently dense to ensure adequate representation in the LAB system if the same interpolation procedure is used. We can determine the criterion for sufficiency by using the integral check of Eq. (28) or (29). Since the calculated LAB cross sections would become infinite at the angle $\phi_{\max}$, the LAB cross section can be calculated only at values of $\phi < \phi_{\max}$. Since the LAB differential cross section is rising rapidly in the near neighborhood of $\phi_{\max}$, the density of values of $\cos \phi_j$ must be large there. To quantize these qualitative statements of required density of $\cos \phi_j$, the following purely empirical methods have resulted in adequate transformations:

- If $K_3^2 < 1$, construct a grid for $\cos \phi_j$ in increments of 0.01 over the range $0 \leq \phi \leq \pi$.
 - If $K_3^2 \geq 1$, divide the range from 0 to $\phi_{\max}$ so that $\Delta \cos \phi = (1 - \cos \phi_{\max})/200$, and use for the first value $\cos \phi_{\max} + \Delta \cos \phi$.
 - If the integrals of the calculated differential cross sections do not agree to a specified tolerance:
 - If $K_3^2 < 1$, halve the increment in $\cos \phi$.
 - If $K_3^2 > 1$ and the LAB integral is too large, recalculate the first value at $\cos \phi_{\max} + 3/2 \Delta \cos \phi$; if $K_3^2 > 1$ and the LAB integral is too small, halve the increment in $\cos \phi$.
- Having determined the set $\cos \phi_j$, calculate the corresponding value for $\cos \theta$ from

$$\cos \theta = \frac{-\sin^2 \phi \Gamma^2 K_3^2 \pm \cos \phi (\Gamma^2 \sin^2 \phi (1 - K_3^2) + \cos^2 \phi)^{1/2}}{\Gamma^2 \sin^2 \phi + \cos^2 \phi} \quad (36)$$

Whether to use both signs in front of the square root or only the + sign depends on the value of K_3^2 .

- If $K_3^2 > 1$, $\phi_{\max} = \tan^{-1} [1/\Gamma(K_3^2 - 1)^{1/2}]$, and both signs should be used.
- If $K_3^2 < 1$, ϕ can take on any value $0 \leq \phi \leq \pi$, and only the + sign is used.
- If $K_3^2 = 1$, $\phi_{\max} = \pi/2$, and only the + sign is used (see Fig. 3) because $\sigma_2(\phi) = 0$. If the value $K_3 = 1$ is substituted into Eq. (36) and the - sign is selected, $\cos \theta = -1$. If $\cos \theta = -1$ is substituted into Eq. (25) [Eq. (38) below], $\sigma_2(\phi) = 0$.

If the LAB angular distributions are given, using the $\cos \theta$ values derived from (34) will usually give an adequate representation of the COM angular distributions.

An alternative method of calculation is to expand a given COM distribution into many more points than are given and use

$$\cos \phi = \frac{\Gamma(\cos \theta + K_3)}{[\sin^2 \theta + \Gamma^2(\cos \theta + K_3)^2]^{1/2}} \quad (37)$$

to obtain the LAB cosines. This method is straightforward, but an adequate definition of the required expanded grid of $\sigma(\cos \theta)$ and $\cos \theta$ is not easy to obtain.

Using the values obtained from Eqs. (30) through (36) [or (37)], we can calculate the required cross sections and/or secondary energies,

$$\begin{aligned} \sigma(\cos \theta) &= \sigma(\cos \phi) \frac{1 + K_3 \cos \theta}{[\sin^2 \theta - \Gamma^2(\cos \theta + K_3)^2]^{3/2}} \\ \sigma(\cos \phi) &= \sigma(\cos \theta) \frac{[\sin^2 \theta - \Gamma^2(\cos \theta + K_3)^2]^{3/2}}{1 + K_3 \cos \theta} \end{aligned} \quad (38)$$

From Eq. (20), the total energy of P_3 in the LAB system is

$$W_3(\phi) = \bar{W}_3 \Gamma(1 + B\bar{\beta}_3 \cos \theta) ,$$

and, from Eq. (1), the kinetic energy E_3 of P_3 is

$$E_3(\phi) = W_3(\phi) - m_3$$

and

$$\bar{E}_3 = \bar{W}_3 - m_3 .$$