THE FUNDAMENTAL THEORY OF ELASTIC WAVE SCATTERING BY DEFECTS IN
ELASTIC MATERIALS • INTEGRAL EQUATION METHODS FOR APPLICATION
TO ULTRASONIC FLAW DETECTION*

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May 1976

Report #2654
Issued by The Materials Science Center
ERDA Technical Report #C00-3161-42

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Abstract

The use of ultrasonic methods in non-destructive testing depends on the interpretation of the scattering of sound waves by flaws. The theory of elastic waves and their scattering in non-uniform media is developed in detail in this report, from first principles, and in generality. Both integral equation and differential methods are discussed, with emphasis on the former. General methods for defining scattering cross sections are presented, and conservation theorems are noted. The Born Approximation to the integral equation is presented, and computed results for several experimental situations are discussed. Several corrections to papers in the literature are made, and in particular the exact scattering of a transverse wave by a spherical flaw is compared with the Rayleigh (long wave) limit.

I. INTRODUCTION AND OVERVIEW

Ultrasonic methods are of great importance in the study of structural and elastic properties of materials, particularly for non-destructive testing. A tremendous body of experimental work exists and has been interpreted to varying degrees. However, these interpretations have frequently been based upon theoretical methods which, in terms of modern concepts, have not advanced nearly as far as have the experimental techniques.

Beyond the experimental aspects of signal production and processing, the central question is that of specifying how flaws, i.e., inhomogeneities, scatter ultrasonic waves. Several experimental programs are now under way to lay a basic foundation for engineering applications, by measuring and analyzing the behavior of specific, well-controlled, prototype defects in a few representative materials.

We have addressed ourselves to the theoretical problems encountered in this controlled, experimental study. Two lines of development have been emphasized, representing substantial advances in the theoretical and analytical capability of theory for interpretive purposes: First, integral equation methods adapted from quantum scattering theory are susceptible to a wider variety of practical approximation methods than are the traditional differential equation approach; second, modern computer capability is vastly greater than that available a decade or more ago when much of the last serious study of elastic wave scattering in solids was done. In both respects, new results have been obtained.

Thus, we are writing a series of reports for record and for use by the experimental community; in part, these reports will be a review of theory, but we also report new material which is not to be found elsewhere, to our knowledge. Moreover, in contrast to purely formal studies, we have computed results for several representative (single) flaw situations in titanium, aluminum, and stainless steel.

In this report we "begin at the beginning", by developing the analysis in detail for scattering by single flaws or obstacles. The main results are contained in formulae for scattering cross sections and how they may be computed. Certain of the general relationships are new.

Of considerable interest is the fact that very complete computations can be carried out at a fraction of the cost of an experimental run to obtain similar data.

Another set of introductory remarks, for perspective, concerns the broader experimental situation. Additional scattering problems are encountered at several levels:

- 1) Even the normal "good" material is usually polycrystalline, so that when looked at with sufficient resolution there is a "noise" source not present in single-crystal material.

- 2) When a well-defined flaw in a polycrystalline medium scatters, both incoming and outgoing waves must usually propagate through the host heterogeneous medium, which certainly modifies the signal.

- 3) In some instances, multiple scattering between flaws requires analysis.

In all of these cases, the traditional differential equation methods are intractable, whereas the general integral equation methods which we emphasize have already been found usable in some applications, and show reasonable potential for wider applicability.

We leave these considerations of heterogeneity in the host material for subsequent study and discussion.

II. GENERAL THEORY OF THE SCATTERING OF ULTRASOUND BY FLAWS

A. Description of the Scattering Problem

The basic scattering problem is depicted in Fig. 1. In a material having elastic stiffness constants C_{ijkl} and density ρ , there exists a flaw defining a region of space R and having a surface S . The material inside the flaw has elastic stiffness constants C'_{ijkl} and density ρ' which, in general, differ from those of the host material. For many cases of interest, $C'_{ijkl} = \rho' = 0$.

Power generated by some source is directed toward the flaw. The presence of the flaw, however, prevents the incident power from propagating unhindered, and the flaw selectively scatters part of the incident power into various directions. What is of interest is to measure the amount of power, relative to the incident power, scattered into a given direction. Such a measurement may serve to differentiate between flaw geometries and hence would be a useful NDE measurement. For example, if diffraction effects are ignored, one expects power incident normal to the plane of a large disc (a crack) to scatter more strongly backwards than forwards. Such a large ratio between forward and backward scattering is not expected if the flaw is a sphere (a pore).

To define the scattering problem more precisely, we place the origin of a Cartesian coordinate system at the centroid of the flaw. (We will only consider flaws in which C'_{ijkl} and ρ' are homogeneous in R ; hence, the centroid and center of mass coincide.) We assume that the incident power is directed along the positive z -direction and is monochromatic with an angular frequency ω . The vector $\underline{r}$ in Fig. 1 determines the direction of observation of the scattered power relative to the Cartesian directions.

The physical quantity that measures the scattering effectiveness is the "cross section" of the flaw. This is related to, but not equal to, the effective geometrical cross-sectional area of the flaw. There are several kinds of cross sections; we now define two types.

First, we define an intensity vector I_i associated with a stress field σ_{ij} and a velocity field u_j as

$$I_i = \sigma_{ij} \dot{u}_j \quad (2.1)$$

where the dot above the displacement field u_j denotes differentiation with respect to time. Throughout we use standard tensor index convention: Repeated indices imply summation, and $,_i$ differentiation with respect to the i -th reference coordinate. We write the intensity vector associated with the scattered fields as I_i^S and I_i^O as the intensity vector associated with the incident field. We use the term intensity to mean the rate of energy transfer per unit area normal to the direction of propagation, that is

$$I = I_i n_i \quad (2.2)$$

where n_i is the unit normal to the direction of propagation. For a plane wave incident along the z -axis the incident intensity is calculated by considering the rate per unit area which energy is carried across an infinite plane perpendicular to the z -axis. Thus,

$$I^O = \sigma_{3i}^O \dot{u}_i^O \quad (2.3)$$

The power flux or rate at which energy is carried across surface A with unit normal n_i is

$$P = \int_A I_i n_i dA = \int_A n_i \sigma_{ij} \dot{u}_j dA \quad (2.4)$$

For a given angular frequency, corresponding to a period T , the total cross section $P(u)$ is the ratio of the average power flux scattered into all directions to the average intensity of the incident fields:

$$P(u) = \frac{\langle P^S \rangle}{\langle I^O \rangle} \quad (2.5)$$

where the angular bracket denote time averaging:

$$\langle f(t) \rangle = \frac{1}{T} \int_0^T f(t) dt \quad (2.6)$$

For a given frequency a measure of the fraction of the incident power scattered into a particular direction is the differential cross section $\frac{dP(\omega)}{d\Omega}$, where $d\Omega$ is the differential element of solid angle. From (2.1) the component of the average power flux, relative to the average of the total incident intensity, directed across an element of area dA normal to $\hat{r}_i$ is

$$dP(\omega) = \frac{\langle \hat{r}_i \sigma_{ij}^s \dot{u}_j^s \rangle dA}{\langle I^0 \rangle} \quad (2.7)$$

where $\hat{r}_i$ is a unit vector in the r -direction. However, at distance r from the origin,

$$dA = r^2 d\Omega.$$

Thus

$$\frac{dP(\omega)}{d\Omega} = \frac{\langle r^2 \hat{r}_i \sigma_{ij}^s \dot{u}_j^s \rangle}{\langle I^0 \rangle} \quad (2.8)$$

To make (2.8) more precise, we write

$$\frac{dP(\omega)}{d\Omega} = \lim_{r \rightarrow \infty} \frac{\langle r^2 \hat{r}_i \sigma_{ij}^s \dot{u}_j^s \rangle}{\langle I^0 \rangle} \quad (2.9)$$

The limiting process is necessary since differential cross section measurements are made at distances very far from the scatterer (the flaw). As we discuss later, when $r \rightarrow \infty$, σ_{ij}^s and $\dot{u}_j^s$ are of the order of $1/r$. Therefore, as $r \rightarrow \infty$, (2.9) represents that portion of the scattered power in $d\Omega$ which is independent of the distance from the flaw.

The use of (2.5) or (2.9) requires the calculation of the scattered stress field and the scattered displacement field. We proceed to derive an equation

governing the scattering of elastic waves by a flaw, and then, after specializing to the case where the elastic properties of the host material is isotropic, state a working formula for the total and differential cross sections of the flaw.

B. The Scattering Equation

To derive an equation governing the scattering of elastic waves by the type of flaw described in Sec. II-A, we start by considering two fields u_i and v_i in an unperturbed (flawless) medium that satisfy in the entire region R of Fig. 2,

$$C_{ijkl} u_{k,jl} + f_i = \rho \ddot{u}_i \quad (2.10a)$$

$$C_{ijkl} v_{k,jl} + g_i = \rho \ddot{v}_i \quad (2.10b)$$

where f_i and g_i are the inhomogeneous terms of these partial differential equations. The origin of these equations is, of course, the equation of motion of a displacement field in a medium with elastic stiffness C_{ijkl} and density ρ . If

$$\begin{aligned} u_i(\underline{r}, t) &= u_i(\underline{r}) e^{-i\omega t}, \\ f_i(\underline{r}, t) &= f_i(\underline{r}) e^{-i\omega t}, \\ v_i(\underline{r}, t) &= v_i(\underline{r}) e^{-i\omega t}, \text{ and} \\ g_i(\underline{r}, t) &= g_i(\underline{r}) e^{-i\omega t} \end{aligned} \quad (2.11)$$

then the above reduces to

$$C_{ijkl} u_{k,jl} + \rho \omega^2 u_i + f_i = 0 \quad (2.12a)$$

$$C_{ijkl} v_{k,jl} + \rho \omega^2 v_i + g_i = 0. \quad (2.12b)$$

These equations when combined suitably give

$$v_i f_i - u_i g_i + C_{ijkl} (v_i u_{k,jl} - u_i v_{k,jl}) = 0,$$

which in turn can be put into a more convenient form with the recognition that

$$C_{ijkl}(v_i u_{k,jl} - u_i v_{k,jl}) = C_{ijkl}(v_i u_{k,l} - u_i v_{k,l})_{,j}.$$

Therefore, Eqs. (2.12a) and (2.12b) reduce to

$$v_i f_i - u_i g_i + C_{ijkl}(v_i u_{k,l} - u_i v_{k,l})_{,j} = 0. \quad (2.13)$$

Specifying that

$$g_i = \delta_{im} \delta(\underline{r} - \underline{r}') \quad (2.14a)$$

and, in terms of a Green's function g_{im} discussed after (2.18), that

$$v_i = g_{im}(\underline{r} - \underline{r}') \quad (2.14b)$$

and then substituting these fields into (2.13), we find that

$$f_i g_{im} - u_i \delta_{im} \delta(\underline{r} - \underline{r}') + C_{ijkl}(g_{im} u_{k,l} - u_i g_{km,l})_{,j} = 0.$$

With these choices an integration of the $\underline{r}$ coordinate over a region of space R that is bounded by a surface S whose outward normal is n_j , yields for $\underline{r}'$ in R

$$u_m(\underline{r}') = C_{ijkl} \int_R dV [g_{im}(\underline{r}, \underline{r}') u_{k,l}(\underline{r}) - u_i(\underline{r}) g_{km,l}(\underline{r}, \underline{r}')]_{,j} \\ + \int_R dV f_i(\underline{r}) g_{im}(\underline{r}, \underline{r}'),$$

which with the use of Gauss' theorem reduces to

$$u_m = C_{ijkl} \int_S dS n_j (g_{im} u_{k,l} - u_i g_{km,l}) + \int_R dV f_i g_{im}. \quad (2.15)$$

The surface integral term in (2.15) can be thought of as a displacement field $u_m^0(\underline{r}')$, which depends on the surface displacement and the surface traction

$C_{ijkl} n_j u_{k,l}$, i.e., on the boundary conditions of the problem. It is easy to show that u_m^0 satisfies the homogeneous partial differential equation

$$C_{ijkl} u_{k,jl}^0 + \rho \omega^2 u_i^0 = 0. \quad (2.16)$$

In scattering problems, the boundary conditions on S are usually specified so that

$$u_i^0(\underline{r}, t) = u_i^0 e^{i(\underline{k} \cdot \underline{r} - \omega t)} \quad (2.17)$$

i.e., u_i^0 is the incident wave which is a plane wave of angular frequency ω and wave vector $\underline{k}$. Finally, one writes (2.15) as

$$u_m = u_m^0 + \int_R dV f_i g_{im}, \quad \text{for } \underline{r}' \text{ in } R \quad (2.18)$$

Equation (2.18) is a general result with an interesting interpretation. From (2.10b) and (2.12b) the field $g_{im}(\underline{r}, \underline{r}')$ satisfies

$$C_{ijkl} g_{km,jl} + \rho \omega^2 g_{im} + \delta_{im} \delta(\underline{r} - \underline{r}') = 0. \quad (2.19)$$

g_{im} is the steady-state Green's function of elastic wave. This Green's function has the interpretation of being the i -th component of a displacement field with time dependence $e^{-i\omega t}$ produced at $\underline{r}$ by the m -th component of a unit

force with time dependence $e^{-i\omega t}$ applied at $\underline{r}'$ to a medium with uniform elastic stiffness C_{ijkl} and constant density ρ . Thus, the volume integration in (2.18) represents the total m -th component of the displacement field produced in R by forces whose i -th component at each point in R is f_i .

B. Treatment of Flaws

The relation of (2.18) to the scattering of elastic waves by a flaw is developed in the following manner: Within R the material parameters, the elastic stiffness and density, may now vary with position so in the absence of body forces we can write

$$\sigma_{ij,j} + \rho''(\underline{r}) \omega^2 u_i = 0 \quad (2.20a)$$

or equivalently that

$$(C''_{ijkl}(\underline{r}) u_{k,l})_{,j} + \rho''(\underline{r}) \omega^2 u_i = 0, \quad (2.20b)$$

where $C''_{ijkl}(\underline{r})$ and $\rho''(\underline{r})$ are the elastic stiffness and density fields in R . For a flaw contained in R_2 , we write C''_{ijkl} and ρ'' as

$$C''_{ijkl}(\underline{r}) = C_{ijkl} + \delta C_{ijkl} \theta(\underline{r}) \quad (2.21a)$$

and

$$\rho''(\underline{r}) = \rho + \delta \rho \theta(\underline{r}) \quad (2.21b)$$

thus defining the perturbations

$$\delta C_{ijkl} = C'_{ijkl} - C_{ijkl} \quad (2.22a)$$

and

$$\delta \rho = \rho' - \rho. \quad (2.22b)$$

The definition of $\theta(\underline{r})$ is

$$\theta(\underline{r}) = \begin{cases} 1, & \text{for } \underline{r} \text{ in } R_2 \\ 0, & \text{for } \underline{r} \text{ not in } R_2 \end{cases} \quad (2.23)$$

R_2 specifies the region of the flaw (see Fig. 2). Equation (2.20) becomes

$$C_{ijkl} u_{k,lj} + \rho \omega^2 u_i = -(\delta \rho \omega^2 u_i \theta + \delta C_{ijkl} u_{k,jl} \theta + \delta C_{ijkl} u_{k,l} \theta_{,j}) \quad (2.24)$$

The quantity $\theta_{,j}$ equals $-n_j \varphi$ where n_j is the outward unit normal to S_2 and φ is unity if $\underline{r}$ is on the surface S_2 and zero otherwise.* Thus, outside R_2 , in R_1 , u_i satisfies

$$C_{ijkl} u_{k,jl} + \rho \omega^2 u_i = 0$$

while inside R_2 , u_i satisfies

$$C'_{ijkl} u_{k,jl} + \rho' \omega^2 u_i = 0.$$

The term $\theta_{,j}$ insures continuity of the displacement field and of the normal components of the stress fields across S_2 .

We now identify the f_i in (2.12) and (2.18) with the right-hand side of (2.24) and write

$$u_m = u_m^0 + \delta \rho \omega^2 \int_{R_2} dV g_{im} u_i + \delta C_{ijkl} \int_{R_2} dV g_{im} u_{k,jl} - \delta C_{ijkl} \int_{S_2} dS g_{im} n_j u_{k,l}$$

Applying Green's theorem to the second volume integral, one obtains

$$u_m(\underline{r}') = u_m^0(\underline{r}') + \delta \rho \omega^2 \int_{R_2} dV g_{im}(\underline{r}, \underline{r}') u_i(\underline{r}) - \delta C_{ijkl} \int_{R_2} dV g_{im,j}(\underline{r}, \underline{r}') u_{k,l}(\underline{r}). \quad (2.25)$$

*This statement is not mathematically rigorous, but is operationally correct.

We will always consider the case where the region $R \rightarrow \infty$. In this limit

$$g_{im}(\underline{r}, \underline{r}') = g_{im}(\underline{r} - \underline{r}')$$

which implies that

$$g_{im,j} = -g_{im,j'}$$

where the prime on the j denotes differentiation with respect to x_j' . Now after several cosmetic changes are made to (2.25), we write in final form our basic scattering equation as

$$\begin{aligned} u_i(\underline{r}) = & u_i^0(\underline{r}) + \delta\rho\omega^2 \int_{R_2} dV' g_{im}(\underline{r} - \underline{r}') u_m(\underline{r}') \\ & + \delta C_{jklm} \int_{R_2} dV' g_{ij,k}(\underline{r} - \underline{r}') u_{l,m}(\underline{r}') \end{aligned} \quad (2.26)$$

where the volume integrations are now over $\underline{r}'$. In writing (2.26) we used the reciprocity relation $g_{ij}(\underline{r}, \underline{r}') = g_{ji}(\underline{r}', \underline{r})$.

Equation (2.26) formulates the scattering problem in terms of an integral equation representing the scattering of an incident wave u_i^0 from a volume inhomogeneity. The Green's function g_{ij} is defined over all R but depends on the elastic stiffness and density of R_1 , the host material. The form of incident wave u_i^0 depends on the boundary conditions applied to the exterior surface S , and is also defined over all R .

The elastic wave scattering problem is more commonly formulated in terms of an integral equation representing scattering off the surface S_2 . We use the volume formulation to connect the present scattering problem with techniques developed for scattering problems in nuclear physics. However, the surface

formulation is instructive as it is useful in illustrating several points about the volume formulation, and we develop it in the next section. The two forms of the theory are both exact, and the choice of which to use depends on personal preference. However, we believe that the present report is the first to present the volume formulation and its application in reasonably complete form. We also believe the volume formulation incorporates the volume inhomogeneity in a more direct fashion than the surface formulation.

. An Alternate Scattering Equation: The Surface Formulation

To derive an equation for the scattering of an elastic wave in terms of integrals over only the surface S_2 , we start in a manner similar to the derivation of the scattering equation in terms of integrals over the region R_2 . The principal difference between the derivations is that in the present case the regions R_1 and R_2 are considered separately.

First, in R_1 we consider the set of equations

$$C_{ijkl} u_{k,jl} + \rho \omega^2 u_i + f_i = 0 \quad (2.27a)$$

$$C_{ijkl} v_{k,jl} + \rho \omega^2 v_i + g_i = 0. \quad (2.27b)$$

The u_i , v_i , f_i and g_i in these equations differ from those in Eqs. (2.12a) and (2.12b). The present functions are defined only for $\underline{r}$ in R_1 ; the former functions are defined for $\underline{r}$ in R which includes R_1 .

With the choices of

$$f_i = 0 \quad (2.28a)$$

$$v_i = g_{im}(\underline{r}, \underline{r}') \quad (2.28b)$$

and

$$g_i = \delta_{im} \delta(\underline{r} - \underline{r}'), \quad (2.28c)$$

it follows from Eqs. (2.10-2.15) that

$$u_m = C_{ijkl} \int_{S_1} dS n_l (g_{jm} u_{k,l} - u_j g_{km,l}), \quad \text{for } \underline{r}' \text{ in } R_1 \quad (2.29)$$

S_1 is the surface bounding R_1 ; it comprises surfaces S and S_2 . We point out that along S_2 the outward normal of S_1 is the inward normal of S_2 . Thus, the surface integral in (2.29) has contributions from the exterior surface S (the experimentally accessible surface) and from the interior surface S_2 (the experimentally inaccessible surface of the flaw).

As in Sec. II-B we write

$$u_m^0 = C_{ijkl} \int_S dS n_i (g_{jm} u_{k,l} - u_j g_{km,l})$$

and note that u_m^0 satisfies in R_1

$$C_{ijkl} u_{k,jl}^0 + \rho \omega^2 u_l^0 = 0$$

and depends on the specification of the surface displacements and the surface normal stresses. Thus, in R_1 one has

$$u_m = u_m^0 + C_{ijkl} \int_{S_2} dS n_i (g_{jm} u_{k,l} - u_j g_{km,l}), \quad \text{for } \underline{r}' \text{ in } R_1 \quad (2.30)$$

The integration over S_2 involves the specification of the surface displacement field and the surface normal stress fields. These fields must be consistent with the properties of the material bounded by S_2 , i.e. represent continuous displacement and normal stress or else must be specified by means of a surface impedance.

Next, in R_2 we consider the set of equations

$$C'_{ijkl} u'_{k,jl} + \rho' \omega^2 u'_l + f'_l = 0 \quad (2.31a)$$

$$C'_{ijkl} v'_{k,jl} + \rho' \omega^2 v'_l + g'_l = 0. \quad (2.31b)$$

The primed variables distinguish quantities defined only in R_2 from those defined only in R_1 . With the choices

$$f'_i = 0 \quad (2.32a)$$

$$v'_i = g'_{im}(\underline{r}, \underline{r}') \quad (2.32b)$$

and

$$g'_i = \delta_{im} \delta(\underline{r} - \underline{r}') , \quad (2.32c)$$

it follows from Eqs. (2.10-2.15) that

$$u'_m = C'_{ijkl} \int_{S_2} dS n'_i (g'_{jm} u'_{k,l} - u'_j g'_{kn,l}), \quad \text{for } \underline{r}' \text{ in } R_2 \quad (2.33)$$

Therefore, to find the displacement field outside of S_2 , one has to find two Green's functions (these functions satisfy different boundary conditions in R_1 and R_2), solve (2.33), use the continuity conditions for the displacements and normal stresses across S_2 , and then solve (2.30).

Equations (2.30) and (2.33), each valid in separate regions of R , are the integral equations for the surface formulation of the scattering problem. Two separate equations and two separate Green's functions are needed to describe the displacement field over the entire volume R . On the other hand, the integral equation (2.26) is one equation with one Green's function valid over all R . Although (2.30) and (2.33) are connected by continuity requirements across S_2 , we will show that (2.26) has these requirements built-in.

E. Continuity Conditions

To demonstrate that the continuity of displacement and normal stress is intrinsic Eq. (2.26), we start by considering two fields u_i and g_{ij} defined over all R and satisfying

$$C_{ijkl} u_{k,jl} + \rho \omega^2 u_i = 0$$

$$C_{ijkl} g_{km,jl} + \rho \omega^2 g_{im} + \delta_{im} \delta(\underline{r}-\underline{r}') = 0$$

From these equations it follows that

$$u_m \delta(\underline{r}-\underline{r}') = C_{ijkl} (g_{im} u_{k,l} - u_i g_{km,l}),_{,j}$$

We next integrate this equation only over R_1 and obtain

$$u_m = \int_{R_1} dv C_{ijkl} (g_{im} u_{k,l} - u_i g_{km,l}),_{,j} \quad \text{for } \underline{r}' \text{ in } R_1$$

and

$$0 = \int_{R_1} dv C_{ijkl} (g_{im} u_{k,l} - u_i g_{km,l}),_{,j} \quad \text{for } \underline{r}' \text{ in } R_2.$$

The second equation is a consequence of g_{ij} being defined in $R=R_1+R_2$. (We are not restricting the range of definition of g and u to R_1 ; we are only restricting the range of integration.)

Invoking the divergence theorem, we find

$$u_m = u_m^0 + C_{ijkl} \int_{S_2} dS n_i (g_{jm} u_{k,l} - u_j g_{km,l}),_{,j} \quad \text{for } \underline{r}' \text{ in } R_1 \quad (2.34a)$$

and

$$0 = u_m^0 + C_{ijkl} \int_{S_2} dS n_i (g_{jm} u_{k,l} - u_j g_{km,l}),_{,j} \quad \text{for } \underline{r}' \text{ in } R_2 \quad (2.34b)$$

where u_m^0 as before is defined by

$$u_m^0 = C_{ijkl} \int_{S_1} dS n_i (g_{im} u_{k,l} - u_i g_{km,l})$$

Across S_2 one has

$$u_1 = u'_1$$

$$n_1 = -n'_1$$

$$n_i C_{ijkl} u_{k,jl} = n'_i C'_{ijkl} u'_{k,jl} \quad (\text{normal stresses}) .$$

The prime is merely to distinguish inside R_2 from outside R_2 . The fields u_1 and u'_1 have no relation to the u_1 and u'_1 of section II-D.

Equation (2.34) is thus equal to

$$u_m = u_m^0 - \int_{S_2} dS n'_i (C'_{ijkl} \epsilon_{jm} u'_{k,l} - C_{ijkl} u'_j \epsilon_{km,l}) \quad \text{for } \underline{r}' \text{ in } R_1$$

and

$$0 = u_m^0 - \int_{S_2} dS n'_i (C'_{ijkl} \epsilon_{jm} u'_{k,l} - C_{ijkl} u'_j \epsilon_{km,l}) \quad \text{for } \underline{r}' \text{ in } R_2 .$$

When the divergence theorem is applied to these equations, the following results:

$$\begin{aligned} u_m = u_m^0 - \int_{R_2} dV (\epsilon_{im} C'_{ijkl} u'_{k,lj} + \epsilon_{im,j} C'_{ijkl} u'_{k,l} \\ - u'_i C_{ijkl} \epsilon_{km,jl} - \epsilon_{im,j} C_{ijkl} u'_{k,l}) \quad \text{for } \underline{r}' \text{ in } R_1 \end{aligned} \quad (2.35a)$$

and

$$\begin{aligned} 0 = u_m^0 - \int_{R_2} dV (\epsilon_{im} C'_{ijkl} u'_{k,lj} + \epsilon_{im,j} C'_{ijkl} u'_{k,l} \\ - u'_i C_{ijkl} \epsilon_{km,jl} - \epsilon_{im,j} C_{ijkl} u'_{k,l}) \quad \text{for } \underline{r}' \text{ in } R_2 . \end{aligned} \quad (2.35b)$$

Using Eqs. (2.27), (2.28), (2.30), and (2.31), we find

$$u_m = u_m^0 + \delta \rho \omega^2 \int_{R_2} dV g_{im} u'_i - \delta C_{ijkl} \int_{R_2} dV g_{im,j} u'_{k,l} \quad \text{for } \underline{r}' \text{ in } R_1 \quad (2.36a)$$

and

$$u'_m = u_m^0 + \delta \rho \omega^2 \int_{R_2} dv g_{im} u'_i - \delta C_{ijkl} \int_{R_2} dv g_{im,j} u'_{k,l} \quad \text{for } r' \text{ in } R_2 \quad (2.36b)$$

Thus it follows that over the entire region $R_1 + R_2$

$$u_m = u_m^0 + \delta \rho \omega^2 \int_{R_2} dv g_{im} u'_i - \delta C_{ijkl} \int_{R_2} dv g_{im,j} u'_{k,l} \quad \text{for } r' \text{ in } R = R_1 + R_2 \quad (2.37a)$$

which is just (2.26) before the cosmetic changes were made. It is now clear in (2.37) or (2.26) the continuity of displacement and normal stress fields at the R_1, R_2 interface is accounted for in the volume formulation. In addition, we note that the Green's function appearing in the integration over R_2 is the Green's function defined by an elastic stiffness C_{ijkl} and density ρ over both R_1 and R_2 .

F. The Green's Function

The Green's function satisfies

$$C_{ijkl} g_{km,jl} + \rho \omega^2 g_{im} + \delta_{im} \delta(\underline{r}-\underline{r}') = 0. \quad (2.38)$$

We will always consider the case where $R \rightarrow \infty$; therefore, $g_{im}(\underline{r}, \underline{r}') = g_{im}(\underline{r}-\underline{r}')$.

A convenient way to find g_{im} is from the Fourier transform of (2.38). We define

$$g_{im}(\underline{r}-\underline{r}') = \frac{1}{(2\pi)^3} \int d\underline{k} e^{i\underline{k} \cdot (\underline{r}-\underline{r}')} \bar{g}_{im}(\underline{k})$$

and

$$\delta(\underline{r}-\underline{r}') = \frac{1}{(2\pi)^3} \int d\underline{k} e^{i\underline{k} \cdot (\underline{r}-\underline{r}')}.$$

Therefore, (2.38) implies

$$C_{ijkl} k_j k_l \bar{g}_{km}(\underline{k}) + \rho \omega^2 \bar{g}_{im}(\underline{k}) = \delta_{im}. \quad (2.39)$$

With the definition

$$\Gamma_{ik} = C_{ijkl} k_j k_l + \rho \omega^2 \delta_{ik}$$

it follows

$$\bar{g}_{ij}(\underline{k}) = \Gamma_{ij}^{-1}(\underline{k}).$$

For C_{ijkl} isotropic, i.e.,

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

it is straightforward to show that

$$\bar{g}_{ij} = - \frac{(\lambda + 2\mu)(\alpha^2 - k^2) \delta_{ij} + (\lambda + \mu) k_i k_j}{\mu(\lambda + 2\mu)(\alpha^2 - k^2)(\beta^2 - k^2)} \quad (2.40)$$

where

$$\alpha^2 = \frac{\rho \omega^2}{\lambda + 2\mu}$$

is the square of the wave number associated with the longitudinal speed

$$v_L = \sqrt{(\lambda + 2\mu)/\rho} \text{ and}$$

$$\beta^2 = \frac{\rho \omega^2}{\mu}$$

is the square of the wave number associated with the transverse speed

$v_T = \sqrt{\mu/\rho}$. Finding $g_{ij}(\underline{r}-\underline{r}')$ requires the Fourier inversion of (2.40); but because the inversion involves the integration over all positive k in (2.40), including the singular points $k^2 = \alpha^2$ or β^2 , the inversion requires careful treatment. The standard technique (trick) to evaluate integrals with integrands like (2.40) is to consider the integration variable to be a complex variable and α and β be complex numbers with small imaginary parts. This continuation of the real variables into the complex plane removes the singularities in the integrand off the real axis and permits the evaluation of the integral by contour integration using the theorem of residues. In the present case, we shall see that the solutions as $\text{Im } \alpha$ (or $\text{Im } \beta$) $\rightarrow 0^+$ are not equal and that the two different limits correspond to two different spatial boundary conditions.

With $\text{Im } \alpha$, and $\text{Im } \beta$, $\neq 0$, we wish to evaluate

$$g_{ij}(\underline{R}) = - \frac{1}{(2\pi)^3} \int d\underline{k} e^{i\underline{k} \cdot \underline{R}} \frac{(\lambda + 2\mu)(\alpha^2 - k^2) \delta_{ij} + (\lambda + \mu) k_i k_j}{\mu(\lambda + 2\mu)(\alpha^2 - k^2)(\beta^2 - k^2)}$$

where $\underline{R} = \underline{r} - \underline{r}'$. This integral may also be written as

$$\begin{aligned}
g_{ij} = & \frac{-\delta_{ij}}{(2\pi)^3} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{R}} \frac{1}{\mu} \frac{1}{\beta^2 - k^2} \\
& + \frac{1}{(2\pi)^3} \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_j} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{R}} \frac{1}{\rho\omega^2} \left[\frac{1}{\alpha^2 - k^2} - \frac{1}{\beta^2 - k^2} \right]
\end{aligned} \quad (2.41)$$

where for the second term we used the fact that $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Thus, the basic integral that we must evaluate is of the form

$$I_q(R) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{q^2 - k^2}, \quad \text{Im } q \neq 0. \quad (2.42)$$

This is a common integral occurring in acoustics, optics, quantum mechanics, etc. We could immediately write down the result by borrowing from these fields of study; however, the evaluation is instructive, and we will explicitly carry out some of the details.

Equation (2.42) equals

$$\frac{1}{8\pi R} \int_{-\infty}^{+\infty} dk \, k \frac{e^{ikR} - e^{-ikR}}{(k - q)(k + q)}$$

after integrating over the orientation of $\mathbf{k}$. Since $R > 0$, a contour as in Fig. 3a can be closed in upper half-plane in the term with e^{iqR} and in the lower half-plane as in Fig. 3b in the term with e^{-iqR} . By the residue theorem,

$$I_q(R) = \frac{-1}{4\pi} \frac{e^{iqR}}{R}, \quad \text{Im } q > 0 \quad (2.43a)$$

$$I_q(R) = \frac{-1}{4\pi} \frac{e^{-iqR}}{R}, \quad \text{Im } q < 0. \quad (2.43b)$$

Thus, as a function of complex q , $I_q(R)$ is multivalued with a branch cut along the real axis. The physically interesting solution is (2.43a) for $\text{Im } q \rightarrow 0^+$. It represents, when coupled with the $e^{-i\omega t}$ time dependence, an outgoing spherical wave. We shall use this value of $I_q(R)$ in (2.41); hence, we find

$$g_{ij}(\underline{r}-\underline{r}') = \frac{1}{4\pi\rho\omega^2} \left[\beta^2 \frac{e^{i\beta R}}{R} \delta_{ij} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left(\frac{e^{i\alpha R}}{R} - \frac{e^{i\beta R}}{R} \right) \right]. \quad (2.44)$$

Knowing g_{ij} allows us to investigate explicitly details of the scattered displacement and stress fields and consequently the cross sections for a flaw present in an isotropic material.

6. Scattered Amplitudes and the Cross Sections

We write (2.26) as

$$u_i = u_i^0 + u_i^s \quad (2.45)$$

where

$$u_i^s = \delta \rho \omega^2 \int_{R_2} dV' g_{im} u_m + \delta C_{jklm} \int_{R_2} dV' g_{ij,k} u_{l,m}. \quad (2.46)$$

Computing the cross sections requires the knowledge of u_i^s as $r \rightarrow \infty$. To find this asymptotic value, we consider the behavior of g_{ij} and $g_{ij,k}$ as $r \rightarrow \infty$. We have

$$g_{ij} = \frac{1}{4\pi\rho\omega^2} \left[\delta_{ij} \beta^2 \frac{e^{i\beta R}}{R} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left(\frac{e^{i\alpha R}}{R} - \frac{e^{i\beta R}}{R} \right) \right]$$

and

$$g_{ij,k} = \frac{1}{4\pi\rho\omega^2} \left[\delta_{ij} \beta^2 \frac{\partial}{\partial x_k} \frac{e^{i\beta R}}{R} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \left(\frac{e^{i\alpha R}}{R} - \frac{e^{i\beta R}}{R} \right) \right].$$

As $r \rightarrow \infty$

$$\frac{1}{R} \sim \frac{1}{r} \quad (2.47a)$$

$$R \sim r - (\underline{r}' \cdot \underline{\hat{r}}) \quad (2.47b)$$

$$\frac{\partial}{\partial x_i} \frac{e^{ikR}}{R} \sim ik \frac{e^{ikr}}{r} \hat{r}_i e^{-ik \cdot \underline{r}'} \quad (2.47c)$$

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{e^{ikR}}{R} \sim (ik)^2 \frac{e^{ikr}}{r} \hat{r}_i \hat{r}_j e^{-ik \cdot \underline{r}'} \quad (2.47d)$$

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \frac{e^{ikR}}{R} \sim (ik)^3 \frac{e^{ikr}}{r} \hat{r}_i \hat{r}_j \hat{r}_k e^{-ik \cdot \underline{r}'} \quad (2.47e)$$

where we use the symbol $\sim$ to denote asymptotic equality and

where $\underline{k} \equiv k \hat{r}_1$. With these asymptotic formulas, the far field value of u_1^s is

$$\begin{aligned}
 u_1^s \sim & \frac{1}{4\pi\rho\omega^2} \delta\rho\omega^2 \int_{R_2} dV' u_j \left[\delta_{ij} \beta^2 \frac{e^{i\beta r}}{r} e^{-i\beta \cdot \underline{r}'} \right. \\
 & + \hat{r}_i \hat{r}_j \left(\alpha^2 \frac{e^{i\alpha r}}{r} e^{-i\alpha \cdot \underline{r}'} - \beta^2 \frac{e^{i\beta r}}{r} e^{-i\beta \cdot \underline{r}'} \right) \\
 & + \frac{1}{4\pi\rho\omega^2} \delta C_{jklm} \int_{R_2} dV' u_{l,m} \left[\delta_{ij} \hat{r}_k \beta^2 \frac{e^{i\beta r}}{r} e^{-i\beta \cdot \underline{r}'} + \hat{r}_i \hat{r}_j \hat{r}_k \right. \\
 & \left. \left(\alpha^3 \frac{e^{i\alpha r}}{r} e^{-i\alpha \cdot \underline{r}'} - \beta^3 \frac{e^{i\beta r}}{r} e^{-i\beta \cdot \underline{r}'} \right) \right]. \quad (2.49)
 \end{aligned}$$

We now consider the radial (longitudinal) component of u_1^s :

$$\begin{aligned}
 \hat{r}_i u_1^s \sim & \frac{1}{4\pi\rho\omega^2} \alpha^2 \frac{e^{i\alpha r}}{r} \left[\delta\rho\omega^2 \hat{r}_j \int_{R_2} dV' u_j e^{-i\alpha \cdot \underline{r}'} \right. \\
 & \left. + i\alpha \hat{r}_j \hat{r}_k \delta C_{jklm} \int_{R_2} dV' u_{l,m} e^{-i\alpha \cdot \underline{r}'} \right]. \quad (2.50)
 \end{aligned}$$

Thus, the longitudinal component of u_1^s is a spherical wave having the longitudinal wave number α .

Defining

$$\begin{aligned}
 A_i(\theta, \varphi) = & \frac{1}{4\pi\rho\omega^2} \hat{r}_i \alpha^2 \left[\delta\rho\omega^2 \int_{R_2} dV' \hat{r}_j u_j e^{-i\alpha \cdot \underline{r}'} \right. \\
 & \left. + i\alpha \hat{r}_j \hat{r}_k \delta C_{jklm} \int_{R_2} dV' u_{l,m} e^{-i\alpha \cdot \underline{r}'} \right]. \quad (2.51)
 \end{aligned}$$

We write the longitudinal component of the scattering, or the longitudinal scattering amplitude as

$$\hat{r}_i \hat{r}_j u_j^s \sim \frac{e^{i\alpha r}}{r} A_i(\theta, \varphi). \quad (2.52)$$

We next consider the quantity $u_i^S - \hat{r}_i \hat{r}_j u_j^S$. From Eq. (2.49) this quantity at far fields equals

$$\begin{aligned} \frac{1}{4\pi\rho\omega^2} \beta^2 \frac{e^{i\beta r}}{r} \left[\delta\rho\omega^2 \int_{R_2} dv' (u_i - \hat{r}_i \hat{r}_m u_m) e^{-i\beta \cdot \underline{r}'} \right. \\ \left. + i\beta (\delta_{ij} - \hat{r}_i \hat{r}_j) \hat{r}_k \delta C_{jklm} \int_{R_2} dv' u_{l,m} e^{-i\beta \cdot \underline{r}'} \right] \end{aligned} \quad (2.53)$$

which has the form of a spherical wave with the transverse wave number β . It is easy to show that the vectors

$$u_i - \hat{r}_i \hat{r}_m u_m = (\delta_{ij} - \hat{r}_i \hat{r}_j) u_j \quad (2.54a)$$

and

$$(\delta_{ij} - \hat{r}_i \hat{r}_j) \hat{r}_k \delta C_{jklm} u_{l,m} \quad (2.54b)$$

are transverse (perpendicular) to $\hat{r}_i$. Therefore, Eq. (2.53) represents the far field amplitude of the transverse component of the scattered displacement field. Accordingly, we define a transverse scattered amplitude $B_i(\theta, \varphi)$ as

$$\begin{aligned} B_i(\theta, \varphi) = \frac{1}{4\pi\rho\omega^2} \beta^2 (\delta_{ij} - \hat{r}_i \hat{r}_j) \left[\delta\rho\omega^2 \int_{R_2} dv' u_j e^{-i\beta \cdot \underline{r}'} \right. \\ \left. + i\beta \hat{r}_k \delta C_{jklm} \int_{R_2} dv' u_{l,m} e^{-i\beta \cdot \underline{r}'} \right] \end{aligned} \quad (2.55)$$

so that the transverse component of the scattered displacement field at large r is

$$\frac{e^{i\beta r}}{r} B_i(\theta, \varphi) \quad (2.56)$$

We note

$$A_i B_i = \hat{r}_i B_i = 0. \quad (2.57)$$

Finally, we write as $r \rightarrow \infty$

$$u_i^s \sim \frac{e^{i\alpha r}}{r} A_i + \frac{e^{i\beta r}}{r} B_i. \quad (2.58)$$

Accompanying the displacement field are stress fields σ_{ij}^s which, if C_{ijkl} is isotropic, equal

$$\lambda u_{1,1}^s \delta_{ij} + \mu (u_{i,j} + u_{j,i}).$$

The field $u_{i,j}^s$ for $r \rightarrow \infty$ is easily found from Eqs. (2.49-2.55) with help from

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l} \frac{e^{ikr}}{R} \sim (ik)^4 \frac{e^{ikr}}{r} \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l e^{-ik \cdot r}. \quad (2.59)$$

The result for this isotropic case is

$$u_{i,j}^s \sim i\alpha \frac{e^{i\alpha r}}{r} A_i \hat{r}_j + i\beta \frac{e^{i\beta r}}{r} B_i \hat{r}_j. \quad (2.60)$$

Hence,

$$u_{1,1}^s \sim i\alpha \frac{e^{i\alpha r}}{r} A_1 \hat{r}_1 \quad (2.61a)$$

and

$$(u_{i,j}^s + u_{j,i}^s) \sim i\alpha \frac{e^{i\alpha r}}{r} (A_i \hat{r}_j + A_j \hat{r}_i) + i\beta \frac{e^{i\beta r}}{r} (B_i \hat{r}_j + B_j \hat{r}_i). \quad (2.61b)$$

The scattered stress field is

$$\begin{aligned} \sigma_{ij}^s \sim i\lambda \alpha \frac{e^{i\alpha r}}{r} A_1 \hat{r}_1 \delta_{ij} + i\mu \left[\alpha \frac{e^{i\alpha r}}{r} (A_i \hat{r}_j + A_j \hat{r}_i) \right. \\ \left. + \beta \frac{e^{i\beta r}}{r} (B_i \hat{r}_j + B_j \hat{r}_i) \right]. \end{aligned} \quad (2.62)$$

Having found the far field values of the scattered displacement and stress fields, we can now compute the total and differential cross sections from (2.6) and (2.9). Physically, the fields are real numbers; however, because of (2.11), the scattered fields in (2.58) and (2.62) are complex numbers. Thus, to have the energy flux be a real number, what we really must compute in (2.6) and (2.9) is

$$\frac{1}{2} \langle \hat{r}_i (\sigma_{ij} + \sigma_{ij}^*) (u_j + u_j^*) \rangle.$$

Since we assumed $\sigma_{ij}(\underline{r}, t) = \sigma_{ij}(\underline{r}) e^{-i\omega t}$ and $u_i(\underline{r}, t) = u_i(\underline{r}) e^{-i\omega t}$, the above becomes

$$= \frac{\omega}{2} \text{Im} \cdot \hat{r}_i \sigma_{ij} u_j^* . \quad (2.63)$$

Hence, the differential cross section is

$$\frac{dP(\omega)}{d\Omega} = \lim_{r \rightarrow \infty} \left[-\frac{\omega r^2}{2} \frac{\text{Im}(\hat{r}_i \sigma_{ij} u_j^*)}{\langle P^0 \rangle} \right] \quad (2.64)$$

and the total cross section is simply

$$P(\omega) = \int d\Omega \frac{dP(\omega)}{d\Omega} \quad (2.65)$$

For an incident plane wave displacement field travelling in the z-direction, we write

$$u_i^0 = a_i e^{i(\alpha z - \omega t)} + b_i e^{i(\beta z - \omega t)} \quad (2.64)$$

where a_i is the longitudinal amplitude vector and b_i is the transverse amplitude vector, i.e., $a_1 = a_2 = b_3 = 0$ and a_3, b_1, b_2 are not equal to zero in general.

(We note $a_i b_i = 0$.) From (2.6) and (2.63) it follows, particularly using (2.57), that

$$\langle P^0 \rangle = -\frac{\omega}{2} [\alpha(\lambda + 2\mu) |a_i|^2 + \beta\mu |b_i|^2], \quad (2.65)$$

while from (2.58), (2.62), and (2.63) it follows that

$$\lim_{r \rightarrow \infty} -\frac{4\pi r^2}{2} \langle \hat{r}_i | \text{Im } \sigma_{1j}^s u_j^{s*} \rangle = -\frac{\omega}{2} [\alpha(\lambda + 2\mu) |A_1|^2 + \beta\mu |B_1|^2] . \quad (2.66)$$

Therefore,

$$\frac{dP(\omega)}{d\Omega} = \frac{\alpha(\lambda + 2\mu) |A_1|^2 + \beta\mu |B_1|^2}{\alpha(\lambda + 2\mu) |a_1|^2 + \beta\mu |b_1|^2} . \quad (2.67)$$

This equation is the differential cross section for an arbitrarily shaped flaw with uniform but otherwise arbitrary elastic stiffness constants and density. The only restrictions are that the host material is isotropic and the incident wave is a plane wave.

Equation (2.67) implies several things about the scattering. First, if the incident wave is purely longitudinal ($b_1 = 0$), then

$$\frac{dP(\omega)}{d\Omega} = \left| \frac{A_1}{a_1} \right|^2 + \frac{\alpha}{\beta} \left| \frac{B_1}{a_1} \right|^2 . \quad (2.68)$$

On the other hand, if the incident wave is purely transverse ($a_1 = 0$), then

$$\frac{dP(\omega)}{d\Omega} = \frac{\beta}{\alpha} \left| \frac{A_1}{b_1} \right|^2 + \left| \frac{B_1}{b_1} \right|^2 . \quad (2.69)$$

Thus, the form of the differential cross section depends on whether the incident wave is longitudinally or transversely polarized. Also, even if the incident wave is of a single mode, the differential cross section has contributions from both modes of the scattered wave. This appearance of the second mode in the scattered wave is called mode conversion: part of the power of the incident mode is converted upon scattering into power in the complimentary mode.

To study mode conversion it is convenient to define several other differential cross-sections. If the incident wave is longitudinal, we define

a longitudinal differential cross section as

$$\frac{dP_L(\omega)}{d\Omega} = \left| \frac{A_i}{a_i} \right|^2 \quad (2.70a)$$

and a transverse differential cross section as

$$\frac{dP_T(\omega)}{d\Omega} = \left| \frac{B_i}{a_i} \right|^2 \quad (2.70b)$$

These cross sections measure for a given mode in a given direction the ratio of the time-averaged scattered power to the total time-averaged incident power. Similar definitions can be made for an incident transverse wave by replacing a_i with b_i . We will calculate these cross sections in our investigation of ultrasound scattering as an NDE tool.

We point out that

$$\frac{dP}{d\Omega} \neq \frac{dP_L}{d\Omega} + \frac{dP_T}{d\Omega}$$

because the differential cross section is defined as the ratio of powers.

H. An "Optical Theorem"

From its early origins in light scattering, the theory of scattering has evolved in complexity, but certain general theorems appear and are common to various applications, including quantum mechanics. One such theorem relates total scattering cross section to the difference between the incident flux and the forward scattered flux; the "optical" analog is apparent. We now develop this relationship for scattered elastic waves; because longitudinal and transverse waves may interfere, the theorem is not as simple as in light scattering or in quantum mechanics.

In region R, the displacement field is

$$u_i = u_i^o + u_i^s,$$

and the accompanying stress field is

$$\sigma_{ij} = \sigma_{ij}^o + \sigma_{ij}^s.$$

Consequently, for the intensity vector we have

$$I_i = \sigma_{ij} \dot{u}_j \quad (2.71a)$$

$$= \sigma_{ij}^o \dot{u}_j^o + (\sigma_{ij}^o \dot{u}_j^r + \sigma_{ij}^s \dot{u}_j^o) + \sigma_{ij}^s \dot{u}_j^s \quad (2.71b)$$

$$= I_i^o + I_i^{int} + I_i^s. \quad (2.71c)$$

The first term I_i^o is the intensity vector of the incident (homogeneous) field; the second term I_i^{int} is an intensity vector associated with the interference of the incident and scattered (inhomogeneous) field; and the third term is the intensity vector of the scattered field.

Considering the power passing out of R through S, we see in (2.71) that the first term equals the power input into the system; hence, by energy conservation, the scattered power and the "interference" power must cancel, that is,

$$\int_S dS n_i I_i^s = - \int_S dS n_i I_i^{int}. \quad (2.72)$$

We note that S by assumption is very far from the flaw (the scatterer); thus, on S, the scattered displacement and stress fields are given by Eqs. (2.58) and (2.62)*. For convenience, we take S to be a sphere. The content of (2.72),

*Equation (2.72) remains valid if S is not far from the flaw. What follows however is valid only for S far from the flaw.

therefore, is

$$\lim_{r \rightarrow \infty} r^2 \int d\Omega \hat{r}_i I_i^S = - \lim_{r \rightarrow \infty} r^2 \int d\Omega \hat{r}_i I_i^{\text{int}} \quad (2.73)$$

where $d\Omega$ is an element of solid angle. However, the time average of the left-hand side of the above is proportional to the total cross section (total power scattered in all directions). Therefore, the total cross section is related to the interference of the incident power (travelling in the forward or +z-direction) with the scattered power (travelling radially from the flaw). We now evaluate the right-hand side of Eq. (2.73); we will find that the interference effects are important in two directions and will derive an equation for total cross section independent of Eq. (2.65). As mentioned before, a similar relation called the optical theorem exists for scattering theory in quantum mechanics and in electrodynamics. We adopt that nomenclature.

The time average of the right-hand side of Eq. (2.73) is [c.f. (2.63)]

$$\lim_{r \rightarrow \infty} \frac{wr^2}{2} \int_S d\Omega \hat{r}_i [\text{Im}(\sigma_{ij}^o u_j^{s*} + \sigma_{ij}^s u_j^{o*})] \quad (2.74)$$

where, for the incident fields, we have

$$u_i^o = a_i e^{i\alpha z} + b_i e^{i\beta z} \quad (2.75a)$$

$$\begin{aligned} \sigma_{ij}^o = i[\lambda \alpha e^{i\alpha z} a_k \delta_{k3} \delta_{ij} + \mu \alpha e^{i\alpha z} (a_i \delta_{j3} + a_j \delta_{i3}) \\ + \mu \beta e^{i\beta z} (b_i \delta_{j3} + b_j \delta_{i3})] \end{aligned} \quad (2.75b)$$

while for the scattered fields at large r we have

$$u_i^s \sim A_i \frac{e^{i\alpha r}}{r} + B_i \frac{e^{i\beta r}}{r} \quad (2.75c)$$

$$\begin{aligned} \sigma_{ij}^s \sim i[\lambda\alpha \frac{e^{i\alpha r}}{r} A_k \hat{r}_k \delta_{ij} + \mu\alpha \frac{e^{i\alpha r}}{r} (A_i \hat{r}_j + A_j \hat{r}_i) \\ + \mu\beta \frac{e^{i\beta r}}{r} (B_i \hat{r}_j + B_j \hat{r}_i)] . \end{aligned} \quad (2.75d)$$

Equation (2.74) may be rewritten as

$$\lim_{r \rightarrow \infty} \frac{\omega r^2}{2} \int_S d\Omega \hat{r}_i [\text{Im}(u_j^{s*} C_{ijkl} u_{k,l}^o + u_j^{o*} C_{ijkl} u_{k,l}^s)] . \quad (2.76)$$

It is straightforward to show for any two fields u_i and v_i which satisfy

$$C_{ijkl} u_{j,k,l} + \rho \omega^2 u_i = 0$$

and

$$C_{ijkl} v_{j,k,l} + \rho \omega^2 v_i = 0$$

that for any surface with an outward unit normal $\hat{n}_i$

$$\int dS n_i u_j^* C_{ijkl} v_{k,l} = \int dS n_i v_j^* C_{ijkl} u_{k,l} .$$

Hence, (2.76) equals

$$\lim_{r \rightarrow \infty} \frac{\omega r^2}{2} \int_S d\Omega \hat{r}_i \text{Im}[u_j^{o*} C_{ijkl} u_{k,l}^s + u_j^{s*} C_{ijkl} u_{k,l}^o]$$

or, upon further simplification:

$$\lim_{r \rightarrow \infty} \omega r^2 \int_S d\Omega \hat{r}_i \text{Im} \sigma_{ij}^s u_j^{o*} . \quad (2.77)$$

From (2.75) and with some straightforward algebra, eventually we find that

$$d\Omega \hat{r}_i \sigma_{ij}^s u_j^{o*} = i \int d\Omega \left[(\lambda + 2\mu) \alpha \frac{e}{r} e^{i\alpha r(1 - \cos \theta)} A_k a_k^* + (\lambda + 2\mu) \alpha \frac{e}{r} e^{i\alpha r(\alpha - \beta \cos \theta)} A_k b_k^* \right. \\ \left. + \mu\beta \frac{e}{r} e^{i\beta r(\beta - \alpha \cos \theta)} B_k a_k^* + \mu\beta \frac{e}{r} e^{i\beta r(1 - \cos \theta)} B_k b_k^* \right]. \quad (2.78)$$

We now consider an integral of the form $H = \int d\Omega e^{ikr(1 - \cos \theta)} h(\theta, \varphi)$ where r is large and $h(\theta, \varphi)$ is a smoothly varying function. Because r is large, small changes in θ (or k) cause the exponential to oscillate rapidly; hence, for $h(\theta, \varphi)$ smoothly varying, the integral will be nonzero only in a small cone of solid angle $\delta\Omega$ between $\theta = 0$ and $\theta = \epsilon$ (the forward direction). (See Fig. 4.) Thus, H equals

$$\int_0^{2\pi} d\varphi h(0, \varphi) \int_{1-\epsilon^2/2}^1 dx e^{ikr(1-x)}$$

where ϵ is very small. The right-hand side integral in the above equals

$$\frac{1 - \exp(ikr\epsilon^2/2)}{ikr},$$

and taking r large enough so that $kr\epsilon^2 \gg 1$, we can drop the oscillating term in the above. The result is that

$$H = \frac{2\pi i \tilde{h}(0)}{kr} \quad (2.79a)$$

where we define

$$\tilde{h}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi h(\theta, \varphi). \quad (2.79b)$$

Similar remarks apply to an integral of the form

$$H = \int d\Omega e^{ir(b - a \cos \theta)} h(\theta, \varphi).$$

In this case, however, the exponential oscillates rapidly except in a small solid angle center between $\theta_0 = \cos^{-1} b/a$ and $\cos^{-1} b/a + \epsilon$ (Fig. 5), assuming that $b/a < 1$. Thus,

$$H = \frac{2\pi i h(\theta_0)}{ar}, \quad b < a \quad (2.80a)$$

or

$$H = 0, \quad b > a. \quad (2.80b)$$

Applying (2.79) and (2.80) to (2.78) we have (after a bit of algebra)

$$\begin{aligned} \int d\Omega \hat{r}_i \sigma_{ij}^s u_j^{o*} = & -\frac{2\pi}{r^2} \left[(\lambda + 2\mu) \tilde{A}_k(0) a_k^* + (\lambda + 2\mu) \frac{\alpha}{\beta} \tilde{A}_k(\theta_0) b_k^* \right. \\ & \left. + \mu \tilde{B}_k(0) b_k^* \right] \end{aligned} \quad (2.81)$$

where $\theta_0 = \cos^{-1}(\alpha/\beta)$. Consequently, from (2.73) we can now write

$$\begin{aligned} \lim_{r \rightarrow \infty} r^2 \int d\Omega \hat{r}_i \langle I_1^s \rangle = & -2\pi i \text{Im} \left[(\lambda + 2\mu) \tilde{A}_k(0) a_k^* + \frac{\alpha}{\beta} \tilde{A}_k(\theta_0) b_k^* \right. \\ & \left. + \mu \tilde{B}_k(0) b_k^* \right]. \end{aligned}$$

Therefore, for an incident wave of the form of (2.64)

$$P(\omega) = 4\pi \text{Im} \left[\frac{(\lambda + 2\mu) \tilde{A}_k(0) a_k^* + \mu \tilde{B}_k(0) b_k^* + (\lambda + 2\mu) \tilde{A}_k(\theta_0) b_k^* (\alpha/\beta)}{\alpha(\lambda + 2\mu) |a_k|^2 + \beta\mu |b_k|^2} \right] \quad (2.82)$$

where $\theta_0 = \cos^{-1}(\alpha/\beta)$. The above expression relates the total cross section to the scattered amplitudes in the forward direction ($\theta = 0$) and an off-forward direction $\theta = \theta_0$. This second direction occurs only when the incident wave has no transverse component ($b_i = 0$). The presence of this second direction distinguishes the present statement of an optical theorem from the statement of the optical theorem in the theory of scattering in quantum mechanics. There the

total cross section is proportional to the imaginary part of the scattered cross section only in the forward direction. There is another difference between quantum mechanics and the present problem: The origin of the optical theorem in quantum mechanics is conservation of probability (particles); the origin of the optical theorem in the present case is conservation of energy.

There is a simple geometrical interpretation of the appearance of the off-forward term in optical theorem. In the steady-state at some time $t = 0$, wave fronts of transverse and longitudinal incident displacement plane waves impinge on the flaw. As a result, both a longitudinal and a transverse spherical wave for the stress field are radiated outward and in phase with the incident waves. At some time $t = t_0$, the transverse plane wave interferes with the transverse spherical wave in the forward or $\theta = 0$ direction. Likewise, there is interference between the longitudinal spherical and plane waves. However, interference also exists between the longitudinal spherical wave and transverse plane wave at the angle $\theta_0 = \cos^{-1}(v_T t_0 / v_L t_0) = \cos^{-1}(\alpha/\beta)$ as depicted in Fig. 6. The essential point is that A_1 and b_1 , although they represent different physical polarizations, are not mutually orthogonal.

The optical theorem has a simple physical interpretation. The total scattered intensity equals the amount by which the incident wave is depleted by the scattering. This depletion results from the interference of the incident wave with the scattered wave and is proportional to the scattered amplitude: the depletion of the incident wave is proportional to everything taken out of the incident wave at all angles of scattering.

The utility of the theorem is well known; instead of having to integrate $(dP/d\Omega)$ over all angles to compute $P(u)$, when the forward and θ_0 -scattered amplitudes can be computed, the total scattering then is given by (2.82).

III. THE BORN APPROXIMATION

A. Introduction

We proceed to discuss theoretical methods to describe the scattering by a single flaw of an elastic plane wave propagating in an infinite medium of density ρ and elastic stiffness C_{ijkl} . The scatterer we are considering is a finite, but otherwise arbitrary region of space, R_2 , within which the density is $\rho + \delta\rho$ and the elastic stiffness tensor is $C_{ijkl} + \delta C_{ijkl}$.

The conventional method of solution is the setting up of a differential equation for the displacement field (2.10). However, as we saw in Sec. II-B, we can also formulate the problem in terms of an equivalent integral equation (2.26):

$$u_i = u_i^0 + \delta\rho \omega^2 \int_{R_2} d\underline{r}' g_{ij}(\underline{r}-\underline{r}') u_j(\underline{r}') + \delta C_{jklm} \int_{R_2} d\underline{r}' g_{ij,k}(\underline{r}-\underline{r}') u_{l,m}(\underline{r}') \quad (3.1)$$

where u_i^0 is the incident wave; $\delta\rho$, δC_{ijkl} are defined in (2.21) and (2.22); $g_{ij}(\underline{r}-\underline{r}')$ is the Green's function of (2.47). The notation $f_{,k}$ stands for $\partial f / \partial x'_k$. Equation (3.1) is a general result, valid for an arbitrary region R_2 and for any $\delta\rho$ or δC_{ijkl} . The equation represents the displacement field as the sum of the incident wave u_i^0 , and a scattered wave u_i^s , which is defined by the integral appearing in (3.1). The field u_i^s itself is the sum of two terms: one involves the density variation $\delta\rho$; the other, the elastic stiffness variation δC_{ijkl} . We also notice that to calculate u_i^s at any point $\underline{r}$ we need to know completely the displacement field u_j inside the scattering region.

For the general case of an arbitrary shape $\delta\rho$, and δC_{ijkl} , no exact solution of (3.1) is available. Even in the far field limit, i.e., at points $\underline{r}$ far from the scatterer where the scattered field is relatively simple [see (2.58)], the solution is always in terms of factors like A_1 , B_1 which are defined by (2.51) and (2.55). However, an exact evaluation of these factors

requires complete knowledge of the displacement field inside the scatterer. No such solutions are available in general, so we shall necessarily consider approximate methods.

B. First Born Approximation

Before defining one possible approximation, we first discuss an exact iterative method of solution to (3.1). Equation (3.1) can be rewritten formally as

$$u_1 = u_1^0 + \int_{R_2} dV' G[u_1(\underline{r}'), u_{1,j}(\underline{r}')] \quad (3.2)$$

where $G[u_1(\underline{r}'), u_{1,j}(\underline{r}')] represents the complete integrand of (3.1). We construct an iterative solution in the following manner: we denote the solution after n iterations by $u^{(n)}$, and we initialize the iteration with $u^{(0)} = u^0$, i.e., the incident wave itself. After n iterations we have$

$$u_1^{(n+1)} = u_1^0 + \int_{R_2} dV' G[u_1^{(n)}(\underline{r}'), u_{1,j}^{(n)}(\underline{r}')] \quad (3.3)$$

If this iterative procedure converges, i.e., if $|u^{(n+1)} - u^{(n)}| \rightarrow 0$ as $n \rightarrow \infty$, we obviously get a solution to (3.1). The series generated in this manner is called the Born Series, and it is as intractable as the integral equation itself. However, if we perform only the first iteration, i.e., set $u_1 = u_1^0$ in the integral in (3.1), we are able to calculate easily the far field amplitudes A_i, B_i for an arbitrary shape, $\delta\rho$ and δC_{ijkl} . (However, in application we restrict δC and C to be isotropic.) Mathematically, the solution we get is only the first term of the infinite series (3.3). Physically, the approximation replaces the exact displacement + strain field inside the scatterer by the incident fields. This approximation is called the first Born approximation.

We calculated the scattered wave in first Born approximation for three cases:

- 1) an incident longitudinal plane wave
- 2) an incident transverse plane wave polarized in x-direction.
- 3) an incident transverse plane wave right circularly polarized.

The results for these three cases are as follows:

Case 1. Incident Longitudinal Wave

a) The scattered displacement field is given by

$$\begin{aligned} \frac{u^s(\underline{r})}{|\underline{a}|} = & \hat{r} \frac{\alpha^2 e^{i\alpha r}}{4\pi r} \left[\frac{\delta\rho}{\rho} \cos \theta - \frac{\delta\lambda + 2\delta\mu \cos^2 \theta}{\lambda + 2\mu} \right] S(\alpha, \alpha; \theta, \varphi) \\ & + \hat{\theta} \frac{\beta^2 e^{i\beta r}}{4\pi r} \left[\frac{2\alpha \delta\mu \cos \theta \sin \theta}{\mu\beta} - \frac{\delta\rho \sin \theta}{\rho} \right] S(\alpha, \beta; \theta, \varphi) \end{aligned} \quad (3.4)$$

where $\hat{r}$ and $\hat{\theta}$ are unit vectors in the r - and θ -directions. We recall that the incident wave propagates in the $+z$ -direction and that the origin of our coordinate system is in the scatterer. The scattered wave propagates radially outwards; it has a longitudinal component polarized in the r -direction that propagates with velocity $v_L = \omega/\alpha$. The scattered wave also has a transverse component polarized in the θ -direction and propagates with $v_T = \omega/\beta$. The functions $S(\alpha, \alpha; \theta, \varphi)$ and $S(\alpha, \beta; \theta, \varphi)$ are defined in the following section. They are shape factors, determined by the shape R_2 of the scatterer. Equation (3.4) has the structure of (2.58), and we can easily identify the $\hat{r}$ and $\hat{\theta}$ components of (3.4) with the Cartesian components A_1 and B_1 of (2.58) by using

$$\begin{aligned} \hat{r} &= \hat{x} \sin \theta \cos \varphi + \hat{y} \sin \theta \sin \varphi + \hat{z} \cos \theta \\ \hat{\theta} &= \hat{x} \cos \theta \cos \varphi + \hat{y} \cos \theta \sin \varphi - \hat{z} \sin \theta \\ \hat{\varphi} &= -\hat{x} \sin \varphi + \hat{y} \cos \varphi \end{aligned} \quad (3.5)$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ have the obvious meaning of unit vectors in the x , y , and z -directions.

b) From (2.58) and (3.4) it is easy to see that

$$A_1 = \frac{\alpha^2}{4\pi} \left[\frac{\delta\rho}{\rho} \cos \theta - \frac{\delta\lambda + 2\delta\mu \cos^2 \theta}{\lambda + 2\mu} \right] S(\alpha, \alpha; \theta, \varphi) \hat{r}_1 \quad (3.6)$$

and

$$E_1 = \frac{\beta^2}{4\pi} \left[\frac{\alpha \delta u}{\beta u} \sin 2\theta - \frac{\delta \rho}{\rho} \sin \theta \right] S(\alpha, \theta; \theta, \varphi) \hat{e}_1 \quad (3.7)$$

The scattered stress field (2.62) takes the form

$$\begin{aligned} \sigma_{ij} = & (\lambda \delta_{ij} + 2\mu \hat{r}_i \hat{r}_j) \frac{i\alpha^3}{4\pi} \frac{e^{i\alpha r}}{r} \left(-\frac{\delta\lambda + 2\delta\mu \cos^2 \theta}{\lambda + 2\mu} + \frac{\delta\rho}{\rho} \cos \theta \right) S(\alpha, \alpha; \theta, \varphi) \\ & + \mu \hat{r}_i \hat{r}_j (\delta_{13} \delta_{j3} - 2\cos^2 \theta) \frac{i\alpha\beta^2}{4\pi} \frac{e^{i\beta r}}{r} \left(-\frac{2\delta\mu}{\mu} + \frac{\beta\delta\rho}{\alpha\rho \cos \theta} \right) S(\alpha, \beta; \theta, \varphi) \end{aligned} \quad (3.8)$$

from which we get for the radial direction

$$\begin{aligned} \hat{r} \cdot \underline{\underline{\sigma}} = & \hat{r}(\lambda + 2\mu) \frac{i\alpha^3}{4\pi} \frac{e^{i\alpha r}}{r} \left(\frac{\delta\lambda + 2\delta\mu \cos^2 \theta}{\lambda + 2\mu} - \frac{\delta\rho}{\rho} \cos \theta \right) S(\alpha, \beta; \theta, \varphi) \\ & - \theta \mu \sin \theta \cos \theta \frac{i\alpha\beta^2}{4\pi} \frac{e^{i\beta r}}{r} \left(\frac{2\delta\mu}{\mu} - \frac{\beta\delta\rho}{\alpha\rho \cos \theta} \right) S(\alpha, \beta; \theta, \varphi) . \end{aligned} \quad (3.9)$$

c) The scattering cross section, as specified by (2.9) is given by (for $|\vec{a}| = 1$)

$$\begin{aligned} \frac{dP}{d\Omega} = & \alpha^4 \left[\frac{\delta\rho \cos \theta}{\rho} - \frac{\delta\lambda + 2\delta\mu \cos^2 \theta}{\lambda + 2\mu} \right]^2 |S(\alpha, \alpha; \theta, \varphi)|^2 \\ & + \alpha\beta^3 \left[\frac{2\alpha\delta\mu \cos \theta \sin \theta}{\mu} - \frac{\delta\rho \sin \theta}{\rho} \right]^2 |S(\alpha, \beta; \theta, \varphi)|^2 \end{aligned} \quad (3.10)$$

Case 2. Incident Transverse Wave Polarized in $\hat{x}$ -Direction

a) For this case the displacement field is given by

$$\begin{aligned} \frac{\underline{u}}{|\underline{b}|} = & \frac{e^{i\beta r}}{4\pi r} \beta^2 \left\{ \left[-\frac{\delta\rho \sin \varphi}{\rho} + \frac{\delta\mu \sin \varphi \cos \theta}{\mu} \right] \hat{\varphi} \right. \\ & + \left[\frac{\delta\rho \cos \theta \cos \varphi}{\rho} - \frac{\delta\mu \cos 2\theta \cos \varphi}{\mu} \right] \hat{\theta} \left. \right\} S(\beta, \beta; \theta, \varphi) \\ & + \frac{e^{i\alpha r}}{4\pi r} \alpha^2 \left[\frac{\delta\rho \sin \theta \cos \varphi}{\rho} - \frac{\delta\mu}{\lambda + 2\mu} \frac{\beta}{\alpha} \sin \theta \cos \varphi \right] \hat{r} S(\beta, \alpha; \theta, \varphi) . \end{aligned} \quad (3.11)$$

This expression has the form of (2.58); the A_i and B_i are given by

$$A_i = \frac{\alpha^2}{4\pi} \left[\frac{\delta\rho \sin \theta \cos \varphi}{\rho} - \frac{\delta\mu}{\lambda + 2\mu} \frac{\beta}{\alpha} \sin \theta \cos \varphi \right] S(\beta, \alpha; \theta, \varphi) \hat{r}_i \quad (3.12)$$

$$B_i = \frac{\beta^2}{4\pi} \left\{ \left[-\frac{\delta\rho \sin \varphi}{\rho} + \frac{\delta\mu \sin \varphi \cos \theta}{\mu} \right] \hat{\varphi}_i + \left[\frac{\delta\rho \cos \theta \cos \varphi}{\rho} - \frac{\delta\mu \cos 2\theta \cos \varphi}{\mu} \right] \hat{\theta}_i \right\} S(\beta, \beta; \theta, \varphi) \quad (3.13)$$

where $\hat{r}_i$, $\hat{\theta}_i$, and $\hat{\varphi}_i$ are given by (3.5). The scattered stress field can be easily expressed using (2.62) and A_i , B_i from (3.10) and (3.11). Instead of presenting σ_{ij} , we show the following more useful quantity:

$$\begin{aligned} \hat{r} \cdot \underline{\underline{\sigma}} = & \frac{i\alpha^3}{4\pi} \frac{e^{i\alpha r}}{r} (\lambda + 2\mu) \left[\frac{\delta\rho \sin \theta \cos \varphi}{\rho} - \frac{\delta\mu}{\lambda + 2\mu} \frac{\beta}{\alpha} \sin \theta \cos \varphi \right] S(\beta, \alpha; \theta, \varphi) \hat{r} \\ & + \frac{i\beta^3}{4\pi} \frac{e^{i\beta r}}{r} \mu \left\{ \left[-\frac{\delta\rho \sin \varphi}{\rho} + \frac{\delta\mu \sin \varphi \cos \theta}{\mu} \right] \hat{\varphi} \right. \\ & \left. + \left[\frac{\delta\rho \cos \theta \cos \varphi}{\rho} - \frac{\delta\mu \cos 2\theta \cos \varphi}{\mu} \right] \hat{\theta} \right\} S(\beta, \beta; \theta, \varphi) \end{aligned} \quad (3.14)$$

b) Scattering Cross Section

Using (2.69), (3.10), and (3.11), we have

$$\frac{dP(\omega)}{d\Omega} = \frac{\alpha}{\beta} \frac{\lambda + 2\mu}{\mu} \frac{|A_1|^2}{|b_1|^2} + \frac{|B_1|^2}{|b_1|^2} = \frac{\beta}{\alpha} \frac{|A_1|^2}{|b_1|^2} + \frac{|B_1|^2}{|b_1|^2}$$

in agreement with (2.69). For $|b_1| = 1$ we have

$$\begin{aligned} \frac{dP(\omega)}{d\Omega} = & \beta\alpha^3 \left[\frac{\delta\rho \sin \theta \cos \varphi}{\rho} - \frac{\delta\mu}{\lambda + 2\mu} \frac{\beta}{\alpha} \sin \theta \cos \varphi \right]^2 |S(\beta, \alpha; \theta, \varphi)|^2 \\ & + \beta^4 \left\{ \left[-\frac{\delta\rho \sin \varphi}{\rho} + \frac{\delta\mu \sin \varphi \cos \theta}{\mu} \right]^2 \right. \\ & \left. + \left[\frac{\delta\rho \cos \theta \cos \varphi}{\rho} - \frac{\delta\mu \cos 2\theta \cos \varphi}{\mu} \right]^2 \right\} |S_1(\beta, \beta; \theta, \varphi)|^2 \end{aligned} \quad (3.15)$$

Case 3. Incident transverse plane wave, right circularly polarized.

(a) The incident displacement field has the form

$$\underline{u}^0 = \frac{u}{\sqrt{2}}(\hat{x} + i\hat{y}) e^{i(\beta z - \omega t)} \quad (3.16)$$

The scattered field is expressed in terms of the local polarization vectors

$$\begin{aligned} \hat{r}, \\ \hat{x}^+ = \frac{1}{\sqrt{2}}(\hat{\theta} + i\hat{\phi}), \\ \hat{x}^- = \frac{1}{\sqrt{2}}(\hat{\theta} - i\hat{\phi}) \end{aligned} \quad (3.17)$$

$$\begin{aligned} \frac{u^s}{u} = & \frac{\beta^2 e^{i\beta r}}{4\pi r} e^{i\varphi} \left\{ \left[\frac{\delta\rho}{\rho} \frac{1 + \cos\theta}{2} - \frac{\delta\mu}{\mu} \frac{\cos\theta + \cos 2\theta}{2} \right] \hat{x}^+ \right. \\ & + \left. \left[\frac{\delta\rho}{\rho} \frac{\cos\theta - 1}{2} + \frac{\delta\mu}{\mu} \frac{\cos\theta - \cos 2\theta}{2} \right] \hat{x}^- \right\} S(\beta, \beta; \theta, \varphi) \\ & + \frac{\alpha^2 e^{i\alpha r}}{4\pi r} e^{i\varphi} \left[\frac{\delta\rho}{\rho} \sin\theta - \frac{\beta}{\alpha} \frac{\delta\mu}{\lambda + 2\mu} \sin 2\theta \right] \hat{r} S(\beta, \alpha; \theta, \varphi) \end{aligned} \quad (3.18)$$

This again has the form of equation (2.58), with

$$A_i = \frac{\alpha^2}{4\pi r} e^{i\varphi} \left[\frac{\delta\rho}{\rho} \sin\theta - \frac{\beta}{\alpha} \frac{\delta\mu}{\lambda + 2\mu} \sin 2\theta \right] S(\beta, \alpha; \theta, \varphi) \hat{r}_i \quad (3.19)$$

$$\begin{aligned} B_i = & \frac{\beta^2}{4\pi r} e^{i\varphi} \left\{ \left[\frac{\delta\rho}{\rho} \frac{1 + \cos\theta}{2} - \frac{\delta\mu}{\mu} \frac{\cos\theta + \cos 2\theta}{2} \right] \hat{x}_i^+ \right. \\ & + \left. \left[\frac{\delta\rho}{\rho} \frac{\cos\theta - 1}{2} + \frac{\delta\mu}{\mu} \frac{\cos\theta - \cos 2\theta}{2} \right] \hat{x}_i^- \right\} S(\beta, \beta; \theta, \varphi) \end{aligned} \quad (3.20)$$

where $\hat{r}_i$, $\hat{x}_i^+$ are given by equations (3.17) and (3.5)

(b) The scattered stress field can be found from equation (2.62) and A_i , B_i .

We never use it so we will not list it.

(c) The scattering cross-sections are given by

$$\frac{dP_L(\omega)}{d\Omega} = \alpha^4 |S(\beta, \alpha; \theta, \varphi)|^2 \left[\frac{\delta\rho}{\rho} \sin\theta - \frac{\beta}{\alpha} \frac{\delta u}{\lambda + 2u} \sin 2\theta \right]^2 \quad (3.21)$$

$$\frac{dP^+(\omega)}{d\Omega} = \beta^4 |S(\beta, \beta; \theta, \varphi)|^2 \left[\frac{\delta\rho}{\rho} \frac{1 + \cos\theta}{2} - \frac{\delta u}{u} \frac{\cos\theta + \cos 2\theta}{2} \right]^2 \quad (3.22)$$

$$\frac{dP^-(\omega)}{d\Omega} = \beta^4 |S(\beta, \beta; \theta, \varphi)|^2 \left[\frac{\delta\rho}{\rho} \frac{\cos\theta - 1}{2} + \frac{\delta u}{u} \frac{\cos\theta - \cos 2\theta}{2} \right]^2 \quad (3.23)$$

and the complete differential cross section is

$$\frac{dP(\omega)}{d\Omega} = \frac{\beta}{\alpha} \frac{dP_L(\omega)}{d\Omega} + \frac{dP^+(\omega)}{d\Omega} + \frac{dP^-(\omega)}{d\Omega} \quad (3.24)$$

We call $\frac{dP^+}{d\Omega}$ the right circularly polarized (transverse) differential cross-section; $\frac{dP^-}{d\Omega}$ is the corresponding quantity for the left-handed polarization.

C. The Shape Factor

The scattering caused by any defect depends not only on the elastic constants and density of the flow, but also on its shape. The role of the change in λ , μ , ρ is clearly seen in (3.9-3.13). However, one of the main attractions of the first Born approximation is that even the most complicated shapes can be handled with relative ease. Shape dependence enters through the various shape factors $S(\gamma, \gamma'; \theta, \varphi)$. In these shape factors, γ, γ' are either α or β . Such a term represents type γ' scattered waves, when the incident wave is type γ . Assuming that the incident wave propagates in the $+z$ -direction, we consider the scattering in the direction $\hat{r}$ defined by the angles θ, φ ; we have

$$S(\gamma, \gamma'; \theta, \varphi) = \int_{R_2} dV' e^{i(\gamma^2 - \gamma'^2) \cdot \underline{r}'} \quad (3.25)$$

Denoting $\gamma^2 - \gamma'^2 = \underline{\Delta k}$ (obviously, $\underline{\Delta k}$ depends on θ), we get for various shapes the following results:

1) For a sphere of radius a :

$$S = 4\pi a^3 \frac{\sin(a\Delta k) - a\Delta k \cos(a\Delta k)}{(a\Delta k)^3} \quad (3.26)$$

where $\Delta k = |\underline{\Delta k}|$.

2) For a cylinder of radius b , length $2a$, and cylinder axis aligned along the z -axis:

$$S = 4\pi a b^2 \frac{\sin a(\Delta k)_z}{a(\Delta k)_z} \frac{J_1(b(\Delta k)_x)}{b(\Delta k)_x} \quad (3.27)$$

where $J_1(x)$ is the Bessel function of order 1, and $(\Delta k)_x, (\Delta k)_z$ are the projections of the vector $\underline{\Delta k}$ onto the x and z axes, respectively.

(3) For an ellipsoid with axes $(a_1, a_2, a_3) = \vec{a}$

$$S = 4\pi a_1 a_2 a_3 \frac{\sin y - y \cos y}{y^3} \quad (3.28)$$

where $y = [\Delta k_1^2 a_1^2 + \Delta k_2^2 a_2^2 + \Delta k_3^2 a_3^2]^{\frac{1}{2}}$; Δk_1 is the projection of $\Delta \underline{k}$ onto the $\underline{a}_1$ axis.

D. Results

Having formulated the Born approximation, we next investigate its utility. To this end, we considered the scattering of longitudinal and transverse plane waves from two types of spherical flows: a cavity and an elastic inclusion. Both cases are exactly solvable (see Appendices A and B), and we use these exact solutions to test the Born Approximation.

For an incident longitudinal plane wave the scattered wave has both a longitudinal and a transverse component which is a function of qa and θ . These components make independent contributions to the scattering cross-sections (2.70). For the differential cross-section we computed each separate cross-section as well as the complete differential cross-section (2.68). We also computed the total cross-section.

For an incident transverse plane wave the scattered wave also has both longitudinal and transverse components. These components, in addition to being a function of βa and θ , are also a function of φ . Because of the elastic and geometrical isotropic symmetry, the φ -dependence can be eliminated by decomposing the transverse contribution to the scattering into two independent contributions from right and left circularly polarized scattering. As a consequence, we computed three components - longitudinal, right circularly polarized, and left circularly polarized - of the differential cross-section, in addition to the complete differential cross-section and the total cross-section.

Besides the elimination of one dependent variable, the decomposition of the transverse scattering has a practical advantage: we are able to make three dimensional plots of all cross-sections. We plot the respective cross-sections on the vertical axis as a function of qa (or βa) and the scattering angle θ . The angle $\theta = 0^\circ$ is the forward scattering direction; the angle $\theta = 180^\circ$, the

back scattering direction. Each graph of the differential cross-section is normalized: the maximum height of each surface is the same. Thus, only the shapes of the surfaces are relevant.

The cases studied are summarized in Table I, and the results are presented in Figs. (3.1-10). The densities and Lamé parameters of the materials investigated are listed in Table II. We first present the results for the incident longitudinal plane wave and then results for the incident transverse plane wave.

Table I
Summary of Calculations Performed

	Longitudinal		Transverse *	
	Exact	Born	Exact	Born
Cavity in Ti	x	x	x	x
Cavity in Al	x	x	x	
Cavity in SS	x	x	x	
Al sphere in Ti	x	x	x	x
SS sphere in Ti	x	x	x	x

SS - stainless steel

x - performed

Table II
Material Parameters

	ρ^a	λ^b	μ^b
Ti	4.50	7.76	4.41
Al	2.70	6.30	2.68
SS	7.90	10.48	7.59

a g/cm^3

b 10^{11} dyn/cm^2

* right circularly polarized

1. Incident longitudinal plane wave

L1) Results for the scattering from a spherical cavity in T_1 are presented in Figures (3.1a). The overall agreement between the Born Approximation and the exact solutions is rather poor; however, this lack of agreement is traceable to an inadequate description in the Born Approximation of the very strong scattering in the forward direction for large qa . In Figures (3.1b) is a comparison of the exact and approximate theories in a region of the wavenumber-angle plane that excludes large qa and the forward directions. The agreement between the calculations is reasonably good for $qa \leq 1$ and $90^\circ < \theta < 180^\circ$.

That the Born Approximation should give poor results in the forward direction is expected on purely formal grounds. From Equations (3.10), (3.11), and (3.15) it is apparent that the scattered amplitudes are real numbers. The optical theorem [Eq. (2.82)] on the other hand implies that the total cross-section should be zero for all qa .

L2) Results for the scattering by spherical cavities in Al and stainless steel are presented in Figures (3.2) and (3.3). These results are similar to those for Ti; therefore, we expect the same range of validity of the Born Approximation for them as for Ti.

L3) Results for the scattering from an Al and a stainless steel sphere in Ti are shown in Figures (3.4) and (3.5). The Born Approximation and the exact calculation agree well over the entire range of qa and θ . One expects better agreement here than in the cases of scattering from cavities (L1 and L2) since the scattering is from a defect whose properties are closer to those of the host material. The scattering thus is weaker and consequently is more amenable to perturbation analysis. However, the good agreement even for large qa and in the forward directions is surprising.

2. Incident transverse plane wave (right circularly polarized)

T1) Results for the scattering from a spherical cavity in T1 are presented in Figure (3.6). Again, as in the case for an incident longitudinal plane wave (L1), the overall agreement is poor, but for $0 \leq \beta a \leq 1/2$ and $90^\circ \leq \theta \leq 180^\circ$ the agreement is satisfactory [See Figures (3.6f-i)]. We note that the exact total cross-section lacks the peaky structure present in case L1.

T2) Exact results for the scattering by spherical cavities in Al and stainless steel are presented in Figures (3.7) and (3.8). These results are similar to those for Ti; therefore, we expect the same range of validity of the Born Approximation for them as for Ti.

T3) Results for the scattering from Al and stainless steel spheres in Ti are presented in Figures (3.9) and (3.10). The agreement between the Born approximation and exact theory seems better than for the cavity (cases T1 and T2), but the agreement is not as good as for the case L3, involving the scattering of a longitudinal wave from an Al sphere in Ti. We note however, that the Born Approximation for the present cases is strikingly similar to the Born Approximation for the cavity (T1). That this should be is easily seen from Equation (3.13): the scattering is independent of the λ of the scatter but depends on the ratios $\delta\rho/\rho$ and $\delta\mu/\mu$. For the cavity these ratios are obviously -1 and because of the equality of the ratios the cross-sections are proportional to the square of -1. For the Al and stainless steel spheres these ratios are approximately equal as one can see from Table II, and as for the cavity the cross-sections are proportional to the square of this ratio. Thus the shapes, not the magnitudes, of the cross-section surfaces are expected to look quite similar.

E. Conclusions

We conclude that the Born Approximation gives a useful description of the scattering of a longitudinally transverse polarized plane wave from a spherical cavity or an elastic inclusion at least when $qa \leq 1$ and when the scattering angle is restricted to backward directions. The principal defect of the Born Approximation is an inadequate description of forward scattering for $qa > 1$. More refined approximations for this range of qa are needed.

We believe that the Born Approximation can be useful in some NDE applications, especially ones operating in a reflection mode. Although at present our statement is supported by calculations for the case of a sphere, we believe our conclusions will remain firm for non-spherical flaw geometries when $ql < 1$ where l is some maximum, but characteristic, length of the flaw. One of the most attractive features of the Born Approximation is the case of the inclusion of non-spherical geometries: All that is required is a different shape factor which is easily evaluated for many relevant shapes.

Studies of scattering by spheroidal and cylindrical defects indicate some interesting features that may prove helpful in NDE. Detailed description of the results for various geometries will be presented in the next report. Here we will mention two such features:

1. For a scatterer (both spheroidal and cylindrical) of radius b and width, $2a$, with the incident wave normal to the circular cross-section, we observe that the ratio $R = P_{180}/P_{90}$ of scattered power backwards to that in the 90° direction depends strongly on the ratio b/a . If b/a is large, R is small.

2. Back-scattering seems to contain information that may be useful in distinguishing cylindrical from spheroidal scatterers. While for the spheroid the average (over a range of frequencies) of the back-scattered power remains nearly constant with increasing frequency, we get increased back scattered power with increasing frequency for the cylindrical scatterer.

APPENDIX A: EXACT SCATTERING OF TRANSVERSE ELASTIC WAVES BY A SPHERICAL FLAW

We consider the problem of the scattering of transverse elastic waves by a spherical flaw. The flaw is either a spherical cavity or an elastic sphere whose properties differ from those of the host medium.

This problem was treated by Einspruch, Witterholt and Truell.¹ In their paper, exact equations for the scattered displacement field were derived. In addition, an expression was obtained for the total scattering cross-section in terms of the solution for the scattered field; this expression was also evaluated in the Rayleigh limit. Using the exact equations for the scattered field derived by Einspruch et. al., Kraft and Franzblau² computed the total scattering cross-section for scattering by a spherical cavity and McBride and Kraft³ computed the total scattering cross-section for scattering by an elastic sphere. Independent of the above work, Knopoff⁴ treated the scattering of transverse elastic waves by a fixed rigid sphere. Exact equations for the scattered field were derived, and the scattered field was computed in the far-field region.

We now consider the scattering of an incident monochromatic transverse elastic wave by a spherical obstacle. To solve the problem, we introduce the solutions of the vector Helmholtz equation.⁵ We define the vector operator $\underline{L}$, which is proportional to the angular momentum operator in quantum mechanics, by

$$\underline{L} = \underline{r} \times \frac{1}{i} \underline{\nabla} \quad (\text{A.1})$$

and the vector spherical harmonics by

$$\underline{X}_{\ell,m} = \frac{1}{\sqrt{\ell(\ell+1)}} \underline{L} Y_{\ell,m} \quad (\text{A.2})$$

where $Y_{\ell,m}(\theta,\varphi)$ is a spherical harmonic. The vector spherical harmonics are defined for $\ell \geq 1$ and $|m| \leq \ell$. Letting z_ℓ denote any linear combination of spherical Bessel functions of order ℓ , we can show that the solutions of the vector Helmholtz equation for transverse waves are

$$z_\ell(\beta r) \underline{\mathbf{X}}_{\ell,m} \quad (\text{A.3})$$

and

$$\frac{1}{\beta} \nabla \times (z_\ell(\beta r) \underline{\mathbf{X}}_{\ell,m}) \quad (\text{A.4})$$

where $\ell \geq 1$ and $|m| \leq \ell$. Here β is the wave number for the propagation of transverse waves of angular frequency ω , i.e. $\beta = \omega/v_T$, where v_T is the transverse wave velocity. Similarly it can be shown that the solutions of the vector Helmholtz equation for longitudinal waves are

$$\frac{1}{\alpha} \nabla (z_\ell(\alpha r) Y_{\ell,m}) \quad (\text{A.5})$$

where $\ell \geq 0$ and $|m| \leq \ell$. Here α is the wave number for the propagation of longitudinal waves of angular frequency ω , i.e. $\alpha = \omega/v_L$, where v_L is the longitudinal wave velocity. The above solutions of the vector Helmholtz equation form a complete set; they are the appropriate solutions for a problem in which the boundary conditions are to be applied on a spherical surface.

Let the incident wave be a circularly polarized plane transverse elastic wave of angular frequency ω , propagating in the $+z$ - direction:

$$\underline{\mathbf{u}}^0 = (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) e^{i\beta z} \quad (\text{A.6})$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors in the x - and y - directions. The helicity of the incident wave may be positive or negative, depending on the sign chosen. (The

helicity of the wave describes the sense of the polarization. A right circularly polarized wave has positive helicity; a left circularly polarized wave has negative helicity.) The time dependence of the wave, which is given by a factor of $e^{-i\omega t}$, is omitted.

The incident wave may be expanded in solutions of the vector Helmholtz equation⁵:

$$\underline{u}^0 = \sum_{\ell} i^{\ell} \sqrt{4\pi(2\ell+1)} \left[j_{\ell}(\beta r) \underline{x}_{\ell,+1} \pm \frac{1}{\beta} \nabla \times (j_{\ell}(\beta r) \underline{x}_{\ell,+1}) \right] \quad (\text{A.7})$$

Here the sum over ℓ is from $\ell = 1$ to $\ell = \infty$; this will apply throughout this appendix unless explicitly indicated otherwise.

We now expand the scattered wave in solutions of the vector Helmholtz equation. For scattering by a spherical obstacle, it is sufficient to write

$$\begin{aligned} \underline{u}^s = \sum_{\ell} i^{\ell} \sqrt{4\pi(2\ell+1)} & \left[a_{\ell} h_{\ell}^{(o)}(\beta r) \underline{x}_{\ell,+1} \pm b_{\ell} \frac{1}{\beta} \nabla \times (h_{\ell}^{(o)}(\beta r) \underline{x}_{\ell,+1}) \right. \\ & \left. \pm c_{\ell} \frac{1}{\alpha} \nabla (h_{\ell}^{(o)}(\alpha r) y_{\ell,+1}) \right] \quad (\text{A.8}) \end{aligned}$$

For the case of scattering by an elastic sphere, there will also be a wave excited within the sphere by the incident wave. We put

$$\begin{aligned} \underline{u}^a = \sum_{\ell} i^{\ell} \sqrt{4\pi(2\ell+1)} & \left[d_{\ell} j_{\ell}(\beta' r) \underline{x}_{\ell,+1} \pm e_{\ell} \frac{1}{\beta'} \nabla \times (j_{\ell}(\beta' r) \underline{x}_{\ell,+1}) \right. \\ & \left. \pm f_{\ell} \frac{1}{\alpha'} \nabla (j_{\ell}(\alpha' r) y_{\ell,+1}) \right] \quad (\text{A.9}) \end{aligned}$$

Here, α' and β' denote respectively the longitudinal and transverse wave numbers within the imbedded elastic sphere.

We need to solve for the coefficients, a_{ℓ} , b_{ℓ} , and c_{ℓ} , of the scattered wave. For the case of scattering by a spherical cavity, the boundary condition is that the force exerted on the surface of the cavity vanish. The following

equations are obtained:

$$a_\ell = \frac{\beta_1}{\alpha_1} \quad (\text{A.10})$$

with

$$\begin{aligned} \alpha_1 &= + \frac{d}{dx} \left(\frac{1}{x} h_\ell^{(i)}(x) \right) \\ \beta_1 &= - \frac{d}{dx} \left(\frac{1}{x} j_\ell(x) \right) \end{aligned} \quad (\text{A.11})$$

and

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} b_\ell \\ c_\ell \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \quad (\text{A.12})$$

with

$$\begin{aligned} \gamma_{11} &= i \sqrt{\ell(\ell+1)} \times \frac{d}{dx} \left(\frac{1}{x} h_\ell^{(i)}(x) \right) \\ \gamma_{12} &= y \frac{d^2 h_\ell^{(i)}(y)}{dy^2} + y h_\ell^{(i)}(y) - \frac{x^2}{2y} h_\ell^{(i)}(y) \\ \delta_1 &= -i \sqrt{\ell(\ell+1)} \times \frac{d}{dx} \left(\frac{1}{x} j_\ell(x) \right) \\ \gamma_{21} &= x \frac{d}{dx} \left(\frac{1}{x} \frac{d}{dx} (x h_\ell^{(i)}(x)) \right) + \frac{x}{2} h_\ell^{(i)}(x) \\ \gamma_{22} &= -i \sqrt{\ell(\ell+1)} y \frac{d}{dy} \left(\frac{1}{y} h_\ell^{(i)}(y) \right) \\ \delta_2 &= - \left[x \frac{d}{dx} \left(\frac{1}{x} \frac{d}{dx} (x j_\ell(x)) \right) + \frac{x}{2} j_\ell(x) \right] \end{aligned} \quad (\text{A.13})$$

Here $x = \beta a$, $y = \alpha a$, with a = the radius of the spherical cavity. These equations can be shown to agree with the equations derived by Einspruch et. al..

The Rayleigh limit occurs when the wavelength of the incident wave is much

greater than the size of the scatterer; in other words, $x \ll 1$. In the Rayleigh limit it can be shown that the solutions for a_ℓ , b_ℓ , and c_ℓ are

$$\begin{aligned}
 a_1 &\rightarrow -i \frac{x^5}{45} \\
 a_\ell &\xrightarrow{\ell \geq 2} i \cdot \frac{1}{(2\ell+1)!!(2\ell-1)!!} \cdot \frac{(\ell-1)}{(\ell+2)} \cdot x^{2\ell+1} \\
 b_1 &\rightarrow -i \frac{2}{9} x^3 \\
 b_\ell &\xrightarrow{\ell \geq 2} i \cdot \frac{1}{(2\ell-1)!!(2\ell-3)!!} \cdot \frac{2(\ell+1)(\ell-1)}{[(2\ell^2+1)-2\ell(\ell-1) \cdot (\frac{y}{x})^2]} \cdot x^{2\ell-1} \\
 c_1 &\rightarrow \frac{\sqrt{2}}{9} \cdot \left(\frac{y}{x}\right)^3 \cdot x^3 \\
 c_\ell &\xrightarrow{\ell \geq 2} \sqrt{\ell(\ell+1)} \cdot \frac{1}{(2\ell-1)!!(2\ell-3)!!} \cdot \frac{2(\ell-1)}{[(2\ell^2+1)-2\ell(\ell-1) \cdot (\frac{y}{x})^2]} \\
 &\quad \cdot \left(\frac{y}{x}\right)^{\ell+2} \cdot x^{2\ell-1}
 \end{aligned} \tag{A.14}$$

Here the double factorial sign is defined by $(2\ell+1)!! \equiv (2\ell+1)(2\ell-1)(2\ell-3)\dots 1$ and $(-1)!! \equiv 1$.

For the case of scattering by an elastic sphere, the boundary conditions are that the displacement at the interface and the force acting across the interface are continuous. The equations for a_ℓ , b_ℓ , and c_ℓ are:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} a_\ell \\ d_\ell \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \tag{A.15}$$

with

$$\alpha_{11} = h_\ell(x)$$

$$\alpha_{12} = -j_\ell(x)$$

$$\beta_1 = -j_\ell(x)$$

$$\alpha_{21} = x^2 \frac{d}{dx} \left(\frac{1}{x} h_\ell^{(i)}(x) \right)$$

$$\alpha_{22} = -\frac{\mu'}{\mu} x'^2 \frac{d}{dx'} \left(\frac{1}{x'} j_\ell(x') \right)$$

$$\beta_2 = -x^2 \frac{d}{dx} \left(\frac{1}{x} j_\ell(x) \right)$$

(A.16)

and

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix} \begin{pmatrix} b_\ell \\ c_\ell \\ e_\ell \\ f_\ell \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} \quad (\text{A.17})$$

with

$$Y_{11} = \frac{1}{x} \frac{d}{dx} \left(x h_\ell^{(i)}(x) \right)$$

$$Y_{12} = -i \sqrt{\ell(\ell+1)} \frac{1}{y} h_\ell^{(i)}(y)$$

$$Y_{13} = -\frac{1}{x'} \frac{d}{dx'} \left(x' j_\ell(x') \right)$$

$$Y_{14} = i \sqrt{\ell(\ell+1)} \frac{1}{y'} j_\ell(y')$$

$$\delta_1 = -\frac{1}{x} \frac{d}{dx} \left(x j_\ell(x) \right)$$

$$Y_{21} = i \sqrt{\ell(\ell+1)} \frac{1}{x} h_\ell^{(i)}(x)$$

$$Y_{22} = \frac{d}{dy} h_\ell^{(i)}(y)$$

$$\gamma_{23} = -i\sqrt{\ell(\ell+1)} \frac{1}{x} j_{\ell}(x')$$

$$\gamma_{24} = -\frac{d}{dy'} j_{\ell}(y')$$

$$\delta_2 = -i\sqrt{\ell(\ell+1)} \frac{1}{x} j_{\ell}(x)$$

$$\gamma_{31} = x \frac{d}{dx} \left(\frac{1}{x} \frac{d}{dx} (x h_{\ell}^{(1)}(x)) \right) + \frac{1}{2} x h_{\ell}^{(1)}(x)$$

$$\gamma_{32} = -i\sqrt{\ell(\ell+1)} y \frac{d}{dy} \left(\frac{1}{y} h_{\ell}^{(1)}(y) \right)$$

$$\gamma_{33} = -\frac{\mu'}{\mu} \left[x' \frac{d}{dx'} \left(\frac{1}{x'} \frac{d}{dx'} (x' j_{\ell}(x')) \right) + \frac{1}{2} x' j_{\ell}(x') \right]$$

$$\gamma_{34} = \frac{\mu'}{\mu} \left[i\sqrt{\ell(\ell+1)} y' \frac{d}{dy'} \left(\frac{1}{y'} j_{\ell}(y') \right) \right]$$

$$\delta_3 = - \left[x \frac{d}{dx} \left(\frac{1}{x} \frac{d}{dx} (x j_{\ell}(x)) \right) + \frac{1}{2} x j_{\ell}(x) \right]$$

$$\gamma_{41} = i\sqrt{\ell(\ell+1)} x \frac{d}{dx} \left(\frac{1}{x} h_{\ell}^{(1)}(x) \right)$$

$$\gamma_{42} = y \frac{d^2}{dy^2} h_{\ell}^{(1)}(y) + y h_{\ell}^{(1)}(y) - \frac{x^2}{2y} h_{\ell}^{(1)}(y)$$

$$\gamma_{43} = -\frac{\mu'}{\mu} \left[i\sqrt{\ell(\ell+1)} x' \frac{d}{dx'} \left(\frac{1}{x'} j_{\ell}(x') \right) \right]$$

$$\gamma_{44} = -\frac{\mu'}{\mu} \left[y' \frac{d^2}{dy'^2} j_{\ell}(y') + y' j_{\ell}(y') - \frac{x'^2}{2y'} j_{\ell}(y') \right]$$

$$\delta_4 = -i\sqrt{\ell(\ell+1)} x \frac{d}{dx} \left(\frac{1}{x} j_{\ell}(x) \right) \quad (A.18)$$

In the above equations, $x = \beta a$, $y = \alpha a$, $x' = \beta' a$, and $y' = \alpha' a$, with $a =$ the radius of the elastic sphere. These equations can be compared with the equations derived by Einspruch et. al.. Our results

differ from theirs in the two matrix elements which Einspruch et. al. denote as E_{11} and E_{12} . Using their notation, we find that $E_{11} = A_1(a)$ and $E_{12} = A_2(a)$. This mistake was carried through by McBride and Kraft. In addition, there is a misprint in the equations derived by Einspruch et. al. in the matrix element denoted by E_{33} ; this error was pointed out and corrected by McBride and Kraft.

In the Rayleigh limit, we obtain

$$\begin{aligned}
 a_1 &\longrightarrow -i \cdot \frac{1}{45} \left[1 - \frac{\mu'}{\mu} \left(\frac{x'}{x} \right)^2 \right] x^5 \\
 a_l \xrightarrow{l \geq 2} & i \left(1 - \frac{\mu'}{\mu} \right) \cdot \frac{1}{(2l+1)!!(2l-1)!!} \cdot \frac{(l-1)}{\left[(l+2) + \frac{\mu'}{\mu} (l-1) \right]} \cdot x^{2l+1} \\
 b_1 &\longrightarrow -i \cdot \frac{2}{9} \cdot \left[1 - \frac{\mu'}{\mu} \left(\frac{x'}{x} \right)^2 \right] x^3 \\
 b_l \xrightarrow{l \geq 2} & i \cdot \left(1 - \frac{\mu'}{\mu} \right) \cdot \frac{1}{(2l-1)!!(2l-3)!!} \cdot \\
 & \cdot \frac{2(l+1)(l-1)}{\left\{ \left[(2l^2+1) + \frac{\mu'}{\mu} 2(l+1)(l-1) \right] - 2l(l-1) \left(1 - \frac{\mu'}{\mu} \right) \left(\frac{y}{x} \right)^2 \right\}} \\
 & \cdot x^{2l-1} \\
 c_1 &\longrightarrow \frac{\sqrt{2}}{9} \left[1 - \frac{\mu'}{\mu} \left(\frac{x'}{x} \right)^2 \right] \left(\frac{y}{x} \right)^3 x^3 \\
 c_l \xrightarrow{l \geq 2} & - \left(1 - \frac{\mu'}{\mu} \right) \sqrt{l(l+1)} \cdot \frac{1}{(2l-1)!!(2l-3)!!} \cdot \\
 & \cdot \frac{2(l-1)}{\left\{ \left[(2l^2+1) + \frac{\mu'}{\mu} 2(l+1)(l-1) \right] - 2l(l-1) \left(1 - \frac{\mu'}{\mu} \right) \left(\frac{y}{x} \right)^2 \right\}} \\
 & \cdot \left(\frac{y}{x} \right)^{l+2} \cdot x^{2l-1}
 \end{aligned}$$

(A.19)

It can be shown that, if the properties of the imbedded elastic sphere are the same as those of the surrounding medium, then the exact equations give no scattering, that is, the coefficients of the scattered wave vanish and the coefficients of the wave excited within the sphere are the same as those of the incident wave outside the sphere. In addition, it can be shown that, in the limit where the elastic constants and density of the imbedded elastic sphere tend to zero in such a way that both the transverse and longitudinal wave velocities within the sphere stay finite and non-zero, the exact solution for the wave scattered by an elastic sphere goes into the exact solution for the wave scattered by a spherical cavity. In this limit, the wave excited within the elastic sphere is, in general, non-zero; this is to be expected since the boundary conditions require that the displacement across the interface between the elastic sphere and the surrounding medium be continuous and, therefore, there must be a wave excited within the sphere. Also, in this limit, the wave excited within the elastic sphere exerts no force across the interface; this property can be proven from the result that, in this limit, the wave scattered by an elastic sphere goes into the wave scattered by a spherical cavity and from the boundary conditions.

We now turn to the calculation of the differential and total scattering cross-sections for the scattering of a transverse wave by a spherical obstacle (see secs. II-A and II-E). The differential scattering cross-section $\frac{dP}{d\Omega}$ is defined as the ratio of the time-averaged power scattered in a given direction per unit solid angle to the time-averaged incident power per unit area. $dP/d\Omega$ has the dimensions of area per unit solid angle. The scattered power is to be measured in the far-field region, that is, at points whose distance from the obstacle is much greater than the typical wavelength and the size of the obstacle. The direction of observation of the scattered power is specified by a polar angle θ measured from the direction of propagation of the incident wave

and an azimuthal angle ϕ ; for an arbitrarily shaped obstacle, $dP/d\Omega$ depends upon both angles θ and ϕ . Assuming that the incident wave is monochromatic with frequency ω , the time-averaging is performed over a period $T = 2\pi/\omega$ of the wave.

Let the incident wave be a circularly polarized plane transverse wave of angular frequency ω , propagating in the $+z$ - direction,

$$\underline{u}^o = (\hat{x} \pm i\hat{y}) e^{i\beta z}. \quad (A.20)$$

In general, the scattered wave $\underline{u}^s$ can be decomposed into a longitudinal component and a transverse component. Without loss of generality, we define $\underline{A}(r)$ and $\underline{B}(r)$ by

$$\underline{u}^s = \underline{A} \frac{e^{i\alpha r}}{r} + \underline{B} \frac{e^{i\beta r}}{r} \quad (A.21)$$

where $\underline{A} \frac{e^{i\alpha r}}{r}$ is the longitudinal component and $\underline{B} \frac{e^{i\beta r}}{r}$ is the transverse component of the scattered wave. Here r is the distance of the point of observation from the center of the obstacle. It can be shown that, for an arbitrarily shaped obstacle, the differential scattering cross-section is given by

$$\frac{dP}{d\Omega} = \frac{1}{2} \left[|\underline{B}|^2 + \left(\frac{\beta}{\alpha}\right) |\underline{A}|^2 \right] \quad (A.22)$$

where the scattered wave is evaluated in the far-field region; compare eq. (2.69)*. Note that the transverse and longitudinal components of the scattered wave contribute independently to the differential cross-section. Since at large distances $\underline{A}$ and $\underline{B}$ do not depend on r , $dP/d\Omega$ is independent of r as required.

In the far-field region, the transverse component $\underline{B} \frac{e^{i\beta r}}{r}$ of the scattered wave can be decomposed into a right circularly polarized transverse wave $\underline{B}^+ \frac{e^{i\beta r}}{r}$ and a left circularly polarized transverse wave $\underline{B}^- \frac{e^{i\beta r}}{r}$:

*For the incident transverse wave given by eq. (A.20), $|\underline{b}_1|^2 = 2$.

$$\tilde{B} \longrightarrow \tilde{B}^+ + \tilde{B}^- . \quad (A.23)$$

Finally, we obtain

$$\frac{dP}{d\Omega} = \frac{1}{2} \left[|\tilde{B}^+|^2 + |\tilde{B}^-|^2 + \left(\frac{\beta}{\alpha}\right) |\tilde{A}|^2 \right] . \quad (A.24)$$

Note that the right- and left-handed components of the scattered transverse wave contribute independently to the differential cross-section.

For scattering of transverse waves by a spherical obstacle, the form of the scattered wave is given by eq. (A.8). Using the asymptotic form of the spherical Hankel function, one can show that in the far-field region,

$$\tilde{A} \longrightarrow \frac{1}{\alpha} \left[\pm \sum_{\ell} \sqrt{4\pi(2\ell+1)} \ c_{\ell} \ Y_{\ell,\pm 1} \right] \hat{r} \quad (A.25)$$

$$\begin{aligned} \tilde{B}^+ &\longrightarrow \frac{1}{\beta} \left[-i \sum_{\ell} \sqrt{4\pi(2\ell+1)} \cdot \frac{1}{2} (a_{\ell} \pm b_{\ell}) (\tilde{x}_{\ell,\pm 1} + i \hat{r} \times \tilde{x}_{\ell,\pm 1}) \right] \\ \tilde{B}^- &\longrightarrow \frac{1}{\beta} \left[-i \sum_{\ell} \sqrt{4\pi(2\ell+1)} \cdot \frac{1}{2} (a_{\ell} \mp b_{\ell}) (\tilde{x}_{\ell,\pm 1} - i \hat{r} \times \tilde{x}_{\ell,\pm 1}) \right] . \end{aligned} \quad (A.26)$$

Substituting for $Y_{\ell,\pm 1}$ in eq. (A.25), it follows that, in the far-field region,

$$|\tilde{A}| \longrightarrow \frac{1}{\alpha} A_L \quad (A.27)$$

with

$$A_L = \left| \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\sqrt{\ell(\ell+1)}} \ c_{\ell} \ P_{\ell}^1(\cos\theta) \right| , \quad (A.28)$$

where $P_{\ell}^1(\cos\theta)$ is an associated Legendre polynomial. Note that the amplitude A_L does not depend on the azimuthal angle ϕ nor on whether the incident wave is right or left circularly polarized.

Expressing $\tilde{x}_{\ell,\pm 1}$ in spherical coordinates and substituting in eq. (A.26), we obtain in the far-field region,

$$\begin{aligned}
\tilde{B}^+ &\longrightarrow -i \frac{1}{\beta} \left\{ \sum_{\ell} \frac{2\ell+1}{\ell(\ell+1)} \cdot \frac{1}{2} (a_{\ell} \pm b_{\ell}) \left[\frac{1}{\sin\theta} P_{\ell}^1(\cos\theta) \pm \frac{dP_{\ell}^1(\cos\theta)}{d\theta} \right] \right\} \\
&\quad \cdot e^{\pm i\varphi} (\hat{\theta} + i\hat{\varphi}) \\
\tilde{B}^- &\longrightarrow -i \frac{1}{\beta} \left\{ \sum_{\ell} \frac{2\ell+1}{\ell(\ell+1)} \cdot \frac{1}{2} (a_{\ell} \mp b_{\ell}) \left[\frac{1}{\sin\theta} P_{\ell}^1(\cos\theta) \mp \frac{dP_{\ell}^1(\cos\theta)}{d\theta} \right] \right\} \\
&\quad \cdot e^{\pm i\varphi} (\hat{\theta} - i\hat{\varphi})
\end{aligned} \tag{A.29}$$

where $\hat{\varphi}$ and $\hat{\theta}$ are unit vectors in the φ - and θ - directions. Therefore, in the far-field region,

$$\begin{aligned}
|B^+| &\longrightarrow \frac{1}{\beta} A_{T+} \\
|B^-| &\longrightarrow \frac{1}{\beta} A_{T-}
\end{aligned} \tag{A.30}$$

with

$$\begin{aligned}
A_{T+} &= \left| \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \cdot \frac{1}{\sqrt{2}} (a_{\ell} \pm b_{\ell}) \left[\frac{1}{\sin\theta} P_{\ell}^1(\cos\theta) \pm \frac{dP_{\ell}^1(\cos\theta)}{d\theta} \right] \right| \\
A_{T-} &= \left| \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \cdot \frac{1}{\sqrt{2}} (a_{\ell} \mp b_{\ell}) \left[\frac{1}{\sin\theta} P_{\ell}^1(\cos\theta) \mp \frac{dP_{\ell}^1(\cos\theta)}{d\theta} \right] \right|
\end{aligned} \tag{A.31}$$

Note that the transverse amplitudes do not depend on the azimuthal angle φ . In addition, using an obvious notation,

$$\begin{aligned}
A_{T++} &= A_{T--} \\
A_{T+-} &= A_{T-+}
\end{aligned} \tag{A.32}$$

as required by conservation of parity. (Roughly speaking, conservation of parity means that an experiment cannot be constructed to distinguish between right and

left. Elastic forces in materials satisfy conservation of parity.)

Since the differential scattering cross-section does not depend on the helicity of the incident transverse wave, we may assume for concreteness that the incident wave has positive helicity. Substituting in eq. (A.24), we obtain

$$\frac{dP}{d\Omega} = \frac{1}{2} \cdot \frac{1}{\beta^2} \left[\left(A_{T+}^2 + A_{T-}^2 \right) + \left(\frac{\beta}{\alpha} \right)^3 A_L^2 \right] \quad (A.33)$$

Let us normalize the differential cross-section by dividing by the geometrical cross-section $P_0 = \pi a^2$ of a spherical obstacle of radius a . Finally, we obtain

$$\frac{1}{P_0} \frac{dP}{d\Omega} = \frac{1}{2\pi} \cdot \frac{1}{(\beta a)^2} \left[\left(A_{T+}^2 + A_{T-}^2 \right) + \left(\frac{\beta}{\alpha} \right)^3 A_L^2 \right] \quad (A.34)$$

Note that $\beta/\alpha = v_L/v_T > 1$. It can be shown that A_{T-} vanishes in the forward direction, A_{T+} vanishes in the backward direction, and A_L vanishes in both the forward and backward directions.

The total scattering cross-section P is defined as the ratio of the time-averaged power scattered by the obstacle to the time-averaged incident power per unit area. P has the dimensions of area.

Let the incident wave be a circularly polarized plane transverse elastic wave of angular frequency ω , propagating in the $+z$ -direction,

$$\underline{u}^0 = (\hat{x} \pm i\hat{y}) e^{i\beta z} \quad (A.35)$$

In general, the scattered wave can be expanded in solutions of the vector Helmholtz equation:

$$\begin{aligned}
u^s = & \sum_{\ell=1}^{\infty} \sum_{|m| \leq \ell} i^{\ell} \sqrt{4\pi(2\ell+1)} \left[a_{\ell,m} h_{\ell}^{(1)}(\beta r) x_{\ell,m} + b_{\ell,m} \frac{1}{\beta} \nabla \times (h_{\ell}^{(1)}(\beta r) x_{\ell,m}) \right] \\
& + \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} i^{\ell} \sqrt{4\pi(2\ell+1)} \left[c_{\ell,m} \frac{1}{\alpha} \nabla \cdot (h_{\ell}^{(1)}(\alpha r) y_{\ell,m}) \right] .
\end{aligned} \quad (A.36)$$

With the definition of P , it is straightforward to show that the total cross-section P is given by

$$\begin{aligned}
P = & 2\pi \cdot \frac{1}{\beta^2} \left\{ \sum_{\ell=1}^{\infty} \sum_{|m| \leq \ell} (2\ell+1) (|a_{\ell,m}|^2 + |b_{\ell,m}|^2) \right. \\
& \left. + \left(\frac{\beta}{\alpha} \right)^3 \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} (2\ell+1) |c_{\ell,m}|^2 \right\} .
\end{aligned} \quad (A.37)$$

In deriving this formula, either the exact form of the scattered wave or its approximation in the far-field region may be used, since it is assumed that no power is absorbed by the medium.

Specializing to scattering by a spherical obstacle, we obtain (see eq. (A.8))

$$\frac{P}{P_0} = 2 \cdot \frac{1}{(\beta a)^2} \sum_{\ell} (2\ell+1) (|a_{\ell}|^2 + |b_{\ell}|^2 + \left(\frac{\beta}{\alpha} \right)^3 |c_{\ell}|^2) . \quad (A.38)$$

Here the cross-section is normalized by the geometrical cross-section $P_0 = \pi a^2$ of the spherical obstacle.

The cross-section may be evaluated in the Rayleigh limit. For scattering by a spherical cavity,

$$\frac{P}{P_0} \longrightarrow \left[\frac{8}{27} \left(1 + \frac{1}{2} \eta^3 \right) + 10 \left(\frac{2}{9-4\eta} \right)^2 \left(1 + \frac{2}{3} \eta^5 \right) \right] x^4 . \quad (A.39)$$

Here $x = \beta a$, $\eta = \alpha/\beta$. This disagrees with the result obtained by Einspruch et. al.; in their result, the numerical factor of $2/3$ multiplying η^5 is replaced by $1/6$. For scattering by an elastic sphere,

$$\frac{P}{P_0} \longrightarrow \left\{ \frac{8}{27} \left[1 - \frac{\mu'}{\mu} \left(\frac{x'}{x} \right)^2 \right]^2 \left(1 + \frac{1}{2} \eta^3 \right) + \right. \\ \left. 10 \left(1 - \frac{\mu'}{\mu} \right)^2 \left[3 \left(3 + 2 \frac{\mu}{\mu'} \right) - 4 \left(1 - \frac{\mu}{\mu'} \right) \eta^2 \right]^2 \left(1 + \frac{2}{3} \eta^5 \right) \right\} x^4 \quad (A.40)$$

Here $x = \beta a$, $x' = \beta' a$, $\eta = \alpha/\beta$. This disagrees with the result obtained by Einspruch et. al.. In particular, suppose that the properties of the imbedded elastic sphere are such that the wave velocities are the same in the elastic sphere as in the surrounding medium; such a case would arise, for example, if the elastic constants and density of the imbedded sphere were twice those of the surrounding medium. In this case, the formula of Einspruch et. al. predicts that P vanishes in the Rayleigh limit whereas our result yields a non-zero value. Finally, we point out that the derivation of the Rayleigh limit results, eqs. (A.39) and (A.40), from eq. (A.38) involves only the $\ell = 1$ and $\ell = 2$ terms in eq. (A.38).

*Supported by Rockwell International under contract number F33615-74-C 5180, with the Advanced Research Projects Agency, Arlington, Virginia and Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson AFB, Ohio. (Support has also been received from the Energy Research and Development Administration under contract E(11-1)-3161 and the Materials Science Center at Cornell University. Materials Science Center Report #2654; ERDA Technical Report #COO-3161-42.)

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Appendix B

The scattering of a plane longitudinal wave by a spherical obstacle in an isotropically elastic solid was solved exactly by Ying and Truell.¹ The complexity of the solution however requires numerical computation to extract scattering cross-sections over most frequency ranges of the incident wave. The first numerical computations were performed by Johnson and Truell.²

We checked the analysis and equations of Ying and Truell, and Johnson and Truell for scattering from an elastic sphere and a spherical cavity and found several inaccuracies, most apparently typographical. We now list corrections to several equations in each paper. YT appearing before an equation number indicates that the equation is in the Ying and Truell paper. Similarly, JT refers to equations in the Johnson and Truell paper.

Case 1: Elastic Sphere

a) YT (20d)

Replace the minus before D_{em} by a plus.

b) YT (26a)

Replace $(2m!)$ by $[(2m)!]^2$.

c) JT, last equation, left column, p.3469

Replace $(-1)^m$ by -1 .

Case 2: Spherical Cavity

JT (15a), (15b), (16a), and (16b).

Replace $(-1)^m$ by -1 .

References

1. C. F. Ying and R. Truett, J. Appl. Phys. 27, 1086 (1956).
2. G. Johnson and R. Truett, J. Appl. Phys. 36, 3466 (1965).

$$R = R_1 + R_2$$
$$S_1 = S + S_2$$

Figure 2

Figure 3

a)

b)

Figure 4

Figure 5

DCRM

EXACT

Fig. 3.14

Cavity in TL, incident longitudinal wave

DORN

EXACT

Fig. 3.1b

Cavity in Tl, incident longitudinal wave

BOHN

EXACT

Fig. 3.2

Cavity in Al, incident longitudinal wave

Fig. 3.3

Cavity in stainless steel, incident longitudinal wave

Fig. 2.4

Al sphere in Ti, incident longitudinal wave

FIG. 3.3

EXACT

BORN

Fig. 3.6

EXACT

Fig. 3.7

Cavity in Al, incident right circularly polarized transverse wave

Fig. 3.8

Cavity in stainless steel, incident right circularly polarized transverse wave

Fig. 3.10

Stainless steel in Ti, incident right, circularly polarized transverse wave