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A Laser Driven Grating Linac

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A LASER DRIVEN GRATING LINAC

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ABSTRACT

The fields induced over a grating exposed to plane parallel light are explored. It is shown that acceleration is possible if either the particles travel skew to the grating lines, or if the radiation is falling at a skew angle onto the grating. A general theory of diffraction in this skew case is given. In one particular case numerical solutions are worked out for some deep grating. It is found that accelerating fields larger even than the initial fields can be obtained, the limit being set by resistive losses on the grating surface. Simple calculations are made to see what accelerating fields might be obtained using CO₂ lasers. Accelerations of 2 or 20 GeV per meter seem possible depending on whether the grating is allowed to be destroyed or not. Power requirements, injection and focussing are briefly discussed and no obvious difficulties are seen. It is concluded, therefore, that the proposed mechanism should be considered as a good candidate for the next generation of particle accelerators.

1. ACCELERATION THEORY

The use of a laser to accelerate particles was first proposed by K. Shimoda^{1]} in 1962. He noted that high values of acceleration per meter could be obtained if velocity matching and mode selection were achieved. These requirements are, however, not easy to obtain.

Fields in free space, far from all sources, consist of a sum of all possible traveling electromagnetic waves. Provided the particles to be accelerated are traveling less than the velocity of light, acceleration^{2]} can occur. Once the velocity approaches that of light only waves traveling in the same direction as the particles remain in phase with the particles. Unfortunately, since free radiation is transversely polarized, no continuous acceleration is possible. Despite claims^{3]}, no juggling with holograms, phase plates or focii can change this. In the presence of a magnetic field, the particle's direction can be perturbed in such a way as to allow continuous acceleration^{4]} but this too decreases as the particle's momentum increases and significant perturbations become impractical. In^{5]} or near^{6]} a dielectric, the inverse Cerenkov effect will accelerate, but the field that can be used is limited since the dielectric cannot be allowed to break down. At high fields any dielectric becomes a plasma and the situation becomes very complicated. Acceleration within such a plasma is certainly possible^{7]} but the magnitude of such acceleration remains to be determined.

Acceleration has also been proposed in a vacuum close to a periodic structure. In particular, two papers have attempted^{8]} to employ the inverse Purcell effect^{9]} by illuminating a grating with plane parallel light and passing the particles over the surface of the grating at right angles to the lines (Fig. 1a). Unfortunately, it has been shown by Lawson^{6]} that these geometries also fail to accelerate relativistic particles. In Lawson's paper consideration is also given to the field between two parallel plane structures. It is shown in this case finite acceleration of relativistic particles is possible. Such a geometry seems to be little more practical than a scaled down conventional Linac. It does,

however, show that there is no fundamental reason why a solution cannot be found. Indeed if we simply rotate the grating by an angle ψ with respect to the beam (see Fig. 1b), then acceleration is indeed possible. In this case surface waves may be induced whose velocity c/K will be lower than the velocity of light, but these waves can remain in phase with a highly relativistic particle because of the angle ψ between the wave and particle directions.

An alternative to a skew grating is to employ a skew initial wave (Fig. 2c). In this case although the grating lines are perpendicular to the particle beam, nevertheless the induced surface waves can still be at an angle to the beam and acceleration can again be obtained.

In order to consider diffraction in this skew condition, it will be convenient to introduce the following modified vector notation. Three-dimensional vectors (A_x, A_y, A_z) will be described by the two-dimensional vector (A_x, A_y) together with the Z component A_z . The two-dimensional vectors will be shown \vec{A} , the corresponding z component would then be shown as A_z . We will be considering the fields above a grating placed nominally at $z = 0$. Any such fields can be parameterized by:

$$E = \sum_{n=-\infty}^{n=+\infty} \vec{A}_n e^{j(p_n z + \vec{K}_n \cdot \vec{R} - \omega t)}$$

$$E_z = \sum_{n=-\infty}^{n=+\infty} \frac{-\vec{A}_n \cdot \vec{K}_n}{p_n} e^{j(p_n z + \vec{K}_n \cdot \vec{R} - \omega t)} \quad (1)$$

where

$$\vec{K}_n = \vec{K} + n\vec{G}$$

$$p_n = \pm \sqrt{-|\vec{K}_n|^2 + k_o^2} \quad , \quad k_o = \frac{2\pi}{\lambda}$$

$$|\vec{G}| = 2\pi/S.$$

The n is the order of the diffracted waves. \vec{A}_n is a set of two dimensional complex vectors (A_x, A_y) describing the amplitudes of the modes polarized in the two directions. \vec{G} is a vector pointing along the surface perpendicular to the grating lines, and whose amplitude is as given. \vec{K}_n is a vector along the surface perpendicular to the wave fronts of the mode. \vec{K} is this vector for the incoming wave.

When $|\vec{K}_n| < k_o$, then p_n is real and the mode is a free propagating wave either approaching (p_n negative) or leaving (p_n positive) the surface. Only the initial wave with $n = 0$ is incoming with p_n negative. All others have p_n positive and are the various diffracted modes. To distinguish between the amplitude of the single incoming (p negative) and outgoing (p_o positive) wave, the former will be given without subscript (\vec{A}) and the latter with subscript \vec{A}_o . The sum in equation (1) covers both the incoming \vec{A} and the set of outgoing waves \vec{A}_n ($n = -\infty$ to $+\infty$). When $|\vec{K}_n| > k_o$, then p_n is positive and complex and the mode is a surface or evanescent wave that falls off exponentially from the surface. The surface velocity of these waves have magnitude $c k_o / \vec{K}_n$ and direction \vec{K}_n . If the particle has a direction and velocity \vec{v} , then the condition that the particles remain in phase with a particular mode n is

$$\vec{k}_n \cdot \vec{\beta} = k_0 \quad (2)$$

The case illustrated in Fig. 1c is when $\vec{\beta} \parallel \vec{G}$, i.e. when the particles are traveling perpendicular to the grating lines. For $\beta = 1$ this implies that the projection of the \vec{k}_n vector onto the vector \vec{G} have the length k_0 . This condition is shown in Fig. 2 for $n = +1$. We may now note that there is an infinite set of initial waves \vec{k}' whose first mode will satisfy the condition (2). It can then be shown that the angle ξ between such initial rays and the beam axis is given by

$$\beta \cos \xi = 1 - \frac{n\vec{G} \cdot \vec{\beta}}{k_0} \quad (3)$$

The set of all such rays form a half cone about the beam axis analogous with a Cerenkov cone (see Fig. 3). This fact turns out to be very advantageous since the sum of all such waves will form a line image on the grating such that the direction of the line points along the particle direction. The narrowness of such a line image ($\sim \lambda$) will then assure the maximum local field for a given electromagnetic energy, and thus represents an efficient laser accelerator.

It should be noted that the one initial ray that is perpendicular to the grating lines (AO in Fig. 3) cannot, by Lawson's argument, induce acceleration. In practice rays near to this case would probably be omitted.

It remains now to determine the actual magnitude of the acceleration for given incident electromagnetic energy. This I will do for a particular case.

There are two fundamentally different approaches to obtaining numerical Eigen solutions to the fields above a surface boundary condition. The first and more common is to define the boundary and then adjust the amplitudes of all possible modes until the boundary condition is satisfied. An alternative that I will follow here is to pick a suitable combination of modes and then find the boundary condition (i.e. grating shape) that is consistent with the resulting fields. This approach is particularly easy if incident waves are chosen such that all resulting modes form standing waves. The field lines for these waves can then be drawn and any surface that is perpendicular to these lines is an acceptable shape for the grating.

For simplicity, I will restrict myself to special cases with the following character: the incoming rays will be chosen to be perpendicular to the beam axis. In addition, the beam axis will be chosen to lie perpendicular to the grating lines. Such a case is illustrated in Figs. 4 and 5. The only variables in describing the incoming wave are its angle ϕ to the normal and its state of polarization, which will be taken to be in the beam direction (x). i.e.

$$\begin{aligned} \vec{k} \cdot \vec{\beta} &= 0 & , & & \vec{k} \cdot \vec{G} &= 0 \\ \vec{A} \cdot \vec{\beta} &= 0 & & & & \end{aligned} \quad (4a)$$

If we further require that the grating shape be symmetric with respect to a reversal of the beam direction then

$$\vec{A}_n = \vec{A}_{-n} \quad (4b)$$

and the number of free parameters is reduced by two. If we consider $\beta = 1$ and require

acceleration for $n = 1$, then the condition for acceleration (eqn. 2) reduces to:

$$n|\tilde{G}|S = k_0$$

$$\text{or } S = \lambda 3n \quad (4c)$$

but $\beta = 1$, $n = 1$, so $S = \lambda$ where S is the grating spacing. Finally, I will consider fields induced by two equal and simultaneous incoming waves, A and A' , whose angles to the normal ϕ and ϕ' are equal and opposite. Two sets of induced waves will then be present denoted by \tilde{A}_n and \tilde{A}'_n with the condition that:

$$\tilde{A}_n = \tilde{A}'_n \quad (4d)$$

Condition 4c assures that all modes other than $n = 0$ are surface waves. Energy conservation thus requires that the amplitudes of the outgoing reflected waves ($n = 0$) be equal and opposite to the incoming waves:

$$\tilde{A}_0 = -\tilde{A}, \quad \tilde{A}'_0 = -\tilde{A}' \quad (4e)$$

With all these conditions (4a-e), the resulting fields are assured to be standing waves and the only variables are the set of scalar amplitudes A_n for $n = 1$ to ∞ .

The fields are now given by:

$$E_x = \cos(\omega t) \cos(Ky) \left\{ B \sin(pz) + \sum_{n=1}^{\infty} B_n e^{-q_n z} \cos(nk_0 x) \right\}$$

$$E_y = 0 \quad (5)$$

$$E_z = -\cos(\omega t) \cos(Ky) \left\{ 0 + \sum_{n=1}^{\infty} B_n (nk_0/q_n) e^{-q_n z} \sin(nk_0 x) \right\}$$

where

$$B = 4j|\tilde{A}|$$

$$B_n = 4|\tilde{A}_n|$$

$$p = +\sqrt{k_0^2 - K^2}$$

$$K = |K|$$

$$q_n = jp_n = +\sqrt{K^2 + k_0^2(n-1)^2}$$

Note $q_1 = K$.

B and B_n are now real numbers. All waves vary in the same way with both time and y position. Clearly maximum acceleration is obtained at $y = 0$ and at values of y spaced at intervals of $2\pi/K$. The first term inside the curly bracket is that due to the incoming and outgoing waves. It is only in the x direction, varies sinusoidally with distance above the grating, and is constant along the direction of acceleration. The second term in the brackets includes all the surface waves that fall off exponentially with height above the grating and vary periodically with position in x . The average acceleration of a particle traveling in the x direction at a height (h) above the surface depends only on the mode $n = 1$ and is

$$\left\langle \frac{d^2x}{dt^2} \right\rangle = \frac{B_1}{2} e^{-Kh} \quad (6)$$

It is convenient to express this mean accelerating field as a fraction (ϵ) of the peak field ($B/2$) that would be present in the absence of the grating. Thus

$$\left\langle \frac{d\phi}{dx} \right\rangle = \epsilon \frac{B}{2}$$

$$\epsilon = \frac{B_1}{B} e^{-Kh} \quad (7)$$

All fields vary in the same way with y . Thus, if a line is found that is perpendicular to the fields at one y , the same line will be perpendicular at all other values of y . In other words, we will have found a surface with a cross section independent of y , i.e. a grating. It remains then to consider some individual cases, examine the pattern of x, z fields at $y = 0$ and find lines perpendicular to these fields, thus defining Eigen solutions to the problem.

We are searching for a solution in which the ratio of the accelerating mode to the incoming mode is as large as possible. It is relevant, therefore, to ask why this ratio cannot be infinite. In other words, ask whether there is an Eigen solution with surface modes, including an accelerating mode, and no free propagating waves at all. In such a solution the grating is behaving like a cavity containing accelerating fields which would, if there were no losses, remain indefinitely without the application of any external field. First we can examine the accelerating mode ($n = 1$) alone. This is shown in Fig. 6b. Any surface perpendicular to these field lines contain cuts that extend to infinite depth; clearly not a practical solution. If, however, we add the mode ($n = 3$) with opposite phase, then at once a solution becomes possible. Consider for instance $K = 0.2 k_0$, $B_3 = -0.025 B_1$. The field pattern obtained is shown in Fig. 6c where a surface perpendicular to all lines is indicated. This then is a solution which, if excited, would accelerate and would remain without radiating away its energy. However, since it is not coupled to the incoming wave, it would in fact not be excited in the first place. What is required is a solution similar to the above but with a small admixture of the incoming mode. For instance $K = 0.2 k_0$, $B_3 = -0.025 B_1$, $B = -0.5 B_1$, which gives the field pattern of Fig. 4d. It may be noted that the uncoupled grating, Fig. 6c, had periodicity with half the wave length, and it can be shown that such a periodicity cannot couple to the incoming waves. This new solution is similar but has a small component of one wave length periodicity. It is this component that provides the coupling.

The acceleration at the surface of this grating is given by $\epsilon = 5.0$ and is thus considerably larger than the peak field present without the grating! This result is not surprising when compared to a conventional accelerating RF cavity. If the "Q" of the cavity is higher, then the accelerating fields for given RF power are also higher. The realizable accelerating field is set when the losses in the cavity approach the RF power applied.

In the grating case the losses at the surface can be calculated if they are due purely to resistive effects. They are then given approximately by

$$f = \frac{S_{\text{losses}}}{S_{\text{incoming}}} \approx \left(\frac{k_0}{K} \epsilon \right)^2 \cdot \frac{1}{4} \left(\frac{c}{\lambda \sigma} \right)^{1/2} \quad (8)$$

For a copper grating ($\sigma = 1/1.5 \cdot 10^{-6} \Omega \text{ cm} \approx 6 \cdot 10^{17} \text{ sec}^{-1}$), wave length of 10μ , $K = 0.2 k_0$, and $\epsilon = 5.0$, we obtain the fractional loss $f \approx 100\%$. Thus the value of $\epsilon = 5$ represents the highest value possible. It would be more realistic to limit the fractional loss to approximately 25% and thus ϵ to 2.5. This value will be used for the following examples.

2. PRACTICAL CONSIDERATIONS

2.1 Grating survival

Two quite different limits must be considered here. Firstly: up to what power level will the grating survive such that it can be used for subsequent pulses. Secondly: up to what power level will the grating survive in the sense that acceleration will still occur above its surface. The second limit is quite appropriate for a grating (whereas quite inappropriate for a conventional LINAC) since only a narrow band ($\sim 25\mu$ wide) would be destroyed and the grating could be displaced between pulses and eventually replaced. Alternatively, it is possible that one could use ripples on a liquid metal surface such as mercury or potassium.

The limits clearly depend, and the pulse duration as well, on the instantaneous power level and frequency. If a CO_2 LASER were employed, then the shortest pulse obtainable would be about 30 psec and for such a pulse the first "few pulse" limit is ^{11]} about 10^{11} watts per cm^2 . This figure is extrapolated from experiments with copper gratings at Los Alamos. The second "one pulse" limit is far harder to estimate but could be as high as 10^{13} watts/ cm^2 . Experiment is required to determine this, but I will use these numbers for the subsequent discussion.

The relation between local power density in space and the local field is given by the Poynting Vector:

$$E = 27.5 \sqrt{P} \quad (9)$$

where E is the maximum local field in volts/cm and P is the power density in watts per cm^2 . The value of the accelerating field can, as we have seen, be higher than this by a factor ϵ depending on the losses in the grating. This factor I will take to be 2.5. The accelerations obtained are then 2 GeV/m and 20 GeV/m, for the "few pulse" and "one pulse" limits respectively (see Table I). We may compare these figures with those obtained at SLAC (10 MeV/m) or with the highest obtained in test cavities (about 100 MeV/m). It appears that the grating has the potential for far higher accelerations than is possible with conventional LINACS. In a length of 1 km energies of 2 or 20 TeV would be obtained.

2.2 Power requirements

I will now address the question of the power requirements of the driving LASER. As has been shown above, the conditions for acceleration can be met at a line image from a cylindrical lens system (see Fig. 4). The aperture would consist of two bands, one either side of the vertical since by Lawson's argument the vertical light cannot contribute. The width of the image will be given approximately by $\lambda/2\phi$ where ϕ is the half angle between the two strips. Thus for solutions similar to that shown in Fig. 4d, the width, w, of the line image will be of the order of 25μ . For the two cases we have been considering the total power for a length of 1 km would then be $2.5 \cdot 10^{13}$ and $2.5 \cdot 10^{15}$ watts, respectively. The energies in a 30 picosecond pulse would be 750 and 75,000 joules, respectively. The first specification is really quite modest by today's large CO_2 LASER standards ^{12]} and the second could probably be built for a reasonable cost if many separate beams were passed through the amplifiers, each delayed with respect to the former (multiplexed operation).

2.3 Injection

One of the objections raised to such Laser driven accelerators was that the phase space accepted was so small that a negligible number of particles could be accelerated. This appears not, in fact, to be the case when the phase space density of the proposed SLAC single pass collider^{13]} is considered. The beams in that proposal would contain 10^{11} particles in a 10 picosecond bunch that would be focussed to about 2μ diameter for a length of 1 cm. The acceptance of the LASER accelerator proposed here would be approximately 25μ wide and 6μ high. If suitably matched, the proposed SLAC beam would be a suitable injector.

2.4 Stability and focussing

It is found that vertical and horizontal stability are obtained if the bunches are slightly behind the phase of maximum acceleration. What is needed is a horizontal fixed magnetic field to counteract the RF electrostatic and magnetic forces tending to push the particles away from the surface. Also required is the use of some initial radiation with polarization perpendicular to the particle direction applied with appropriate phase. An example has been worked out^{14]} that at 20 GeV/c provided a vertical betatron wave length of 16 cm and a horizontal wave length of 80 cm. These values are entirely compatible with the emittance of the SLAC beam discussed above.

Longitudinal stability is not obtained in this example but is found not to be needed since the synchrotron wave length is several kilometers. A special buncher would, however, be required at the front end that would employ magnetic fields to lower the synchrotron wave length.

I conclude that acceleration in fields generated over a grating surface is possible and that very high gradients may be feasible. More detailed study including experimental tests should be carried out, but unless such work reveals a major flaw the proposed scheme should be taken as a serious candidate for the next generation of accelerators.

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Table 1

Parameters of two speculative accelerators: a) in which the grating is not destroyed, and b) in which a disposable grating is used.

Example	S	l	W	J	$\delta\mathcal{E}/\delta z$	\mathcal{E}
	<u>Watts/cm²</u>	<u>Km</u>	<u>Watts</u>	<u>joules</u>	<u>GeV/m</u>	<u>TeV</u>
a)	10^{11}	1	$2.5 \cdot 10^{13}$	750	2	2
b)	10^{13}	1	$2.5 \cdot 10^{15}$	75,000	20	20

* * *

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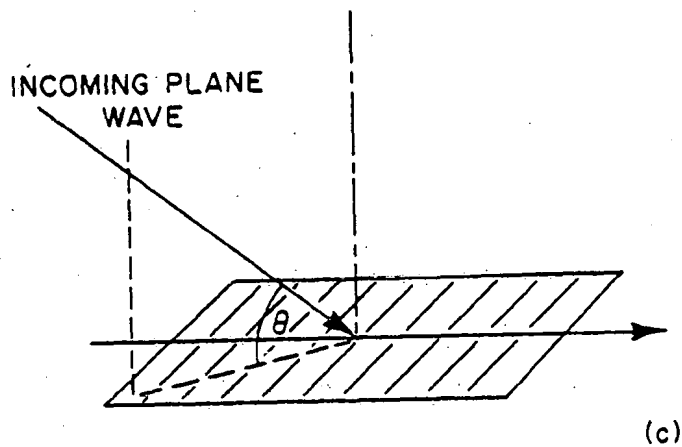
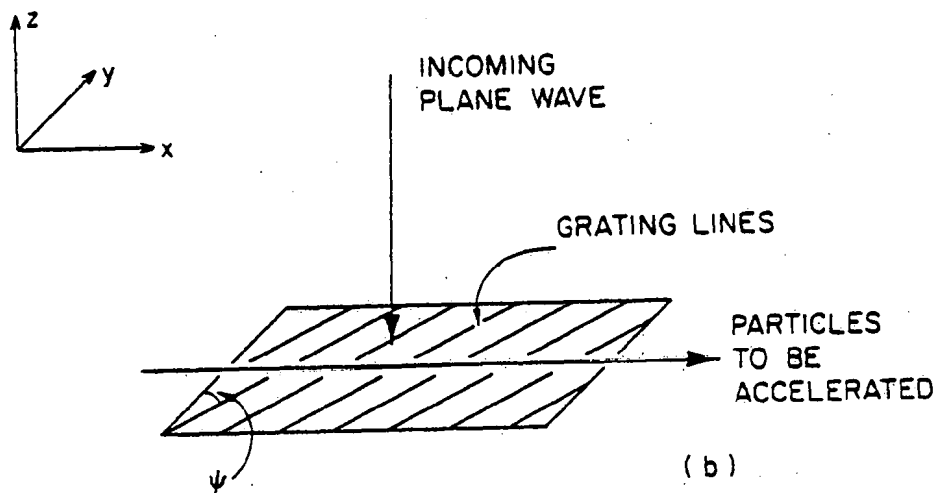
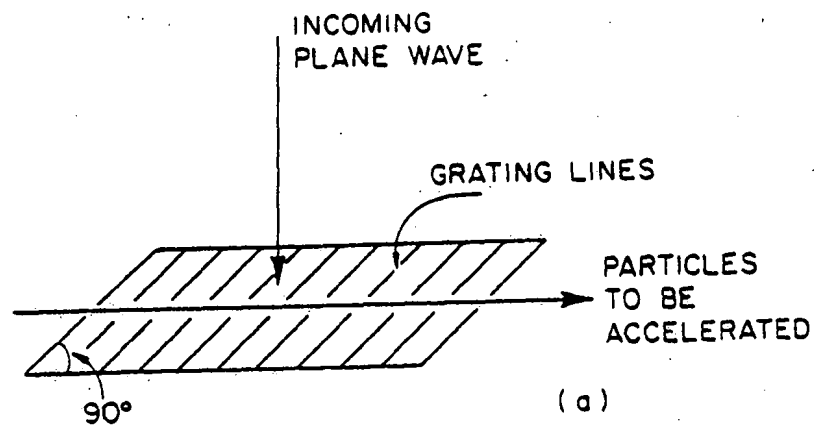
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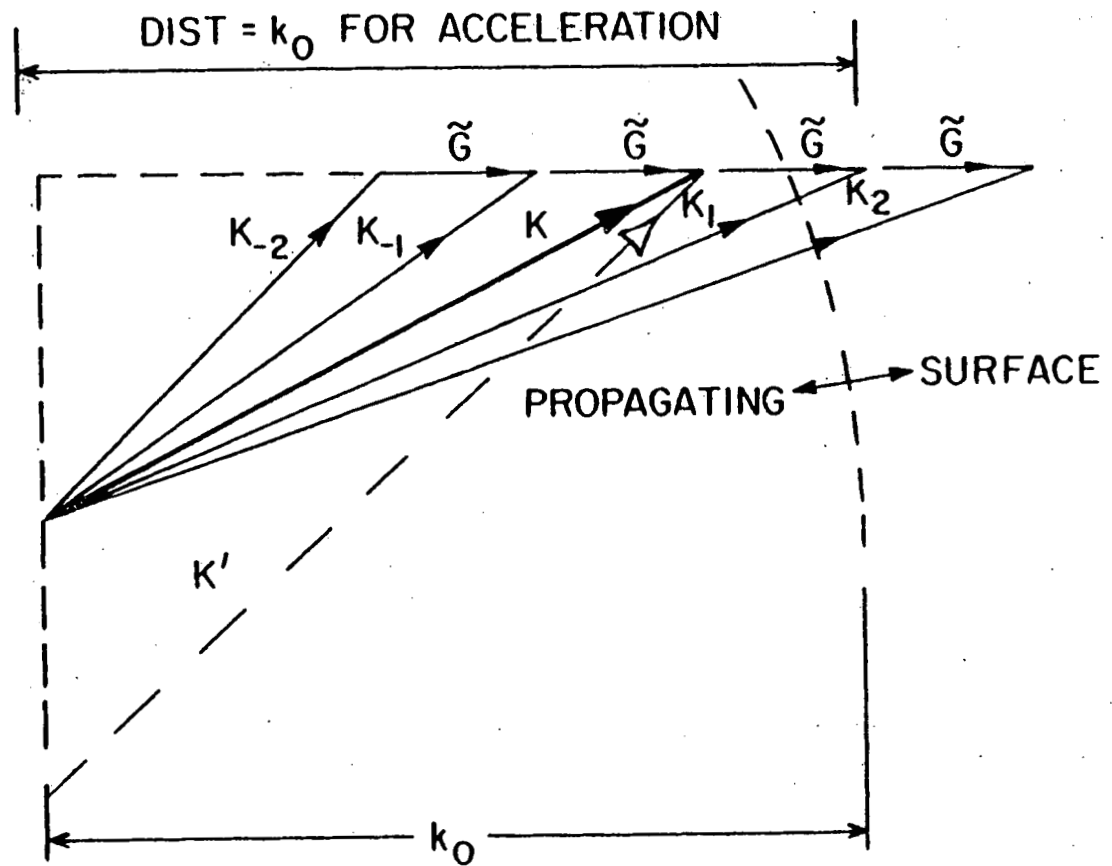
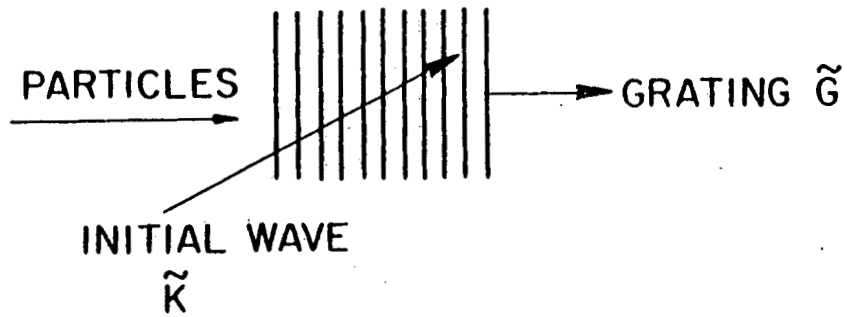
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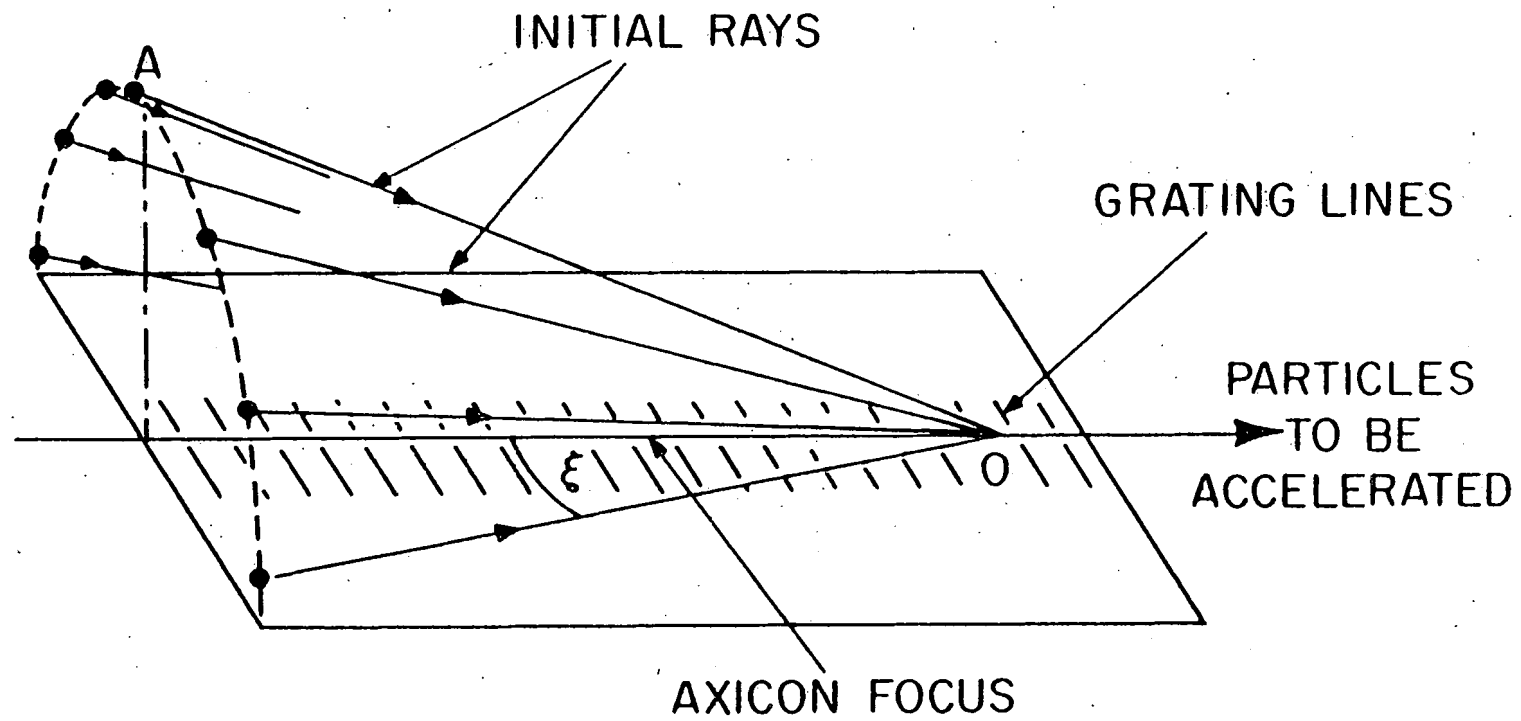
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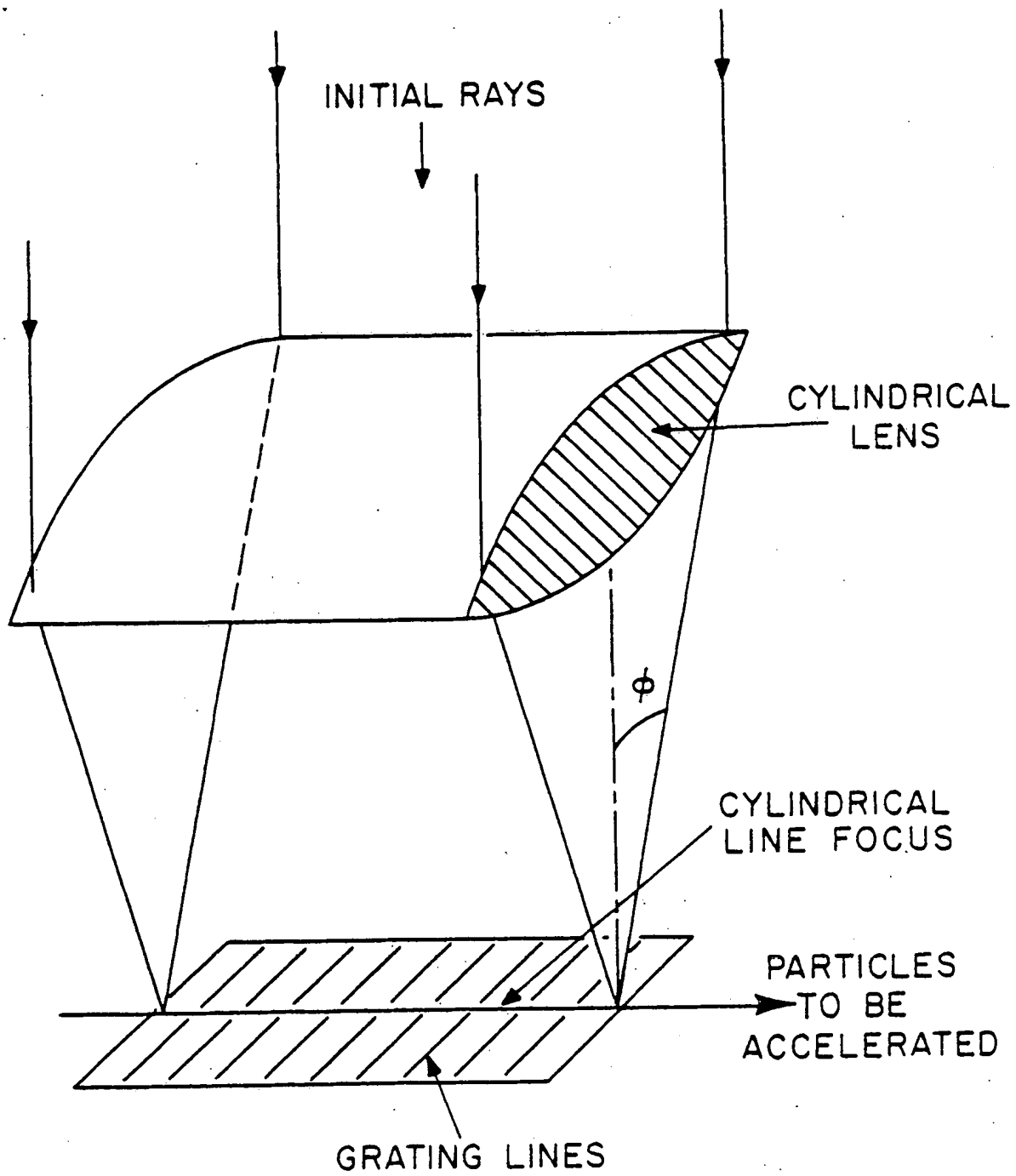
1. Geometries of Grating Accelerators: a) As proposed by Takeda and Matsui; b) with skew grating to allow acceleration of relativistic particles; c) with skew initial wave as alternative to b).



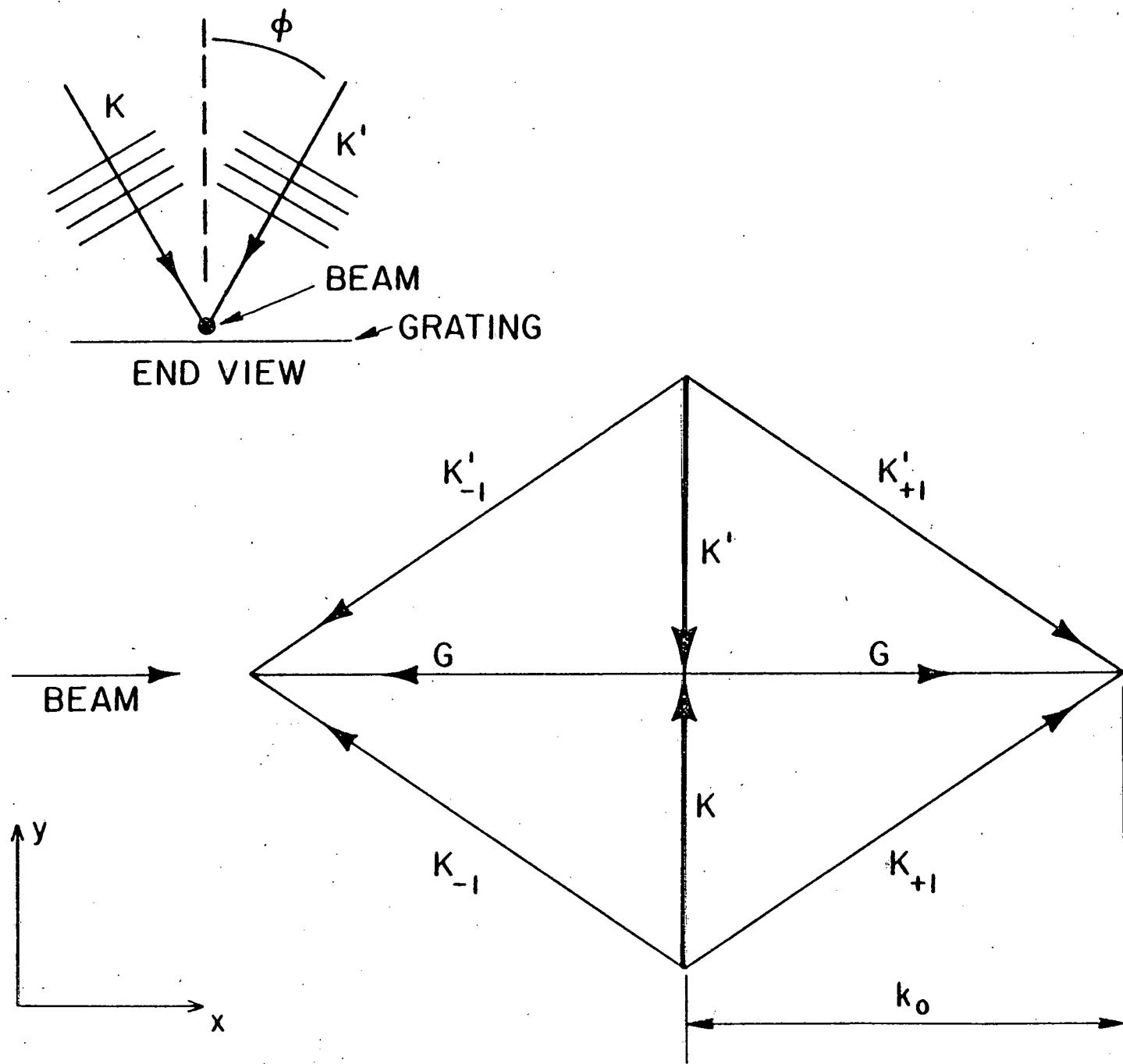
2. Graphical representation of diffracted waves (K_n) induced by a skew initial wave K .



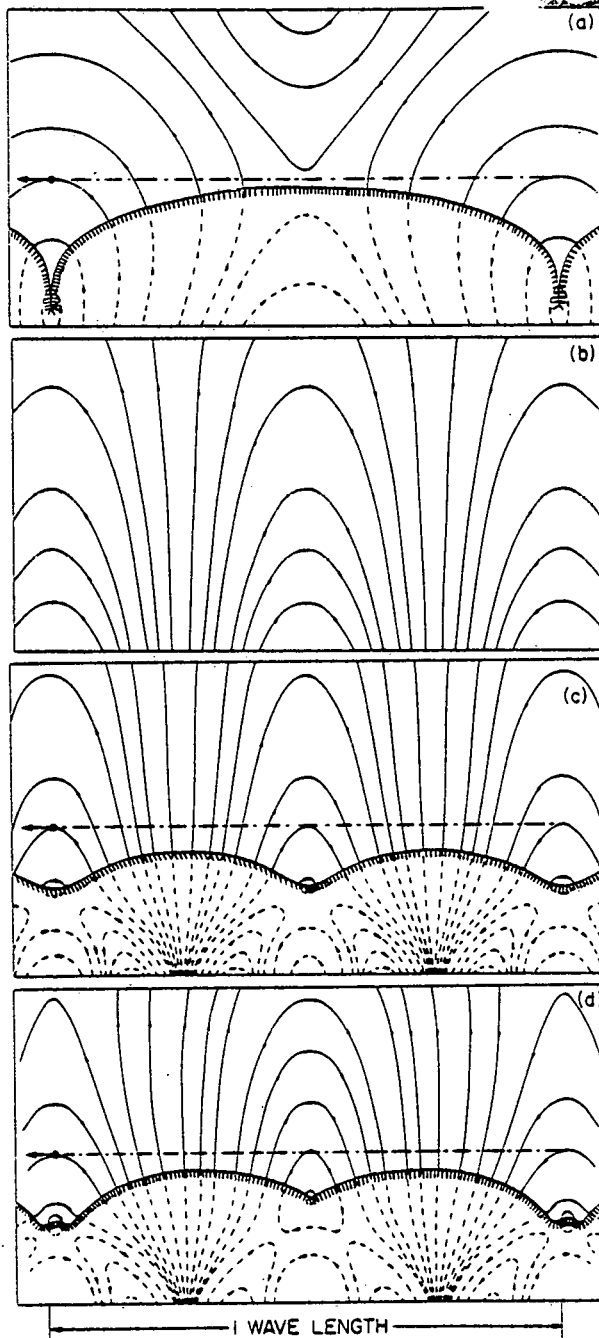
3. General geometry of initial waves inducing acceleration of particles over a grating.



4. Special case of the geometry shown in Figure 3 where $\theta = 90^\circ$.



5. Graphical representation of first diffracted waves (K_1) induced by initial waves in the geometry of Figure 4.



6. The electric field patterns produced by different combinations of modes, together with the shape of the grating surfaces that will support these combinations: a) Case with initial wave ($n = 0$) and the accelerating modes ($n = \pm 1$) only; b) Field lines for the accelerating ($n = \pm 1$) modes alone, There is no grating surface that will support this mode alone; c) Case with accelerating mode ($n = \pm 1$) and a small addition of the third mode ($n = \pm 3$); this solution does not couple to any initial wave; d) Case with a small initial wave ($n = 0$), a strong accelerating mode ($n = \pm 1$) and a small addition of the third mode ($n = \pm 3$), this solution couples to the initial wave and provides good acceleration.