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INCORRECTNESS OF THE USUAL
GYROKINETIC TREATMENT IN
CYLINDRICALLY SYMMETRIC SYSTEMS

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Incorrectness of the Usual Gyrokinetic Treatment
in Cylindrically Symmetric Systems

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It is shown that the usual gyrokinetic theory does not consistently retain all terms of leading order in the expansion parameter $\epsilon = \text{gyroradius}/\text{equilibrium scale length}$. This is illustrated for cylindrically symmetric systems by comparing the perturbed distribution function calculated by the gyrokinetic method with that obtained by explicitly integrating the Vlasov equation over the unperturbed orbit. The integral equation used in some recent treatments of drift waves in sheared-slab geometry is shown to be incorrect. The correct calculation of the ion density perturbation for a collisionless $\beta = 0$ plasma with cylindrical symmetry is presented.

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I. INTRODUCTION

The gyrokinetic equation, obtained by averaging the linearized Vlasov equation over gyrophase angle, is a starting point for much work in the linear stability theory of both "sheared-slab" and more complicated magnetic configurations. In this paper I show that the conventional gyrokinetic treatment (GK) does not consistently retain all terms of the same order in the expansion parameter ϵ ($=$ gyroradius/equilibrium scale length). This is illustrated in Section III by the simple example of a cylindrically symmetric plasma (the result is equally applicable to a sheared-slab model). For this case I show that GK leads to an incorrect perturbed distribution function, when compared with that obtained by explicitly integrating over the unperturbed orbit. The reason is that when the orbit is computed to the order (in ϵ) needed to generate the usual gradient and curvature drifts, an additional shear-related term arises. Although this term is formally of the same order as the usual drift terms, it is absent from GK because of an inconsistent treatment of the ordering.

II. INCONSISTENCY OF THE USUAL GYROKINETIC TREATMENT

For definiteness I will refer to the gyrokinetic formulation by Catto.¹ His method has the virtues of avoiding a WKB approximation for the perturbed electrostatic potential, and of considering arbitrary magnetic geometries.

Catto obtains for the gyroaveraged Vlasov operator [his Eqs. (6) and (8)]

$$\left\langle \frac{\partial}{\partial t} + \vec{v} \cdot \nabla + \left(\frac{Ze}{Mc} \right) \vec{v} \times \vec{B} \cdot \nabla \right\rangle_{\phi} + \frac{\partial}{\partial t} + (v_{||} \hat{n} + \vec{v}_d) \cdot \nabla_R \quad (1)$$

where $\vec{R}$ is the guiding-center position and

$$\vec{v}_d = \hat{n} \times [(v_{\perp}^2 / 2\Omega) \nabla \ln B + (v_{||}^2 / \Omega) \hat{n} \cdot \nabla \hat{n}] \quad (2)$$

is the usual drift velocity. For the case of collisionless drift instabilities, the right side of (1) leads to a perturbed ion distribution function with the usual resonant denominator

$$\omega - \vec{k} \cdot (v_{||} \hat{n} + \vec{v}_d) = \omega - k_{||} v_{||} - k_{\perp} v_d \quad (3)$$

[see Catto's Eq. (16)].

At what point is the magnetic field unit vector $\hat{n}$ to be defined? Since $\hat{n}$ varies over the fast (Ω^{-1}) time scale by only a fractional amount ϵ , the question may appear irrelevant to the order of accuracy required here. This is not the case. Suppose we define $k_{\perp}(x)$ and $k_{||}(x)$ with reference to the direction $\hat{n}(x)$ of the magnetic field at some x . Then, for $\delta x \sim$ gyroradius, we have $\vec{k} \cdot \hat{n}(x + \delta x) = k_{||}(x) + k_{\perp}(x) \delta x / L_S$ where L_S is the shear length. Near the mode-rational surface we have $k_{||} \sim k_{\perp} \epsilon$; therefore the correction term $k_{\perp} \delta x / L_S \sim k_{\perp} \epsilon$ is of the same order as $k_{||}(x)$. Changing by an amount $\sim \rho_i$ the position at which $k_{||}$ in Eq. (3) is evaluated, thus introduces a correction term that is itself of the same order as each of the three terms in Eq. (3).

To obtain the perturbed distribution function $f_1(\vec{r}, \vec{v}, t)$, the usual GK procedure is to evaluate $\hat{n}$ (or $k_{||}$) at $\vec{r}$. [Catto's Eq. (18) for the perturbed ion density is obtained in this way.] For the slab model with $v_d = 0$, this implies that there is

no ion resonant contribution at the mode-rational surface $x = 0$ (since the resonant $v_{||} = \omega/k_{||}(x) \rightarrow \infty$ there). But this conclusion is false. As we shall see, the correct ion resonant denominator (for $\vec{v}_d = 0$) is

$$[\omega - k_{||}(x_{gc})v_{||}] \approx [\omega - k_{||}(x)v_{||} - k_{\perp}v_{||}v_{\perp} \cos \alpha / (\Omega L_g)], \quad (4)$$

where $x_{gc} \approx x + \Omega^{-1}v_{\perp} \cos \alpha$ and $k_{\perp} = \vec{k} \cdot \hat{n} \times \hat{x}$. The last term, $k_{\perp}v_{||}v_{\perp} \cos \alpha / (\Omega L_g)$, is of kinematic origin and is absent from the usual GK result. Physically, the ions at position x at time t have guiding-center positions ranging over $\sim [x - \rho_i, x + \rho_i]$; they therefore sample a range of $k_{||}$ values (see Linsker^{2,3} and Bakshi⁴).

III. GYROKINETIC VERSUS PRESENT ANALYSIS FOR A CYLINDRICAL SYSTEM

To see explicitly the difference between the usual GK approach and the present result, consider the cylindrically symmetric configuration $\vec{B}(r) = [0, B_{\theta}(r), B_z(r)]$ with shear and gradient scale lengths L_g and L_B respectively. (Definitions and details of the calculation are given in the Appendix.) I choose the conventional ordering [here $\rho_i = \Omega^{-1}(T/M)^{1/2}$]:

$$\begin{aligned} \epsilon &= \rho_i/r \\ \rho_i/(\text{all equilibrium gradient scale lengths}) &\sim O(\epsilon) \\ \omega/\Omega &\sim O(\epsilon) \\ k_{\perp}\rho_i &\sim O(1) \\ k_{||}\rho_i &\sim O(\epsilon). \end{aligned} \quad (5)$$

The unperturbed distribution function can be taken to be

$$f_0(v^2, r_0) = \left(\frac{M}{2\pi T}\right)^{3/2} n_0 \exp\left(-\frac{Mv^2}{2T}\right) \quad (6)$$

where n_0 and T are functions of r_0 , the guiding center position.

Now choose an electrostatic potential perturbation of the form

$$\phi(\vec{r}, t) = \int_{-\infty}^{\infty} dk_x \hat{\phi}(k_x) \exp[i(k_x x + k_\theta \theta + k_z z - \omega t)] \quad (7)$$

where x is the radial distance from the mode-rational surface. When integrating over the unperturbed orbit, it is crucial to calculate the orbital velocities through order $(v\epsilon)$ corrections; otherwise the usual magnetic drift terms will not emerge, and neither will the additional kinematic term in Eq. (4). [Cf. Krall and Rosenbluth⁵ for an orbit integration that does not take into account these order $(v\epsilon)$ corrections.] A detailed calculation yields for the perturbed distribution function at time $t = 0$:

$$f_1(\vec{r}, t = 0; v_{||}, v_\perp, \alpha) = -\frac{ef_0}{T} \left\{ \phi(\vec{r}, t = 0) + i \left[\omega - \omega_* \left(1 - (3/2)\eta + \frac{Mv^2}{2T}\eta \right) \right] \exp[i(k_\theta \theta + k_z z)] H \right\}, \quad (8)$$

where $\eta = d \ln T / d \ln n_0$, $\omega_* = k_\perp T / (e B L_n)$, $L_n^{-1} = d \ln n_0 / dr$, and

$$H = \int_{-\infty}^0 dt' \int_{-\infty}^{\infty} dk_x \hat{\phi}(k_x) \exp i[k_x x' + k_\theta(\theta' - \theta) + k_z(z' - z) - \omega t'] \quad (9)$$

where the unperturbed orbit is x' , θ' , $z'(t')$ with $x' = x$, $\theta' = \theta$, $z' = z$ at $t' = 0$. As shown in the Appendix,

$$H = -i \int_{-\infty}^{\infty} dk_x \hat{\phi}(k_x) \exp[ik_x x + i v_\perp \Omega^{-1} (k_x \cos \alpha - k_\perp \sin \alpha)] \\ \times J_0[v_\perp \Omega^{-1} (k_\perp^2 + k_x^2)^{1/2}] \\ \times [k_\parallel(x) v_\parallel - \omega + k_\perp (\Omega L_s)^{-1} (v_\perp^2 F_g - v_\parallel^2 F_c + v_\perp v_\parallel \cos \alpha)]^{-1}, \quad (10)$$

where $F_g = (1/2) L_s / L_B$ and $F_c = (B_\theta^2 L_s) / (B^2 r)$ are the gradient and curvature drift coefficients, and J_0 is a Bessel function. The last term, $v_\perp v_\parallel \cos \alpha$, is the additional kinematic term discussed earlier. Note that all terms in the denominator are formally of the same order in ϵ .

To see the effect of this kinematic term on the perturbed ion density, let $F_c = F_g = 0$ for simplicity. (The case of arbitrary $F_{c,g}$ is considered in the Appendix.) Then

$$n_1(\vec{r}, t = 0) = -\frac{en_0}{T} \left\{ \phi(r, t = 0) + \exp[i(k_\theta \theta + k_z z)] [1 - (\omega_*/\omega)] \right. \\ \left. \times \int_{-\infty}^{\infty} dk_x \hat{\phi}(k_x) \exp(ik_x x) \Gamma_0 F(x, k_x) \right\}, \quad (11)$$

$$F(x, k_x) = \int_{-\infty}^{\infty} d\xi \left\{ \zeta' Z(\zeta') F_1(\xi - x, k_x) \right. \\ \left. + \zeta'^2 [1 + \zeta' Z(\zeta')] F_2(\xi - x, k_x) \right\}, \quad (12)$$

$$F_{1,2} = (2\pi)^{-1} \int_{-\infty}^{\infty} dk'_x \Gamma_0^{-1} \exp(-b_1) I_0(b_2) \\ \times \exp[i(k_x - k'_x)(\xi - x)] G_{1,2}(k_x, k'_x), \quad (13)$$

$$G_1 = 1 + \left(\frac{\omega_* \eta}{\omega - \omega_*} \right) \left[\frac{1}{2} + b_1 - b_2 \frac{I_1(b_2)}{I_0(b_2)} \right], \quad (14)$$

$$G_2 = - \left(\frac{\omega_* \eta}{\omega - \omega_*} \right).$$

Here $\zeta' = 2^{1/2} \rho_i \lambda / |\xi|$, $\lambda = (1/2) \omega |L_S / (\omega_* L_n)|$,

$$b = (k_1^2 + k_x^2) \rho_i^2, \quad b' = (k_1^2 + k_x'^2) \rho_i^2,$$

$$b_1 = (1/2) (b + b'), \quad b_2 = (bb')^{1/2},$$

$\Gamma_j = \exp(-b) I_j(b)$, and I_j are Bessel functions.

If the kinematic term were absent, one would have F replaced by

$$\begin{aligned} \tilde{F}(x, k_x) = & \left\{ 1 + \left(\frac{\omega_* \eta}{\omega - \omega_*} \right) \left[\frac{1}{2} + b - b \frac{\Gamma_1}{\Gamma_0} \right] \right\} \zeta Z(\zeta) \\ & - \left(\frac{\omega_* \eta}{\omega - \omega_*} \right) \zeta^2 [1 + \zeta Z(\zeta)] \end{aligned} \quad (15)$$

where $\zeta = 2^{1/2} \rho_i \lambda / |x|$. Eq. (15) agrees with Catto's Eq. (18). The kinematic term therefore causes $n_1(x)$ to depend upon the Z function evaluated not only at x , but globally, with a complicated (wavenumber-dependent) weighting determined by F_1 and F_2 .

What is the practical effect of this modification? Consider for simplicity the case $\eta = 0$ (no temperature gradient). Then

$$F(x, k_x) = \int_{-\infty}^{\infty} d\xi \zeta' Z(\zeta') F_1(\xi - x, k_x), \quad (16)$$

F_1 is given by Eq. (13) with $G_1 = 1$, and the usual GK treatment gives $\tilde{F}(x) = \zeta Z(\zeta)$. Figures 1-3 show the dependence of $\tilde{F}$ and F on x , for selected values of k_1 , k_x , and λ . We find that near $x = 0$ F and $\tilde{F}$ differ significantly when $|\lambda|$ is of order unity. This can occur either when shear is strong ($L_S/L_N \sim 1$) and $|\omega| \sim \omega_*$, or when $|\omega| \ll \omega_*$ as can happen for ion drift waves with $T_e \sim T_i$.

The high-frequency limit of Eq. (13) (with $G_1 = 1$) is the Gaussian weight function

$$F_1(\xi - x; k_x \rightarrow \infty) = \pi^{-1/2} \exp [-(\xi - x)^2 / (2\rho_i^2)] .$$

If the corresponding $F(x; k_x \rightarrow \infty)$ were plotted in Figures 1-3, it would virtually coincide with $F(x; k_x = 0)$ for the values of λ depicted. Note also that for given $k_x \neq 0, \infty$, $F(x, k_x)$ is assymmetric in x , although we do have $F(x, k_x) = F(-x, -k_x)$.

For the case of collisionless drift waves in sheared-slab geometry, an integral equation based on the usual GK treatment (using $\tilde{F}$, without the additional kinematic term) has been studied numerically by Tang, Rewoldt, and Frieman,⁶ and analytically by Chen, et al.,⁷ and by Lee, Chen, and Nevins.⁸ The present work suggests that when $|\lambda| \sim$ order unity, the drift eigenmode shape for $|x| \lesssim \rho_i$ may depend strongly upon whether the function F or the incorrect $\tilde{F}$ is used for the perturbed ion density. At least for the electron drift mode, the effect discussed here is important only for strongly sheared systems. Thus although recent analytic stability work by Lee, Chen, and Nevins⁸ uses the incorrect form $\tilde{F} = \zeta Z(\zeta)$ from Eq. (15) with $\eta = 0$, there is at present no reason to suspect that the "universal" mode in slab geometry is ever unstable. A general stability proof using the correct Eq. (16) would however be desirable.

The effect of the additional kinematic term on the ion drift mode (for which $|\lambda| \sim 1$ can occur even with weak shear, e.g. $L_S/L_n \sim 20 - 40$) is being studied. Further work on the solutions of the quasineutrality equation (and Ampere's law, for the electromagnetic case) in cylindrical or slab geometry, using an integral formulation and including the kinematic term neglected by the GK method, will be presented elsewhere. (The method of solution is briefly outlined in Ref. 3.)

IV. CONCLUSION

In the above analysis, the correct perturbed distribution function has been found by explicitly integrating over the unperturbed orbit calculated to the necessary accuracy. We see that one can recover the correct results from Eq. (3) by making the ad hoc replacement of $k_{||}(x)$ by $k_{||}(x_{gc})$. The usual GK analysis gives no prescription of this sort, because it does not properly account for the fractional variations of order ϵ in $\vec{k} \cdot (\vec{v}_{||} \hat{n})$ on the fast time scale. We have seen that for the ordering (5) these variations are of the same order as are the terms ω , $\vec{k} \cdot (\vec{v}_{||} \hat{n})$, and $\vec{k} \cdot \vec{v}_d$ themselves, and therefore cannot be neglected.

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APPENDIX

The perturbed ion density will be derived by integrating the Vlasov equation over the unperturbed orbit, for a cylindrically symmetric collisionless system with $\beta = 0$ and zero equilibrium electric field.

The equilibrium magnetic field is written as $\vec{B}(r) = (0, B_\theta(r), B_z(r))$, with $B_\theta = B \sin \psi$, $B_z = B \cos \psi$, pitch $P(r) = rB_z/B_\theta$. Define $L_B = B/B'$, $L_\psi = [\psi' - (1/2)r^{-1} \sin 2\psi]^{-1} = -(r^2 + P^2)/(rP')$. (Primes denote d/dr .) In terms of the canonical momenta p_θ and p_z one can define new constants of the motion $r_0, v_{||0}$ implicitly by ($c \equiv 1$)

$$p_\theta - er_0 A_\theta(r_0) = mr_0 v_{||0} \sin \psi(r_0)$$

$$p_z - eA_z(r_0) = mv_{||0} \cos \psi(r_0).$$

The linearized Vlasov equation is

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \nabla f_1 + \frac{e}{m} \vec{v} \times \vec{B} \cdot \nabla_v f_1 - \frac{e}{m} \nabla \phi \cdot \nabla_v f_0 = 0. \quad (A1)$$

The perturbation ϕ and equilibrium distribution function f_0 are defined in the main text. We have

$$f_1(\vec{r}, \vec{v}, t = 0) = \frac{e}{m} \int_{-\infty}^0 \nabla' \phi \cdot \nabla_v f_0 dt' \quad (A2)$$

for the particle orbit $(\vec{r}', \vec{v}'; t')$ with $\vec{r}'(0) = \vec{r}$ and $\vec{v}'(0) = \vec{v}$.

Parametrize the orbit by $s = t'[1 + O(\epsilon)]$ and define ψ, L_B , and $\Omega = eB/M$ at the fixed position r . Define the coordinate system $(\hat{r}, \hat{\eta}, \hat{\zeta})$ by $\hat{\zeta} = \hat{B}(r)$, $\hat{\eta} = \hat{\zeta} \times \hat{r}$. Then the $\hat{\eta}$ and $\hat{\zeta}$ components of particle velocity are, to the order needed to give the magnetic drifts,

$$\begin{aligned} v'_{\hat{\eta}}(s) &= v_{\perp} \cos(\Omega s + \alpha) \\ &+ \Omega^{-1} \{ (1/2) v_{\perp}^2 (L_B^{-1} - r^{-1} \cos^2 \psi) - r^{-1} v_{\parallel}^2 \sin^2 \psi + v_{\perp} v_{\parallel} \psi' \cos \alpha \} \\ &- (1/4) \Omega^{-1} v_{\perp} v_{\parallel} (r^{-1} \sin 2\psi + 2\psi') \cos(\Omega s + \alpha) \\ &+ (1/2) \Omega^{-1} v_{\perp}^2 (r^{-1} \cos^2 \psi - L_B^{-1}) \cos 2(\Omega s + \alpha), \end{aligned} \quad (A3)$$

$$\begin{aligned} v'_{\hat{\zeta}}(s) &= v_{\parallel} + \epsilon \tilde{v}_{\parallel} \\ &+ \Omega^{-1} [r^{-1} v_{\perp} v_{\parallel} \sin^2 \psi - v_{\perp}^2 \psi' \cos \alpha] \cos(\Omega s + \alpha) \\ &+ (1/8) \Omega^{-1} v_{\perp}^2 (r^{-1} \sin 2\psi + 2\psi') \cos 2(\Omega s + \alpha), \end{aligned} \quad (A4)$$

with

$$s = t' \left\{ 1 + \Omega^{-1} \left[(1/4) v_{\parallel} \left(3r^{-1} \sin 2\psi + 2\psi' \right) + L_B^{-1} v_{\perp} \cos \alpha \right] \right\}.$$

(Here $\tilde{v}_{\parallel}$ is arbitrary, of order v_{thermal} , and reflects the order- ϵ freedom in the definition of $v_{\parallel}$.)

For arbitrary k_θ, k_z , of dimensions 1 and (length)⁻¹ respectively, define at fixed r :

$$\begin{aligned} k_{||} &= k_z \cos \psi + r^{-1} k_\theta \sin \psi \\ k_{\perp} &= -k_z \sin \psi + r^{-1} k_\theta \cos \psi. \end{aligned} \quad (A5)$$

Then, to the required order,

$$\begin{aligned} &k_z [z'(s) - z] + k_\theta [\theta'(s) - \theta] \\ &= k_{\perp} v_{\perp} \Omega^{-1} [\sin(\Omega s + \alpha) - \sin \alpha] \\ &+ s \left[k_{||} v_{||} + k_{\perp} \Omega^{-1} \left((1/2) L_B^{-1} v_{\perp}^2 - r^{-1} v_{||}^2 \sin^2 \psi + L_s^{-1} v_{||} v_{\perp} \cos \alpha \right) \right]. \end{aligned} \quad (A6)$$

The terms multiplying $sk_{\perp}\Omega^{-1}$ are respectively the VB drift, the curvature drift, and the additional kinematic term discussed in the main text.

To obtain Eq. (10) from Eqs. (9) and (A6) note that, to the required order, $k_x [x'(s) - x] \approx k_x v_{\perp} \Omega^{-1} [\cos \alpha - \cos(\Omega s + \alpha)]$, and use $\exp(ix \sin y) = \sum_{n=-\infty}^{\infty} J_n(x) e^{iny}$ and the ordering $\omega \ll \Omega$.

To derive Eq. (11)-(14) from Eqs. (8)-(10), and to generalize to include gradient and curvature drifts, proceed as follows. The density perturbation is

$$\begin{aligned}
 n_1(\vec{r}, t = 0) &= -\frac{en_0}{T} \left\{ \phi(\vec{r}, t = 0) \right. \\
 &+ \exp[i(k_\theta \theta + k_z z)] \int dk_x \hat{\phi}(k_x) \\
 &\times [\omega - \omega_*(1 - (3/2)\eta - \eta \partial/\partial\gamma)] D(\gamma = 1) \Big\}, \quad (A7)
 \end{aligned}$$

where

$$\begin{aligned}
 D(\gamma) &= -\frac{L_S}{2^{1/2} k_\perp v_T} \pi^{-(3/2)} \int dv_{||} v_\perp dv_\perp d\alpha \\
 &\times \exp[-\gamma(v_{||}^2 + v_\perp^2) + i v_\perp (k_x \cos \alpha - k_\perp \sin \alpha) + i k_x x] \\
 &\times J_0(v_\perp y) [F_C v_{||}^2 + A v_{||} + C]^{-1}, \quad (A8)
 \end{aligned}$$

$$A = -(x + v_\perp \cos \alpha),$$

$$C = \frac{\omega L_S}{2^{1/2} k_\perp v_T} - v_\perp^2 F_g, \quad (A9)$$

$$y = (k_\perp^2 + k_x^2)^{1/2}, \quad v_T = (T/M)^{1/2}.$$

In (A8) and (A9) $v_{||}$ and $v_\perp$ are in units of $2^{1/2} v_T$; x, k_x^{-1} , and $k_\perp^{-1}$ are in units of $2^{1/2} v_T / \Omega$; and the velocity contours are real for $\text{Im } \omega > 0$ and defined by analytic continuation for $\text{Im } \omega \leq 0$.

I first simplify the $v_\perp$ and α integrations by introducing the factor $1 = \int d\xi \delta(\xi - x - v_\perp \cos \alpha) = \int dk'_x (2\pi)^{-1} \times \exp[ik'_x(-\xi + x + v_\perp \cos \alpha)]$. Using $A = -\xi$ and $ik_x v_\perp \cos \alpha = ik_x(\xi - x)$, the α integration yields

$$D(\gamma) = (2\pi)^{-1} \left| \frac{L_S}{2^{1/2} k_\perp v_T} \right| \int dk'_x d\xi v_\perp dv_\perp$$

$$\begin{aligned} & \times \exp[-\gamma v_{\perp}^2 + i k'_x x + i(k_x - k'_x)\xi] \\ & \times 2 J_0(v_{\perp} y) J_0(v_{\perp} y') D_1(\gamma) \end{aligned} \quad (A10)$$

with $y' = (k_{\perp}^2 + k_x'^2)^{1/2}$ and

$$D_1(\gamma) = -\pi^{-1/2} \operatorname{sgn}(k_{\perp} L_s) \int dv_{\parallel} \exp(-\gamma v_{\parallel}^2) [F_C v_{\parallel}^2 + A v_{\parallel} + C]^{-1}. \quad (A11)$$

Consider first the case $F_C = 0$. Define

$$\zeta' = |\xi|^{-1} \left[\omega \left| \frac{L_s}{2^{1/2} k_{\perp} v_T} \right| - v_{\perp}^2 F_g \operatorname{sgn}(k_{\perp} L_s) \right]. \quad (A12)$$

Then

$$D_1(\gamma) = |\xi|^{-1} Z(\gamma^{1/2} \zeta'). \quad (A13)$$

If $F_C \neq 0$, the $v_{\parallel}$ integration may still be done readily, provided $\operatorname{Im} \omega > 0$. (If $\operatorname{Im} \omega \leq 0$, the complex contour introduces additional complications.) Integrating by the method of partial fractions, one obtains for $\operatorname{Im} \omega > 0$,

$$\begin{aligned} D_1(\gamma) &= [F_C (F_b - F_a)]^{-1} \operatorname{sgn}(k_{\perp} L_s) [Z(F_a \gamma^{1/2}) + Z(-F_b \gamma^{1/2})], \\ F_{a,b} &= (1/2) F_C^{-1} [\xi \pm (\xi^2 - 4 F_C C)^{1/2}], \end{aligned} \quad (A14)$$

where the signs are chosen such that $\text{Im } F_a > 0$ and $\text{Im } F_b < 0$.

The v_1 integration may be done analytically if $F_g = 0$.

In that case D_1 is independent of v_1 and one has⁹

$$\begin{aligned} & \int_0^\infty 2v_1 \, dv_1 \exp(-\gamma v_1^2) J_0(v_1 y) J_0(v_1 y') \\ &= \gamma^{-1} \exp(-b_1/\gamma) I_0(b_2/\gamma) \end{aligned} \quad (A15)$$

where $b_1 = (1/2)(b + b')$, $b_2 = (bb')^{1/2}$, $b = (1/2)y^2$, and $b' = (1/2)y'^2$. From Eqs. (A13) and (A15) one then easily obtains Eqs. (11) to (14) of the main text, applicable when $F_c = F_g = 0$. [When $F_g \neq 0$, the v_1 integration is done numerically.]

REFERENCES

- ¹P. J. Catto, Plasma Physics 20, 719 (1978). References to earlier gyrokinetic treatments are cited therein.
- ²R. Linsker, 1980 Sherwood Meeting, paper 3B26 (Tucson, AZ; April 23-25, 1980).
- ³R. Linsker, in Proceedings of the Reversed Field Pinch Theory Workshop (Los Alamos Scientific Laboratory, April 28 - May 2, 1980), to be published.
- ⁴After completing the calculations described herein, I learned that the existence of the kinematic term has been noted earlier by P. Bakshi (1977 Sherwood Meeting), and that the $\exp(-b_1)I_0(b_2)$ behavior in Eq. (13) was derived by B. Coppi, M. N. Rosenbluth, and R. Z. Sagdeev, Physics of Fluids 10, 582 (1967).
- ⁵N. A. Krall and M. N. Rosenbluth, Physics of Fluids 8, 1488 (1965).
- ⁶W. M. Tang, G. Rewoldt, and E. A. Frieman, "Integral Equation Analysis of Drift Wave Eigenmodes in a Sheared Slab Geometry," Report PPPL-1627 (Princeton University, Feb. 1980).
- ⁷L. Chen, C. Z. Cheng, E. A. Frieman, P. N. Guzdar, P. K. Kaw, W. W. Lee, W. M. Nevins, C. R. Oberman, H. Okuda, G. Rewoldt, P. H. Rutherford, W. M. Tang, R. B. White, in Plasma Physics and Controlled Nuclear Fusion Research 1978, v. 1, p. 763 (IAEA, Vienna, 1979).
- ⁸Y. C. Lee, L. Chen, and W. M. Nevins, "Stability of Drift-Wave Eigenmodes with Arbitrary Radial Wavelengths," Report PPPL-1544 (Princeton University, May 1979), to be published in Nuclear Fusion.

⁹G. N. Watson, Theory of Bessel Functions, p. 395 (Cambridge, 1958).

(PPPL-802182)
 Fig. 1. The functions $F(x, k_x)$ Eq. (16) and $F(x) = \zeta Z(\zeta)$, $\zeta = 2^{1/2} \rho_1 \lambda / |x|$, for $\lambda = 3$ and $k_1 \rho_1 = 2^{1/2}$. Plotted are the real and imaginary parts of F ; $F(x; k_x \rho_1 = 0)$, labelled by '0', and $F(x; k_x \rho_1 = 2^{1/2})$, dashed and labelled by '2'.

(PPPL-802183)

Fig. 2. Same as Fig. 1, but for $\lambda = 1$.

(PPPL-802184)

Fig. 3. Same as Fig. 1, but for $\lambda = 2 - i$.