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APPLICATIONS OF A REVIVED EULER-SHOHAT
PERTURBATION EXPANSION TECHNIQUE
IN THE STUDY OF OCEAN CIRCULATION

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ABSTRACT

Consideration of geophysical vertical plane flows in estuaries and on continental shelves results in nonlinearly coupled partial differential equations for flow and density variables. For the case of steady flow, the equations are reduced to ordinary differential equations by the use of similarity transformations. The remaining nonlinearly coupled ordinary differential equations are solved using a revised Euler-Shohat perturbation technique. An existence theorem for this application of the technique is stated and proved.

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1. Introduction

Mathematical solutions of the equations governing the balances of heat, momentum, mass and energy in the marine environment are in their totality, and at best, impossible to obtain. Consequently, many types of mathematical approaches, running the gamut from purely analytical to entirely numerical, have been tried and accomplished. Examples of this general and all encompassing statement are too numerous to even refer to. Between the analytical and numerical ends of the modeling spectrum are the quasi-analytical, quasi-numerical, approximation techniques, and within this band of the modeling spectrum lie the perturbation expansion methods. Excellent, state-of-the-art reviews and cookbook recipes for perturbation kinds of approaches to problem solving are given in Cole (1968), Van Dyke (1964), ^X(Ames, (1965 and 1972) Kantorovich and Krylov (1964) Nayfeh (1973), an infinite series by Bellman, et. al. (1962; 1963 a, b, c, d; 1964 a, b; 1965 a, b; 1966 a, b, c, d; 1967; 1965 a, b; 1966 a, b, c, d; 1967; 1968 a, b, c; and a host of others.

Herein, a type of perturbation solution to non-linearly coupled, non-linear partial differential equations governing some aspects estuarine and continental shelf physics which has been considered by several investigators, including Hansen and Rattray (1972), Pietrafesa (1973) and Pietrafesa and Rattray (1974), has yielded seemingly meaningful results but

has been the subject of some controversy concerning its mathematical validity. This note will attempt to explore the usefulness and validity of these past results.

2. System of Equations

$$-fv = -\frac{1}{\rho_c} P_x + (Au_z)_z \quad (1)$$

$$fu = (Av_z)_z \quad (2)$$

$$P'_z = g\rho_c(1 - \alpha T) \quad (3)$$

$$u_x + w_z = 0 \quad (4)$$

and

$$UT_x + wT_z = (K_H T_z)_z + (K_L T_x)_x \quad (5)$$

is the system of interest, as an example, where the subscripts x and z denote partial differentiations with respect to those spatial variables, the subscript c denotes a constant value and H and L denote vertical and horizontal components of the diffusion coefficients.

One could try to solve these equations taking either a separation transformation approach (Ames, 1965) and or a boundary layer type approach.

If, we attempt a separation transformation approach to the equations (1-5), we could introduce

the following system of variables:

$$\zeta = (x/L) \quad (6)$$

$$\eta = (z/h_L)\zeta^\alpha, \quad (7)$$

$$h(x) = h_L \zeta^\alpha \quad (8)$$

$$\phi(\eta) = \frac{\psi(x,z)}{\psi_L} \zeta^{\alpha_1}, \quad (9)$$

$$\phi(\eta) - \theta(0)\zeta^{-\lambda} = \frac{(\rho(x,z) - \bar{\rho} + 1)\zeta^{-\lambda}}{\rho} \quad (10)$$

where

$$\bar{\rho} = \frac{\rho(L, h_L) + \rho(L, 0)}{2} \quad (11)$$

and

$$\rho = 2\bar{\rho} \quad (12)$$

$$A_v = A_{vc}(x)\zeta^{-\alpha_3}, \quad (13)$$

$$K_L = K_{Lc}(x)\zeta^{-\alpha_4}, \quad (14)$$

$$K_v = K_{vc}(x)\zeta^{-\alpha_5}, \quad (15)$$

$$V(\eta) = \frac{v(x,z)}{\hat{V}} \zeta^{-\alpha_7}. \quad (16)$$

$$T_{ew} = i \cdot \underline{t}(x, 0)\zeta^{-\alpha_8}, \quad \text{ew, east-west component} \quad (17)$$

$$T_{nw} = j \cdot \underline{t}(x, 0)\zeta^{-\alpha_9}, \quad \text{nw, north-west component} \quad (18)$$

$$Q(x) = \frac{-a_T q \zeta^{-9}}{b_v S_o} + b_v S_o (E - P)\zeta^{-\alpha_9}, \quad (19)$$

and

$$m \frac{\partial Z_o}{\partial y} = \frac{\partial Z}{\partial y} \zeta^{-\alpha_{11}} \text{ where } m = 0, 1. \quad (20)$$

Substituting these transformed variables into the system of equations (1) through (5), with the addition of a y-independent barotropic pressure gradient in the downstream direction, i.e., a constant surface slope into the y-momentum equation yields the succeeding system of non-linearly coupled ordinary differential equations and appropriate conditions for the separation transformation of ζ :

$$\frac{d^5 \phi}{d\eta^5} + Ta \frac{d\phi}{d\eta} = -Ra ((\alpha + \lambda) \frac{d\theta}{d\eta} + \alpha \eta \frac{d^2 \theta}{d\eta^2}) + \frac{gh_L^5 f}{\psi_L A_v^2} m \frac{\partial Z_o}{\partial y}, \quad (21)$$

$$(I^2 + \alpha^2 \eta^2) \frac{d^2 \theta}{d\eta^2} + (\alpha \eta (3\lambda - 1) - M(\lambda - 2\alpha - 1)) \quad (22)$$

$$\phi) \frac{d\theta}{d\eta} + (\lambda (2\lambda - \alpha - 1) + M\lambda \frac{d\phi}{d\eta}) \theta = 0, \text{ and}$$

$$\frac{dV}{d\eta} + \frac{h_L g \hat{\rho}}{f L \rho_0 \hat{V}} \lambda \theta + \alpha \eta \frac{d\theta}{d\eta} - \frac{A_v \psi_L}{h_L^3 f} \frac{d^4 \phi}{d\eta^4} = 0, \text{ with} \quad (23)$$

$$\alpha_3 + 4\alpha = \lambda - 1 - \alpha_1, \quad \alpha_4 - 2 - 2\alpha - \alpha_5 = 0,$$

$$\alpha_3 + \lambda - 1 - \alpha_1 = 0, \quad \alpha_3 + 2\alpha = 0, \quad (24)$$

and

$$\alpha_1 - 1 - \alpha - \alpha_5 = 0, \quad \alpha_7 - \alpha_3 = \alpha_1 + 3\alpha,$$

with the added condition that if $m = 1$ then $\alpha - \lambda - 1 = 0$.

The behavior of the system is characterized by five dimensionless coefficients. $I^2 = L^2 K_v / h_L^2 K_L$ is a relative isotropy number for turbulent mixing and effectively becomes a ratio of horizontal to vertical mixing time scales. $Ra = g \rho h_L^4 / A_v \psi_L L \rho_0$ is a modified form of the Rayleigh number, well-known in thermal convection problems. This parameter expresses the ratio of buoyancy to frictional forces and is important in determining the effects of stratification in modifying or controlling the vertical plane circulatory dynamics. $M = \psi_L L / K_L h_L$ is a measure of the strength of circulation relative to horizontal mixing; the horizontal and vertical distribution of hydrographic properties is a function of M . $Ta = h_L^4 f^2 / A_v^2$ is a ratio of rotational to frictional forces and is equivalent to an inverse square of the vertical Ekman number. The fourth root of Ta is equal to the ratio of the total depth to the depth of frictional influence. Ta is also known as the Taylor number. $\delta = h_L / L$ is an aspect ratio.

Note that L here denotes the width of the surface feature and h is the depth of the level of "no-motion".

There has been some conjecture in the literature concerning the possibility of being able to solve this system of equations in their coupled form. One cannot easily follow typical perturbation expansion techniques

(Nayfeh, 1973) because none of the parameters, which appear in multiplicative association with the terms which are coupled in ϕ and θ and their derivatives, are "small".

The following is a mathematical justification and proof of the existence and validity of a solution to the complete system of equations and boundary conditions which have been presented. There is a mathematical rationale presented for use of the Shohat Renormalization Technique (Ames, 1965) to uncouple and solve the system of equations in an iterative, perturbative scheme.

we now consider the boundary conditions:

$$\psi_x(x,0) = 0 \quad (25)$$

$$A_v v_z(x,0) = - \frac{gA_v}{f} \rho_x(0,x) + A_v \left(\frac{A_v}{f} \psi_{zz} \right)_{zz} \quad (26)$$

$$\psi_{zz}(x,0) = - \frac{i \cdot I(x,0)}{A_v} \quad (27)$$

$$K_v \rho_z(x,0) = Q \text{ where } Q = Q_c \zeta^{\alpha_6}, \quad (28)$$

where q is the flux of heat across the surface and E-P is the net evaporation-precipitation balance,

$$\psi_z(x,h) = 0 \quad (29)$$

$$\psi_x(x,h) = 0 \quad (30)$$

$$v(x,0) = v_{gs}, \text{ where } v_{gs} = v_{gsc} \zeta^{\alpha_{10}} \text{ and} \quad (31)$$

$$K_{V\rho_z}(x,h) - K_{Lh\rho_x}(x,h) = B, \text{ where } B = B_c \zeta^{\alpha_{12}}. \quad (32)$$

These conditions could then be transformed in our new variables to be:

$$\frac{d\theta(0)}{d\eta} = Q$$

$$\phi(1) = 0 \quad (33)$$

$$\frac{d\phi}{d\eta}(1) = 0 \quad (34)$$

$$\phi(0) = 0 \quad (35)$$

$$\frac{d^2\phi}{d\eta^2}(0) = -\tau_x \quad (36)$$

$$\frac{d^4\phi(0)}{d\eta^4} = \tau_y + \lambda R \theta(0) \quad (37)$$

$$(I^2 + \alpha^2) \frac{d\theta}{d\eta}(1) + \alpha\lambda\theta(1) = 0 \quad (38)$$

$$\phi(1) = a_0 \quad (39)$$

with the additional constraint that $\theta(1)-\theta(0)=2$. This latter constraint simply sets the vertical plane condition for the number of isopycnals crossing the seaward wall of the regime of interest. This condition is derived from: $\rho(L,h)-\rho(L,0)=\Delta\rho=a$ specified constant. Note: $R, S, T, \alpha, I, \tau_x, \tau_y, \lambda, M$ and a_0 are fixed constants.

We shall now prove a mathematical theorem.

3. Mathematical Theorem

If $I \neq 0$ and if $|a_0|$ is sufficiently small, then there exists a solution (ϕ, θ) of the boundary value problem 21-39. More over, the solution can be obtained by a method of successive approximations.

PROOF: This proof exploits the homogeniety of equation 22 and the boundary condition. Indeed if we write $\theta = a_0 \theta^*$, then 21-31 becomes

$$\frac{d^5 \phi}{d\eta^5} + T_a \frac{d\phi}{d\eta} = -a_0 R((\alpha + \lambda) \frac{d\theta^*}{d\eta} + \alpha \eta \frac{d^2 \theta^*}{d\eta^2}) + S \quad (40)$$

$$(I^2 + \alpha^2 \eta^2) \frac{d^2 \theta^*}{d\eta^2} + (\alpha \eta (3\lambda - 1) - M(\lambda - 2\alpha - 1)\phi) \frac{d\theta^*}{d\eta} \quad (41)$$

$$+ (\lambda(2\lambda - \alpha - 1) + M\lambda \frac{d}{d\eta}) \theta^* = 0$$

$$\phi(1) = 0 \quad (42)$$

$$\frac{d\phi}{d\eta}(1) = 0 \quad (43)$$

$$\phi(0) = 0 \quad (44)$$

$$\frac{d^2 \phi}{d\eta^2}(0) = -\tau_x \quad (45)$$

$$\frac{d^4 \phi}{d\eta^4}(0) = \tau_y + \lambda R a_0 \theta^*(0) \quad (46)$$

$$(I^2 + \alpha^2) \frac{d\theta^*}{d\eta}(1) + \alpha \lambda = 1 \quad (47)$$

$$\theta^*(1) = 1. \quad (48)$$

Let θ_0 denote the solution of 40, 42-46 when $a_0 = 0$. It is clear that this latter linear fifth order boundary-value problem has a solution ϕ_0 . For any continuous function f on $(0,1)$, let $||f||$ denote $\sup f(\eta)$. Let θ_0^* be the solution of 41, 47, 48 with ϕ replaced by ϕ_0 . It is clear that θ_0^* also exists and so let k be chosen so that $||\theta_0^*|| \leq k$, $||\frac{d\theta_0^*}{d\eta}|| \leq k$ and $||\frac{d^2\theta_0^*}{d\eta^2}|| \leq k$. Now since $I \neq 0$, by continuity there exists a $\delta > 0$ such that the solution θ^* of 41, 47, 48 satisfies $||\theta^*|| \leq 2k$, $||\frac{d\theta^*}{d\eta}|| < 2k$ and $||\frac{d^2\theta^*}{d\eta^2}|| < 2k$, provided ϕ satisfies $||\phi - \phi_0|| < \delta$ and $||\frac{d\phi}{d\eta} - \frac{d\phi_0}{d\eta}|| \leq \delta$. Similarly, then there exists a $\Delta > \delta$ such that the solution ϕ of 40, 42-46 satisfies $||\phi - \phi_0|| \leq \delta$, $||\frac{d\phi}{d\eta} - \frac{d\phi_0}{d\eta}|| \leq \delta$, provided θ^* satisfies $||\theta^*|| \leq 2k$, $||\frac{d\theta^*}{d\eta}|| \leq 2k$, $||\frac{d^2\theta^*}{d\eta^2}|| \leq 2k$ and $|a_0| \leq \Delta$.

We now let a_0 satisfy $|a_0| \leq \Delta$ and consider the two spaces A and A^* of all functions ϕ and θ^* , respectively, with continuous second deviations on $(0,1)$ and satisfying $||\phi - \phi_0|| \leq \delta$, $||\frac{d\phi}{d\eta} - \frac{d\phi_0}{d\eta}|| \leq \delta$, $||\theta^*|| \leq 2k$,

$||\frac{d\theta^*}{d\eta}|| < 2k$, $||\frac{d^2\theta^*}{d\eta^2}|| \leq 2k$. By the above considerations, if θ_1^* is selected from A^* and 40, 42-46 is solved for ϕ_1 with $\theta^* = \theta_1^*$, then ϕ_1 lies in A . Alternatively, if ϕ_2 is selected from A and 41, 47, 48 is solved for

θ_2^* with $\phi = \phi_2$, then θ_2^* lies in A^* . By Granwall's lemma, there exists an L such that if ϕ_1 and ϕ_2 belong to A , then the corresponding solutions θ_1^* and θ_2^* of 41, 47, 48 with $\phi = \phi_1$ and $\phi = \phi_2$ respectively, satisfy

$$\begin{aligned} & ||\theta_1^* - \theta_2^*|| + ||\frac{d\theta_1^*}{d\eta} - \frac{d\theta_2^*}{d\eta}|| + ||\frac{d^2\theta_1^*}{d\eta^2} - \frac{d^2\theta_2^*}{d\eta^2}|| \quad (49) \\ & \leq L (||\phi_1 - \phi_2|| + ||\frac{d\phi_1}{d\eta} - \frac{d\phi_2}{d\eta}||). \end{aligned}$$

Similarly, there exists an H such that if θ_1^* and θ_2^* belong to A^* , then the corresponding solution ϕ_1 and ϕ_2 of 40, 42-46 with $\theta^* = \theta_1^*$ and $\theta^* = \theta_2^*$, respectively, satisfy

$$\begin{aligned} & ||\phi_1 - \phi_2|| + ||\frac{d\phi_1}{d\eta} - \frac{d\phi_2}{d\eta}|| \quad (50) \\ & \leq |a_0| H (||\theta_1^* - \theta_2^*|| + ||\frac{d\theta_1^*}{d\eta} - \frac{d\theta_2^*}{d\eta}|| + ||\frac{d^2\theta_1^*}{d\eta^2} - \frac{d^2\theta_2^*}{d\eta^2}||). \end{aligned}$$

Thus if a_0 also satisfies $|a_0|HL < 1$, then the mapping from A into A^* defined by first selecting θ^* in A^* , solving 40, 42-46 for the corresponding ϕ and then solving 41, 47, 48 with this ϕ for the resulting θ^* , becomes a contraction by virtue of the inequalities 49 and 50. It therefore has a unique fixed point θ^* in P^* which also satisfies 47 and 48 corresponding to this

fixed-point is the solution ϕ of 40 - 48 in A, and this pair of functions constitute a solution of the system because of the contraction the fixed-point (and solution functions) can be obtained by a method of successive approximations based on the map described above. This completes the proof of the theorem.

Now suppose (ϕ, θ) is a solution of 21-39 with $a_0 \neq 0$. Then $(\phi^* = \frac{\phi}{a_0 R}, \theta^* = \frac{\theta}{a_0})$ is a solution of the system

$$\frac{d^5 \phi^*}{d\eta^5} + T_a \frac{d\phi^*}{d\eta} = -((\alpha + \lambda) \frac{d\theta^*}{d\eta} + \frac{d^2 \theta^*}{d\eta^2}) + S^* \quad (51)$$

$$(I^2 + \alpha^2 \eta^2) \frac{d^2 \theta^*}{d\eta^2} + (\alpha \eta (3\lambda - 1) - M^* (\lambda - 2d - 1) \phi^*) \frac{d\theta^*}{d\eta} +$$

$$(\lambda (2\lambda - \alpha - 1) + M^* \lambda \frac{d\phi^*}{d\eta}) \theta^* = 0 \quad (52)$$

$$\phi^* (1) = 0 \quad (53)$$

$$\frac{d\phi^*}{d\eta} (1) = 0 \quad (54)$$

$$\phi^* (0) = 0 \quad (55)$$

$$\frac{d^2 \phi^*}{d\eta^2} (0) = -T_x \quad (56)$$

$$\frac{d^4 \phi^*}{d\eta^4} = T_y + \lambda \theta^* (0) \quad (57)$$

$$(I^2 + \alpha^2) \frac{d\theta^*(1)}{d\eta} + \alpha\lambda = 0 \quad (58)$$

$$\theta^*(1) = 1 \quad (59)$$

where

$$S = a_0 R S^*, \tau_x = a_0 R T_x, \tau_y = a_0 R T_y, M = M^*/a_0 R, \quad (60)$$

and conversely. If we solve this system 51-60 for (ϕ^*, θ^*) with fixed values of $S^*, T_a, \alpha, I, T_x, T_y$, and M^* (so that a_0 does not enter directly into the system) as functions of λ (as well as η), then we can satisfy the difference boundary condition

$$\theta(1) - \theta(0) = 2 \quad (61)$$

simply by writing

$$a_0 = \frac{2}{1 - \theta^*(0)}, \quad (62)$$

assuming only that λ is chosen so that $\theta^*(0) \neq 1$.

Because of 63 and this condition can be assumed if λ is chosen so that the final boundary condition

$$\frac{d\theta}{d\eta}(0) = 0, \text{ i.e.} \quad (63)$$

$$\frac{d\theta^*(0)}{d\eta} = 0, \quad (64)$$

corresponds to the stationary value of $\theta^*(\eta)$ less than but nearest to 1. This last condition determines the correct value of the eigen value λ can be met if

$\theta^*(\eta)$ has at least one stationary value for $\eta < 1$ with suitable λ . Once a_0 and λ are determined (by 62 and 64), the values of the parameters S/R , T_x/R , τ_y/R and RM are obtained directly from (60).

The above discussion suggests a rationale for the computational procedure using, say successive approximations, as defined in the proof of the existence theorem. An alternative computational procedure can be based on power series expansion of ϕ^* , θ^* and λ in terms of a perturbation parameter ϵ , where

$$M^* = \frac{F\epsilon}{1-\epsilon} \quad (65)$$

and F is a convenient unit (for M^*). The rationale for this procedure stems from the fact that the solution functions ϕ^* and θ^* of the system 51-64 as well as λ , are analytic in the parameter M^* (near any particular solution set), provided only that the condition

$$\frac{\partial}{\partial \lambda} \left(\frac{d\theta^*(0)}{d\eta} \right) \neq 0 \quad (66)$$

is satisfied. This situation is an instance of the implicit-function theorem for which 66 insures that λ , determined by 64 so also are ϕ^* and θ^* . The circumstance $\frac{\partial}{\partial \lambda} \left(\frac{d\theta^*(0)}{d\eta} \right) = 0$ is not to be expected in the general case (i.e., except for very special parameter values) and could be checked computationally in any event. So by 65 the solution functions ϕ^* , θ^* and the eigen value

λ can be expected to be analytic in ϵ .

Equation 65 is an instance of the well-known Euler transformation which is employed to accelerate the rate of convergence of power series expansions or even institute convergence for divergent (formal) power series. It is commonly referred to as the Euler summability method. In the simplest case for example, if some function f of the complex variable $z = M^*$ is analytic for $\text{Re}(z) > -F/2$, then f has a power series expansion about $z = 0$, say

$$f(z) = \sum_{n \geq 0} a_n z^n, \quad (67)$$

which may have a finite radius of convergence δ determined by the nearest complex singularity and perhaps as small as $F/2$). If this be the case, the expansion 67 can not be used to compute values of f except for $|z| < \delta$. On the other hand, if f is considered a function of the transformed variable ϵ , with $z = F\epsilon/(1-\epsilon)$ as in 65, then f has a power series expansion

$$f\left(\frac{F\epsilon}{1-\epsilon}\right) = \sum_{n \geq 0} b_n \epsilon^n, \quad (68)$$

(the b_n can be obtained from the a in 67) which is valid for $|\epsilon| < 1$. The region $|\epsilon| < 1$ corresponds precisely to the region $\text{Re}(z) > -F/2$ and the expansion 68 can be used to compute the values of f for all such z . In

particular, the expansion 68 in ϵ can be used to compute (indirectly) the values of f for all real $z = M^*$, even though the expansion 67 can be used to compute (directly) the values of f for real values of $z = M^*$ only if $|z| < \delta$. Applying these considerations to the solution functions ϕ^* , θ^* and eigen value λ (which can be expected to be analytic in M^*) suggests that we might consider expansions

$$\begin{aligned}\phi^* &= \sum_{n \geq 0} \phi_n^* \epsilon^n = \phi_0^* + \phi_1^* \epsilon + \phi_2^* \epsilon^2 + \dots \\ \theta^* &= \sum_{n \geq 0} \theta_n^* \epsilon^n = \theta_0^* + \theta_1^* \epsilon + \theta_2^* \epsilon^2 + \dots\end{aligned}\quad (69)$$

$$\lambda = \sum_{n \geq 0} \lambda_n \epsilon^n = \lambda_0 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \dots$$

together with 81, and substitute them into the system 51-60 and then solve (successively) the resulting infinite set of linear boundary-value problems determined by the order of ϵ . The zeroth order system becomes uncoupled with θ_0^* determined by

$$(I^2 + \alpha^2 \eta^2) \frac{d^2 \theta_0^*}{d\eta^2} + \alpha \eta (3\lambda_0 - 1) \frac{d\theta_0^*}{d\eta} + \lambda_0 (2\lambda_0 - \alpha - 1) \theta_0^* = 0$$

$$\theta_0^*(1) = 1, \quad \frac{d\theta_0^*}{d\eta}(1) = -\frac{\alpha \lambda_0}{I^2 + \alpha^2}, \quad \frac{d\theta_0^*}{d\eta}(0) = 0. \quad (70)$$

It is clear that for $|\lambda_0|$ sufficiently large, the solution function $\theta_0^*(\eta)$ is oscillatory and so this boundary value problem is solvable. The case $\alpha=0$ is

fairly typical and for which we have

$$\theta_o^*(\eta) = \cos\left(\frac{2\lambda_o^2 - \lambda_o}{I}\right)(\eta-1)$$

and

$$\frac{d\theta_o^*(0)}{d\eta} = \frac{2\lambda_o^2 - \lambda_o}{I} \sin \frac{2\lambda_o^2 - \lambda_o}{I},$$

Hence

$$2\lambda_o^2 - \lambda_o = \pi I, \text{ i.e., } \lambda_o = \frac{1}{4}(1 + 1 + 8\pi^2 I^2).$$

In any case, $\theta_o^*(\eta)$ can be expanded as a power series

$$\sum_{\eta \geq 0} c_\eta (\eta-1)^\eta, \text{ convergent for } |\eta-1| \leq 1 + (I/\alpha)^2, c_o = 1,$$

$$c_1 = \alpha \lambda_o / (I^2 + \alpha^2) \text{ and}$$

$$-c_{\eta+2} = \frac{(2\alpha^2 \eta(\eta+1) + \alpha(3\lambda_o - 1)(\eta+1))c_{\eta+1} + (\alpha^2 \eta(\eta-1) + \alpha(3\lambda_o - 1)\eta)c_\eta}{(\eta+2)(\eta+1)(I^2 + \alpha^2)}$$

for $\eta \geq 0$. In particular, this series representation of $\theta_o^*(\eta)$, and similarly of $\frac{d\theta_o^*(\eta)}{d\eta}$, is valid for $\eta = 0$

and can be used to determine λ_o . With $\theta_o^*(\eta)$ and λ_o determined by (69), $\phi_o^*(\eta)$ is then obtained from the linear system

$$\frac{d^5 \theta_o^*}{d\eta^5} + T \frac{d\phi_o^*}{d\eta} = -((\alpha + \lambda_o) \frac{d\theta_o^*}{d\eta} + \alpha \eta \frac{d^2 \theta_o^*}{d\eta^2}) + S^* \quad (71)$$

$$\phi_o^*(1) = 0, \frac{d\phi_o^*}{d\eta}(1) = 0, \phi_o^*(0), \frac{d^2\phi_o^*}{d\eta^2}(0) = -T_x,$$

$$\frac{d^4\phi_o^*}{d\eta^4}(0) = T_y + \lambda_o \theta_o^*(0).$$

With ϕ_o^* , θ_o^* and λ_o determined, similar linear boundary value problems are then solved for ϕ_1^* , θ_1^* and λ_1 and the process is continued. Such a computational procedure is certainly valid for small $|\epsilon|$ and can be expected to hold for other values of $|\epsilon|$, as explained above, for any reasonable ranges of parameter values.

Thus, an overlooked, powerful perturbation method has been shown to have both mathematical validity as well as physical application to some problems in the marine environment.

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